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Positive (p, n)-intermediate scalar curvature and cobordism

Matthew Burkemper [a], Catherine Searle [a,*], Mark Walsh [b]

[a] Department of Mathematics, Statistics, and Physics, Wichita State University, Wichita, KS, United States of America
[b] Department of Mathematics and Statistics, Maynooth University, Ireland

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ABSTRACT

In this paper we consider a well-known construction due to Gromov and Lawson, Schoen and Yau, Gajer, and Walsh which allows for the extension of a metric of positive scalar curvature over the trace of a surgery in codimension at least 3 to a metric of positive scalar curvature which is a product near the boundary. We extend this construction for (p, n)-intermediate scalar curvature for $0 \leqslant p \leqslant n-2$ for surgeries in codimension at least $p+3$. We then use it to generalize a well known theorem of Carr. Letting $\mathcal{R}^{s_{p,n}>0}(M)$ denote the space of positive (p, n)-intermediate scalar curvature metrics on an n-manifold M, we show for $0 \leqslant p \leqslant 2n-3$ and $n \geqslant 2$, that for a closed, spin, $(4n-1)$-manifold M admitting a metric of positive $(p, 4n-1)$-intermediate scalar curvature, $\mathcal{R}^{s_{p,4n-1}>0}(M)$ has infinitely many path components.

1. Introduction

A central question in modern geometry is the following: given a smooth n-dimensional manifold M and a preferred curvature condition C, can we specify a Riemannian metric g on M so that g satisfies C? The scalar curvature is the weakest invariant of the curvature tensor and so, unsurprisingly, the greatest success has been achieved in classifying which manifolds admit metrics of positive scalar curvature (psc-metrics). By contrast, the problem of classifying which manifolds admit metrics whose sectional curvature is strictly positive, or finding new examples of manifolds admitting such metrics, is formidably hard.

Such classification problems require obstructive tools for ruling out certain manifolds from consideration and constructive tools for building metrics in the case where no obstructions exist. Despite its relative weakness as a curvature constraint, there exist many smooth manifolds which do not admit metrics of positive scalar curvature. For example, work of Schrödinger and Lichnerowicz [23] and Hitchin [18], shows that if M is a closed, spin manifold of dimension n admitting a psc-metric, a certain invariant, $\alpha(M) \in KO^{-n}(pt)$, representing the index of the Dirac operator and generalizing the $\hat{A}$-genus, must vanish. This obstructive tool is complemented by a powerful constructive result due to Gromov and Lawson [16] and Schoen and Yau [27]: suppose M and M' are smooth manifolds and M' is obtained from M via surgery in codimension at least three, then any psc-metric g on M can be used to construct a psc-metric g' on M'. Combining these respective obstructive and constructive tools led to considerable progress in classifying which manifolds admitted psc-metrics. In particular, Stolz [28] showed that a closed, smooth, simply connected manifold M admits a psc-metric if and only if M is either non-spin, or M is spin and $\alpha(M) = 0$.

* Corresponding author.
E-mail addresses: matthew.burkemper@wichita.edu (M. Burkemper), catherine.searle@wichita.edu (C. Searle), Mark.Walsh@mu.ie (M. Walsh).

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The surgery technique breaks down when one attempts to apply it to stronger notions like positive Ricci or sectional curvatures. Indeed, not only does the construction not work for positive Ricci or positive sectional curvature, there are topological obstructions to the existence of metrics of positive Ricci curvature, and hence positive sectional curvature, on manifolds arising from surgeries which are admissible in the positive scalar curvature setting.

It is thus natural to consider intermediate curvatures interpolating between scalar and Ricci, and scalar and sectional curvatures. These intermediate curvatures are the k-positive Ricci curvature, where $k \in \{1, 2, \cdots, n\}$, defined by Wolfson in [34], and the (p, n)-intermediate scalar curvatures (originally, the p-curvature), where $p \in \{0, 1, \cdots, n-2\}$ defined by Labbi in [21]; see Definition 2.1. Analogs of the Surgery Theorem for these respective curvatures have been proven in [34] and [21]. In particular, it is shown in [21] that if M and M' are smooth n-dimensional manifolds and M' is obtained from M via surgery in codimension at least $p+3$, then any metric, g, on M admitting positive (p, n)-intermediate scalar curvature can be used to construct a metric, g', on M' which also admits positive (p, n)-intermediate scalar curvature. This is a generalization of the Surgery Theorem of [16] and [27] stated above, where the scalar curvature corresponds to the case when $p = 0$.

In this paper, we focus our attention on the (p, n)-intermediate scalar curvature. Our main result, Theorem A, concerns an analogous strengthening of a more general positive scalar curvature construction. The surgery technique of [16] and [27] can be enhanced to give rise to a psc-metric on the trace of a surgery in codimension $\geqslant 3$ by work of Gajer [14] and Walsh [30]. In particular, the resulting metric satisfies a Riemannian product structure on a collar neighborhood of the boundary. In Theorem A, we extend the result of [14] and [30] from the case when $p = 0$ to more general p as follows.

Theorem A. *Let M be a smooth n-manifold, $\phi : S^k \times D^{n-k} \to M$, a smooth embedding, and $\{\bar{M}_\phi; M, M_\phi\}$, the trace of the surgery on ϕ. Suppose furthermore that $n-k \geqslant 3$ and $p \in \{0, 1, \cdots, n-k-3\}$. Then for any metric g on M with positive (p, n)-intermediate scalar curvature, there are metrics g_ϕ on M_ϕ and $\bar{g}_\phi$ on $\bar{M}_\phi$ satisfying:*

1. *The metrics g_ϕ and $\bar{g}_\phi$ have respectively positive (p, n) and $(p, n+1)$-intermediate scalar curvatures on M_ϕ and $\bar{M}_\phi$; and*
2. *Near the boundary components M and M_ϕ, $\bar{g}_\phi$ takes the form of the respective product metrics $\bar{g}_\phi = g + dt^2$ and $\bar{g}_\phi = g_\phi + dt^2$.*

An important application of this theorem is in exhibiting non-triviality in the topology of the space of Riemannian metrics of positive (p, n)-intermediate scalar curvature on a smooth manifold. We denote by $\mathcal{R}(M)$, the space of all Riemannian metrics on the smooth manifold M. This space has a standard C^∞ topology; see Chapter 1 of Tuschmann and Wraith [29] for the specific construction. For each $p \in \{0, 1, \cdots n-2\}$, we consider the subspace $\mathcal{R}^{s_{p,n}>0}(M)$ of Riemannian metrics of positive (p, n)-intermediate scalar curvature on M. In the case when $p = 0$, this is precisely $\mathcal{R}^{s>0}(M)$, the space of psc-metrics on M.

More generally, one may consider, for any curvature condition C, the subspace $\mathcal{R}^C(M) \subset \mathcal{R}(M)$ of Riemannian metrics which satisfy C. In recent years, there has been substantial interest in understanding the topology of the space, $\mathcal{R}^C(M) \subset \mathcal{R}(M)$, for a variety of manifolds M and curvature conditions C. Much of this has also involved the corresponding moduli spaces obtained as a quotient of $\mathcal{R}^C(M)$ by the action of appropriate subgroups of the group of self-diffeomorphisms of M, Diff(M). Recall that Diff(M) acts on $\mathcal{R}^C(M)$ by means of pulling back metrics. The most progress has occurred in the case when C denotes positive scalar curvature; see for example results due to Botvinnik, Ebert, and Randall-Williams [2], Botvinnik, Hanke, Schick, and Walsh [3], Coda-Marquez [6], Crowley and Schick [7], Ebert and Randall-Williams [9] and [10], Ebert and Wiemeler [11], Frenck [12], Hanke, Schick, and Steimle [17], and Walsh [33]. There are numerous results for other curvature conditions such as negative sectional curvature or positive Ricci curvature, see, for example, [29].

Theorem A can be used to exhibit non-triviality in the topology of this space for many manifolds and many $p > 0$ by extending existing results for positive scalar curvature, that is, when $p = 0$. We will not provide a comprehensive account of this here but rather, in Theorem B, an example which illustrates this point.

Theorem B. *Let M be a smooth closed spin manifold of dimension $4n-1$, $n \geqslant 2$, which admits an $s_{p,4n-1} > 0$ curvature metric for some $p \in \{0, 1, \cdots, 2n-3\}$. Then the space $\mathcal{R}^{s_{p,4n-1}>0}(M)$ has infinitely many path components.*

Theorem B generalizes Theorem 4 of Carr [5], for positive scalar curvature. In particular, Theorem 4 of [5] is the $p = 0$ case of Theorem B when $M = S^{4n-1}$ for $n \geqslant 2$. Note that extending the theorem from the case when $M = S^{4n-1}$ to an arbitrary closed, simply-connected, spin manifold admitting psc-metrics is not difficult. Indeed, it follows as an immediate corollary of the main theorem of [11]. The main work of the proof is in dealing with the spherical case.

Note that Theorem 4 of [5] was generalized to positive Ricci curvature by Wraith in [35] for M, a homotopy $(4n-1)$-sphere bounding a parallelizable manifold, showing that the moduli space of $\mathcal{R}^{\text{Ric}>0}(M)$ contains infinitely many path-components. However, there is no clear implication relating positive (p, n)-intermediate scalar curvature to positive Ricci curvature. In fact, the methods used to prove Theorem B are quite different from those used in [35], where the Kreck-Stolz s-invariant is used to distinguish between the path components of the moduli space of $\mathcal{R}^{\text{Ric}>0}(M)$.

Finally, using our Theorem A, we obtain the following generalization of results in [5] and in Mantione and Torres [24]. The $p = 0$ case of this result is Corollary 2 of [5] in the case of orientable manifolds, and forms Theorems 6 and 7 of [24] for non-orientable manifolds with an added assumption on the group G. The principal constructive technique behind these results is Theorem 3 of [5], which our Theorem A generalizes from the $p = 0$ case.

Corollary C. *For any finitely presented group G, there is a closed, smooth, orientable Riemannian n-manifold* $(M^n(G), g)$ *such that* $\pi_1(M^n(G)) = G$, *and g has positive* (p, n)*-intermediate scalar curvature provided* $0 \leqslant p \leqslant n-4$. *The same result holds for non-orientable* (M^n, g), *provided G contains a subgroup of order two.*

During the writing of this paper we discovered that Theorem B also follows as a case of a recent theorem of Frenck and Kordass [13]. Theorems A and B of [13] extend some powerful techniques of [2] for positive scalar curvature to the cases of positive (p, n)-intermediate scalar curvature and positive k-Ricci curvature. Our work has independent value, as Theorem A above provides a geometrically explicit construction of an $(s_{p,n} > 0)$-metric over the trace of an appropriate surgery, something which is not done in [13]. In doing so, we also provide detailed curvature calculations for sectional and intermediate scalar curvature of various warped product metrics that are of value in their own right. Likewise, our proof of Theorem B differs from theirs in that ours gives an explicit construction for representative elements of distinct path components of $\mathcal{R}^{s_{p,4n-1}>0}(M)$.

1.1. Organization

The paper is organized as follows. In Section 2 we establish preliminaries. In Section 3 we establish an isotopy-concordance result for positive (p, n)-intermediate scalar curvature. In Section 4, we determine the intermediate scalar curvature of a warped product metric. In Section 5, we apply these calculations to the standard metrics on the sphere and the disk. In Section 6, we prove Theorem A and in Section 7, we prove Theorem B.

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2. Preliminaries

2.1. (p, n)-Intermediate Scalar Curvature

We consider a generalization in [21] of the sectional and scalar curvatures which we call the (p, n)-intermediate scalar curvature and denote $s_{p,n}$. This was originally referred to as the p-curvature, s_p. However, we will often deal with cylinders and more general manifolds with boundary, where dimensions n and $n-1$ arise in tandem. Thus, we adopt this term to aid the reader in distinguishing between the p-curvature on the ambient n-manifold and the p-curvature on an embedded submanifold or boundary component of dimension $n-1$.

Let M be a smooth n-dimensional manifold, possibly with non-empty boundary. For any $x \in M$, $k \in \{0, 1, \cdots, n\}$, we denote by $\mathrm{Gr}_k(T_xM)$, the Grassmann manifold of k-dimensional subspaces of the tangent space T_xM and by $\mathrm{Gr}_k(M)$, the corresponding Grassmann bundle of k-dimensional subspaces obtained as the union of $\mathrm{Gr}_k(T_xM)$ over $x \in M$. We now define the (p, n)-*intermediate scalar curvature* of a Riemannian metric on M as follows.

Definition 2.1. Let (M, g) be an n-dimensional Riemannian manifold, with possibly non-empty boundary, and let $p \in \{0, 1, \cdots, n-2\}$. The (p, n)-*intermediate scalar curvature* of M is the function $s_{p,n} : \mathrm{Gr}_p(M) \to \mathbb{R}$ defined for $x \in M$, P a p-plane in T_xM and $\{e_1, \cdots, e_{n-p}\}$, an orthonormal basis of the orthogonal complement $P^\perp$ of P in T_xM, by

$$s_{p,n}(x, P) := \sum_{i,j} K_x(e_i, e_j),$$

where $K_x(e_i, e_j)$ is the sectional curvature at x of the subspace of T_xM spanned by the vectors e_i and e_j.

It follows that $s_{p,n}(x, P)$ is the scalar curvature at x of the locally specified $(n-p)$-dimensional submanifold of M given by restricting the exponential map of g at x to the subspace $P^\perp \subset T_xM$. In particular, it is well defined for any choice of orthonormal basis $\{e_1, \cdots, e_{n-p}\}$ for $P^\perp$. When $p = 0$, $P^\perp = T_xM$ and so $s_{0,n}(x) := s_{0,n}(x, \bar{0})$ is precisely the scalar curvature of the Riemannian manifold (M, g) at the point x. When $\dim P = p = n-2$, $s_{n-2,n}(x, P)$ is twice the sectional curvature at x of the plane $P^\perp \subset T_xM$ with respect to (M, g). The (p, n)-intermediate scalar curvatures for $0 \leqslant p \leqslant n-2$ are therefore a collection of curvatures interpolating between the scalar curvature, when $p = 0$, and twice the sectional curvature when $p = n-2$. For any given value of $p < n-2$, the (p, n)-intermediate scalar curvature is a trace of the $(p+1, n)$-intermediate scalar curvature.

Of particular interest is the case when the (p,n)-intermediate scalar curvature is positive on a manifold. We say that a Riemannian metric g on M is a *metric of positive (p,n)-intermediate scalar curvature*, that is, an *$(s_{p,n} > 0)$-metric*, if for any $x \in M$ and any $P \subset T_xM$, $s_{p,n}(x,P) > 0$. It is obvious that if the (p,n)-intermediate scalar curvature of (M,g) is positive for $p > 0$, then the $(p-1,n)$-intermediate scalar curvature is positive as well. By continuing to take traces, it is clear that this holds for any integer $0 \leqslant q \leqslant p$. We summarize this hierarchy in the following proposition.

Proposition 2.2. *If a Riemannian manifold (M,g) has positive (p,n)-intermediate scalar curvature, then it has positive (q,n)-intermediate scalar curvature for $0 \leqslant q \leqslant p$.*

Note that the converse is not true. For any dimension n and any $0 \leqslant p < n-2$, we can construct a Riemannian manifold that has positive (p,n)-intermediate scalar curvature, but not positive $(p+1,n)$-intermediate scalar curvature. In fact we only need to look at products of spheres.

Example 2.3. Let M be the n-dimensional Riemannian product manifold of m standard round spheres of radius one and of dimension at least one. Then M has positive (p,n)-intermediate scalar curvature if and only if $p < n-m$.

This example can be extended to include other factors with positive sectional curvature.

2.2. Isotopy and Concordance

Various notions of isotopy and concordance arise throughout Mathematics. Here, we are only concerned with metrics of positive (p,n)-intermediate scalar curvature and we define these notions in this case.

Definition 2.4. Two metrics g_0, g_1 on an n-dimensional manifold M with positive (p,n)-intermediate scalar curvature are said to be *$(s_{p,n} > 0)$-isotopic* if they are connected by a path $t \mapsto g_t$ in the space of positive (p,n)-intermediate scalar curvature metrics on M, $t \in [0,1]$. The connecting path is called an *$(s_{p,n} > 0)$-isotopy.*

Definition 2.5. The metrics g_0 and g_1 on M are said to be *$(s_{p,n} > 0)$-concordant* if, for some $L > 0$, there is a metric $\bar{g}$ on the cylinder $M \times [0, L+2]$, of positive $(p,n+1)$-curvature, and satisfying

$$\bar{g}|_{M\times[0,1]} = g_0 + dt^2 \quad \text{and} \quad \bar{g}|_{M\times[L+1,L+2]} = g_1 + dt^2.$$

The metric $\bar{g}$ is known as an *$(s_{p,n} > 0)$-concordance.*

We will frequently shorten $(s_{p,n} > 0)$-isotopy and $(s_{p,n} > 0)$-concordance to just isotopy and concordance. It is straightforward to show that both isotopy and concordance determine equivalence relations on the space of positive (p,n)-intermediate scalar curvature metrics on the manifold.

The problem of whether or not a given pair of concordant metrics are in turn isotopic is notoriously difficult and we do not consider it here. The converse problem however is much more tractable. It has long been known in the case of metrics of positive scalar curvature that isotopic metrics are concordant. This indeed holds more generally, as we demonstrate in Proposition 3.3 below.

3. Isotopy Implies Concordance

We start with an isotopy g_r on M. To create a concordance from this isotopy, it seems natural to turn this into the metric $g_r + dr^2$ on $M \times [0,1]$. However, this metric does not necessarily have positive $(p,n+1)$-intermediate scalar curvature since, even though the metric g_r on the slice $M \times \{r\}$ has positive curvature, there may be negative curvature coming from the r direction. Therefore we will introduce a function $f : \mathbb{R} \to [0,1]$ and consider a new metric $g_{f(t)} + dt^2$ on $M \times \mathbb{R}$. This function will allow us to control the changes in the t-direction so we can minimize the contributions of negative curvature while keeping the positive contributions from the metric $g_{f(t)}$ on the slice $M \times \{f(t)\}$.

We begin with a lemma about the Riemann curvature of such a metric.

Lemma 3.1. *Let M be an n-dimensional manifold, g_r with $r \in [0,1]$ a smooth path of metrics on M, and $f : \mathbb{R} \to [0,1]$ a smooth function. Define the metric $\bar{g} = g_{f(t)} + dt^2$ on $M \times \mathbb{R}$. If $(x_0, t_0) \in M \times R$ has local coordinates $(x_1, \dots, x_n, t)$ where $(x_1, \dots, x_n)$ are normal coordinates for x_0 with respect to the metric $g_{f(t_0)}$ on M, then the Riemannian curvatures $\bar{R}$ of $\bar{g}$ at (x_0, t_0) satisfy*

$$\bar{R}^{\ell}_{ijk} = R^{\ell}_{ijk} + O(|f'|^2),\ \bar{R}^{t}_{ijk} = O(|f'|),\ \text{and}\ \bar{R}^{t}_{itk} = O(|f'|^2) + O(|f''|).$$

In particular, the sectional curvatures $\bar{K}$ of $\bar{g}$ are then given by

$$\bar{K}_{ij} = K_{ij} + O(|f'|^2)\ \text{and}\ \bar{K}_{it} = O(|f''|) + O(|f'|^2).$$

Proof. Denote the coordinate vector fields by $\partial_1, \dots, \partial_n, \partial_t$. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of $g_{f(t_0)}$ and $\bar{g}$ respectively and let $\bar{\Gamma}^k_{ij}$ and Γ^k_{ij} be the Christoffel symbols of $\bar{\nabla}$ and ∇, respectively, at (x_0, t_0). We denote the Riemannian and sectional curvatures, respectively, by $\bar{R}^\ell_{ijk}$ and $\bar{K}_{ij}$ for $\bar{g}$ and R^ℓ_{ijk} and K_{ij} for $g_{f(t_0)}$.

Since $\bar{K}_{ij} = \sum_\ell \bar{g}_{i\ell}\bar{R}^\ell_{ijj}$, and $\bar{K}_{it} = -\bar{g}_{tt}\bar{R}^t_{iti} - \sum_{\ell\neq t}\bar{g}_{\ell t}\bar{R}^\ell_{iti} = -\bar{R}^t_{iti}$, the results for the sectional curvatures follow immediately.

Since $\bar{g}_{ij} = (g_{f(t_0)})_{ij}$, we have $\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}$ for $i, j, k \in \{1, \dots, n\}$. Moreover, we have the constant values $\bar{g}_{it} = 0$ for all $i \in \{1, \dots, n\}$ and $\bar{g}_{tt} = \bar{g}^{tt} = 1$. Therefore

$$\bar{\Gamma}^t_{it} = \bar{\Gamma}^k_{tt} = \bar{\Gamma}^t_{tt} = 0,$$
$$\bar{\Gamma}^t_{ij} = -\frac{1}{2}\partial_t\bar{g}_{ij}, \text{ and } \bar{\Gamma}^k_{it} = \frac{1}{2}\sum_\ell \bar{g}^{k\ell}\partial_t\bar{g}_{i\ell}.$$

Setting $r = f(t)$, we obtain

$$\partial_t\bar{g}_{ij} = \frac{\partial(g_r)_{ij}}{\partial r}f'(t),$$

and using normal coordinates at the point (x_0, t_0) this gives us that

$$\bar{\Gamma}^t_{ij} = -\frac{1}{2}\left[\frac{\partial(g_r)_{ij}}{\partial r}\right]_{(x_0, f(t_0))} f'(t_0) = O(|f'|),$$
$$\bar{\Gamma}^k_{it} = \frac{1}{2}\left[\frac{\partial(g_r)_{ik}}{\partial r}\right]_{(x_0, f(t_0))} f'(t_0) = O(|f'|).$$

We now calculate the Riemannian curvatures. For $i, j, k, \ell \in \{1, \dots, n\}$, since $\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}$ and m varies over $1, \dots, n$ and t, we have

$$\begin{aligned}
\bar{R}^\ell_{ijk} &= \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \sum_{m\neq t}(\Gamma^\ell_{im}\Gamma^m_{jk} - \Gamma^\ell_{jm}\Gamma^m_{ik}) + \bar{\Gamma}^\ell_{it}\bar{\Gamma}^t_{jk} - \bar{\Gamma}^\ell_{jt}\bar{\Gamma}^t_{ik} \\
&= R^\ell_{ijk} - \frac{1}{4}\left[\frac{\partial(g_r)_{i\ell}}{\partial r}\frac{\partial(g_r)_{jk}}{\partial r} - \frac{\partial(g_r)_{j\ell}}{\partial r}\frac{\partial(g_r)_{ik}}{\partial r}\right]_{(x_0, f(t_0))}(f'(t_0))^2 \\
&= R^\ell_{ijk} + O(|f'|^2).
\end{aligned}$$

For the Riemannian curvatures involving t, we only need to compute $\bar{R}^t_{ijk}$ and $\bar{R}^t_{itk}$ for $i, j, k \in \{1, \dots, n\}$. First,

$$\begin{aligned}
\bar{R}^t_{ijk} &= \partial_i\bar{\Gamma}^t_{jk} - \partial_j\bar{\Gamma}^t_{ik} + \sum_{m\neq t}(\bar{\Gamma}^t_{im}\bar{\Gamma}^m_{jk} - \bar{\Gamma}^t_{jm}\bar{\Gamma}^m_{ik}) + \bar{\Gamma}^t_{it}\bar{\Gamma}^t_{jk} - \bar{\Gamma}^t_{jt}\bar{\Gamma}^t_{ik} \\
&= -\frac{1}{2}\left[\frac{\partial^2(g_r)_{jk}}{\partial x_i\partial r} - \frac{\partial^2(g_r)_{ik}}{\partial x_j\partial r}\right]_{(x_0, f(t_0))} f'(t_0) \\
&= O(|f'|).
\end{aligned}$$

To calculate $\bar{R}^t_{itk}$ at the point (x_0, t_0), a straightforward calculation gives us

$$\partial_t\bar{\Gamma}^t_{ik} = -\frac{1}{2}\left[\frac{\partial^2(g_r)_{ik}}{\partial r^2}\right]_{(x_0, f(t_0))}(f'(t_0))^2 - \frac{1}{2}\left[\frac{\partial(g_r)_{ik}}{\partial r}\right]_{(x_0, f(t_0))} f''(t_0).$$

Therefore, we have

$$\begin{aligned}
\bar{R}^t_{itk} &= \frac{1}{2}\left[\frac{\partial(g_r)_{ik}}{\partial r}\right]_{(x_0, f(t_0))} f''(t_0) \quad + \left[\frac{1}{2}\frac{\partial^2(g_r)_{ik}}{\partial r^2} - \frac{1}{4}\sum_{m\neq t}\frac{\partial(g_r)_{im}}{\partial r}\frac{\partial(g_r)_{km}}{\partial r}\right]_{(x_0, f(t_0))}(f'(t_0))^2 \\
&= O(|f''|) + O(|f'|^2). \quad \square
\end{aligned}$$

We are now ready to estimate the $(p, n+1)$-intermediate scalar curvature on $(M \times \mathbb{R}, \bar{g})$.

Lemma 3.2. *Let M be a compact n-dimensional manifold and g_r with $r \in [0, 1]$ a smooth path of $(s_{p,n} > 0)$-metrics on M. Then there exists a positive constant $C \leqslant 1$ so that for every smooth function $f : \mathbb{R} \to [0, 1]$ with $|f'|, |f''| \leqslant C$ the metric $\bar{g} = g_{f(t)} + dt^2$ on $M \times \mathbb{R}$ has positive $(p, n+1)$-intermediate scalar curvature, $0 \leqslant p \leqslant n-2$.*

Proof. Choose a point $z_0 = (x_0, t_0) \in M \times \mathbb{R}$. We use $\bar{R}$, $\bar{K}$, $\bar{s}_{p,n}$ to denote Riemann, sectional, and intermediate scalar curvatures for the metric $\bar{g}$ and R, K, $s_{p,n}$ for those curvatures in the metric $g_{f(t_0)}$. Fix a value $0 \leqslant p \leqslant n-2$.

Since M is compact, the Grassmannian $\mathrm{Gr}_p(M)$ is also compact. By the continuity of the curvature, this means for each r there is a positive lower bound for the (p,n)-intermediate scalar curvature of the metric g_r. As the r vary over the compact interval $[0,1]$, we can choose a positive lower bound B_p for the (p,n)-intermediate scalar curvatures of all the metrics and thus a positive lower bound B_{p-1} for the $(p-1,n)$-intermediate scalar curvatures of the metrics g_r as well.

Let P be a p-plane in $T_{z_0}(M \times \mathbb{R})$ so that $P^\perp$ is $(n-p+1)$-dimensional. We denote the hyperplane $T_{z_0}(M \times \{t_0\})$ by T. Consider the intersection $Q = P^\perp \cap T$. If $P^\perp$ is properly contained in T then $Q = P^\perp$. If $P^\perp$ is not properly contained in T, then the span of both must be the entire $(n+1)$-dimensional $T_{z_0}(M \times \mathbb{R})$, and so by Grassmann's identity we have

$$\begin{aligned}\dim(Q) &= \dim(P^\perp \cap T) = \dim(P^\perp) + \dim(T) - \dim(P^\perp + T)\\ &= (n-p+1) + n - (n+1) = n-p.\end{aligned}$$

This leads to three cases: Case 1, where $P^\perp$ is contained in T, Case 2, where P is contained in T, and Case 3, where neither P nor $P^\perp$ is contained in T.

Case 1: $P^\perp$ is contained in T.

In this case, we can take an orthonormal basis $\{e_1, \ldots, e_{n-p+1}\}$ for $P^\perp$ and extend it to orthonormal bases $\{e_1, \ldots, e_n\}$ for T and $\{e_1, \ldots, e_n, e_t\}$ for $T_{z_0}(M \times \mathbb{R})$. Using the exponential map for $g_{f(t_0)}$, we get local coordinates $(x_1, \ldots, x_n, x_t)$ at z_0 where $(x_1, \ldots, x_n)$ are normal coordinates at x_0. Then, using Lemma 3.1, we compute

$$\begin{aligned}\bar{s}_{p,n+1}(P) &= \sum_{i,j=1}^{n-p+1} \bar{K}_{ij}\\ &= \sum_{i,j=1}^{n-p+1} \left[K_{ij} + O(|f'|^2)\right]\\ &= s_{p-1,n}(P \cap T) + O(|f'|^2).\end{aligned}$$

As $s_{p-1,n}(P \cap T) \geqslant B_{p-1} > 0$, then for a small enough value of C, we can force the contributions of f to not be too negative and allow $\bar{s}_{p,n+1}(P)$ to remain positive.

Case 2: P is contained in T.

If $P \subseteq T$ then $T^\perp \subseteq P^\perp$, and since $T^\perp$ is 1-dimensional, let e_t be a unit vector spanning $T^\perp$. Then taking an orthonormal basis $\{e_1, \ldots, e_{n-p}\}$ for $Q = P^\perp \cap T$, we have an orthonormal basis $\{e_1, \ldots, e_{n-p}, e_t\}$ for $P^\perp$. We can extend this to orthonormal bases $\{e_1, \ldots, e_n\}$ for T and $\{e_1, \ldots, e_n, e_t\}$ for $T_{z_0}(M \times \mathbb{R})$ and again we get local coordinates $(x_1, \ldots, x_n, x_t)$ at z_0 where $(x_1, \ldots, x_n)$ are normal coordinates at x_0. Then we compute using Lemma 3.1

$$\begin{aligned}\bar{s}_{p,n+1}(P) &= \sum_{i,j=1}^{n-p} \bar{K}_{ij} + 2\sum_{i=1}^{n-p} \bar{K}_{it}\\ &= \sum_{i,j=1}^{n-p} \left[K_{ij} + O(|f'|^2)\right] + 2\sum_{i=1}^{n-p}\left[O(|f'|^2) + O(|f''|)\right]\\ &= s_{p,n}(P) + O(|f'|^2) + O(|f''|).\end{aligned}$$

As $s_{p,n}(P) \geqslant B_p > 0$, then for a small enough value of C, we can again force the contributions of f to allow $\bar{s}_{p,n+1}(P)$ to remain positive.

Case 3: Neither P nor $P^\perp$ is contained in T.

In this case $Q = P^\perp \cap T$ is $(n-p)$-dimensional. We depict this situation in Fig. 3.1. Take $\{e_1, \ldots, e_{n-p}\}$ to be an orthonormal basis for Q, and choose $v \in P^\perp \setminus Q$ so that $\{e_1, \ldots, e_{n-p}, v\}$ is an orthonormal basis for $P^\perp$. Then obviously v is not in T, but it also is not perpendicular to T, or we would have $P \subseteq T$. Therefore v has a nonzero projection onto T, which we denote by αe_{n-p+1}, where e_{n-p+1} is a unit vector. Putting this together with the previously defined e_i, we extend to orthonormal bases $\{e_1, \ldots, e_n\}$ for T and $\{e_1, \ldots, e_n, e_t\}$ for $T_{z_0}(M \times \mathbb{R})$ and again we get local coordinates $(x_1, \ldots, x_n, x_t)$ at z_0 where $(x_1, \ldots, x_n)$ are normal coordinates at x_0. Note that by construction, we have $v = \alpha e_{n-p+1} + \beta e_t$, as the projection of v onto T is a multiple of e_{n-p+1}.

We compute the curvatures for $1 \leqslant i \leqslant n-p$ using Lemma 3.1

$$\begin{aligned}\bar{K}(e_i, v) &= \bar{g}(\bar{R}(v, e_i)e_i, v)\\ &= \bar{g}(\bar{R}(\alpha e_{n-p+1} + \beta e_t, e_i)e_i, \alpha e_{n-p+1} + \beta e_t)\end{aligned}$$

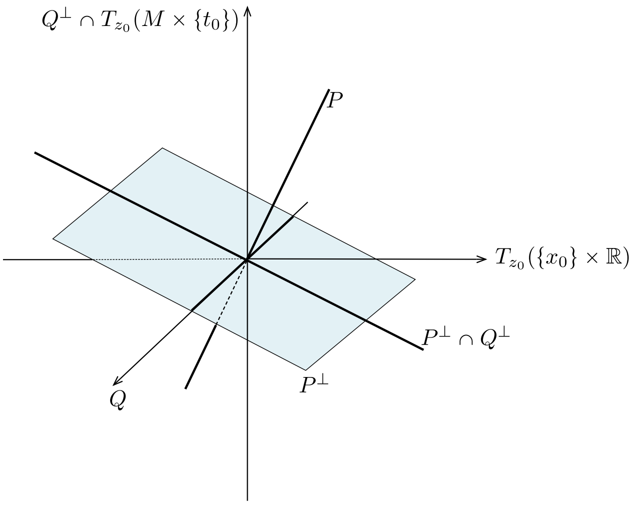

Fig. 3.1. The arrangement of the subspace $P \subset T_{z_0}(M \times \mathbb{R})$ in Case 3.

$$
\begin{aligned}
&= \alpha^2 \bar{R}^{n-p+1}_{(n-p+1)ii} + 2\alpha\beta \bar{R}^{t}_{(n-p+1)ii} + \beta^2 \bar{R}^t_{tii} \\
&= \alpha^2 \big[R^{n-p+1}_{(n-p+1)ii} + O(|f'|^2)\big] + 2\alpha\beta O(|f'|) + \beta^2\big[O(|f'|^2) + O(|f''|)\big] \\
&= \alpha^2 K_{i(n-p+1)} + O(|f'|) + O(|f'|^2) + O(|f''|).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\bar{s}_{p,n+1}(P) &= \sum_{i,j=1}^{n-p} \bar{K}_{ij} + 2\sum_{i=1}^{n-p} \bar{K}(e_i, v) \\
&= \sum_{i,j=1}^{n-p} \big[K_{ij} + O(|f'|^2)\big] + 2\sum_{i=1}^{n-p}\big[\alpha^2 K_{i(n-p+1)} + O(|f'|) + O(|f'|^2) + O(|f''|)\big] \\
&= \sum_{i,j=1}^{n-p} K_{ij} + 2\alpha^2 \sum_{i=1}^{n-p} K_{i(n-p+1)} + O(|f'|) + O(|f'|^2) + O(|f''|).
\end{aligned}
$$

Set $A = \sum_{i,j=1}^{n-p} K_{ij}$ and $B = 2\sum_{i=1}^{n-p} K_{i(n-p+1)}$. Then $A = s_{p,n}(Q^\perp) \geqslant B_p > 0$ and

$$
A + B = \sum_{i,j=1}^{n-p+1} K_{ij} = s_{p-1,n}(P \cap T) \geqslant B_{p-1} > 0.
$$

Since $\alpha^2 > 0$, for $B \geqslant 0$, we have $A + \alpha^2 B \geqslant A \geqslant B_p$. For $B < 0$ since $\alpha^2 < 1$, we have $A + \alpha^2 B > A + B \geqslant B_{p-1}$. In either case, $A + \alpha^2 B \geqslant \min\{B_p, B_{p-1}\}$, and hence for a small enough choice of C, we can force the contributions of f' and f'' to allow $\bar{s}_{p,n+1}(P)$ to remain positive as well. □

Rephrasing this result in the language of isotopy and concordance, we obtain the following result.

Proposition 3.3. *Let M be a smooth compact manifold of dimension n. Then, for any $p \in \{0, 1, \cdots, n-2\}$, metrics which are $(s_{p,n} > 0)$-isotopic on M are also $(s_{p,n} > 0)$-concordant.*

Proof. Let g_0 and g_1 be two $(s_{p,n} > 0)$-isotopic metrics on M with the isotopy g_r for $r \in [0,1]$. By Lemma 3.2, there is a $C \leqslant 1$ such that for every smooth function $f : \mathbb{R} \to [0,1]$ with $|f'|, |f''| \leqslant C$, the metric $\bar{g} = g_{f(t)} + dt^2$ on $M \times R$ has positive $(p, n+1)$-intermediate scalar curvature.

Let $\mu : (-\infty, \infty) \to [0, 1]$ be the function

$$\mu(t) = \begin{cases} 0 & t \leqslant 0 \\ \frac{e^{-1/t}}{e^{-1/t}+e^{-\frac{1}{1-t}}} & t \in (0,1) \\ 1 & t \geqslant 1 \end{cases}$$

that smoothly transitions from 0 to 1 over the interval $[0, 1]$. For any $L > 0$, a translation and rescaling gives us the function $\mu_L(t) = \mu\left(\frac{t-1}{L}\right)$ that smoothly transitions from 0 to 1 over the interval $[1, L+1]$. The derivatives μ' and μ'' are bounded and we have $\mu_L'(t) = \frac{1}{L}\mu'\left(\frac{t-1}{L}\right)$ and $\mu_L''(t) = \frac{1}{L^2}\mu''\left(\frac{t-1}{L}\right)$. Therefore we can choose L sufficiently large to force $|\mu_L'|, |\mu_L''| \leqslant C$.

Taking f to be the restriction of μ_L to the interval $[0, L+2]$, the manifold $M \times [0, L+2]$ with metric $\bar{g} = g_{f(t)} \times dt^2$ has positive $(p, n+1)$-intermediate scalar curvature. Since $\bar{g} = g_0 \times dt^2$ for $0 \leqslant t \leqslant 1$ and $\bar{g} = g_1 \times dt^2$ for $L+1 \leqslant t \leqslant L+2$, then by definition, this is a $(s_{p,,n} > 0)$-concordance between g_0 and g_1. □

4. Curvature of Warped Product Metrics

We first fix notation. Let (B^b, g_B) and (F^n, g_F) be, respectively, b- and n-dimensional Riemannian manifolds and consider their product, $M = B \times F$ with the warped product metric $g = g_B + \beta^2 g_F$, where $\beta : B \to (0, \infty)$ is a smooth function. We assume that $b, n \geqslant 1$. We denote by π_B and π_F, the corresponding projections from M to B and to F, and for a point $x \in M$, set

$$\check{x} := \pi_B(x) \quad \text{and} \quad \hat{x} = \pi_F(x).$$

At each point $x \in M$, the maps, π_B and π_F, induce derivative maps

$$(\pi_B)_* : T_xM \to T_{\check{x}}B \quad \text{and} \quad (\pi_F)_* : T_xM \to T_{\hat{x}}F.$$

The warped product structure of the metric gives us the following horizontal and vertical spaces at $x \in M$

$$\mathscr{H}_x := T_x(B \times \{\pi_F(x)\}) \quad \text{and} \quad \mathscr{V}_x := T_x(\{\pi_B(x)\} \times F).$$

In particular, the restriction of the derivative map $(\pi_B)_*$ to the horizontal space $\mathscr{H}_x$ is the isometry

$$(\pi_B)_*|_{\mathscr{H}_x} : (\mathscr{H}_x, g_x|_{\mathscr{H}_x}) \to (T_{\check{x}}B, (g_B)_{\check{x}}).$$

We denote the *vertical* and *horizontal distributions* of the submersion by $\mathscr{V}$ or $\mathscr{H}$. The notation, $\mathscr{V}$, $\mathscr{H}$, serves a dual purpose as we also use it to mean the projection onto the vertical or horizontal subspace. Let $u \in T_xM$ be some tangent vector. Then,

$$u_F := \mathscr{V}(u) \in \mathscr{V}_x \quad \text{and} \quad u_B := \mathscr{H}(u) \in \mathscr{H}_x,$$

denote the corresponding orthogonal projections. Note that the vectors u_B and u_V in $\mathscr{V}_x$ and $\mathscr{H}_x$ are distinct from their corresponding isometric images under $(\pi_F)_*$ and $(\pi_B)_*$. In the case of the derivative maps, we write:

$$\hat{u} := (\pi_F)_*(u) \quad \text{and} \quad \check{u} = (\pi_B)_*(u).$$

4.1. The Riemann Curvature Tensor

In computing the curvature tensor, we will make use of well-known formulas of Gray [15] and O'Neill [25], (see also Theorem 9.28 in [1]). These formulas involve the tensors, $\mathbf{A}, \mathbf{T} : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM)$, defined for vectors fields E_1, E_2 on M as follows:

$$\mathbf{A}_{E_1}E_2 = \mathscr{H}(\nabla_{\mathscr{H}(E_1)}\mathscr{V}(E_2)) + \mathscr{V}(\nabla_{\mathscr{H}(E_1)}\mathscr{H}(E_2)),$$
$$\mathbf{T}_{E_1}E_2 = \mathscr{H}(\nabla_{\mathscr{V}(E_1)}\mathscr{V}(E_2)) + \mathscr{V}(\nabla_{\mathscr{V}(E_1)}\mathscr{H}(E_2)).$$

Note that here with the warped product metric the horizontal distribution for each $x \in M$ is naturally identified with $T_{\check{x}}B$ and in particular, this horizontal distribution is *integrable* (see Chapter 19 in Lee [22]) so the $\mathbf{A}$ tensor above vanishes in our case (see Proposition 9.24 in [1]).

We recall the formulas from [1] involving the $\mathbf{T}$ tensor.

Theorem 4.1. *[1] Let manifold $M = B \times F$ be equipped with the warped product metric $g = g_B + \beta^2 g_F$. Let X, Y be a pair of horizontal vector fields and U, V, a pair of vertical vector fields tangent to M. Then*

$$\mathbf{T}_X U = \mathbf{T}_X Y = 0, \qquad \mathbf{T}_U V = \mathscr{H}(\nabla_U V) = \mathbf{T}_V U, \qquad \mathbf{T}_U X = \mathscr{V}(\nabla_U X),$$

and

$$g(\mathbf{T}_U V, X) = -g(\mathbf{T}_U X, V).$$

We adapt Theorem 9.28 in [1], using the fact that $\mathbf{A}$ is everywhere zero.

Theorem 4.2. *[1] Let the manifold $M = B \times F$ be equipped with the warped product metric $g = g_B + \beta^2 g_F$. Let $\pi_B : (M, g) \to (B, g_B)$ denote a Riemannian warped product submersion for some warping function $\beta : B \to (0, \infty)$. Let X, Y, Z, Z' be horizontal and U, V, W, W' vertical vector fields tangent to M. Finally, let R_B and R_F denote the respective Riemann curvature tensors for (B, g_B) and (F, g_F). Then the Riemann curvature tensor, R, of g satisfies the following properties:*

1. $R(U, V, W, W') = g(R_F(\hat{U}, \hat{V})\hat{W}, \hat{W}') + g(\mathbf{T}_U W, \mathbf{T}_V W') - g(\mathbf{T}_V W, \mathbf{T}_U W')$;
2. $R(U, V, W, X) = g((\nabla_U \mathbf{T})_V W, X) - g((\nabla_V \mathbf{T})_U W, X)$;
3. $R(X, U, Y, V) = g(\mathbf{T}_U X, \mathbf{T}_V Y) - g((\nabla_X \mathbf{T})_U V, Y)$;
4. $R(U, V, X, Y) = g(\mathbf{T}_U X, \mathbf{T}_V Y) - g(\mathbf{T}_V X, \mathbf{T}_U Y)$;
5. $R(X, Y, Z, U) = 0$; *and*
6. $R(X, Y, Z, Z') = g(R_B(\check{X}, \check{Y})\check{Z}, \check{Z}')$.

We now introduce some conventions we will use to make our computations easier to follow. On the manifold $B \times F$, we assume coordinate vector fields $\partial_1, \cdots, \partial_{b+n}$, such that $\partial_1, \ldots, \partial_b$ are horizontal and $\partial_{b+1}, \ldots, \partial_{b+n}$ are vertical and $g_x(\partial_i, \partial_j) = 0$ for $i \neq j$, and $g_x(\partial_i, \partial_j) = \beta(\check{x})^2 \delta_{ij}$ for $i, j \in \{b+1, \cdots, b+n\}$. We adopt the convention that the indices λ, μ, ν will be used for the base directions $\{1, \ldots, b\}$ and the indices i, j, k, ℓ will be used for the fiber directions $\{b+1, \ldots, b+n\}$. For an index that varies over all of $\{1, \ldots, b+n\}$, we use s.

The following lemma gives us the Christoffel symbols for a warped product metric. We note that there are six cases to consider depending on whether the coordinates are in the base or the fiber. The proof is a straightforward computation that we leave to the reader (see Burkemper [4] for more details).

Lemma 4.3. *Let $(B \times F, g_B + \beta^2 g_F)$ be as described above. Then at a point $x \in B \times F$, the Christoffel symbols are given by the following equations*

$$\begin{aligned} &\Gamma^{\nu}_{\lambda\mu} = \check{\Gamma}^{\nu}_{\lambda\mu}, && \Gamma^{k}_{ij} = \hat{\Gamma}^{k}_{ij}, \\ &\Gamma^{k}_{\lambda\mu} = 0, && \Gamma^{\nu}_{\lambda j} = 0, \\ &\Gamma^{k}_{\lambda j} = \frac{\beta_\lambda}{\beta}\delta_{jk}, && \text{and } \Gamma^{\nu}_{ij} = -\beta\beta_\nu g_B^{\nu\nu}\delta_{ij}. \end{aligned} \tag{4.1}$$

From now on, we let $v_{\mathbf{B}}$ and $v_{\mathbf{F}}$ denote the horizontal and vertical parts of v. Using Properties 1, 5, and 6 of Theorem 4.2 and the symmetries of the curvature tensor, we obtain the following lemma, giving us an expression for the Riemann curvature tensor of a 2-plane, $P \subset T_xM$ generated by v and w in terms of the horizontal and vertical parts of each vector.

Lemma 4.4. *Let $(B \times F, g_B + \beta^2 g_F)$ be as described above. Given $x \in B \times F$, consider an arbitrary 2-dimensional subspace $P \subset T_xM$. We let $v, w \in P \subset T_xM$ be an arbitrary pair of linearly independent vectors. Then*

$$\begin{aligned} R(v, w, w, v) = {} & R_B(\check{v}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{v}_{\mathbf{B}}) + \beta^2 R_F(\hat{v}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{v}_{\mathbf{F}}) + g(\mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{F}}, \mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{F}}) - g(\mathbf{T}_{v_{\mathbf{F}}} v_{\mathbf{F}}, \mathbf{T}_{w_{\mathbf{F}}} w_{\mathbf{F}}) \\ & + g((\nabla_{v_{\mathbf{B}}} \mathbf{T})_{w_{\mathbf{F}}} w_{\mathbf{F}}, v_{\mathbf{B}}) - g(\mathbf{T}_{w_{\mathbf{F}}} v_{\mathbf{B}}, \mathbf{T}_{w_{\mathbf{F}}} v_{\mathbf{B}}) + g((\nabla_{w_{\mathbf{B}}} \mathbf{T})_{v_{\mathbf{F}}} v_{\mathbf{F}}, w_{\mathbf{B}}) - g(\mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{B}}, \mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{B}}) \\ & - 2g((\nabla_{w_{\mathbf{F}}} \mathbf{T})_{v_{\mathbf{F}}} w_{\mathbf{F}}, v_{\mathbf{B}}) + 2g((\nabla_{v_{\mathbf{F}}} \mathbf{T})_{w_{\mathbf{F}}} w_{\mathbf{F}}, v_{\mathbf{B}}) + 2g((\nabla_{w_{\mathbf{F}}} \mathbf{T})_{v_{\mathbf{F}}} v_{\mathbf{F}}, w_{\mathbf{B}}) - 2g((\nabla_{v_{\mathbf{F}}} \mathbf{T})_{w_{\mathbf{F}}} v_{\mathbf{F}}, w_{\mathbf{B}}) \\ & + 2g(\mathbf{T}_{w_{\mathbf{F}}} v_{\mathbf{B}}, \mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{B}}) - 2g(\mathbf{T}_{v_{\mathbf{F}}} v_{\mathbf{B}}, \mathbf{T}_{w_{\mathbf{F}}} w_{\mathbf{B}}) + 2g(\mathbf{T}_{w_{\mathbf{F}}} v_{\mathbf{B}}, \mathbf{T}_{v_{\mathbf{F}}} w_{\mathbf{B}}) - 2g((\nabla_{v_{\mathbf{B}}} \mathbf{T})_{w_{\mathbf{F}}} v_{\mathbf{F}}, w_{\mathbf{B}}). \end{aligned} \tag{4.2}$$

Using Theorem 4.1 and Display (4.1), we obtain the following expressions for the components of the $\mathbf{T}$ tensor at the point x:

$$\mathbf{T}_{\partial_i}\partial_j = -\sum_{\lambda} \beta\beta_\lambda g_B^{\lambda\lambda}\delta_{ij}\partial_\lambda, \text{ and } \mathbf{T}_{\partial_i}\partial_\lambda = \frac{\beta_\lambda}{\beta}\partial_i.$$

With these expressions, we can compute the inner products involving the **T** tensor in Display (4.2). We obtain

$$\begin{aligned}
g(\mathbf{T}_{\partial_i}\partial_j, \mathbf{T}_{\partial_k}\partial_\ell) &= \beta^2\delta_{ij}\delta_{k\ell}\sum_\lambda(\beta_\lambda)^2 g^B_{\lambda\lambda},\\
g(\mathbf{T}_{\partial_i}\partial_\lambda, \mathbf{T}_{\partial_j}\partial_\mu) &= \beta_\lambda\beta_\mu\delta_{ij},\\
g(\mathbf{T}_{\partial_i}\partial_\lambda, \mathbf{T}_{\partial_j}\partial_\mu) &= \beta_\lambda\beta_\mu\delta_{ij},\\
g((\nabla_{\partial_\lambda}\mathbf{T})_{\partial_i}\partial_j, \partial_\mu) &= -\delta_{ij}\left[\left(\beta\beta_{\lambda\mu}-\beta_\lambda\beta_\mu\right)+\sum_\nu\beta\beta_\nu\left(\partial_\lambda(g_B^{\mu\nu})+g_B^{\nu\nu}\check{\Gamma}^\mu_{\lambda\nu}\right)g^B_{\mu\mu}\right], \text{ and}\\
g((\nabla_{\partial_i}\mathbf{T})_{\partial_j}\partial_k, \partial_\lambda) &= 0.
\end{aligned}\tag{4.3}$$

Setting $v=\sum_{s=1}^{b+n} v_s\partial_s$ and $w=\sum_{s=1}^{b+n} w_s\partial_s$, then $v_{\mathbf{B}}=\sum_\lambda v_\lambda\partial_\lambda$, $w_{\mathbf{B}}=\sum_\lambda w_\lambda\partial_\lambda$, $v_{\mathbf{F}}=\sum_i v_i\partial_i$, and $w_{\mathbf{F}}=\sum_i w_i\partial_i$. With these conventions, from Display (4.3), we obtain

$$\begin{aligned}
g(\mathbf{T}_{v_\mathbf{F}}w_\mathbf{F}, \mathbf{T}_{v_\mathbf{F}}w_\mathbf{F}) &= \sum_{i,j,\lambda} v_iw_iv_jw_j\,\beta^2\beta_\lambda^2 g^B_{\lambda\lambda},\\
g(\mathbf{T}_{v_\mathbf{F}}v_\mathbf{F}, \mathbf{T}_{w_\mathbf{F}}w_\mathbf{F}) &= \sum_{i,j,\lambda} v_i^2w_j^2\,\beta^2\beta_\lambda^2 g^B_{\lambda\lambda},\\
g(\mathbf{T}_{w_\mathbf{F}}v_\mathbf{B}, \mathbf{T}_{w_\mathbf{F}}v_\mathbf{B}) &= \sum_{i,\lambda,\mu} w_i^2v_\lambda v_\mu\,\beta_\lambda\beta_\mu,\\
g(\mathbf{T}_{v_\mathbf{F}}w_\mathbf{B}, \mathbf{T}_{v_\mathbf{F}}w_\mathbf{B}) &= \sum_{i,\lambda,\mu} v_i^2w_\lambda w_\mu\,\beta_\lambda\beta_\mu,\\
g(\mathbf{T}_{v_\mathbf{F}}v_\mathbf{B}, \mathbf{T}_{w_\mathbf{F}}w_\mathbf{B}) &= \sum_{i,\lambda,\mu} v_iw_iv_\lambda w_\mu\,\beta_\lambda\beta_\mu,\\
g(\mathbf{T}_{w_\mathbf{F}}v_\mathbf{B}, \mathbf{T}_{v_\mathbf{F}}w_\mathbf{B}) &= \sum_{i,\lambda,\mu} v_iw_iv_\lambda w_\mu\,\beta_\lambda\beta_\mu,\\
g((\nabla_{w_\mathbf{F}}\mathbf{T})_{v_\mathbf{F}}w_\mathbf{F}, v_\mathbf{B}) &= 0,\\
g((\nabla_{v_\mathbf{F}}\mathbf{T})_{w_\mathbf{F}}w_\mathbf{F}, v_\mathbf{B}) &= 0,\\
g((\nabla_{w_\mathbf{F}}\mathbf{T})_{v_\mathbf{F}}v_\mathbf{F}, w_\mathbf{B}) &= 0,\\
g((\nabla_{v_\mathbf{F}}\mathbf{T})_{w_\mathbf{F}}v_\mathbf{F}, w_\mathbf{B}) &= 0,\\
g((\nabla_{v_\mathbf{B}}\mathbf{T})_{w_\mathbf{F}}w_\mathbf{F}, v_\mathbf{B}) &= -\sum_{i,\lambda,\mu} w_i^2v_\lambda v_\mu\left[\beta\beta_{\lambda\mu}-\beta_\lambda\beta_\mu+\sum_\nu\beta\beta_\nu\left(\partial_\lambda(g_B^{\mu\nu})+g_B^{\nu\nu}\check{\Gamma}^\mu_{\lambda\nu}\right)g^B_{\mu\mu}\right],\\
g((\nabla_{w_\mathbf{B}}\mathbf{T})_{v_\mathbf{F}}v_\mathbf{F}, w_\mathbf{B}) &= -\sum_{i,\lambda,\mu} v_i^2w_\lambda w_\mu\left[\beta\beta_{\lambda\mu}-\beta_\lambda\beta_\mu+\sum_\nu\beta\beta_\nu\left(\partial_\lambda(g_B^{\mu\nu})+g_B^{\nu\nu}\check{\Gamma}^\mu_{\lambda\nu}\right)g^B_{\mu\mu}\right], \text{ and}\\
g((\nabla_{v_\mathbf{B}}\mathbf{T})_{w_\mathbf{F}}v_\mathbf{F}, w_\mathbf{B}) &= -\sum_{i,\lambda,\mu} v_iw_iv_\lambda w_\mu\left[\beta\beta_{\lambda\mu}-\beta_\lambda\beta_\mu+\sum_\nu\beta\beta_\nu\left(\partial_\lambda(g_B^{\mu\nu})+g_B^{\nu\nu}\check{\Gamma}^\mu_{\lambda\nu}\right)g^B_{\mu\mu}\right].
\end{aligned}\tag{4.4}$$

Substituting these equalities from Display (4.4) into Display (4.2), we obtain the following result.

Proposition 4.5. *Let $(B\times F, g_B+\beta^2 g_F)$ be as described above. Given $x\in B\times F$, consider an arbitrary 2-dimensional subspace $P\subset T_xM$. We let $v, w\in P\subset T_xM$ be an arbitrary pair of linearly independent vectors, written in components as $v=\sum_{s=1}^{b+n} v_s\partial_s$ and $w=\sum_{s=1}^{b+n} w_s\partial_s$. Then*

$$
\begin{aligned}
R(v,w,w,v) &= R_B(\check{v}_{\mathbf{B}},\check{w}_{\mathbf{B}},\check{w}_{\mathbf{B}},\check{v}_{\mathbf{B}}) + \beta^2 R_F(\hat{v}_{\mathbf{F}},\hat{w}_{\mathbf{F}},\hat{w}_{\mathbf{F}},\hat{v}_{\mathbf{F}}) \\
&\quad - \sum_{i<j,\lambda}(v_i w_j - v_j w_i)^2\beta^2\beta_\lambda^2 g^B_{\lambda\lambda} \\
&\quad - \sum_{i,\lambda,\mu}(w_i^2 v_\lambda v_\mu + v_i^2 w_\lambda w_\mu - 2v_i w_i v_\lambda w_\mu)\left[\beta\beta_{\lambda\mu} + \sum_\nu \beta\beta_\nu\big(\partial_\lambda(g_B^{\mu\nu}) + g_B^{\nu\nu}\check{\Gamma}^\mu_{\lambda\nu}\big)g^B_{\mu\mu}\right].
\end{aligned}
\tag{4.5}
$$

Note that in the case of a product metric, $\beta = 1$. Since this makes each of the derivatives of β zero, Equation (4.5) reduces to the well-known formula

$$
R(v,w,w,v) = R_B(\check{v}_{\mathbf{B}},\check{w}_{\mathbf{B}},\check{w}_{\mathbf{B}},\check{v}_{\mathbf{B}}) + R_F(\hat{v}_{\mathbf{F}},\hat{w}_{\mathbf{F}},\hat{w}_{\mathbf{F}},\hat{v}_{\mathbf{F}}). \tag{4.6}
$$

4.2. The Sectional Curvature of the Specific Warped Product

We now restrict our attention to a metric of the form

$$
g = dr^2 + \omega(r,t)^2 dt^2 + \beta(r)^2 ds_n^2
$$

defined on the product manifold $M = B \times F$ where $B = (0,b_1)\times(0,b_2)$ and $F = S^n$ for $n \geqslant 2$ with the standard round metric, where $\beta : (0,b_1) \to (0,\infty)$ and $\omega : (0,b_1)\times(0,b_2) \to (0,\infty)$ are smooth warping functions. We are interested in computing the $(p, n+2)$-intermediate scalar curvatures of (M,g). We begin by computing some of the sectional curvatures using our earlier work.

Lemma 4.6. *The sectional curvatures of $M = B \times F$ as above, are given by*

$$
K_{rt} = -\frac{\omega_{rr}}{\omega}, \quad K_{ri} = \frac{-\beta_{rr}}{\beta}, \quad K_{ti} = -\frac{\omega_r\beta_r}{\omega\beta}, \quad K_{ij} = \frac{1-\beta_r^2}{\beta^2}. \tag{4.7}
$$

Proof. The metric on the base is given by $g_B = dr^2 + \omega(r,t)^2dt^2$. We compute the Christoffel symbols and obtain

$$
\begin{aligned}
&\Gamma^r_{rr} = 0,\\
&\Gamma^t_{rr} = 0,\\
&\Gamma^r_{rt} = 0,\\
&\Gamma^r_{tt} = -\omega\omega_r,\\
&\Gamma^t_{tt} = \frac{\omega_t}{\omega}, \text{ and}\\
&\Gamma^t_{rt} = \frac{\omega_r}{\omega}.
\end{aligned}
\tag{4.8}
$$

Note that since we are dealing with coordinate vector fields, we have $\nabla_{\partial_r}\partial_t = \nabla_{\partial_t}\partial_r$. Thus,

$$
\begin{aligned}
&\nabla_{\partial_r}\partial_r = 0,\\
&\nabla_{\partial_t}\partial_t = -\omega\omega_r\partial_r + \frac{\omega_t}{\omega}\partial_t, \text{ and}\\
&\nabla_{\partial_r}\partial_t = \frac{\omega_r}{\omega}\partial_t.
\end{aligned}
$$

Since $\nabla_{\partial_r}\partial_r = 0$, we have $\nabla_{\partial_t}\nabla_{\partial_r}\partial_r = 0$, and a calculation gives $\nabla_{\partial_r}\nabla_{\partial_t}\partial_r = \frac{\omega_{rr}}{\omega}\partial_t$. Therefore

$$
\langle R(\partial_t,\partial_r)\partial_r,\partial_t\rangle = -\frac{\omega_{rr}}{\omega}\omega^2. \tag{4.9}
$$

We now expand out the sums over the base directions r and t in our Riemann curvature equation (4.5). Since we know the form of the base metric and its inverse matrix, and that the function β only depends on r, so that $\beta_t = 0$. Equation (4.5) then simplifies to

$$\begin{aligned}
R(v,w,w,v) = {} & R_B(\check{v}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{v}_{\mathbf{B}}) + \beta^2 R_F(\hat{v}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{v}_{\mathbf{F}}) \\
& - \sum_{i<j,\lambda} (v_i w_j - v_j w_i)^2 \beta^2 \beta_r^2 \\
& - \sum_i (w_i^2 v_r v_r + v_i^2 w_r w_r - 2 v_i w_i v_r w_r) \left[\beta\beta_{rr} + \beta\beta_r \check{\Gamma}_{rr}^r \right] \\
& - \sum_i (w_i^2 v_r v_t + v_i^2 w_r w_t - 2 v_i w_i v_r w_t) \beta\beta_r \check{\Gamma}_{rr}^t \omega^2 \\
& - \sum_i (w_i^2 v_t v_r + v_i^2 w_t w_r - 2 v_i w_i v_t w_r) \beta\beta_r \check{\Gamma}_{tr}^r \\
& - \sum_i (w_i^2 v_t v_t + v_i^2 w_t w_t - 2 v_i w_i v_t w_t) \beta\beta_r \check{\Gamma}_{tr}^t \omega^2 .
\end{aligned}$$

Using the Christoffel symbols we computed in Display (4.8), we then obtain

$$\begin{aligned}
R(v,w,w,v) = {} & R_B(\check{v}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{v}_{\mathbf{B}}) + \beta^2 R_F(\hat{v}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{v}_{\mathbf{F}}) - \sum_{i<j} (v_i w_j - v_j w_i)^2 \beta^2 \beta_r^2 \\
& - \sum_i (v_i w_r - w_i v_r)^2 \beta\beta_{rr} - \sum_i (v_i w_t - w_i v_t)^2 \beta\beta_r \omega\omega_r .
\end{aligned}$$

The fibers F are unit spheres of dimension at least 2 with $K_F = 1$, so we have

$$\begin{aligned}
R_F(\hat{v}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}, \hat{v}_{\mathbf{F}}) &= |\hat{v}_{\mathbf{F}} \wedge \hat{w}_{\mathbf{F}}|_F^2 K_F(\hat{v}_{\mathbf{F}}, \hat{w}_{\mathbf{F}}) \\
&= \sum_{i<j} (v_i w_j - v_j w_i)^2 .
\end{aligned}$$

Similarly, the base B is two-dimensional and from Lemma 4.6 its sectional curvature is given by $K_{rt} = -\frac{\omega_{rr}}{\omega}$. So using the metric $g_B = dr^2 + \omega^2 dt^2$, we have

$$\begin{aligned}
R_F(\check{v}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{w}_{\mathbf{B}}, \check{v}_{\mathbf{B}}) &= |\check{v}_{\mathbf{B}} \wedge \check{w}_{\mathbf{B}}|_B^2 K_F(\check{v}_{\mathbf{B}}, \check{w}_{\mathbf{B}}) \\
&= -\omega\omega_{rr} (v_t w_r - v_r w_t)^2 .
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
R(v,w,w,v) = {} & -\omega\omega_{rr} (v_t w_r - v_r w_t)^2 + \sum_{i<j} (v_i w_j - v_j w_i)^2 \beta^2 (1 - \beta_r^2) \\
& - \sum_i (v_i w_r - w_i v_r)^2 \beta\beta_{rr} - \sum_i (v_i w_t - w_i v_t)^2 \beta\beta_r \omega\omega_r .
\end{aligned} \tag{4.10}$$

We now compute the specific sectional curvatures when the vectors v and w are in the coordinate directions. First we start with $v = \partial_t$ and $w = \partial_i$, so that $v_t = 1$, $v_r = v_j = 0$ for all j directions in the fiber, $w_r = w_t = 0$, $w_i = 1$, and $w_j = 0$ for all $j \neq i$. Substituting this information into Equation (4.10), we get

$$R_{tiit} = -(v_t w_i)^2 \beta\beta_r \omega\omega_r = -\beta\beta_r \omega\omega_r . \tag{4.11}$$

Next, let $v = \partial_r$ and $w = \partial_i$, so that $v_r = 1$, $v_t = v_j = 0$ for all j directions in the fiber, $w_r = w_t = 0$, $w_i = 1$, and $w_j = 0$ for all $j \neq i$. Substituting this information into Equation (4.10), we get

$$R_{riir} = -(v_r w_i)^2 \beta\beta_{rr} = -\beta\beta_{rr} . \tag{4.12}$$

Finally, let $v = \partial_i$ and $w = \partial_j$ for $i \neq j$ so that $v_r = v_t = w_r = w_t = 0$ and $v_k = w_k = 0$, except for $v_i = 1$ and $w_j = 1$. Since $K_F = 1$, we have

$$R_{ijji} = (v_i w_j)^2 \beta^2 (1 - \beta_r^2) = \beta^2 (1 - \beta_r^2) . \tag{4.13}$$

The result now follows from Equations (4.9), (4.11), (4.12), and (4.13). □

4.3. Intermediate Scalar Curvature

We will now use the sectional curvatures we have computed in the previous section to derive a formula for the $(p, n+2)$-intermediate scalar curvatures of our metric. Throughout this section, the vectors v and w always represent a pair of **unit length orthogonal vectors.** Let $0 \leqslant p \leqslant n$ and let P be a p-plane in T_xM. Then $P^\perp$ is a $(n-p+2)$-plane. Since the dimension of $P^\perp + T_xS^n$ can be at most the dimension of T_xM,

$$\begin{aligned}\dim(P^\perp \cap T_xS^n) &= \dim(P^\perp) + \dim(T_xS^n) - \dim(P^\perp + T_xS^n)\\ &\geqslant (n-p+2) + n - (n+2)\\ &= n-p.\end{aligned}$$

Therefore, we know that there are at least $n-p$ linearly independent vectors in $P^\perp$ tangent to the sphere.

Without loss of generality, we will assume that the coordinate vector fields on the sphere are $\partial_1, \dots, \partial_n$ where $\partial_1, \dots, \partial_{n-p}$ are in $P^\perp$. Completing an orthogonal basis for $P^\perp$, we have at most two unit length orthogonal vectors v, w that are not in T_xS^n. We consider three cases.

Case 1. The projections of v and w into T_xS^n span a 0-dimensional subspace. This means that v and w do not have any fiber component and must be spanned by ∂_r and ∂_t. However, this means that ∂_r and ∂_t must be in $P^\perp$ and we can just assume that these satisfy $v = \partial_r$ and $w = \frac{1}{\omega}\partial_t$. In particular, $P^\perp$ has the orthogonal basis $\{\partial_r, \partial_t, \partial_1, \dots, \partial_{n-p}\}$. We compute the $(p, n+2)$-intermediate scalar curvature,

$$\begin{aligned}s_{p,n+2}(P) &= 2\sum_{i<j} K_{ij} + 2\sum_{i=1}^{n-p} K_{ri} + 2\sum_{i=1}^{n-p} K_{ti} + 2K_{rt}\\ &= (n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2} - 2(n-p)\frac{\beta_{rr}}{\beta} - 2(n-p)\frac{\omega_r\beta_r}{\omega\beta} - \frac{2\omega_{rr}}{\omega}.\end{aligned}$$

Case 2. The projections of v and w into T_xS^n span a 1-dimensional subspace. This subspace is orthogonal to the directions $\partial_1, \dots, \partial_{n-p}$, and so without loss of generality, we can assume that it is spanned by ∂_{n-p+1}. Setting $k = n-p+1$ for brevity, this means that the vectors v and w have the form

$$v = v_r\partial_r + \frac{v_t}{\omega}\partial_t + \frac{v_k}{\beta}\partial_k, \qquad w = w_r\partial_r + \frac{w_t}{\omega}\partial_t + \frac{w_k}{\beta}\partial_k.$$

Using Equation (4.10), for $1 \leqslant i \leqslant n-p$, we obtain

$$\begin{aligned}K(v, \partial_i) = R(v, \partial_i, \partial_i, v) &= v_k^2(1-\beta_r^2) - v_r^2\frac{\beta_{rr}}{\beta} - v_t^2\frac{\beta_r\omega_r}{\beta\omega} \text{ and}\\ K(w, \partial_i) = R(w, \partial_i, \partial_i, w) &= w_k^2(1-\beta_r^2) - w_r^2\frac{\beta_{rr}}{\beta} - w_t^2\frac{\beta_r\omega_r}{\beta\omega}.\end{aligned}$$

Similarly,

$$K(v, w) = R(v, w, w, v) = -(v_tw_r - v_rw_t)^2\frac{\omega_{rr}}{\omega} - (v_kw_r - w_kv_r)^2\frac{\beta_{rr}}{\beta} - (v_kw_t - w_kv_t)^2\frac{\beta_r\omega_r}{\beta\omega}.$$

The $(p, n+2)$-intermediate scalar curvature is then given by

$$\begin{aligned}s_{p,n+2}(P) =& 2\sum_{i<j} K_{ij} + 2\sum_{i=1}^{n-p} K(v, \partial_i) + 2\sum_{i=1}^{n-p} K(w, \partial_i) + 2K(v, w)\\ =&(n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2}\\ &+ 2(n-p)\left(v_k^2(1-\beta_r^2) - v_r^2\frac{\beta_{rr}}{\beta} - v_t^2\frac{\beta_r\omega_r}{\beta\omega}\right)\\ &+ 2(n-p)\left(w_k^2(1-\beta_r^2) - w_r^2\frac{\beta_{rr}}{\beta} - w_t^2\frac{\beta_r\omega_r}{\beta\omega}\right)\\ &- 2\left((v_tw_r - v_rw_t)^2\frac{\omega_{rr}}{\omega} + (v_kw_r - w_kv_r)^2\frac{\beta_{rr}}{\beta} + (v_kw_t - w_kv_t)^2\frac{\beta_r\omega_r}{\beta\omega}\right)\\ =&(n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2}\end{aligned}$$

$$+2(n-p)\left((v_k^2+w_k^2)(1-\beta_r^2)-(v_r^2+w_r^2)\frac{\beta_{rr}}{\beta}-(v_t^2+w_t^2)\frac{\beta_r\omega_r}{\beta\omega}\right)$$
$$-2\left((v_tw_r-v_rw_t)^2\frac{\omega_{rr}}{\omega}+(v_kw_r-w_kv_r)^2\frac{\beta_{rr}}{\beta}+(v_kw_t-w_kv_t)^2\frac{\beta_r\omega_r}{\beta\omega}\right).$$

Case 3. The projections of v and w onto T_xS^n span a 2-dimensional subspace. This subspace is orthogonal to the directions $\partial_1,\dots,\partial_{n-p}$, and so without loss of generality, we can assume that it is spanned by ∂_{n-p+1} and ∂_{n-p+2}. Setting $k=n-p+1$ so that $k+1=n-p+2$, this means that the vectors v and w have the form

$$v=v_r\partial_r+\frac{v_t}{\omega}\partial_t+\frac{v_k}{\beta}\partial_k+\frac{v_{k+1}}{\beta}\partial_{k+1},\qquad w=w_r\partial_r+\frac{w_t}{\omega}\partial_t+\frac{w_k}{\beta}\partial_k+\frac{w_{k+1}}{\beta}\partial_{k+1}.$$

Using Equation (4.10), for $1\leqslant i\leqslant n-p$, we get

$$K(v,\partial_i)=R(v,\partial_i,\partial_i,v)=(v_k^2+v_{k+1}^2)(1-\beta_r^2)-v_r^2\frac{\beta_{rr}}{\beta}-v_t^2\frac{\beta_r\omega_r}{\beta\omega},\ \text{and}$$
$$K(w,\partial_i)=R(w,\partial_i,\partial_i,w)=(w_k^2+w_{k+1}^2)(1-\beta_r^2)-w_r^2\frac{\beta_{rr}}{\beta}-w_t^2\frac{\beta_r\omega_r}{\beta\omega}.$$

Furthermore, we have

$$\begin{aligned}K(v,w)=R(v,w,w,v)=&-(v_tw_r-v_rw_t)^2\frac{\omega_{rr}}{\omega}+(v_kw_{k+1}-v_{k+1}w_k)^2(1-\beta_r^2)\\&-[(v_kw_r-w_kv_r)^2+(v_{k+1}w_r-w_{k+1}v_r)^2]\frac{\beta_{rr}}{\beta}\\&-[(v_kw_t-w_kv_t)^2+(v_{k+1}w_t-w_{k+1}v_t)^2]\frac{\beta_r\omega_r}{\beta\omega}.\end{aligned}$$

The $(p,n+2)$-intermediate scalar curvature is then given by

$$\begin{aligned}s_{p,n+2}(P)&=2\sum_{i<j}K_{ij}+2\sum_{i=1}^{n-p}K(v,\partial_i)+2\sum_{i=1}^{n-p}K(w,\partial_i)+2K(v,w)\\&=(n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2}\\&+2(n-p)\left((v_k^2+v_{k+1}^2)(1-\beta_r^2)-v_r^2\frac{\beta_{rr}}{\beta}-v_t^2\frac{\beta_r\omega_r}{\beta\omega}\right)\\&+2(n-p)\left((w_k^2+w_{k+1}^2)(1-\beta_r^2)-w_r^2\frac{\beta_{rr}}{\beta}-w_t^2\frac{\beta_r\omega_r}{\beta\omega}\right)\\&+2\Bigg[-(v_tw_r-v_rw_t)^2\frac{\omega_{rr}}{\omega}+(v_kw_{k+1}-v_{k+1}w_k)^2(1-\beta_r^2)\\&\quad-[(v_kw_r-w_kv_r)^2+(v_{k+1}w_r-w_{k+1}v_r)^2]\frac{\beta_{rr}}{\beta}\\&\quad-[(v_kw_t-w_kv_t)^2+(v_{k+1}w_t-w_{k+1}v_t)^2]\frac{\beta_r\omega_r}{\beta\omega}\Bigg].\end{aligned}$$

Since we know the warping functions β and ω are strictly positive by definition, and most coefficients in our final formulas for the intermediate scalar curvatures involve nonnegative squared terms, we summarize these three cases in the following proposition.

Proposition 4.7. *Let $M=B\times F$ with $B=(0,b_1)\times(0,b_2)$ and $F=S^n$, having a metric of the form*

$$g=dr^2+\omega(r,t)^2dt^2+\beta(r)^2ds_n^2$$

where $\beta:(0,\beta_1)\to(0,\infty)$ and $\omega:(0,b_1)\times(0,b_2)\to(0,\infty)$ are smooth warping functions. If $x\in M$ and P is a p-plane in T_xM for $p\in\{0,\cdots,n\}$, then the $(p,n+2)$-intermediate scalar curvature of P has the form

$$s_{p,n+2}(P)=(n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2}+A^2(1-\beta_r)^2-B^2\frac{\omega_{rr}}{\omega}-C^2\frac{\beta_{rr}}{\beta}-D^2\frac{\beta_r\omega_r}{\beta\omega},$$

for some bounded real-valued functions A, B, C and D depending only on the plane P.

5. Standard Metrics on the Sphere and the Disk

In this section we recall some well known metrics on the disk and sphere. In particular, we recall the so-called *torpedo* and *boot metrics*. Such metrics are described in detail in section 3 of [32] although in the context of positive scalar curvature. Here we will establish conditions whereby these metrics have positive (p, n)-intermediate scalar curvature for appropriate $p \in \{0, 1, \cdots n-2\}$.

5.1. Introducing the Metrics

Before we get into a formal construction of the metrics, we give a very brief description with the aid of Figs. 5.1 and 5.2. An important point to note is that each space will be topologically the disk D^{n+2}, but each will be distinguished by its metric.

A (δ, λ)-*torpedo metric* on a disk, D^{n+2}, (where $n \geqslant 0$) is metriccally a cylinder, of length λ, of a round $(n+1)$-sphere of radius δ near the boundary of the disk, before closing up as a round $(n+2)$-dimensional hemisphere at the center. We denote such a metric, $g_{\text{torp}}^{n+2}(\delta)_\lambda$ and it is depicted in the first picture in Fig. 5.1.

Restricting such a metric to an upper half-disk D_+^{n+2} results in a *half-*(δ, λ)*-torpedo metric*, denoted $g_{\text{torp}+}^{n+2}(\delta)_\lambda$. This is the second picture in Fig. 5.1.

By carefully gluing to a cylinder of torpedo metrics, $g_{\text{torp}}^{n+1}(\delta)_\lambda + dt^2$, on $D^{n+1} \times [0, 1]$ a half-torpedo metric, $g_{\text{torp}+}^{n+2}(\delta)_\lambda$, on D_+^{n+2} along a D^{n+1} contained in the boundary of D_+^{n+2}, we obtain a metric denoted $g_{\text{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$ on the manifold with corners, D_{stretch}^{n+2}, obtained by attaching and smoothing the underlying manifolds D_+^{n+2} and $D^{n+1} \times [0, 1]$. We refer to the metric $g_{\text{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$ as the *toe metric* for its shape and its role in the next construction. This is the third picture in Fig. 5.1.

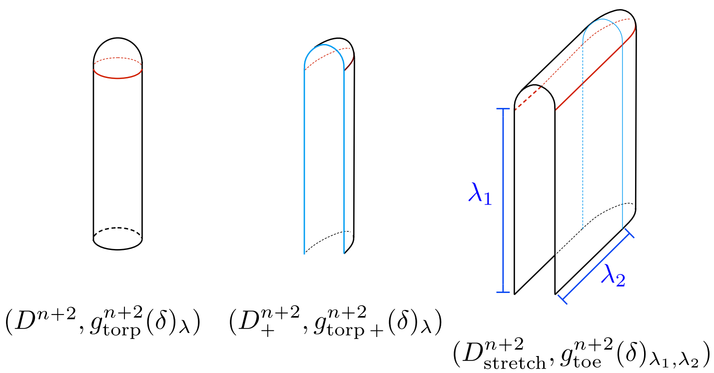

Fig. 5.1. Various metrics on the disk D^{n+2}.

Finally, we introduce an $(n+2)$*-dimensional* δ*-boot metric*. Briefly, this metric is constructed in 4 steps as follows.

Step 1. Beginning with some torpedo metric, $g_{\text{torp}}^{n+1}(\delta)_\lambda$, trace out a cylinder of torpedo metrics before bending the cylinder around an angle of $\frac{\pi}{2}$ to finish as a Riemannian cylinder perpendicular to the first part of the cylinder in the direction suggested by the third image of Fig. 5.1. The resulting object has two cylindrical ends with different metrics. One is of the form $dr^2 + g_{\text{torp}}^{n+1}(\delta)_\lambda$ and the other $dt^2 + g_{\text{torp}}^{n+1}(\delta)_\lambda$, where r and t are orthogonal coordinates depicted in Fig. 5.2.

Step 2. In order to control any negative sectional curvatures arising from the bending itself, we control the bending with a parameter $\Lambda > 0$. Essentially, the bending takes place along a quarter-circle of radius $\Lambda > 0$. A large choice of Λ ensures that negative curvatures arising from the bending are small.

Step 3. Away from the "caps" of the torpedos, this metric takes the form $dr^2 + dt^2 + \delta^2 ds_n^2$. This part can easily be extended to incorporate the corner depicted in the third image of Fig. 5.1 and so that the necks of the torpedo "ends" have any desired lengths, l_1 and l_4. These distances along with Λ determine the distances l_2 and l_3, which are pictured in Fig. 5.2.

Step 4. Finally, we smoothly "cap-off" the cylindrical end which takes the form $dt_1^2 + g_{\text{torp}}^{n-1}(\delta)_{l_1}$, by attaching a half-torpedo metric, $g_{\text{torp}+}^{n+2}(\delta)_{l_1}$. This is the so-called "toe" of the boot metric.

The resulting metric is denoted $g_{\text{boot}}^{n+2}(\delta)_{\Lambda,\bar{l}}$ where $\Lambda > 0$ is the bending constant discussed above and $\bar{l} = (l_1, l_2, l_3, l_4) \in \mathbb{R}_+^4$ determines the various neck-lengths. While the choices of l_1 and l_4 are arbitrary, the constants l_2 and l_3, as mentioned above, are determined by Λ, l_1 and l_4.

In the remainder of this section we will establish some results about the $(p, n+2)$-intermediate scalar curvature of these metrics.

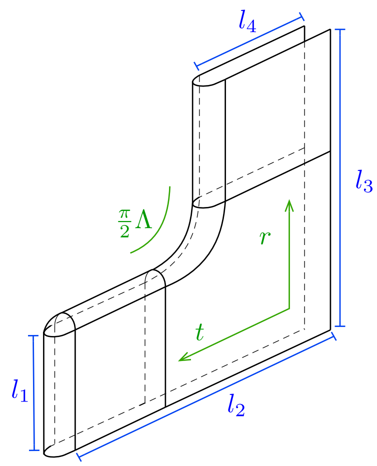

Fig. 5.2. The boot metric $g_{\mathrm{boot}}^{n+2}(\delta)_{\Lambda,\bar{l}}$.

5.2. The Torpedo Metric

We consider a pair of smooth functions $\alpha, \beta : [0, b] \to [0, \infty)$, where $b > 0$, which satisfy the following conditions.

$$\alpha(r) = \alpha_0 - \int_0^r \sqrt{1 - \beta_r(u)^2}du, \quad \text{where} \quad \alpha_0 = 1 + \int_0^{\frac{b}{2}} \sqrt{1 - \beta_r(u)^2}du, \text{ and} \tag{5.1}$$

$$\begin{array}{ll} \text{(i)} & \beta(r) > 0, \text{ for all } r \in (0, b); \\ \text{(ii)} & \beta(0) = 0, \quad \beta_r(0) = 1, \quad \beta^{(even)}(0) = 0, \text{ and;} \\ \text{(iii)} & \beta(b) = 0, \quad \beta_r(b) = -1, \quad \beta^{(even)}(b) = 0. \end{array} \tag{5.2}$$

The important point to note here is that α and β satisfy $(\alpha_r)^2 + (\beta_r)^2 = 1$. In particular, if $\beta(r) = \sin r$ on $(0, b) = (0, \pi)$, then $\alpha(r) = 1 + \cos r$. We now consider the map, F_β, defined by:

$$\begin{aligned} F_\beta : (0, b) \times S^{n+1} &\longrightarrow \mathbb{R}^{n+2} \times \mathbb{R}, \\ (r, \theta) &\longmapsto (\beta(r)\theta, \alpha(r)), \end{aligned}$$

and recall Proposition 3.1 of [32].

Proposition 5.1. *[32] For any smooth functions $\alpha, \beta : [0, b] \to [0, \infty)$ satisfying the conditions laid out in Displays (5.2) and (5.1), the map F_β above is an embedding.*

Pulling back the Euclidean metric on $\mathbb{R}^{n+2} \times \mathbb{R}$ via F_β induces a metric, g_β, which we compute to be

$$\begin{aligned} g_\beta &:= F_\beta^*(dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2 + dx_{n+3}^2) \\ &= dr^2 + \beta(r)^2 ds_{n+1}^2, \end{aligned}$$

where ds_{n+1}^2 is the standard round metric of radius 1 on S^{n+1}. The following proposition is proved in Chapter 1, Section 3.4 of Petersen [26].

Proposition 5.2. *[26] Provided the smooth function $\beta : [0, b] \to [0, \infty)$ satisfies the conditions laid out in Display (5.2), the metric g_β extends uniquely to a rotationally symmetric metric on S^{n+2}. Furthermore, if we drop condition (iii) of Display (5.2) and simply insist that $\beta(b) > 0$, this metric is now a smooth rotationally symmetric metric on the disk D^{n+2}.*

In particular, by setting $\beta(r) = \delta \sin \frac{r}{\delta}$ for $r \in [0, \delta\pi]$, we obtain for g_β the standard round metric of radius δ on S^{n+2}.

Let $\mu : \mathbb{R} \to [0, 1]$ be the function

$$\mu(r) = \begin{cases} 0 & r \leqslant 0 \\ \frac{e^{-1/r}}{e^{-1/r} + e^{-\frac{1}{1-r}}} & r \in (0, 1) \\ 1 & r \geqslant 1 \end{cases}$$

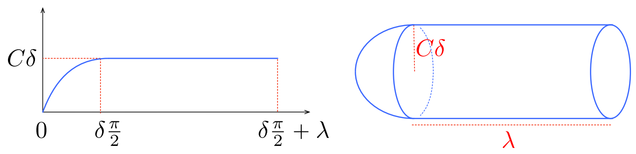

Fig. 5.3. A torpedo function $\eta_{\delta,\lambda}$ and the resulting torpedo metric $g^{n+2}_{\text{torp}}(\delta)_\lambda$ on the disk.

that smoothly transitions from 0 to 1. For any $\delta > 0$ and $\lambda \geqslant 0$, let $\eta_{\delta,\lambda} : [0, \delta\frac{\pi}{2} + \lambda] \to [0, 1]$ be the smooth function with derivative

$$\frac{\partial \eta_{\delta,\lambda}}{\partial r}(r) = \cos\left(\frac{r}{\delta}\right)\mu\left(2 - \frac{4r}{\delta\pi}\right).$$

This function satisfies conditions (i) and (ii) of (5.2) as well as the following:

(i) $\eta_{\delta,\lambda}(r) = \delta \sin\frac{r}{\delta}$ when $r \leqslant \delta\frac{\pi}{4}$;
(ii) $\eta_{\delta,\lambda}(r) = C\delta$ when $r \geqslant \delta\frac{\pi}{2}$ for $C \approx 0.916$;
(iii) $\eta_{\delta,\lambda}(r) \leqslant \delta \sin\frac{r}{\delta}$ and $\frac{\partial\eta_{\delta,\lambda}}{\partial r}(r) \leqslant \cos\frac{r}{\delta}$ for $\delta\frac{\pi}{4} < r < \delta\frac{\pi}{2}$;
(iv) $\frac{\partial^2}{\partial r}(\eta_{\delta,\lambda}(r)) \leqslant 0$; and
(v) $\frac{\partial^{(k)}\eta_{\delta,\lambda}}{\partial r}(\delta\frac{\pi}{2}) = 0$ for all $k \geqslant 1$.

The function $\eta_{\delta,\lambda}$ is known as a *torpedo function*. As it satisfies conditions (i) and (ii) of Display (5.2) and has $\eta_{\delta,\lambda}(\frac{\pi}{2} + \lambda) > 0$, it gives rise to a smooth metric on D^{n+2}. The resulting metric is called a *torpedo metric of radius* δ and *neck length* λ (or (δ, λ)-torpedo metric). It is denoted $g^{n+2}_{\text{torp}}(\delta)_\lambda$ and given by the formula:

$$g^{n+2}_{\text{torp}}(\delta)_\lambda = dr^2 + \eta_{\delta,\lambda}(r)^2 ds^2_{n+1},$$

where $r \in [0, \frac{\pi\delta}{2} + \lambda]$. Such a metric is rotationally symmetric metric on the disk D^{n+2} and roughly, a round hemisphere of radius δ near the center of the disk and takes a radius δ cylindrical form on the annular region where $r \in [\frac{\pi\delta}{2}, \frac{\pi\delta}{2} + \lambda]$; see Fig. 5.3.

Proposition 5.3. *Suppose $n \geqslant 1$ and p satisfies $0 \leqslant p \leqslant n - 1$. For any $\delta > 0, \lambda \geqslant 0$ the metric $g^{n+2}_{\text{torp}}(\delta)_\lambda$ has positive $(p, n+2)$-intermediate scalar curvature. Moreover, this curvature can be bounded below with an arbitrarily large positive constant by choosing δ sufficiently small.*

Proof. Excluding the point at $r = 0$, the metric $g^{n+2}_{\text{torp}}(\delta)_\lambda$ is the warped product metric $dr^2 + \beta^2 ds^2_{n+1}$, where $\beta = \eta_{\delta,\lambda}$. Recall the Riemann curvatures of a warped product from Equation (4.5). Here the base is the one-dimensional $B = (0, \frac{\pi\delta}{2})$ and the fibers are $(n+1)$-spheres $F = S^{n+1}$ with constant sectional curvature equal to 1. Therefore Equation (4.5) reduces to

$$R(v, w, w, v) = \sum_{i<j}(v_i w_j - v_j w_i)^2 \beta^2\left(1 - (\beta_r)^2\right) - \sum_i (v_r w_i - v_i w_r)^2 \beta\beta_{rr}.$$

If P is a p-plane in T_xM, then $P^\perp$ has dimension $n - p + 2$. Since the dimension of $P^\perp + T_xS^{n+1}$ can be at most the dimension of T_xM,

$$\begin{aligned}\dim(P^\perp \cap T_xS^{n+1}) &= \dim(P^\perp) + \dim(T_xS^{n+1}) - \dim(P^\perp + T_xS^{n+1}) \\ &\geqslant (n - p + 2) + (n+1) - (n+2) \\ &= n - p + 1.\end{aligned}$$

Therefore, we have two cases.

Case 1. The projection of $P^\perp$ into T_xM has dimension $n - p + 1$. This means there is a direction in $P^\perp$ orthogonal to T_xS^{n+1}, and so we can take $\{\partial_1, \dots, \partial_{n-p+1}, \partial_r\}$ as an orthonormal basis for $P^\perp$ where $\partial_1, \dots, \partial_{n-p+1}$ are tangent to the sphere. Therefore the $(p, n+2)$-intermediate curvature is given by

$$\begin{aligned}s_{p,n+2}(P) &= \sum_{i,j=1}^{n-p+1} K(\partial_i, \partial_j) + 2\sum_{i=1}^{n-p+1} K(\partial_i, \partial_r) \\ &= (n-p+1)(n-p)\frac{1-\beta_r^2}{\beta^2} - 2(n-p+1)\frac{\beta_{rr}}{\beta}.\end{aligned} \tag{5.3}$$

Case 2. The projection of $P^\perp$ into T_xS^{n+1} has dimension $n-p+2$. Then we can take an orthonormal basis $\{\partial_1, \dots, \partial_{n-p+1}, v\}$ for $P^\perp$ where $\partial_1, \dots, \partial_{n-p+1}$ are tangent to the sphere. We can write $v = v_{n-p+2}\partial_{n-p+2} + v_r\partial_r$ where ∂_{n-p+2} is a unit vector in T_xS^{n+1} orthogonal to the other ∂_i. Therefore the $(p, n+2)$-intermediate sectional curvature is given by

$$\begin{aligned} s_{p,n+2}(P) &= \sum_{i,j=1}^{n-p+1} K(\partial_i, \partial_j) + 2\sum_{i=1}^{n-p+1} K(\partial_i, v) \\ &= (n-p+1)(n-p)\frac{1-\beta_r^2}{\beta^2} \\ &\quad + 2(n-p+1)\left[v_{n-p+2}^2\left(1-\beta_r^2\right) - v_r^2\frac{\beta_{rr}}{\beta}\right]. \end{aligned} \tag{5.4}$$

Since $\beta > 0$, $\beta_r^2 \leqslant 1$, and $\beta_{rr} \leqslant 0$, we have that $-\frac{\beta_{rr}}{\beta} \geqslant 0$ and $1-\beta_r^2 \geqslant 0$, so the second terms of Equations (5.3) and (5.4) are nonnegative. Consider $K_{ij} = \frac{1-\beta_r^2}{\beta^2}$. For $0 < r \leqslant \frac{\pi\delta}{4}$, we have $\beta(r) = \eta_{\delta,\lambda}(r) = \delta\sin\frac{r}{\delta}$ and $\beta'(r) = \cos\frac{r}{\delta}$. Therefore,

$$K_{ij} = \frac{1-\cos^2(\frac{r}{\delta})}{\delta^2\sin^2(\frac{r}{\delta})} = \frac{\sin^2(\frac{r}{\delta})}{\delta^2\sin^2(\frac{r}{\delta})} = \frac{1}{\delta^2}.$$

On the other hand, when $r \geqslant \frac{\pi\delta}{2}$, we have $\beta(r) = \eta_{\delta,\lambda}(r) = C\delta$ and $\beta_r(r) = 0$. Hence in this case, as $C < 1$, we immediately have $K_{ij} = \frac{1}{C^2\delta^2} > \frac{1}{\delta^2}$. In the transition region $\frac{\pi\delta}{4} < r < \frac{\pi\delta}{2}$, since $\eta_{\delta,\lambda}(r) \leqslant \delta\sin\frac{r}{\delta}$ and $\frac{\partial\eta_{\delta,\lambda}}{\partial r}(r) \leqslant \cos\frac{r}{\delta}$, then

$$K_{ij} \geqslant \frac{1-\cos^2(\frac{r}{\delta})}{\delta^2\sin^2(\frac{r}{\delta})} = \frac{1}{\delta^2}.$$

This means that as long as $r > 0$, the second terms of Equations (5.3) and (5.4) are non-negative, while the first terms consist of $K_{ij} \geqslant \frac{1}{\delta^2}$. Since $p \leqslant n-1$, then $n-p > 0$, and so the first terms are both strictly positive and can be bounded below by an arbitrarily large positive constant by choosing δ sufficiently small.

This leaves just the point at $r = 0$. Since $\beta(r) = \eta_{\delta,\lambda}(r) = \delta\sin\frac{r}{\delta}$ when r is near 0, and the sectional curvature is continuous, we can compute the curvature at this point using limits. Note that $\beta(0) = 0$, $\beta_r(0) = 1$, and $\beta_{rr}(0) = 0$. Since $\beta_{rrr}(r) = -\frac{1}{\delta^2}\cos\frac{r}{\delta}$ for r near 0, we also have $\beta_{rrr}(0) = -\frac{1}{\delta^2}$. Then we apply L'Hôpital's rule, and we get the sectional curvatures

$$\lim_{r\to 0}\frac{1-(\beta_r)^2}{\beta^2} = \lim_{r\to 0} -\frac{2\beta_{rr}\beta_r}{2\beta_r\beta} = \lim_{r\to 0} -\frac{\beta_{rr}}{\beta} = \lim_{r\to 0} -\frac{\beta_{rrr}}{\beta_r} = \frac{1}{\delta^2}.$$

Hence at $r = 0$, all sectional curvatures are in fact equal to the positive value $\frac{1}{\delta^2}$. Therefore $s_{p,n+2}(P) = 2(n-p+2)(n-p+1)\frac{1}{\delta^2}$ is positive, so that $g_{\mathrm{torp}}^{n+2}(\delta)_\lambda$ has positive $(p, n+2)$-intermediate scalar curvature that can be bounded below by an arbitrarily large positive constant by choosing δ sufficiently small. □

We also need to consider the product of a torpedo metric with an interval in the construction of the boot metric. The proof of the following proposition follows directly from Proposition 4.7 by setting $\omega(r, t) = 1$ and $B = (0, b_1) \times (0, 1)$.

Proposition 5.4. *Suppose $n \geqslant 1$ and p satisfies $0 \leqslant p \leqslant n-2$. For any $\delta > 0, \lambda \geqslant 0$ the metric $g_{\mathrm{torp}}^{n+1}(\delta)_\lambda + dt^2$ on the product $D^{n+1} \times (0, 1)$ has positive $(p, n+2)$-intermediate scalar curvature.*

5.3. The Toe Metric

In this section we discuss the construction of the toe metric $g_{\mathrm{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$. In Lemma 2.1 of Walsh [31] it is shown that the metric smoothing, necessary in constructing $g_{\mathrm{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$, can be done so as to preserve positive scalar curvature. In the following Lemma, we show that the construction described in [31] actually preserves positive $(p, n+2)$-curvature for $p \leqslant n-2$.

Lemma 5.5. *Suppose $n \geqslant 2$ and $0 \leqslant p \leqslant n-2$. For any $\delta, \lambda_1, \lambda_2 > 0$, the metric $g_{\mathrm{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$ has positive $(p, n+2)$-curvature. Moreover, this curvature can be bounded below with an arbitrarily large positive constant by choosing δ sufficiently small.*

Proof. The strategy of proof involves describing the metric as in Lemma 2.1 of [31] and then computing its $(p, n+2)$-curvature. Thus, we regard $g_{\mathrm{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2}$ as obtained by, firstly, tracing out a cylinder of torpedo metrics, $g_{\mathrm{torp}}^{n+1}(\delta)$, in one direction before, secondly, bending by an angle of $\frac{\pi}{2}$ to finish with another cylinder in an orthogonal direction. It is then easy to extend the "rectangular" part of this metric which takes the form $dr^2 + dt^2 + \delta^2 ds_n^2$ to obtain any desired pair of neck-lengths, $\lambda_1, \lambda_2 > 0$.

Along the region where the bending has taken place, this metric takes the form

$$g_{\text{toe}}^{n+2}(\delta)_{\lambda_1,\lambda_2} = dr^2 + \omega(r,t)^2 dt^2 + \beta(r)^2 ds_n^2,$$

for certain smooth warping functions $\omega : [0,b] \times [-2, \frac{\pi}{2}+2] \to [1,\infty)$ and $\beta : [0,b] \to [0,\infty)$. The function β is a torpedo function and so we can assume, for all $r \in [0,b]$, $\beta(r) = \eta_{\delta,\lambda}(r)$ as in Section 5.2, for some appropriate neck-length λ. Note that, there is a corresponding real-valued function, α, on $[0,b]$ which satisfies the Condition (5.1) above. The other warping function, ω, is now constructed to satisfy the following conditions:

$$\omega(r,t) = \begin{cases} 1 & t \in [-2,-1] \\ \mu(-t) + (1-\mu(-t))\alpha(r) & t \in [-1,0] \\ \alpha(r) & t \in [0, \frac{\pi}{2}] \\ \mu(t-\frac{\pi}{2}) + (1-\mu(t-\frac{\pi}{2}))\alpha(r) & t \in [\frac{\pi}{2}, \frac{\pi}{2}+1] \\ 1 & t \in [\frac{\pi}{2}+1, \frac{\pi}{2}+2], \end{cases}$$

where $\mu : [0,1] \to [0,1]$ is a cut-off function satisfying $\mu(t) = 0$ when t is near 0, $\mu(t) = 1$ when t is near 1 and $\mu'(t) \geqslant 0$ for all $t \in [0,1]$.

Given the form of the metric, recall from Proposition 4.7, the $(p, n+2)$-curvatures, $s_{p,n+2}(P)$, of this metric for a p-plane P in the tangent space have the form

$$s_{p,n+2}(P) = (n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2} + A^2(1-\beta_r)^2 - B^2\frac{\omega_{rr}}{\omega} - C^2\frac{\beta_{rr}}{\beta} - D^2\frac{\beta_r\omega_r}{\beta\omega},$$

for some numbers A, B, C, and D depending on the plane P.

Since $p \leqslant n-2$, the coefficient on the first term is non-zero. Positivity of this term follows from the definition of β as a δ-torpedo function, applying L'Hôpital's rule at $r = 0$. Indeed, as in Proposition 5.3, this term can be made arbitrarily large by choosing $\delta > 0$ sufficiently small. Similarly, the second term is nonnegative. The remaining terms involve either $-\omega_r\beta_r$, $-\omega_{rr}$ or $-\beta_{rr}$. By construction, ω_{rr} and β_{rr} are both non-positive. Hence, the corresponding terms above are also non-negative. Finally, ω_r is a non-negative constant multiple of α_r. Moreover, when α_r and β_r are non-zero, they have opposite signs, so the remaining terms above are also non-negative. This completes the proof. □

5.4. The Boot Metric

We now consider, $g_{\text{boot}}^{n+2}(\delta)_{\Lambda,\bar{l}}$, the boot metric introduced above. As our only concern here is establishing conditions for positivity of the $(p, n+2)$-curvature of such a metric, the values of the particular components of the vector $\bar{l}$ are unimportant and so we will suppress them from the notation, writing the metric simply as $g_{\text{boot}}^{n+2}(\delta)_\Lambda$. This metric can be regarded as consisting of four pieces:

$$g_{\text{boot}}^{n+2}(\delta)_\Lambda := \begin{cases} g_{\text{toe}}^{n+2}(\delta) & \text{on } R_1 = D_{\text{stretch}}^{n+2} \\ g_{\text{bend}}^{n+2}(\delta)_\Lambda & \text{on } R_2 \cong D^{n+1} \times [0,1] \\ g_{\text{torp}}^{n+1}(\delta) + dt^2 & \text{on } R_3 \cong D^{n+1} \times [0,1] \\ dr^2 + dt^2 + \delta^2 ds_n^2 & \text{on } R_4 \cong D^2 \times S^n, \end{cases} \tag{5.5}$$

as depicted in Fig. 5.4 below.

While the first, second and fourth components of this metric above are clearly defined, the third component $g_{\text{bend}}^{n+2}(\delta)_\Lambda$ requires some description. A detailed account of this construction is given in Section 5 of [32] and so we will be brief. As mentioned above, the metric component, $g_{\text{bend}}^{n+2}(\delta)_\Lambda$, is obtained by bending a cylinder of torpedo metrics $g_{\text{torp}}^{n+1}(\delta) + dt^2$ around a quarter circle. Importantly, the bend is in the opposite direction to that employed in Lemma 5.5, and, unlike in that case creates negative curvature. Provided we perform the bending slowly enough, that is provided the quarter circle has sufficiently large radius, we can minimize such negative curvature. The parameter Λ is the radius of this quarter circle. In section 5, page 892, of [32], the metric $g_{\text{bend}}^{n+2}(\delta)_\Lambda$ is defined as follows:

$$g_{\text{bend}}^{n+2}(\delta)_\Lambda := dr^2 + \omega_\Lambda(r,t)^2 dt^2 + \beta(r)^2 ds_n^2,$$

where $r \in [0,b]$, $t \in [-3, \frac{\pi}{2}+3]$. Here $\beta : [0,b] \to [0,\infty)$ is defined as:

$$\beta(r) = \eta_\delta(b-r),$$

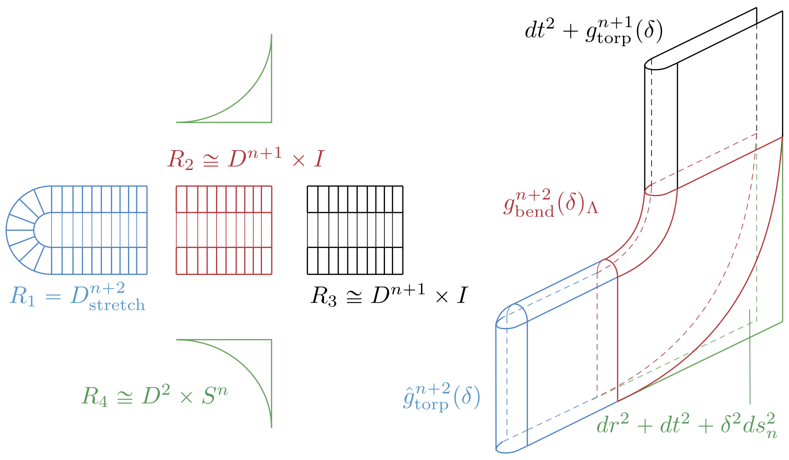

Fig. 5.4. The various components of the boot metric, to the left are the "assembly instructions" for the boot and to the right is the assembled boot.

with respect to a torpedo function η_δ as defined in Equation 5.2, where we suppress the neck length λ as it is unimportant. The corresponding smooth function α, is constructed from β to satisfy $\alpha_r^2 + \beta_r^2 = 1$, as defined in Equation (5.1). The function

$$\omega_\Lambda : [0, b] \times \left[-3, \frac{\pi}{2} + 3\right] \longrightarrow [1, \infty)$$

is now defined to satisfy the following properties:

(i)

$$\omega_\Lambda(r,t) = \begin{cases} 1 & \text{if } -3 \leqslant t \leqslant -2 \\ \Lambda & \text{if } -\frac{3}{2} \leqslant t \leqslant -\frac{1}{2} \\ \Lambda + \alpha(r) & \text{if } t \in [0, \frac{\pi}{2}] \\ \Lambda & \text{if } \frac{\pi}{2} + \frac{1}{2} \leqslant t \leqslant \frac{\pi}{2} + \frac{3}{2} \\ 1 & \text{if } \frac{\pi}{2} + 2 \leqslant t \leqslant \frac{\pi}{2} + 3; \end{cases}$$

(ii) $\omega_\Lambda(r,t) \geqslant \Lambda - \max\{|\alpha(r)| : r \in [0, b]\}$ when $t \in [-\frac{3}{2}, \frac{\pi}{2} + \frac{3}{2}]$;

(iii) $\dfrac{\partial w_\Lambda}{\partial r}(r,t) = 0$ when $t \in [-3, -\frac{1}{2}] \cup [\frac{\pi}{2} + \frac{1}{2}, \frac{\pi}{2} + 3]$;

(iv) $\left|\dfrac{\partial^{(k)} w_\Lambda}{\partial r^{(k)}}(r,t)\right| \leqslant \left|\dfrac{\partial^{(k)} \alpha}{\partial r^{(k)}}(r)\right|$ for all $k \in \{0, 1, 2, \cdots\}$; and

(v) $\dfrac{\partial^2 w_\Lambda}{\partial r^2} \leqslant 0$.

In Fig. 5.5 we provide a schematic description of this function on its rectangular domain $[0, b] \times [-3, \frac{\pi}{2} + 3]$, to aid the reader. The white regions in this picture indicate the smooth transition by way of a cut-off function in exactly the spirit of the cut-off function μ used when defining the warping function ω in the proof of Lemma 5.5.

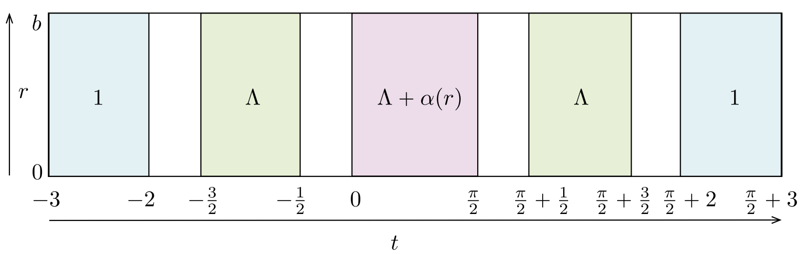

Fig. 5.5. The function ω_Λ.

Lemma 5.6. *For $n \geqslant 2$ and any $\delta > 0$ there is a positive constant Λ for which the metric $g^{n+2}_{\text{bend}}(\delta)_\Lambda$ has positive $(p, n+2)$-intermediate scalar curvature for all $p \in \{0, \cdots, n-2\}$.*

Proof. Making use of Proposition 4.7, with ω replaced by ω_Λ, we obtain the following formula for the intermediate scalar curvatures $s_{p,n+2}(P)$ for some P, a p-dimensional subspace of the tangent space.

$$s_{p,n+2}(P)=(n-p)(n-p-1)\frac{1-\beta_r^2}{\beta^2}+A^2(1-\beta_r)^2-B^2\frac{(\omega_\Lambda)_{rr}}{\omega_\Lambda}-C^2\frac{\beta_{rr}}{\beta}-D^2\frac{\beta_r(\omega_\Lambda)_r}{\beta\omega_\Lambda},$$

where A, B, C, and D are real valued functions dependent on the plane P, as defined in Proposition 4.7.

As with the proof of Lemma 5.5, the condition on p and the fact that β is a torpedo function, albeit pointing in the opposite direction, means that the first term is positive and the second term is nonnegative. As $\beta_{rr}\leqslant 0$ and $(\omega_\Lambda)_{rr}\leqslant 0$, the third and fourth terms are also nonnegative.

Unlike the case of Lemma 5.5 however, we do not get such a nice relationship between the signs of the first derivatives of β and α. This is because $\beta_r\geqslant 0$ in this case. Thus, there may be some negativity arising from the $\frac{\beta_r(\omega_\Lambda)_r}{\beta\omega_\Lambda}$ term. This negativity arises only in the region where $t\in[-\frac{1}{2},\frac{\pi}{2}+\frac{1}{2}]$, since $\frac{\partial\omega_\Lambda}{\partial r}=0$ off this region. By choosing Λ sufficiently large, this negativity can be minimized.

The first potential problem is that β tends to zero as r tends to zero. However, on this region ($t\in[-\frac{1}{2},\frac{\pi}{2}+\frac{1}{2}]$), we have $\omega_\Lambda=\Lambda+\alpha$. When r is near zero, α and β satisfy

$$\alpha(r)=\delta\cos(\frac{r}{\delta})\text{ and }\beta(r)=\delta\sin(\frac{r}{\delta}).$$

Thus when r is near zero, the quotient

$$\frac{(\omega_\Lambda)_r}{\beta}=-\frac{1}{\delta}.$$

The functions α, β, and their derivatives, all of which are bounded, are fixed and independent of Λ. Since the factor of D^2 in this term depends continuously on the choice of plane P which varies over the compact Grassmannian, $\mathrm{Gr}_p(TR_2)$, there is some choice of Λ sufficiently large to minimize the negative impact of these terms and ensure overall positivity of $s_{p,n+2}(P)$. □

Corollary 5.7. *For $n\geqslant 2$ and any $\delta>0$ there is a positive constant, Λ, for which the boot metric $g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda$ has positive $(p,n+2)$-intermediate scalar curvature for all $p\in\{0,\cdots,n-2\}$.*

Proof. Here we simply utilize the description in Equation (5.5), of $g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda$ and its four component metrics. On R_1, the metric $g^{n+2}_{\mathrm{toe}}(\delta)$ has positive $(p,n+2)$-intermediate scalar curvature by Lemma 5.5. On R_2, the metric $g^{n+2}_{\mathrm{bend}}(\delta)_\Lambda$ has positive $(p,n+2)$-intermediate scalar curvature by Lemma 5.6. On R_3, the metric $dt^2+g^{n+1}_{\mathrm{torp}}(\delta)$ has positive $(p,n+2)$-intermediate scalar curvature by Proposition 5.4.

Finally, on R_4, we have the metric $dr^2+dt^2+\delta^2ds_n^2$. By Proposition 4.7, using $\omega=1$ and $\beta=\delta$, we have for any p-plane,

$$s_{p,n+2}(P)=(n-p)(n-p-1)+A^2$$

for some function A of the plane P. Since $n-p\geqslant 2$, this is positive for all planes P so we have positive $(p,n+2)$-intermediate scalar curvature on R_3. □

5.5. The Product of the Boot with a Sphere

The next proposition generalizes the result of Corollary 5.7 to the product of a boot metric and a round sphere.

Proposition 5.8. *For $n\geqslant 2$, $m\geqslant 0$ and any $\delta>0$, there is a positive constant Λ for which the product metric $g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda+ds_m^2$ has positive $(p,n+m+2)$-intermediate scalar curvature for all $p\in\{0,\dots,n-2\}$.*

Proof. Let M be the product $B\times S^m$ where B is a disk with a boot metric $g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda$ and S^m is the unit round sphere for some $m\geqslant 0$ with the standard product metric

$$g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda+ds_m^2$$

Let $p\in\{0,\dots,n-2\}$ so that the boot $(B,g^{n+2}_{\mathrm{boot}}(\delta)_\Lambda)$ has positive $(p,n+2)$-intermediate scalar curvature for large enough Λ. If $x\in M$ and P is a p-plane in T_xM, then the orthogonal complement $P^\perp$ has dimension $n+m+2-p$. Since the sum $P^\perp+T_xB$ is at most the entire tangent space T_xM, we have

$$\begin{aligned}\dim(P^\perp\cap T_xB)&=\dim(P^\perp)+\dim(T_xB)-\dim(P^\perp+T_xB)\\&\geqslant(n+m+2-p)+(n+2)-(n+m+2)\\&=n+2-p.\end{aligned}$$

Therefore, we can take an orthonormal basis for $P^\perp$ consisting of vectors $\{e_1, \dots, e_{n+2-p}\}$ from T_xB and $\{v_1, \dots, v_m\}$ the other m vectors. Therefore the $(p, n+m+2)$-intermediate scalar curvature of P is

$$s_{p,n+m+2}(P) = \sum_{i,j} K(e_i, e_j) + 2\sum_{i,j} K(e_i, v_j) + \sum_{i,j} K(v_i, v_j). \tag{5.6}$$

If we denote the sectional curvatures of the boot and sphere by K_B and K_S respectively, and we have orthonormal vectors $v = v^B + v^S$ and $w = w^B + w^S$ in T_xM where v^B, w^B are tangent to the boot and v^S, w^S are tangent to the sphere, then recall the formula,

$$K(v, w) = \left|v^B \wedge w^B\right|^2 K_B(v^B, w^B) + \left|v^S \wedge w^S\right|^2 K_S(v^S, w^S).$$

For the first summand of Equation (5.6), since all of the ∂_i are entirely in T_xB and are still orthonormal there,

$$\sum_{i,j} K(e_i, e_j) = \sum_{i,j} K_B(e_i, e_j) = s_{n+2,p}(Q),$$

where Q is the p-plane in T_xB that is the orthogonal complement of span$(\{e_1, \dots, e_{n+2-p}\})$. As a $(p, n+2)$-curvature of the boot, so long as we choose a large enough Λ, this summand is strictly positive.

Moving on to the second summand of Equation (5.6), since v_j is orthogonal to e_i, which is already contained in T_xB, the projection v_j^B remains orthogonal to e_i. Therefore,

$$K(e_i, v_j) = \left|e_i \wedge v_j^B\right|^2 K_B(e_i, v_j^B) = \|v_j^B\|^2 K_B(e_i, v_j^B).$$

For the third summand, the only immediate simplification we can make is to use the fact that the sphere has constant curvature equal to 1,

$$\begin{aligned} K(v_i, v_j) &= \left|v_i^B \wedge v_j^B\right|^2 K_B(v_i^B, v_j^B) + \left|v_i^S \wedge v_j^S\right|^2 K_S(v_i^S, v_j^S) \\ &= \left|v_i^B \wedge v_j^B\right|^2 K_B(v_i^B, v_j^B) + \left|v_i^S \wedge v_j^S\right|^2. \end{aligned}$$

The second term of this only adds positivity, so we focus on the first term. That is, we need to deal with the sectional curvatures of the boot $K_B(v, w)$. There are four cases depending on where the projection of the point x is in Fig. 5.4. However by construction, R_1, R_3, and R_4 have nonnegative sectional curvature, so when the projection of x is in any of these regions, we have that the first summand of Equation (5.6) is strictly positive, while the other two are nonnegative. Therefore $s_{p,n+m+2}(P) > 0$ in these cases. All that remains is to consider the case when x projects into the R_2 region.

Since the metric on R_2 is given by

$$g^{n+2}_{\text{bend}}(\delta)_\Lambda = dr^2 + \omega_\Lambda(r,t)^2 dt^2 + \beta(r)^2 ds_n^2,$$

we have from Equation (4.10), with $\omega = \omega_\Lambda$,

$$\begin{aligned} \sum_{i,j} K_B(e_i, v_j^B) = \sum_{i,j} \frac{1}{\|v_j^B\|^2} \Bigg[&-(e_t v_r - e_r v_t)^2 \frac{(\omega_\Lambda)_{rr}}{\omega_\Lambda} + \sum_{k<\ell} (e_k v_\ell - e_\ell v_k)^2 (1-\beta_r^2) \\ &- \sum_k (e_k v_r - e_r v_k)^2 \frac{\beta_{rr}}{\beta} - \sum_k (e_k v_t - e_t v_k)^2 \frac{\beta_r (\omega_\Lambda)_r}{\beta\omega_\Lambda} \Bigg], \end{aligned}$$

and

$$\begin{aligned} \sum_{i,j} K_B(v_i^B, w_j^B) = \sum_{i,j} \frac{1}{|v_i^B \wedge w_j^B|^2} \Bigg[&-(v_i w_j - v_j w_i)^2 \frac{(\omega_\Lambda)_{rr}}{\omega_\Lambda} + \sum_{k<\ell} (v_k w_\ell - v_\ell w_k)^2 (1-\beta_r^2) \\ &- \sum_k (v_k w_r - v_r w_k)^2 \frac{\beta_{rr}}{\beta} - \sum_k (v_k w_t - v_t w_k)^2 \frac{\beta_r (\omega_\Lambda)_r}{\beta\omega_\Lambda} \Bigg]. \end{aligned}$$

Incorporating these two expressions into Equation (5.6), we have

$$\begin{aligned} s_{p,n+m+2}(P) &= \sum_{i,j} K_B(e_i, e_j) + 2\sum_{i,j} \|v_i^B\|^2 K_B(e_i, v_j^B) + \sum_{i,j} |v_i^B \wedge w_j^B|^2 K_B(v_i^B, w_j^B) + \sum_{i,j} |v_i^S \wedge w_j^S|^2 \\ &= s_{p,n+2}(Q) - A^2 \frac{(\omega_\Lambda)_{rr}}{\omega_\Lambda} + B^2(1-\beta_r^2) - C^2 \frac{\beta_{rr}}{\beta} - D^2 \frac{\beta_r(\omega_\Lambda)_r}{\beta\omega_\Lambda} + E^2, \end{aligned}$$

for real-valued functions A, B, C, D, E ranging over the choice of plane P. As in the proof of Lemma 5.6, the second, third, and fourth terms are all nonnegative, as is the sixth term. The first term is the $(p, n+2)$-intermediate scalar curvature of the boot metric, and by Corollary 5.7 this is positive provided we choose Λ large enough.

However, as in the proof of Lemma 5.6 we can make sure that the negativity contributed by the fifth term is bounded, and so ensure overall positivity of $s_{p,n+m+2}(P)$. $\square$

6. Proof of Theorem A

6.1. The Surgery Theorem of [16] and [27] for Positive (p, n)-Intermediate Scalar Curvature

We begin with a smooth manifold M of dimension n. Suppose $\phi : S^k \times D^{\ell+1} \to M$ is an embedding, where $n = k + \ell + 1$. Recall that a *surgery* on M, with respect to the embedding ϕ, is the construction of a manifold, M_ϕ obtained by removing the image of ϕ from M and using the restricted map $\phi|_{S^k \times S^\ell}$ to attach $D^{k+1} \times S^\ell$ along the common boundary.

The *trace of the surgery on* ϕ is the cobordism between M and M_ϕ, obtained by gluing the cylinder $M \times [0, 1]$ to the disk product $D^{k+1} \times D^{\ell+1}$ via the embedding ϕ. This is done by attaching $M \times \{1\}$ to the boundary component $S^k \times D^{\ell+1}$ through the composition $i \circ \phi : S^k \times D^{\ell+1} \to M \times \{1\}$ where $i : M \to M \times \{1\}$ is the inclusion $i(x) = (x, 1)$. After appropriate smoothing, we obtain the elementary cobordism $\bar{M}_\phi$.

Returning to the embedding, ϕ, we consider the family of rescaling maps

$$\begin{aligned} \sigma_\rho : S^k \times D^{\ell+1} &\longrightarrow S^k \times D^{\ell+1} \\ (x, y) &\longmapsto (x, \rho y), \end{aligned}$$

where $\rho \in (0, 1]$. We then set $\phi_\rho := \phi \circ \sigma_\rho$ and $N_\rho := \phi_\rho(S^k \times D^{\ell+1})$ and $N := N_1$. Thus, for any metric g on M and any $\rho \in (0, 1]$, $\phi_\rho^* g$ is the metric obtained by taking the restriction metric $g|_{N_\rho}$, pulling it back via ϕ to obtain the metric $\phi^* g|_{N_\rho}$ on $S^k \times D^{\ell+1}(\rho)$ and finally, via the obvious rescaling map σ_ρ, pulling it back to obtain a metric on $S^k \times D^{\ell+1}$.

The positive scalar curvature Surgery Theorem of [16] and [27] was generalized for positive (p, n)-intermediate scalar curvature in [21]. At its heart is the following theorem from [21].

Theorem 6.1. *[21] Let M be a smooth n-dimensional manifold and $\phi : S^k \times D^{\ell+1} \to M$ an embedding with $k + \ell + 1 = n$. We further assume that $\ell \geqslant 2$ and $p \in \{0, 1, \cdots, \ell - 2\}$. Then for any metric g on M with positive (p, n)-intermediate scalar curvature, there is a metric g_{std} with positive (p, n)-intermediate scalar curvature such that:*

1. *In the neighborhood $N_{1/2} = \phi_{1/2}(S^k \times D^{\ell+1})$, g_{std} pulls back to the metric*
$$\phi_{1/2}^* g_{\text{std}} = ds_k^2 + g_{\text{torp}}^{\ell+1}; \text{ and}$$
2. *Outside $N = \phi(S^k \times D^{\ell+1})$, $g_{\text{std}} = g$.*

The metric g_{std} is thus *prepared for surgery* or *standardized* on $N_{1/2}$. By removing part of the standard piece taking the form

$$(S^k \times D^{\ell+1}, ds_k^2 + g_{\text{torp}}^{\ell+1}),$$

and replacing it with

$$(D^{k+1} \times S^\ell, g_{\text{torp}}^{k+1} + ds_\ell^2),$$

we obtain a metric g' on M' with positive (p, n)-intermediate scalar curvature.

In order to prove our main theorem, we require one more fact: that the metrics g and g_{std} above are $(s_{p,n} > 0)$-isotopic. Proofs of this fact for the case of positive scalar curvature, that is when $p = 0$, can be found in Theorem 2.3 of [30] and in Ebert and Frenck [8]. More recently, Kordass [19], proved this fact for a variety of general curvature conditions including positive (p, n)-intermediate scalar curvature. In particular, Theorem 3.1 from [19], which we state below, is a special case of his results.

Lemma 6.2. *[19] Let M be a smooth n-dimensional manifold and let $\phi : S^k \times D^{\ell+1} \to M$ be an embedding with $k + \ell + 1 = n$, $\ell \geqslant 2$ and $p \in \{0, 1, \cdots, \ell - 2\}$. Then for any metric g on M with positive (p, n)-intermediate scalar curvature, there is an isotopy through $(s_{p,n} > 0)$-metrics, g_t, $t \in [0, 1]$ which satisfies the following conditions:*

1. *$g_t|_{M \setminus N} = g|_{M \setminus N}$ for all $t \in [0, 1]$, and*
2. *$g_0 = g$ and $g_1 = g_{\text{std}}$, where g_{std} is the metric obtained from g by Theorem 6.1 above and satisfies*
$$\phi_{1/2}^* g_{\text{std}} = ds_k^2 + g_{\text{torp}}^{\ell+1}.$$

We will now make use of this lemma to prove Theorem A, which we restate for the sake of the reader.

Theorem 6.3. *Let M be a smooth n-dimensional manifold, $\phi : S^k \times D^{\ell+1} \to M$, a smooth embedding, and $\{\bar{M}_\phi; M, M_\phi\}$, the trace of the surgery on ϕ. Suppose that $\ell \geqslant 2$ and $p \in \{0, 1, \cdots \ell - 2\}$. Then for any metric g on M with positive (p, n)-intermediate scalar curvature, there are metrics g_ϕ on M_ϕ and $\bar{g}_\phi$ on $\bar{M}_\phi$ satisfying:*

1. *The metrics g_ϕ and $\bar{g}_\phi$ have respectively positive (p, n) and $(p, n+1)$-intermediate scalar curvature on M_ϕ and $\bar{M}_\phi$, and*
2. *Near the boundary components M and M_ϕ of $\bar{M}_\phi$, $\bar{g}_\phi$ takes the form of the respective product metrics $\bar{g}_\phi = g + dt^2$ and $\bar{g}_\phi = g_\phi + dt^2$.*

Proof. We begin by employing Lemma 6.2 to obtain an isotopy, g_t for $t \in [0, 1]$, between $g_0 = g$ and $g_1 = g_{\text{std}}$, as defined above. Corollary 3.3 gives us a concordance, $\bar{g}$, on a cylinder $M \times [0, L+2]$ for some $L > 0$ which satisfies the following conditions:

$$\bar{g}|_{M\times[0,1]} = g + dt^2 \quad \textit{and} \quad \bar{g}|_{M\times[L+1,L+2]} = g_{\text{std}} + dt^2.$$

This concordance is schematically depicted in Fig. 6.1.

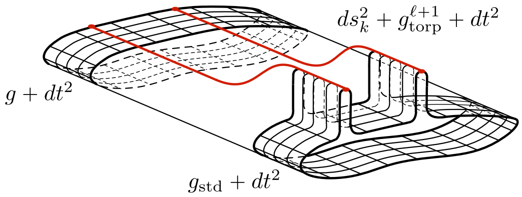

Fig. 6.1. The concordance $\bar{g}$.

Consider a boot metric, $g_{\text{boot}}^{\ell+2}(1)_{\Lambda,\bar{l}}$, as depicted in Fig. 5.2, now with $\delta = 1$. By Corollary 5.7, we may choose Λ to ensure positive $(p, n+1)$-curvature for $p \in \{0, \cdots, \ell - 2\}$. Recall here that $\bar{l}$ is a quadruple (l_1, l_2, l_3, l_4) determining the lengths of various sides of the boot metric, see Fig. 5.2. We set $l_1 = l_4 = 1$. Recall that l_2 and l_3 depend on Λ and may be large, so without loss of generality, we may assume that both are greater than 1. To simplify notation, we refer to this metric henceforth as $g_{\text{boot}}^{\ell+2}$.

We next extend the collar $M \times [L+1, L+2]$ to $M \times [L+1, L+1+l_3]$ and extend the metric $\bar{g}$ as $g_{\text{std}} + dt^2$ on this larger cylinder. Consider the restriction of the metric, $\bar{g}$, to the region, $N_{1/2} \times [L+1, L+l_3]$. Here, $\bar{g}$ takes the form

$$\bar{g}|_{N_{1/2}\times[L+1,L+1+l_3]} = ds_k^2 + g_{\text{torp}}^{\ell+1} + dt^2.$$

We replace $\bar{g}|_{N_{1/2}\times[L+1,L+1+l_3]}$ on $N_{1/2} \times [L+1, L+1+l_3]$ with the metric

$$ds_k^2 + g_{\text{boot}}^{\ell+2},$$

as depicted in Fig. 6.2, to obtain the metric

$$\bar{g}_{\text{boot}} := \begin{cases} \bar{g}|_{M\times[0,L+1]} & \text{on } M \times [0, L+1], \\ \bar{g}|_{(M\setminus N_{1/2})\times[L+1,L+1+l_3]} & \text{on } (M \setminus N_{1/2}) \times [L+1, L+1+l_3], \text{ and} \\ ds_k^2 + g_{\text{boot}}^{\ell+2} & \text{on } N_{1/2} \times [L+1, L+1+l_3]. \end{cases}$$

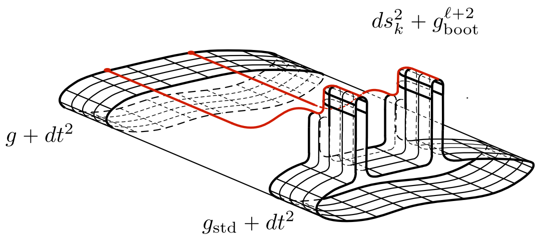

Fig. 6.2. The metric resulting from "attaching boots", $\bar{g}_{\text{boot}}$.

This metric has positive $(p, n+1)$-curvature for all $p \in \{0, 1, \cdots \ell - 2\}$. In particular, on $N_{1/2} \times [L+l_3, L+1+l_3]$, this metric takes the form of a product $g_{\text{std},l_2} + dt^2$, where

$$g_{\text{std}} = ds_k^2 + g_{\text{torp}}^{\ell+1}(1)_{l_2}.$$

In other words, this metric is a cylinder of standard metrics with torpedo necks stretched from length 1 to length l_2. The metric $\bar{g}_{\text{boot}}$ is readymade for surgery.

Returning to the region, $N_{1/2} \times [L+1, L+1+l_3]$, where the metric $\bar{g}_{\text{boot}}$ takes the form $ds_k^2 + g_{\text{boot}}^{\ell+1}$, we recall that the boot factor $g_{\text{boot}}^{\ell+1}$ takes the form of a toe, $\hat{g}_{\text{toe}}^{\ell+1}(1)$, see Fig. 5.4, on a subregion. As depicted in Fig. 6.3, we replace $ds_k^2 + \hat{g}_{\text{toe}}^{\ell+2}(1)$ with $g_{\text{torp}}^{k+1} + g_{\text{torp}}^{\ell+1}$ by "cutting off the toes" and introducing a "handle".

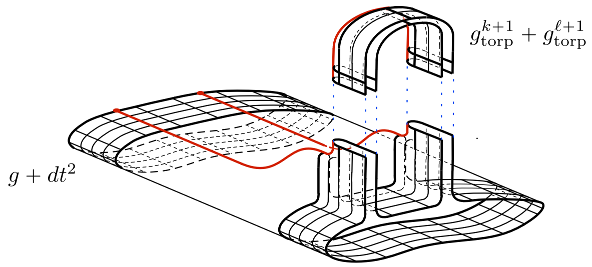

Fig. 6.3. Replacing the toes with a handle.

After gluing on this handle, we obtain the desired metric $\bar{g}_\phi$ on the trace $\bar{M}_\phi$ of the surgery as shown in Fig. 6.4.

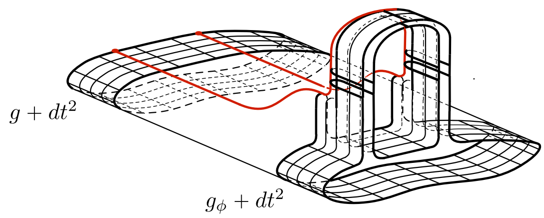

Fig. 6.4. The space $(\bar{M}_\phi, \bar{g}_\phi)$ obtained as the trace of the surgery.

This completes the proof of Theorem A. □

7. The Proof of Theorem B

The proof of Theorem B follows closely that of [5] for the case of positive scalar curvature. There is also a very readable summary of the method of [5] in section 4.2 of [29]. We will therefore provide only the relevant information and refer the reader to these sources for further detail. As before, we restate the theorem to aid the reader.

Theorem 7.1. *Let M be a smooth, closed, spin manifold of dimension $4n-1$, $n \geqslant 2$, which admits an $(s_{p,4n-1} > 0)$-curvature metric for some $p \in \{0, 1, \cdots, 2n-3\}$. Then $\mathcal{R}^{s_{p,4n-1}>0}(M)$ has infinitely many path components.*

Proof. We begin by considering the case where M is the sphere, S^{4n-1} for some $n \geqslant 2$. In section 4 of [5], the author constructs an infinite collection, X_q, $q \in \{1, 2, \cdots\}$, with the following properties:

(i) X_q is a smooth $4n$-dimensional manifold with boundary ∂X_q diffeomorphic to S^{4n-1}, the standard sphere;
(ii) X_q is homotopy equivalent to a finite wedge of $2n$-spheres; and
(iii) For $q_0 \neq q_1$, the closed manifold $\bar{X}_{q_0,q_1} := X_{q_0} \cup (S^{4n-1} \times I) \cup X_{q_1}$, where $I = [0,1]$ and obtained by gluing along common boundaries, has $\alpha(\bar{X}_{q_0,q_1}) \neq 0$. Thus $\bar{X}_{q_0,q_1}$ admits no metrics of positive scalar curvature.

These manifolds are constructed using the technique of *plumbing* disk bundles of the tangent bundle TS^{4n} with respect to certain graphs, see, for example, section 4.2.1 of [29]. It is well known that performing this construction with respect to the graph which is the Dynkin diagram of the exceptional Lie group E_8 results in a smooth manifold, X_{E_8}, whose boundary is homeomorphic to the sphere S^{4n-1}; see, for example, Ch. VI, Sec. 12 in Kozinski [20]. In the case of X_q, the plumbing construction is in accordance with graphs based on $q\theta_{4n-1}$ copies of the Dynkin diagram for E_8, where θ_{4n-1} is the order of the boundary, ∂X_{E_8}, in the group, Θ_{4n-1}, of homeomorphic smooth $(4n-1)$-spheres.

Let $W_q := X_q \setminus D^{4n}$ denote the result of removing a disk, D^{4n}, from the interior of X_q. Thus, we obtain a cobordism, $\{W_q; \partial D^{4n}, \partial X_q\}$, where each boundary component is diffeomorphic to S^{4n-1}. Making use of an appropriate Morse function, we may decompose this into a finite union of elementary cobordisms:

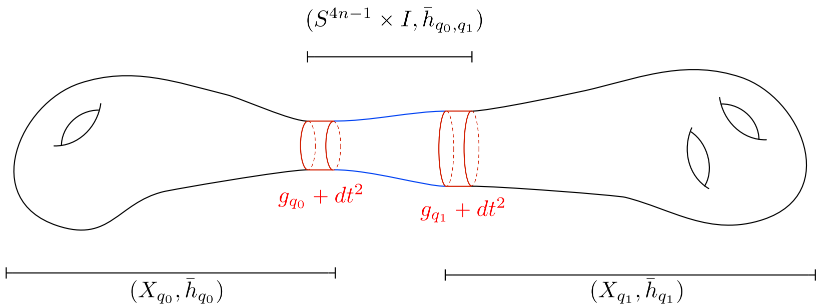

Fig. 7.1. The implied Riemannian manifold $\bar{X}_{q_0,g_1}, \bar{g}_{q_0,g_1}$.

$$\{W_q; S_0^{4n-1} = \partial D^{4n}, S_1^{4n-1} = \partial X_q\} = \{Y_1; S_0^{4n-1}, S_1^{4n-1}\} \cup \{Y_2; S_1^{4n-1}, S_2^{4n-1}\} \cup \cdots \cup \{Y_m; S_{m-1}^{4n-1}, S_m^{4n-1}\},$$

where S_0^{4n-1} is the boundary of the $4n$-dimensional disk we removed from the interior of X_q and $S_m^{4n-1} = \partial X_q$. Property (ii) above means that each elementary cobordism is the trace of a surgery on a $(2n-1)$-dimensional embedded sphere. Thus, each surgery takes place in codimension $(4n-1)-(2n-1) = 2n$.

Equipping S_0^{4n-1} with the round metric, ds^2_{4n-1}, and fixing collars near the boundaries of the individual elementary cobordisms, we repeatedly apply the geometric trace construction of Theorem A to obtain a Riemannian metric, $\bar{g}_q$ on W_q. Given that all surgeries are in codimension $2n$, we can ensure that W_q with the resulting metric, $\bar{g}_q$, has $s_{p,4n} > 0$ for $p = 2n-3$ and hence all $p \in \{0, 1, \cdots 2n-3\}$. Moreover, $\bar{g}_q$ takes a product structure near the boundary and the restriction $g_q = \bar{g}|_{\partial X_q}$ is a metric on S^{4n-1}, with $s_{p,4n-1} > 0$.

Suppose now that we apply this procedure for a pair $q_0 \neq q_1$. For each $i = 0, 1$, we can now form $s_{p,4n} > 0$ metrics, $\bar{h}_{q_i}$ on X_{q_i}, from $\bar{g}_i$ on W_{q_i}, by simply replacing the previously removed interior $4n$-disks and equipping the replacement disks with torpedo metrics g^{4n}_{torp}. Thus,

$$(X_{q_i}, \bar{h}_{q_i}) = (W_{q_i}, \bar{g}_{q_i}) \cup (D^{4n}, g^{4n}_{\text{torp}}).$$

The product structure near the round boundary $(4n-1)$-spheres mean these attachments are smooth.

If g_{q_0} and g_{q_1} lie in the same path component of $\mathcal{R}^{s_{p,4n-1}>0}(S^{4n-1})$, there is an $(s_{p,4n-1} > 0)$-isotopy and so, by Lemma 3.2, an $(s_{p,4n-1} > 0)$-concordance between them. We denote this concordance, $\bar{h}_{q_0,q_1}$. This allows us to construct the closed Riemannian manifold

$$(\bar{X}_{q_0,q_1}, \bar{g}_{q_0,q_1}) := (X_{q_0}, \bar{h}_{q_0}) \cup (S^{4n-1} \times I, \bar{h}_{q_0,q_1}) \cup (X_{q_1}, \bar{h}_{q_1}),$$

depicted in Fig. 7.1.

The metric $\bar{g}_{q_0,g_1}$ is a union of $(s_{p,4n} > 0)$-metrics and so is itself an $(s_{p,4n} > 0)$-metric for $p \in \{0, 1, \cdots, 2n-3\}$. However this is impossible since, by Property 3 above, $\bar{X}_{q_0,q_1}$ does not admit positive scalar curvature metrics when $q_0 \neq q_1$, and hence admits no $s_{p,n} > 0$ metric for any $p \geqslant 0$. Thus, there is no isotopy between g_{q_0} and g_{q_1} and, by implication, the space $\mathcal{R}^{s_{p,4n-1}}(S^{4n-1})$ has infinitely many path components. This proves Theorem B for the case $M = S^{4n-1}$.

We now consider the more general case of a closed smooth spin manifold, M, of dimension $4n-1$, which admits an $(s_{p,4n-1} > 0)$-metric, g_M, for $p \in \{0, 1, \cdots, 2n-3\}$. For any $(s_{p,4n-1} > 0)$-metric, g, on the sphere S^{4n-1}, the connected sum metric, $g_M \# g$, obtained by applying the surgery construction in Theorem A, is an $(s_{p,4n-1} > 0)$-metric also. We will show that for any pair $q_0 \neq q_1$, the metrics $g_M \# g_{q_0}$ and $g_M \# g_{q_1}$, where g_{q_0} and g_{q_1} are the metrics constructed above, lie in different path components of $\mathcal{R}^{s_{p,4n-1}>0}(M)$.

For each $i = 0, 1$, let W_{q_i} denote the manifold obtained above by removing a $4n$-dimensional disk, D^{4n}, from the interior of X_{q_i}. Thus, $\partial W_{q_i} = S_{i0}^{4n-1} \sqcup S_{i1}^{4n-1}$, where $S_{i0}^{4n-1} = \partial D^{4n}$ and $S_{i1}^{4n-1} = \partial X_{q_i}$. For $i, j = 0, 1$, we let the maps: $\tau_{ij} : S^{4n-1} \times [0, \epsilon) \longrightarrow W_{q_i}$, where $\tau_{ij}(S^{4n-1} \times \{0\}) = S_{ij}^{4n-1}$, denote the disjoint collar neighborhood embeddings, employed in the metric construction of Theorem A. Choose path embeddings, $\gamma_i : [0, 1] \to W_{q_i}$, $i = 0, 1$, satisfying the following compatibility conditions:

- the endpoints of γ_i satisfy $\gamma_i(0) \in S_{i0}^{4n-1}$ and $\gamma_i(1) \in S_{i1}^{4n-1}$,
- when t is near 0, $\tau_{i0}^{-1} \circ \gamma_i(t) = t$ while, when t is near 1, $\tau_{i1}^{-1} \circ \gamma_i(t) = 1-t$.

Finally, we specify a path $\gamma : I \to M \times I$, defined by $\gamma(t) = (x_0, t)$ for some fixed point $x_0 \in M$. By removing small tubular neighborhoods around γ_i and γ, we perform a slicewise-connected sum along these embedded paths to obtain $Z_{q_i} := W_{q_i} \#(M \times I)$, as depicted in Fig. 7.2. We will assume that in Z_{q_i}, our slicewise connected sum construction associates S_{ij}^{4n-1} with $M \times \{j\}$. Thus, Z_{q_i} is a manifold whose boundary is a disjoint union of two diffeomorphic copies of M.

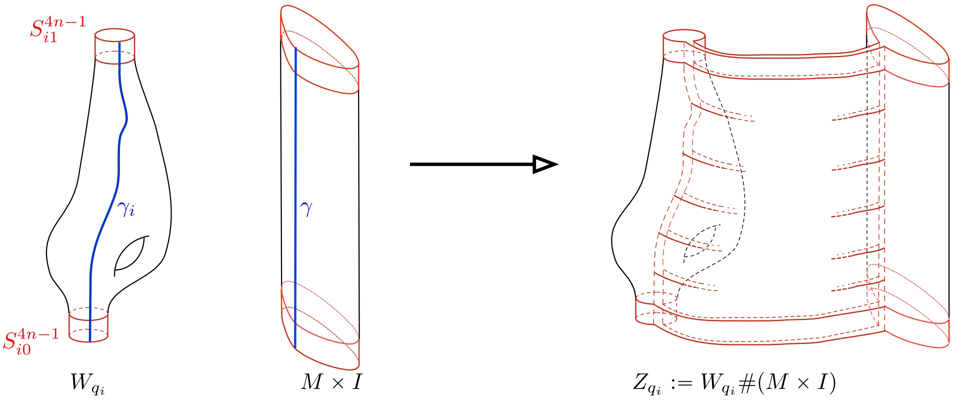

Fig. 7.2. Constructing the manifold Z_{q_i} as a slicewise connected sum (with respect paths γ_i and γ) of X_{q_i} and $M \times I$.

By Theorem A, with respect to the collars τ_{ij}, we have $(s_{p,n} > 0)$-metrics, $\bar{g}_{q_i}$ on W_{q_i} for each $i = 0, 1$. Each metric, $\bar{g}_{q_i}$, has a product structure near the boundary and, we assume, restricts as ds^2_{4n-1} near S^{4n-1}_{i0} and $g_{q_i} + dt^2$ near S^{4n-1}_{i1}. Employing the technique of Theorem A over the slicewise connected sum leads to a metric, $\bar{g}_{Z_{q_i}}$, on Z_{q_i} which has the following properties:

- The manifold Z_{q_i} with the metric $\bar{g}_{Z_{q_i}}$ has $s_{p,4n} > 0$ and has a product structure near the boundary.
- The metric $\bar{g}_{Z_{q_i}}$ takes the form, $ds^2_{4n-1} \# g_M$, on the $S^{4n-1}_{i0} \# (M \times \{0\}) \cong M$ boundary component.

The first of these properties is ensured by the earlier compatibility conditions on the paths, γ_i, which ensure the connected sum construction is constant near the boundary.

Thus, we can form the Riemannian manifold with boundary

$$(Z_{q_0,q_1}, \bar{g}_{Z_{q_0,q_1}}) := (Z_{q_0}, \bar{g}_{Z_{q_0}}) \cup (Z_{q_1}, \bar{g}_{Z_{q_1}}),$$

by gluing together the S^{4n-1}_{00} and S^{4n-1}_{10} boundary components where both metrics agree. By construction Z_{q_0,q_1} with this metric has $s_{p,4n} > 0$, has a product structure near its boundary and restricts respectively on its two spherical boundary components, S^{4n-1}_{01} and S^{4n-1}_{11}, as g_{q_0} and g_{q_1}.

As before, if g_{q_0} and g_{q_1} are $(s_{p,4n-1} > 0)$-isotopic, there is an $(s_{p,4n-1} > 0)$-concordance on the cylinder, $(M \times I, \bar{h}_{q_0,q_1})$. Attaching this cylinder to Z_{q_0,q_1}, so that each boundary of the cylinder is attached to one of the boundary components of Z_{q_0,q_1}, gives rise to a closed, $(s_{p,4n} > 0)$-Riemannian manifold:

$$(Y_{q_0,q_1}, \bar{g}_{Y_{q_0,q_1}}) := (Z_{q_0,q_1}, \bar{g}_{Z_{q_0,q_1}}) \cup (S^{4n-1} \times I, \bar{h}_{q_0,q_1}).$$

Now Y_{q_0,q_1} is easily seen to be diffeomorphic to the connected sum $X_{q_0,q_1} \# (M \times S^1)$. The additive property of the α-invariant over connected sums implies that:

$$\alpha(Y_{q_0,q_1}) = \alpha(X_{q_0,q_1} \# (M \times S^1)) = \alpha(X_{q_0,q_1}) + \alpha(M \times S^1) = \alpha(X_{q_0,q_1}) + 0 \neq 0.$$

The summand, $\alpha(M \times S^1)$, vanishes since M admits a psc-metric and hence, so does $M \times S^1$. Thus, Y_{q_0,q_1} admits no metric of positive scalar curvature and so we have a contradiction. Therefore, the metrics $g_M \# g_{q_0}$ and $g_M \# g_{q_1}$ lie in different path components of $\mathcal{R}^{s_{p,4n-1}>0}(M)$. $\square$

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