Gain-Scheduled Control: Relaxing Slow Variation Requirements by Velocity-Based Design

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Abstract

This paper investigates the application of the gain-scheduling approach to flight control tasks where the conditions required by conventional techniques need not be satisfied. The conditions required by conventional techniques are progressively relaxed and the design of gain-scheduled controllers under a range of conditions is addressed. In particular, the paper considers gain-scheduled control design in situations where the vehicle is manoeuvring aggressively far from equilibrium, the airspeed need not be slowly varying and/or scheduling on the instantaneous incidence angle is required.

Nomenclature

q body axis pitch rate (rad/s), α angle of incidence (rad), V airspeed (m/s)

η_Z normal acceleration (m/s^2), Z normal force (N), M pitching moment (N)

m mass of the missile (204.02 kg), I_{yy} pitching moment of inertia (247.437 kg m^2)

$\overline{q}$ dynamic pressure ($\overline{q} = \frac{1}{2}\rho V^2$, ρ is the air density), S characteristic area (0.0409 m^2)

c characteristic distance (0.228 m), $\overline{M}$ mach number, δ effective elevator deflection (rad)[2]

C_L aerodynamic force coefficient, C_M aerodynamic moment coefficient, f denotes $180/\pi$

π indexes equilibrium points of missile, π_o denotes a specific equilibrium point, ρ denotes the scheduling variable

$\rho_0, \delta_0, q_0, \alpha_0$ and η_{Z_0} are, respectively, the values of ρ, δ, q, α and η_Z at the equilibrium point π_o

Δ (and sometimes δ) denotes a perturbation quantity (e.g. $\Delta q = q - q_0$)

$\wedge$ denotes an estimated quantity, overstrike $\cdot$ denotes d/dt

$$M_\alpha(\rho_o) = \frac{1}{I_{yy}}\frac{\partial M}{\partial \alpha}\bigg|_{\rho_o}, \quad M_\delta(\rho_o) = \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta}\bigg|_{\rho_o}, \quad Z_\alpha(\rho_o) = \frac{1}{mV}\frac{\partial Z}{\partial \alpha}\bigg|_{\rho_o}, \quad Z_\delta(\rho_o) = \frac{1}{mV}\frac{\partial Z}{\partial \delta}\bigg|_{\rho_o}$$

$\mathbf{x}$, $\mathbf{w}$, $\mathbf{x}^i$, $\mathbf{w}^i$, ω, $\mathbf{z}$ denote state-vectors, $\mathbf{y}$ a system output, $\mathbf{r}$, u, ν system inputs

ε (with various subscripts) is used to denote residual terms

1. Introduction

Gain-scheduling control is widely employed in flight control applications, where high performance has to be achieved over a broad operating envelope. In the gain-scheduling design approach (see, for example, Refs 1 and 2), a non-linear controller is constructed by continuously interpolating, in some manner, between the members of a family of linear controllers. Each linear controller is, typically, associated with a specific equilibrium operating point of the aircraft and is designed to ensure that, locally to the equilibrium operating point, the performance requirements are met. By employing a series expansion linearisation which, locally to the equilibrium operating point, has similar dynamics to the aircraft, linear techniques may be used to resolve this local design task. Continuity is, therefore, maintained with established linear design techniques for which a considerable body of experience has been accumulated.

Whilst the traditional gain-scheduling approach described above is extremely successful in most flight control applications (Ref 3), conventional theoretical techniques for analysing the dynamics of gain-scheduled systems are poorly developed and provide little support for the gain-scheduling design approach (Ref 4). Moreover, the trend is towards aircraft and missile configurations where the conventional gain-scheduling conditions may not always be satisfied (see, for example, Refs 5 and 6). Gain-scheduled controllers are traditionally designed on the basis of the dynamics relative to a family of trim conditions assuming that the airspeed is slowly varying. However, during aggressive manoeuvring the vehicle may be far from equilibrium with rapidly varying airspeed (Ref 3 p523). In addition, the requirement to operate

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[2] The effective elevator deflection, δ, may represent the combined action of several control surfaces rather than simply the elevator alone. Note, in order to clarify the analysis, the dynamics of the associated actuation systems are neglected.

at high angles of attack can necessitate scheduling on instantaneous incidence angle rather than, for example, conventional flap scheduling on averaged incidence (Ref 3 p523). There is, consequently, interest in the literature in alternative nonlinear control design approaches such as dynamic inversion (see, for example, Refs 5 and 7). However, owing to the substantial body of experience which has been accumulated with gain-scheduling methods both with regard to meeting performance requirements and also such practical issues as safety certification, there is a strong incentive to retain the gain-scheduling approach. The aim of this paper is, therefore, to investigate the application of the gain-scheduling approach to flight control design tasks when the conditions required by conventional techniques need not be satisfied.

The velocity-based analysis and design framework, recently proposed in Refs 4, 8 and 9, associates a linear system with every operating point of a nonlinear system, not just the equilibrium operating points. This approach thereby relaxes the restriction to near equilibrium operation whilst maintaining the continuity with linear methods which is a principle advantage of the conventional gain-scheduling approach. Moreover, the velocity-based approach does *not* inherently involve a slow variation requirement unlike the conventional gain-scheduling approach which requires a slow variation restriction to keep the system close to the equilibrium operating points. It therefore provides a natural and unified framework for gain-scheduling analysis and design which addresses many of the shortcomings of conventional gain-scheduling analysis and design and is an appropriate framework within which to study the relaxation of the restrictions associated with the conventional gain-scheduling approach.

The paper is organised as follows. In section 2, a missile example is described which motivates the later discussion. The velocity-based gain-scheduling approach is briefly summarised in section 3 and applied to the missile example in section 4 to study the influence on the controller of progressively weaker assumptions regarding the rate of variation of the scheduling variable. The conclusions are summarised in section 5.

2. Motivating example

Consider a missile (similar to that studied by Refs 10 and 11) with longitudinal dynamics

$$\dot{q} = \frac{M}{I_{yy}}, \quad \dot{\alpha} = \frac{Z}{mV} + q, \quad \eta_Z = Z/m \tag{1}$$

with

$$Z = \overline{q}\, SC_L, \quad M = \overline{q}\, ScC_M \tag{2}$$

Consideration is restricted to the glide phase of the missile trajectory where the mass is constant and thrust is not applied. The aerodynamic force and moment coefficients, for mach numbers between 1 and 4, are

$$C_L = (-0.4 + 0.033\,\overline{M}\,)f\,\alpha - 0.034f\,\delta, \quad C_M = -0.3f\,\alpha - 0.206f\,\delta \tag{3}$$

The pitch rate and the normal acceleration are measured but the angle of incidence cannot be reliably measured. The requirement is to design a controller for the skid-to-turn missile which achieves a uniform normal acceleration step response over the entire flight envelope with a rise time (to 95% of final value) of around 0.3 seconds and overshoot less than 5%. Whilst this is, of course, not a complete performance specification, it is adequate for the present study.

The conventional gain-scheduling design approach requires appropriate linearisations of the dynamics which approximate, locally to specific equilibrium flight conditions, the nonlinear dynamic behaviour of the missile. Let the equilibrium operating points be parameterised by π; for example, $\begin{bmatrix} \delta & V & \overline{q} & \overline{M} \end{bmatrix}^T$. Adopting the standard short-period approximation (see, for example, Ref 3 p78, Ref 12), assume that, locally to a specific equilibrium operating point at which π equals π_o, the variations in the forward velocity, dynamic pressure and mach number associated with the pitch motion are negligible. The non-linear dynamics of the missile, (1), may then be approximated, locally to the specific equilibrium operating point π_o, by the series expansion linearisation,

$$\Delta\dot{q} = M_\alpha(\rho_0)\Delta\alpha + M_\delta(\rho_0)\Delta\delta, \quad \Delta\dot{\alpha} = Z_\alpha(\rho_0)\Delta\alpha + Z_\delta(\rho_0)\Delta\delta + \Delta q \tag{4}$$

$$\Delta\eta_Z = Z_\alpha(\rho_o)V(\rho_o)\Delta\alpha + Z_\delta(\rho_o)V(\rho_o)\Delta\delta \tag{5}$$

together with the input, output and state transformations

$$\Delta\delta = \delta - \delta_0, \quad q = q_0 + \Delta q, \quad \alpha = \alpha_0 + \Delta\alpha, \quad \eta_Z = \eta_{Z_0} + \Delta\eta_Z \tag{6}$$

where $\rho = \begin{bmatrix} V & \overline{q} & \overline{M} \end{bmatrix}^T$ and $\rho_0, \delta_0, q_0, \alpha_0$ and η_{Z_0} are, respectively, the values of ρ, δ, q, α and η_Z at the equilibrium operating point π_o and

$$M_\alpha(\rho_o) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\alpha}\bigg|_{\rho_o}, \quad M_\delta(\rho_o) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\delta}\bigg|_{\rho_o}, \quad Z_\alpha(\rho_o) = \frac{1}{mV}\frac{\partial Z}{\partial\alpha}\bigg|_{\rho_o}, \quad Z_\delta(\rho_o) = \frac{1}{mV}\frac{\partial Z}{\partial\delta}\bigg|_{\rho_o} \tag{7}$$

Differentiating, (4) may be reformulated as the linear short period equation

$$\Delta\ddot{\alpha} - Z_\alpha(\rho_o)\Delta\dot{\alpha} - M_\alpha(\rho_o)\Delta\alpha = M_\delta(\rho_o)\Delta\delta + Z_\delta(\rho_o)\Delta\dot{\delta} \tag{8}$$

Since the magnitude of Z_δ is much less than the magnitude of M_δ (specifically, $|Z_\delta/M_\delta|$ is less than 0.003 over the entire flight envelope), (8) may, for control design purposes, be simplified to,

$$\Delta\ddot\alpha - Z_\alpha(\rho_o)\Delta\dot\alpha - M_\alpha(\rho_o)\Delta\alpha = M_\delta(\rho_o)\Delta\delta \tag{9}$$

which is valid locally to the specific equilibrium operating point π_o. Hence, the linearised plant dynamics at the equilibrium operating point π_o are described by (9) and (5) together with the input, output and state transformations (6). It should be noted that, whilst the input, output and state transformations (6) are different at every equilibrium operating point, the linearised dynamics, (9) and (5), are the same at equilibrium operating points for which ρ equals ρ_o. Hence, (9) and (5) define a linear dynamic family parameterised by ρ.

For each member of the linear family, (9) and (5), a linear controller is designed to meet the performance specification. A cascaded inner-outer loop controller configuration is employed with an attitude inner loop and a normal acceleration outer loop; see Appendix A for details. The resulting linear controller family, expressed in state-space form, is

$$
\begin{bmatrix} \Delta\dot{\hat\alpha} \\ \Delta\dot x_1 \\ \Delta\dot x_2 \\ \Delta\dot x_3 \\ \Delta\dot x_4 \end{bmatrix} =
\begin{bmatrix}
Z_\alpha(\rho_o)-Z_\delta(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} & Z_\delta(\rho_o) & 0 & 0 & Z_\delta(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)}\dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} \\[2mm]
(K_2+Z_\alpha(\rho_o))\dfrac{K_1}{M_\delta(\rho_o)} & -K_2 & 1 & 0 & -(K_2+Z_\alpha(\rho_o))\dfrac{K_1}{M_\delta(\rho_o)}\dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} \\[2mm]
M_\alpha(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} & 0 & 0 & 0 & -M_\alpha(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)}\dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} \\[2mm]
0 & 0 & 0 & -2\varsigma w_n & 0 \\[1mm]
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} \Delta\hat\alpha \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix}
+\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} \Delta q \\ \Delta\eta_{Z_d}-\Delta\eta_Z \end{bmatrix} \tag{10}
$$

$$
\Delta\delta = \begin{bmatrix} -\dfrac{K_1}{M_\delta(\rho_o)}\dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} & 1 & 0 & 0 & \dfrac{K_1}{M_\delta(\rho_o)}\dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} \end{bmatrix}
\begin{bmatrix} \Delta\hat\alpha \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} \tag{11}
$$

where $\Delta\eta_{Z_d}$ is the perturbation in the demanded normal acceleration and $K_1 = 2500$, $K_2 = 70$, $w_n = 10$ rad/s, $\zeta = 0.7$. It should be noted that, since the controllers are designed on the basis of the linearisation of the missile dynamics at trim flight conditions, the linear controllers act on perturbed quantities. The input, output and state transformations relating the perturbed quantities to their measured counterparts depend on the specific equilibrium flight condition considered. Hence, a family of input, output and state transformations, parameterised by the equilibrium flight condition, is also associated with the family of linear controller transfer functions.

The family of linear controllers is parameterised by ρ; that is, by the airspeed, dynamic pressure and mach number of the missile. Perhaps the most widespread approach to constructing a non-linear controller realisation is to simply substitute these scheduling variables into the linear controller family, (10)-(11); that is, simply vary the gains of the controller according to the airspeed, dynamic pressure and mach number (see, for example, Ref 3 p298, Refs 10 and 12) whilst relying on the feedback action to accommodate the input, output and state transformations. The resulting non-linear controller, denoted controller realisation A, is depicted in figure 1. The response, obtained using non-linear simulations, of the non-linear controller to a sequence of step demands in normal acceleration is shown in figure 2 (the mach number time history, shown in figure 3, is employed[3],). Evidently, the non-linear controller achieves the performance specification of a uniform normal acceleration step response over the flight envelope with rise time of around 0.3 seconds and overshoot less than 5%.

The conventional gain-scheduling approach followed in the preceding discussion does not uniquely define the nonlinear controller. State-space representations of a linear controller which are related by a non-singular linear state transformation are linearly equivalent; that is, they have the same transfer functions. Hence, each member of the family of linear controllers, (10)-(11), has infinitely many state-space representations. For example, an alternative controller family is

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[3] Only a very simple representation of the missile is employed and, in particular, the translational dynamics are not modelled. The airspeed, dynamic pressure and mach number time histories are generated by auxiliary equations in a similar manner to Ref 10. Nevertheless, the model is sufficient to illustrate the dependence of the closed-loop dynamics on the rate of variation of these quantities.

$$
\begin{bmatrix} \Delta\dot{\hat{\alpha}} \\ \Delta\dot{\tilde{x}}_1 \\ \Delta\dot{\tilde{x}}_2 \\ \Delta\dot{\tilde{x}}_3 \\ \Delta\dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} Z_\alpha(\rho_o) - Z_\delta(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} & Z_\delta(\rho_o) & 0 & 0 & Z_\delta(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} \\ (K_2 + Z_\alpha(\rho_o))\dfrac{K_1}{M_\delta(\rho_o)} & -\hat{K}_2 & 1 & 0 & -(K_2 + Z_\alpha(\rho_o))\dfrac{K_1}{M_\delta(\rho_o)} \\ M_\alpha(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} & 0 & 0 & 0 & -M_\alpha(\rho_o)\dfrac{K_1}{M_\delta(\rho_o)} \\ 0 & 0 & 0 & -2\varsigma w_n & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{\alpha} \\ \Delta\tilde{x}_1 \\ \Delta\tilde{x}_2 \\ \Delta\tilde{x}_3 \\ \Delta\tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \dfrac{w_n^2}{Z_\alpha(\rho_o)V(\rho_o)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\eta_{Z_d} - \Delta\eta_Z \end{bmatrix} \quad (12)
$$

$$
\Delta\delta = \begin{bmatrix} -\dfrac{K_1}{M_\delta(\rho_o)} & 1 & 0 & 0 & \dfrac{K_1}{M_\delta(\rho_o)} \end{bmatrix} \begin{bmatrix} \Delta\hat{\alpha} \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta\alpha_d \end{bmatrix} \quad (13)
$$

The linear family, (12)-(13), employs a different choice of state from (10)-(11). However, it is emphasised that the members of the families of linear controllers defined by (10)-(11) and (12)-(13) are dynamically equivalent; that is, they have the same transfer functions. In this example, the linear family, (12)-(13), is obtained by simply altering the location of controller outer loop gain which in realisation A is scheduled. The non-linear controller realisation constructed by directly scheduling the controller gains of the linear family, (12)-(13), with respect to airspeed, dynamic pressure and mach number, denoted controller realisation B, is depicted in figure 4. The response of this non-linear controller to a sequence of step demands in normal acceleration is also shown in figure 2. Evidently, the performance of non-linear controller realisations A and B is markedly different. Whilst the non-linear controller with realisation A achieves the performance specification, the response with the non-linear controller with realisation B fails to satisfy the rise time and overshoot specification. Indeed, with this controller realisation the response is quite oscillatory and, perhaps, divergent.

Clearly, the performance attained in this example is sensitive to the choice of controller realisation. Since the controllers are different realisations for the same family of linear controller transfer functions the dynamic behaviour, and consequently the closed-loop performance, of the controllers is similar when the scheduling variable varies sufficiently slowly (trivially, when the rate of change is zero). However, since the airspeed, dynamic pressure and mach number are directly related to the normal acceleration of the missile, the variations in the scheduling variables are not *a priori* slowly varying relative to the dynamics of the controller; in particular, with respect to the dynamics of the acceleration outer loop controller (as demonstrated by the difference in the response of realisation A and realisation B). Moreover, whilst the controllers are designed on the basis of the missile dynamics relative to trim conditions, the missile motion in this example is not confined to small perturbations about trim but rather involves aggressive manoeuvring which takes the missile far from equilibrium. The assumptions underlying the conventional gain-scheduling procedure are clearly not met. Nevertheless, the controller realisation A *does* meet the performance specification. Hence, the potential clearly exists to extend the gain-scheduling design methodology to accommodate situations, such as the present example, where the conventional approach is no longer warranted.

It should be noted that the selection of an appropriate gain-scheduling controller realisation is considered by Refs 13-15. The utility of the former approach is, however, somewhat limited in general (Refs 16 and 17). In the latter approach, a controller realisation is sought which leads to the weakest slow variation requirement within the context of conventional gain-scheduling; that is, on the basis of the plant dynamics relative to the equilibrium operating points. However, the slow variation requirement can, in general, be further weakened by exploiting knowledge of the plant dynamics at non-equilibrium operating points (Ref 8). In the present missile example, the controller realisations do not belong to either of the classes studied by Refs 13-15.

3. Velocity-based gain-scheduling

Despite the widespread application of gain-scheduled controllers, conventional theoretical techniques for analysing the dynamics of gain-scheduled systems are rather poorly developed. A detailed review is presented in Ref 4. Since the conventional gain-scheduling design approach is based on combining linear controllers designed on the basis of plant linearisations about a number of equilibrium operating points, the resulting nonlinear controller is only appropriate, in general, in the vicinity of the equilibrium operating points. This, in turn, implies an inherent slow variation requirement to keep the system trajectories close to the equilibrium operating points. Nonetheless, the stability of the closed-loop dynamics, locally to a specific equilibrium operating point, can be analysed by series expansion linearisation theory. In addition, provided the family of equilibrium operating points can be parameterised by the input to the closed loop system, frozen-input theory may be employed to analyse certain dynamic properties near the family of equilibrium operating points. However, the latter analysis is restricted to a small neighbourhood of the equilibrium operating points. Furthermore, it is based on a frozen-input representation of the controlled system which is quite distinct from the mixed

series-expansion/frozen-scheduling variable representation[4] employed in the conventional gain-scheduling design procedure. Similarly, whilst the *frozen-scheduling variable* linearisation is employed during the design procedure, the *series expansion* linearisation of the controller is employed when analysing the dynamic behaviour locally to a single equilibrium operating point. Hence, the analysis of the dynamic properties of the controlled system near a specific equilibrium operating point, does not reduce to either the series expansion analysis or the mixed series-expansion/frozen-scheduling variable analysis employed in the design procedure. The foregoing analysis tools therefore provide limited support for the conventional gain-scheduling design approach and are unsuitable for investigating the application of gain-scheduling to flight control tasks when the aircraft motion may take it far from equilibrium and the scheduling variable may not be slowly varying.

The velocity-based analysis and design framework, proposed recently in Refs 4,8 and 9, associates a linearisation with every operating point of a nonlinear system, not just the equilibrium operating points. The approach thereby relaxes the restriction to near equilibrium operation whilst maintaining the continuity with linear methods which is a principle advantage of conventional gain-scheduling. Moreover, the velocity-based approach does *not* inherently involve a slow variation requirement. It therefore provides an appropriate framework within which to investigate the relaxation of the restrictions associated with conventional gain-scheduling. In contrast to the conventional analysis methods discussed above, the velocity-based framework provides a unified framework for gain-scheduling analysis and design which utilises a single type of linearisation, namely the velocity-based linearisation, for both analysis and design. It is briefly summarised below.

Consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{14}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable with bounded, Lipschitz continuous derivatives. The set of equilibrium operating points of the nonlinear system, (14), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which

$$\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0 \tag{15}$$

Let $\Phi:\Re^n\times\Re^m$ denote the space consisting of the product of the state, $\mathbf{x}$, with the input, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear system, (14), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the system to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

Suppose that the nonlinear system, (14), is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, in Φ and at time, t_1, the trajectory has reached the point, $(\mathbf{x_1}, \mathbf{r_1})$. From Taylor series expansion theory, the subsequent behaviour of the nonlinear system can be approximated, locally to the point, $(\mathbf{x_1}, \mathbf{r_1})$, by the first order representation,

$$\delta\dot{\hat{\mathbf{x}}} = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) + \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\delta\hat{\mathbf{x}} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\delta\mathbf{r} \tag{16}$$

$$\delta\hat{\mathbf{y}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\delta\hat{\mathbf{x}} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\delta\mathbf{r} \tag{17}$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_1}, \quad \hat{\mathbf{y}} = \mathbf{G}(\mathbf{x_1}, \mathbf{r_1}) + \delta\hat{\mathbf{y}}, \quad \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x_1}, \quad \dot{\hat{\mathbf{x}}} = \delta\dot{\hat{\mathbf{x}}} \tag{18}$$

provided $\mathbf{x_1}+\delta\hat{\mathbf{x}} \subseteq N_\mathbf{x}$ $\mathbf{r_1}+\delta\mathbf{r}\subseteq N_\mathbf{r}$, where the neighbourhoods, $N_\mathbf{x}$ and $N_\mathbf{r}$, of, respectively, $\mathbf{x_1}$ and $\mathbf{r_1}$ are sufficiently small. It should be noted that the expansion is carried out relative to the fixed operating point, $(\mathbf{x_1},\mathbf{r_1})$, as opposed to a trajectory passing through $(\mathbf{x_1},\mathbf{r_1})$, and the derivative of $\mathbf{x_1}$ is, therefore, zero. It is emphasised that (16)-(18) is a well-defined system in its own right, distinct from the nonlinear system (14), with state $\delta\hat{\mathbf{x}}$. Furthermore, the point, $(\mathbf{x_1},\mathbf{r_1})$, need not be an equilibrium operating point and, indeed, may lie far from the locus of equilibrium operating points. When $\delta\hat{\mathbf{x}}(t_1)$ is zero, the solution, $\hat{\mathbf{x}}(t)$, to (16)-(18), initially at time t_1, is tangential to the solution, $\mathbf{x}(t)$, of (14). Indeed, locally to time t_1, $\hat{\mathbf{x}}(t)$ provides a second-order approximation to $\mathbf{x}(t)$ and $\hat{\mathbf{y}}(t)$ provides a first-order approximation to $\mathbf{y}(t)$ (Ref 4).

The solution, $\hat{\mathbf{x}}(t)$, to the first-order series expansion, (16)-(18), provides a valid approximation only while the solution, $\mathbf{x}(t)$, to the nonlinear system remains in the vicinity of the operating point, $(\mathbf{x_1}, \mathbf{r_1})$. However, the solution, $\mathbf{x}(t)$, to the nonlinear system need not stay in the vicinity of a single operating point. Consider the time interval, [0,T]; the initial time can, without loss of generality, always be taken as zero. An approximation to $\mathbf{x}(t)$ is obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is the solution to the first-order series expansion relative to the operating point reached at the initial time for the sub-interval (with the initial conditions for each sub-interval chosen to ensure continuity of the approximate solution). The number of local solutions employed is dependent on the duration of the sub-intervals, but the local solutions are accurate to second order; that is, the

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[4] In the conventional gain-scheduling approach, a nonlinear controller is obtained by combining, in some manner, the members of a family of linear controllers. A typical procedure is simply to let the scheduling variable vary over the linear controller family whilst neglecting the input, output and state transformations relating the input, output and state of the nonlinear plant and controller to the perturbed quantities with respect to which the members of the linear plant and controller families are described. Hence, although the *series expansion* linearisations of the plant are employed in the design procedure, the corresponding local controller designs are *frozen-scheduling variable* linearisations of the resulting nonlinear controller.

approximation error is proportional to the duration of the sub-interval cubed. Hence, as the number of sub-intervals increases, the approximation error associated with each rapidly decreases and the overall approximation error also decreases. Indeed, the overall approximation error tends to zero as the maximum size of the sub-intervals tends to zero (Ref 4). Hence, the family of first-order series expansions, with members defined by (16)-(18), can provide an arbitrarily accurate approximation to the solution of the nonlinear system. Moreover, this approximation property holds throughout Φ and is not confined to the vicinity of a single equilibrium operating point or even of the locus of equilibrium operating points.

However, the state, input and output transformations, (18), depend on the operating point relative to which the series expansion is carried out. When the solution to the nonlinear system is confined to a neighbourhood about a single operating point, the transformations, (18), are static and the dynamic behaviour is described by the system, (16)-(17), alone. However, when the solution to the nonlinear system traces a trajectory which is not confined to a neighbourhood about a single operating point, the transformations, (18), are no longer static and the dynamic behaviour is no longer described solely by the system, (16)-(17). Instead, the dynamic behaviour is described by (16)-(18). Combining (16) and (17) with the local input, output and state transformations, (18), each member, (16)-(18), of the family of first-order representations may be reformulated as,

$$\dot{\hat{\mathbf{x}}} = \{\mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) - \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{x_1} - \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r_1}\} + \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r} \tag{19}$$

$$\hat{\mathbf{y}} = \{\mathbf{G}(\mathbf{x_1}, \mathbf{r_1}) - \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{x_1} - \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\mathbf{r_1}\} + \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r} \tag{20}$$

In contrast to the representation, (16)-(17), the state, input and output are now the same for all members of the reformulated family. The dynamics, (19)-(20), of an individual member of the family are affine rather than linear even when $(\mathbf{x_1}, \mathbf{r_1})$ is an equilibrium operating point. The inhomogeneous terms in (19)-(20) may, in general, be extremely large and can dominate the solution.

On differentiating (19)-(20)

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{21}$$

$$\dot{\hat{\mathbf{w}}} = \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \tag{22}$$

$$\dot{\hat{\mathbf{y}}} = \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \tag{23}$$

The system, (21)-(23), is dynamically equivalent to the system, (19)-(20), in the sense that with appropriate initial conditions, namely,

$$\hat{\mathbf{x}}(t_1) = \mathbf{x_1}, \quad \hat{\mathbf{w}}(t_1) = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}), \quad \hat{\mathbf{y}}(t_1) = \mathbf{G}(\mathbf{x_1}, \mathbf{r_1}) \tag{24}$$

the solution, $\hat{\mathbf{x}}$, to (21)-(23), is the same as the solution, $\hat{\mathbf{x}}$, to (19)-(20). However, in contrast to (19)-(20), the transformed system, (21)-(23), is linear. The relationship between the nonlinear system and its velocity-based linearisation, (21)-(23), is direct. Differentiating (14), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \tag{25}$$

$$\dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{26}$$

$$\dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{27}$$

Dynamically, (25)-(27), with appropriate initial conditions corresponding to (24), and (14) are equivalent (have the same solution, $\mathbf{x}$). (When $\mathbf{w} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$ is invertible for every $(\mathbf{x}, \mathbf{r})$, so that $\mathbf{x}$ may be expressed as a function of $\mathbf{w}, \mathbf{r}$ and $\mathbf{y}$, then the transformation relating (25)-(27) to (14) is, in fact, algebraic). When

$$\hat{\mathbf{x}}(t_1) = \mathbf{x}(t_1), \quad \hat{\mathbf{w}}(t_1) = \mathbf{w}(t_1), \quad \hat{\mathbf{y}}(t_1) = \mathbf{y}(t_1) \tag{28}$$

$\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{y}}(t)$ still provide a second- and first-order approximation to, respectively, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ and $\hat{\mathbf{w}}(t)$ provides a first-order approximation to $\mathbf{w}(t)$ (Ref 4). Clearly, the velocity-based linearisation, (21)-(23), is simply the frozen form of (25)-(27) at the operating point, $(\mathbf{x_1}, \mathbf{r_1})$.

There exists a velocity-based linearisation, (21)-(23), for every point in Φ. Hence, a velocity-based linearisation family, with members defined by (21)-(23), can be associated with the nonlinear system, (14). Similarly to the family of first-order expansions, the solutions to the members of the family of velocity-based linearisations, (21)-(23), can be pieced together (with the initial conditions for each sub-interval chosen to ensure continuity of $\hat{\mathbf{x}}$, $\hat{\mathbf{w}}$ and $\hat{\mathbf{y}}$) to approximate the solution to the nonlinear system, (25)-(27) to an arbitrary degree of accuracy (Ref 4). Hence, the velocity-based linearisation family embodies the entire dynamics of the nonlinear system, (14), with no loss of information and provides an alternative representation of the nonlinear system. Whilst the velocity-based representation is equivalent to the direct representation, (14), in the sense that they each embody the entire dynamics of the nonlinear system, they are not necessarily equivalent with respect to other considerations. In particular, the direct relationship between the velocity-form of the nonlinear system and the velocity-based linearisation family and the linearity of the members of the latter family provides continuity with established linear theory which, for example, facilitates analysis (Ref 4) and controller design (Refs 5,6).

With regard to controller design, the velocity-based linearisation of the feedback combination of a plant and controller is simply the feedback combination of the velocity-based linearisations of the plant and controller (Ref 5). The following velocity-based gain-scheduling design procedure is, therefore, appropriate.

1. Determine the velocity-based linearisation family associated with the nonlinear plant dynamics.

2. Based on the velocity-based linearisation family of the plant, determine the required velocity-based linearisation family of the controller such that the resulting closed-loop family achieves the performance requirements. Since each member of the plant family is linear, conventional *linear* design methods can be utilised to design each corresponding member of the controller family. Of course, it is necessary to ensure that the members of the controller family are, in some appropriate sense, compatible with one another.

3. Implement a nonlinear controller with the velocity-based linearisation family designed at step 2. This step is discussed in detail in Refs 4 and 5. When the controller contains integral action, the controller velocity-based linearisation family may be implemented directly in the velocity-based form, (25)-(27).

This design procedure retains a divide and conquer approach and maintains the continuity with linear design methods which is an important feature of the conventional gain-scheduling approach. However, in contrast to the conventional gain-scheduling approach, the resulting nonlinear controller is valid throughout the operating envelope of the plant, not just in the vicinity of the equilibrium operating points. This extension is a direct consequence of employing the velocity-based linearisation framework rather than the conventional series expansion linearisation about an equilibrium operating point.

Stability of the nonlinear closed-loop system is guaranteed provided the members of the closed-loop velocity-based linearisation family are uniformly stable, the mapping from $\mathbf{w}$ to $\mathbf{x}$ is bounded (*e.g.* when $\nabla_\mathbf{x}\mathbf{F}(\mathbf{x}, \mathbf{r})$ is integrable and the integral uniformly invertible) and the class of inputs and initial conditions is restricted to limit the rate of evolution of the solution trajectories to be sufficiently small (Ref 4). In addition, provided that the rate of evolution is sufficiently slow, the nonlinear system inherits the stability robustness of the members of the velocity-based linearisation family. Nevertheless, it is emphasised that the velocity-based gain-scheduling methodology does *not* inherently involve a slow variation requirement: rather, a number of factors which are a function of the controller design, and which the designer is free to adjust, contribute towards any limitation on the rate of evolution of the trajectories. A primary factor which contributes towards any slow variation requirement is the degree of similarity between the dynamics of the velocity-based linearisations, of the closed-loop system, associated with different operating points. This determines the degree of nonlinearity of the closed-loop system and is largely dependent on the performance specification. A secondary factor is the choice of controller realisation which, in contrast to the foregoing, is often subject to relatively few restrictions. Whilst the transfer function of each member of the controller velocity-based linearisation family is unchanged by a non-singular state transformation which is different for each member of the family, the dynamics of the nonlinear controller are changed. This issue is largely neglected in the conventional gain-scheduling approach and may also be ignored in the velocity-based gain-scheduling approach provided the rate of variation of the solution trajectories is sufficiently slow, since the controller dynamics are then insensitive to the choice of state (Appendix B). However, by choosing the controller realisation appropriately, any limitation on the rate of evolution of the trajectories can be relaxed. When, for example, the controller is designed such that closed-loop velocity-based linearisations have the same input-output dynamics at every operating point and compatible states, there is no restriction whatsoever on the rate of evolution and the gain-scheduled controller may be interpreted as a dynamic inversion controller (Ref 9). Of course, the velocity-based gain-scheduling approach is not confined to the design of dynamic inversion controllers. Since a velocity-based linearisation is associated with every operating point, the controller can instead be designed such that the closed-loop velocity-based linearisations vary, in some appropriate manner, across the operating envelope; for example, in the context of flight control, a uniform response is often undesirable in piloted aircraft and a degradation in the response to warn of the onset of stall may be preferred.

4. Velocity-based analysis of missile example

By differentiating, the nonlinear missile dynamics, (1), may be reformulated in the velocity-based form

$$\dot{\mathbf{x}} = \mathbf{w} \tag{29}$$

$$\dot{\mathbf{w}} = \begin{bmatrix} 0 & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \alpha}(\rho) \\ 1 & \dfrac{1}{mV(\rho)}\dfrac{\partial Z}{\partial \alpha}(\rho) \end{bmatrix} \mathbf{w} + \begin{bmatrix} \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \overline{q}}(\rho) & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \overline{M}}(\rho) & 0 & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \delta}(\rho) \\ \dfrac{1}{mV(\rho)}\dfrac{\partial Z}{\partial \overline{q}}(\rho) & \dfrac{1}{mV(\rho)}\dfrac{\partial Z}{\partial \overline{M}}(\rho) & -\dfrac{Z}{mV^2(\rho)} & \dfrac{1}{mV(\rho)}\dfrac{\partial Z}{\partial \delta}(\rho) \end{bmatrix} \dot{\mathbf{r}} \tag{30}$$

$$\dot{\eta}_Z = \left[0 \quad \frac{1}{m}\frac{\partial Z}{\partial \alpha}(\rho)\right]\mathbf{w} + \left[\frac{1}{m}\frac{\partial Z}{\partial \overline{q}}(\rho) \quad \frac{1}{m}\frac{\partial Z}{\partial \overline{M}}(\rho) \quad 0 \quad \frac{1}{m}\frac{\partial Z}{\partial \delta}(\rho)\right]\dot{\mathbf{r}} \tag{31}$$

where $\mathbf{x} = [q \ \alpha]^T$, $\mathbf{w} = [\dot{q} \ \dot{\alpha}]^T$, $\mathbf{r} = [\overline{q} \ \overline{M} \ V \ \delta]^T$.

4.1 Classical situation

Before considering more general situations, the conventional gain-scheduling design approach is analysed within the velocity-based framework. Assume that the aerodynamic moment and force are linear with respect to incidence and elevator angle and that $\frac{\partial M}{\partial \alpha}, \frac{\partial M}{\partial \delta}, \frac{\partial Z}{\partial \alpha}$ and $\frac{\partial Z}{\partial \delta}$ do not depend on the angle of incidence, α, or the elevator angle, δ; that is,

$$M = \frac{\partial M}{\partial \alpha}(\rho)\alpha + \frac{\partial M}{\partial \delta}(\rho)\delta + M_\varepsilon(\rho), \qquad Z = \frac{\partial Z}{\partial \alpha}(\rho)\alpha + \frac{\partial Z}{\partial \delta}(\rho)\delta + Z_\varepsilon(\rho) \tag{32}$$

where the scheduling variable, ρ, does not depend on the incidence and elevator angles. This assumption is clearly satisfied in the present missile example, in which the elements of the scheduling variable ρ are airspeed, dynamic pressure and mach number, and is almost always satisfied for conventional aircraft and missile configurations in normal operation out of stall (see, for example, Ref 3 p103).

Under these conditions, the missile dynamics, (29)-(31), may be reformulated as

$$\dot{\mathbf{x}} = \mathbf{w}, \qquad \dot{\mathbf{w}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{w} + \begin{bmatrix} M_\delta(\rho) \\ Z_\delta(\rho) \end{bmatrix}\dot{\delta} + \varepsilon_{\mathbf{w}}, \qquad \dot{\eta}_Z = [0 \quad Z_\alpha(\rho)V(\rho)]\mathbf{w} + Z_\delta(\rho)V(\rho)\dot{\delta} + \varepsilon_\eta \tag{33}$$

where

$$\varepsilon_{\mathbf{w}} = \begin{bmatrix} \frac{d}{dt}\left(\frac{M}{I_{yy}}\right) - M_\alpha(\rho)\dot{\alpha} - M_\delta(\rho)\dot{\delta} \\ \frac{d}{dt}\left(\frac{Z}{mV}\right) - Z_\alpha(\rho)\dot{\alpha} - Z_\delta(\rho)\dot{\delta} \end{bmatrix}, \qquad \varepsilon_\eta = \frac{d}{dt}\left(\frac{Z}{m}\right) - Z_\alpha(\rho)V(\rho)\dot{\alpha} - Z_\delta(\rho)V(\rho)\dot{\delta} \tag{34}$$

and $M_\alpha(\rho)$, $M_\delta(\rho)$, $Z_\alpha(\rho)$ and $Z_\delta(\rho)$ are obtained by allowing the scheduling variable, ρ, to vary in (7). Assume that the scheduling variable, ρ, is sufficiently slowly varying that $\varepsilon_{\mathbf{w}}$ and ε_η may be neglected in (33); that is, the nonlinear missile dynamics are described by

$$\dot{\mathbf{x}} = \mathbf{w}, \qquad \dot{\mathbf{w}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{w} + \begin{bmatrix} M_\delta(\rho) \\ Z_\delta(\rho) \end{bmatrix}\dot{\delta}, \qquad \dot{\eta}_Z = [0 \quad Z_\alpha V]\mathbf{w} + Z_\delta V\dot{\delta} \tag{35}$$

The members of the family of velocity-based linearisations associated with the missile dynamics are simply the frozen forms of (35). The assumption that $\varepsilon_{\mathbf{w}}$ and ε_η may be neglected is essentially a requirement that the short period approximation is accurate.

In the velocity-based gain-scheduling approach, a family of linear controllers is designed corresponding to the family of plant velocity-based linearisations. The velocity-based linearisation family includes linearisations of the plant at both non-equilibrium and equilibrium operating points. In contrast, in the conventional gain-scheduling approach a family of linear controllers is designed corresponding to the family of series expansion linearisations of the plant relative to the *equilibrium operating points only*. Nevertheless, the conventional series expansion linearisation, (4)-(5), relative to the equilibrium operating point at which δ, q, α, η_Z and ρ are, respectively, equal to $\delta_0, q_0, \alpha_0, \eta_{Z_0}$ and ρ_0, may be expressed in state-space form as

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & M_\alpha(\rho_0) \\ 1 & Z_\alpha(\rho_0) \end{bmatrix}\begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} M_\delta(\rho_0) \\ Z_\delta(\rho_0) \end{bmatrix}\Delta\delta, \qquad \Delta\eta_Z = [0 \quad Z_\alpha(\rho_0)V(\rho_0)]\begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + Z_\delta(\rho_0)V(\rho_0)\,\Delta\delta \tag{36}$$

together with the input, output and state transformations

$$\Delta\delta = \delta - \delta_0, \quad q = q_0 + \Delta q, \quad \alpha = \alpha_0 + \Delta\alpha, \quad \eta_Z = \eta_{Z_0} + \Delta\eta_Z \tag{37}$$

Clearly, in this example, the members of the conventional series expansion linearisation family are closely related to the members of the velocity-based linearisation family even though the states, inputs and outputs are different. In particular, the velocity-based linearisation family can be determined directly, by inspection, from the series expansion linearisation family provided that there exists an equilibrium operating point corresponding to every value in the range of ρ. This correspondence is certainly not the case in general. For example, the incidence and elevator angles are related at equilibrium operating points but not at non-equilibrium operating points; that is, α_o is a function of δ, but α in general is not a function of δ and so every pair (α, δ) does not correspond to a pair $(\alpha_o(\delta), \delta)$. Hence, when the incidence and elevator angles are also elements of the scheduling variable, the velocity-based linearisation family contains members which do not correspond to any of those of the series expansion linearisation family. Nevertheless, in the present missile

example, the equilibrium operating points are parameterised by the scheduling variable which is quite typical of flight control applications where the elements of the scheduling variable are airspeed, dynamic pressure and mach number. Consequently, the dynamics at the equilibrium operating points embody the dynamics at every operating point and a gain-scheduled controller designed on the basis of the equilibrium dynamics may be valid even when operating far from equilibrium; for example, during aggressive manoeuvring.

In the conventional gain-scheduling approach, a family of linear controllers is designed, for example (10)-(11) or (12)-(13), of the form

$$\Delta \dot{\mathbf{x}} = \mathbf{A}(\rho_o) \, \Delta \mathbf{x} + \mathbf{B}(\rho_o) \, \Delta \mathbf{r} \tag{38}$$

$$\Delta \delta = \mathbf{c}(\rho_o) \, \Delta \mathbf{x} + \mathbf{d}(\rho_o) \, \Delta \mathbf{r} \tag{39}$$

Exploiting the close relationship between the plant series expansion and velocity-based linearisation families, it follows that the controller velocity-based linearisation family, obtained by substituting for the state, input and output in (38)-(39),

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{40}$$

$$\dot{\hat{\mathbf{w}}} = \mathbf{A}(\rho_1) \, \hat{\mathbf{w}} + \mathbf{B}(\rho_1) \, \dot{\mathbf{r}} \tag{41}$$

$$\dot{\hat{\delta}} = \mathbf{c}(\rho_1) \, \hat{\mathbf{w}} + \mathbf{d}(\rho_1) \, \dot{\mathbf{r}} \tag{42}$$

is appropriate for the velocity-based gain-scheduling approach. In the conventional gain-scheduling approach, the linear controller family, (38)-(39), is typically implemented (ignoring the input, output and state transformations associated with the perturbation quantities $\Delta \mathbf{r}$, $\Delta \delta$, $\Delta \mathbf{x}$) directly as the nonlinear controller

$$\dot{\mathbf{x}} = \mathbf{A}(\rho) \, \mathbf{x} + \mathbf{B}(\rho) \, \mathbf{r}, \qquad \delta = \mathbf{c}(\rho) \, \mathbf{x} + \mathbf{d}(\rho) \, \mathbf{r} \tag{43}$$

It should be noted that the scheduling variable, ρ, acts as an implicit input to the nonlinear controller, (43), in addition to the explicit input, $\mathbf{r}$. The velocity-based linearisation of the nonlinear controller, (43), at the operating point ($\mathbf{x}_1$, $\mathbf{r}_1$, ρ_1), is

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{44}$$

$$\dot{\hat{\mathbf{w}}} = \mathbf{A}(\rho_1) \, \hat{\mathbf{w}} + \mathbf{B}(\rho_1) \, \dot{\mathbf{r}} + \varepsilon_{\hat{\mathbf{w}}} \tag{45}$$

$$\dot{\hat{\delta}} = \mathbf{c}(\rho_1) \, \hat{\mathbf{w}} + \mathbf{d}(\rho_1) \, \dot{\mathbf{r}} + \varepsilon_{\hat{\delta}} \tag{46}$$

where

$$\varepsilon_{\hat{\mathbf{w}}} = \nabla_\rho \big(\mathbf{A}(\rho)\mathbf{x}\big)\big|_{\substack{\rho=\rho_1 \\ \mathbf{x}=\mathbf{x}_1}} \dot{\rho} + \nabla_\rho \big(\mathbf{B}(\rho)\mathbf{r}_1\big)\big|_{\substack{\rho=\rho_1 \\ \mathbf{r}=\mathbf{r}_1}} \dot{\rho} \;,\quad \varepsilon_{\hat{\delta}} = \nabla_\rho \big(\mathbf{c}(\rho)\mathbf{x}\big)\big|_{\substack{\rho=\rho_1 \\ \mathbf{x}=\mathbf{x}_1}} \dot{\rho} + \nabla_\rho \big(\mathbf{d}(\rho)\mathbf{r}\big)\big|_{\substack{\rho=\rho_1 \\ \mathbf{r}=\mathbf{r}_1}} \dot{\rho} \tag{47}$$

Assuming that the rate of variation of ρ is sufficiently slow, $\varepsilon_{\hat{\mathbf{w}}}$ and $\varepsilon_{\hat{\delta}}$ are small and may be neglected in (44)-(46); that is, the velocity-based linearisation family of the implemented nonlinear controller is, indeed, the required family, (40)-(42). It should be noted that whilst the transfer function of each member of the linear controller family is unchanged by a non-singular state transformation which is different for each member of the family, the dynamics of the corresponding nonlinear controller are changed, see Appendix B. However, the difference in dynamics can be made arbitrarily small by restricting the rate of variation of the scheduling variable, ρ. Since this issue is neglected in the conventional gain-scheduling approach, assume that the rate of variation of ρ is sufficiently slow that the nonlinear controller dynamics are insensitive to the choice of state in the velocity-based linearisation family and so to the choice of controller realisation.

Clearly, the velocity-based analysis provides a rigorous basis for the conventional gain-scheduling approach of designing a family of linear controllers on the basis of the family of series expansion linearisations and employing a direct controller realisation of the form, (43). Strong support for the utility of the velocity-based paradigm in the context of the conventional gain-scheduling design approach is provided by its ability to provide an analytic basis for a number of, previously apparently undesirable, aspects of the conventional approach including (i) the practice of neglecting the input, output and state transformations associated with the linearised descriptions used in analysis and design and (ii) the use of frozen scheduling-variable controller linearisations in the design procedure. The latter linearisation neglects the variations in the scheduling variable and differs from the series expansion linearisation of the nonlinear gain-scheduled controller at the relevant equilibrium operating point. Moreover, the analysis shows that since the dynamics at the equilibrium operating points embody the dynamics at every operating point, a gain-scheduled controller designed on the basis of the equilibrium dynamics may be valid, albeit inadvertently, even when operating far from equilibrium. The latter result is quite encouraging since it indicates that the utility of gain-scheduled controllers is considerably greater than suggested by conventional gain-scheduling analysis.

4.2 Airspeed, dynamic pressure and/or mach number not "slowly varying"

The analysis of section 4.1 establishes a rigorous basis for the conventional gain-scheduling design approach as typically applied in flight control applications. However, a number of conditions are required for the conventional approach to be valid. These conditions include the requirements that the short period approximation is accurate, the aerodynamic moment and force are linear with respect to incidence and elevator angle, the equilibrium operating points parameterise the scheduling variable ρ and, in addition, ρ varies sufficiently slowly that the nonlinear controller

dynamics are insensitive to the choice of state and the velocity-based linearisation family of the controller is directly related to the conventional series expansion linearisation family. The short period approximation is well established and is known to be accurate for a wide variety of flight configurations and operating conditions (Refs 3 and 12).   Similarly, during conventional operation out of stall, the aerodynamic moment and force are frequently approximately linear in incidence and elevator angle (see, for example, Ref 3 p103) and the equilibrium operating points parameterise ρ provided scheduling with respect to incidence is not required. However, the validity of the final assumption, that ρ is sufficiently slowly varying, is less clear.

It follows from the analysis of section 4.1 that when the rate of variation of ρ is sufficiently slow, the transfer functions of the closed-loop velocity-based linearisations are the same with controller realisations A and B and the dynamics of the nonlinear controllers are similar. However, whilst the airspeed, dynamic pressure and mach number usually vary slowly in comparison to the incidence angle, these scheduling variables are directly related to normal acceleration. Hence, particularly during aggressive manoeuvring, the scheduling variable ρ is not *a priori* slowly varying with respect to the dynamics of the acceleration control loop. This is illustrated by the simulation results presented in section 3 where the closed-loop dynamics are significantly different with controller realisations A and B and inspection of figures 2 and 4 reveals that the only difference between the controller realisations lies in the position of the scheduled gain in the acceleration outer loop. The sensitivity of the performance to changes in the controller realisation provides, therefore, a direct and immediately relevant indication of the strength of the controller nonlinearity induced by the variation of the scheduling variables. It should be noted that, in a similar manner, simulation studies indicate that the performance in this example is (unlike the acceleration outer loop controller) insensitive to the choice of realisation of the incidence inner loop controller.

Since the dependence of the controller dynamics on the choice of realisation is a purely nonlinear effect, conventional gain-scheduling analysis provides little insight into this issue.   In the present example, the requirement is for uniform closed-loop dynamics across the operating envelope; that is, for a dynamic inversion type of controller. Of course, general nonlinear control design methods such as Feedback Linearisation (Refs 7 and 18) or the velocity-based gain-scheduling approach proposed in Ref 9 can be employed to design a dynamic inversion controller. However, in the present flight control context, a rather simpler approach is possible by exploiting the specific structure of the missile and controller dynamics.   Firstly, the controller is designed so that the dynamics of the incidence inner loop are sufficiently fast, compared to those of the acceleration outer loop, that the incidence angle may be assumed to accurately track the demanded incidence angle; that is, α equals $\alpha_d + \varepsilon_\alpha$ where ε_α is small[5]. Secondly, it follows from (32) that

$$\eta_Z = Z_\alpha(\rho)V\alpha + Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m \tag{48}$$

where it is assumed, when designing the controller, that Z_δ and Z_ε are sufficiently small that $Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m$ may be treated as a minor disturbance (in fact Z_ε is zero in the present example).   In controller realisation A, the incidence demand satisfies

$$\alpha_d = \frac{w_n^2}{Z_\alpha(\rho)V} x_4 \tag{49}$$

where x_4 is related to the normal acceleration error by linear dynamics with the transfer function

$$x_4(s) = \frac{1}{s(s + 2\varsigma w_n)}(\eta_{Z_d}(s) - \eta_Z(s)) \tag{50}$$

Hence,

$$\eta_Z = w_n^2 x_4 + Z_\alpha(\rho)V\varepsilon_\alpha + Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m \tag{51}$$

and it follows that the normal acceleration dynamics, in pseudo-transfer function form, are

$$\eta_Z(s) = \frac{w_n^2}{s^2 + 2\varsigma w_n s + w_n^2} \eta_{Z_d}(s) + \frac{s^2 + 2\varsigma w_n s}{s^2 + 2\varsigma w_n s + w_n^2}\left(Z_\delta(\rho)V\delta + \frac{Z_\varepsilon(\rho)}{m} + Z_\alpha(\rho)V\varepsilon_\alpha\right) \tag{52}$$

where s denotes the d/dt operator. By assumption, $Z_\delta(\rho)V\delta$, $Z_\varepsilon(\rho)/m$ and $Z_\alpha(\rho)V\varepsilon_\alpha$ are sufficiently small that they may be neglected in (52) and the normal acceleration is linearly related to the acceleration demand, η_{Z_d} , as required. Hence, by adopting an appropriate controller realisation, namely realisation A, the conventional gain-scheduling approach can be extended in a straightforward manner to accommodate situations where the airspeed, dynamic pressure and mach number are not "slowly varying".   Conversely, a poor choice of controller realisation can lead to unnecessary restrictions on the operating envelope.

It is noted that the requirement, (48), may be relaxed to one that the acceleration is of the form

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[5] Since α is not measured, the inner loop ensures that the estimated incidence angle, $\hat{\alpha}$, tracks α_d. It follows from (48) that $\dot{\alpha} - \dot{\hat{\alpha}} = Z_\alpha(\rho)(\alpha - \hat{\alpha})$ and so provided Z_α is negative, $\hat{\alpha}$ is an accurate (after some initial transients) estimate of α.

$$\eta_Z = z_\alpha(\rho)\mathrm{V}\alpha + z_\varepsilon(\rho)\mathrm{V} \tag{53}$$

provided that, in the foregoing analysis, $Z_\alpha\mathrm{V}$ is replaced by $z_\alpha\mathrm{V}$, and $Z_\delta\mathrm{V}\delta$ and Z_ε/m are replaced by $z_\varepsilon\mathrm{V}$. The condition, (53), is quite weak; for example, employing a partial series expansion of Z with respect to α,

$$\eta_Z = \frac{Z}{m} = \frac{1}{m}\left(\frac{\partial Z}{\partial\alpha} + \frac{\partial^2 Z}{\partial\alpha^2}\alpha + \cdots\right)\alpha + \left(\frac{Z}{m}\right)_0 \tag{54}$$

where $(Z/m)_0$ is the acceleration when α is zero and it is assumed that Z is sufficiently differentiable. The term $(Z/m)_0$ is simply a trim offset which may be treated as a constant disturbance and neglected for the purposes of control design. The infinite series expansion, (54), can, of course, be truncated provided the resulting approximation to η_Z is sufficiently accurate over the range of incidence angles associated with the allowable flight envelope.

4.3 Aerodynamic force and moment nonlinearly related to incidence angle

In the foregoing analysis, it is assumed that the aerodynamic moment and force are linear with respect to the incidence and elevator angles; that is, $\dfrac{\partial M}{\partial\alpha}, \dfrac{\partial M}{\partial\delta}, \dfrac{\partial Z}{\partial\alpha}, \dfrac{\partial Z}{\partial\delta}$ do not depend on incidence or elevator angle and scheduling with respect to incidence and elevator angle is not necessary. This is frequently the case during conventional operation out of stall (McLean 1990 p103) and scheduling with respect to the instantaneous incidence angle is traditionally avoided (Snell *et al.* 1989) (as opposed to, for example, flap scheduling with respect to average incidence obtained using a low-pass filter with very low bandwidth). However, scheduling with respect to the instantaneous incidence angle is likely to be required in, for example, future super-manoeuvrable aircraft (Refs 5 and 6).

When $\dfrac{\partial M}{\partial\alpha}, \dfrac{\partial M}{\partial\delta}, \dfrac{\partial Z}{\partial\alpha}, \dfrac{\partial Z}{\partial\delta}$ depend on the incidence and/or elevator angles, the analysis of sections 4.1 and 4.2 remains valid provided that the scheduling variable now includes the incidence and elevator angles. However, since the incidence and elevator angles almost always vary considerably more rapidly than the airspeed, dynamic pressure and mach number, it follows that the scheduling variable cannot now be assumed to be slowly varying. In particular, it cannot be assumed to be slowly varying with respect to the incidence inner loop dynamics. Hence, the pitch dynamics can be expected to be sensitive to the choice of realisation of both the inner and outer loop controllers. Moreover, whilst the incidence and elevator angles are related at equilibrium operating points, this is not the case at non-equilibrium operating points. Hence, when the incidence and elevator angles are elements of the scheduling variable, the velocity-based linearisation family contains members which do not correspond to any of those of the series expansion linearisation family and a gain-scheduled controller designed on the basis of the equilibrium dynamics may not be valid when operating far from equilibrium; for example, during aggressive manoeuvring. This observation provides insight into some of the well-known difficulties associated with scheduling on a primary flight variable such as incidence angle which need not be slowly varying (see, for example, refs 5 and 10).

To illustrate the performance degradation which can occur when scheduling with respect to incidence and elevator angle, consider a missile which is similar to that studied in section 2 except that its aerodynamic moment and force coefficients are described by

$$C_M = -0.3f\alpha - 0.108f\sin(3\delta) \tag{55}$$

$$C_L = 0.000103f^3\alpha^3 - 0.00945f^2|\alpha|\alpha + (-0.339 + 0.056\overline{M})f\alpha - 0.034f\delta \tag{56}$$

The response, obtained using non-linear simulations, with non-linear controller realisation A (with values of the aerodynamic derivatives corresponding to (55) and (56)) to a sequence of step demands in normal acceleration is shown in figure 6. It can be seen that the missile acceleration is unstable and diverges until the simulation fails due to numeric overflow. It should be noted that the airspeed, dynamic pressure and mach number are held constant in the simulation in order to ensure that the performance degradation is purely a consequence of the nonlinearity of the aerodynamic moment and force with respect to the incidence and elevator angles. Evidently, the conventional gain-scheduling approach is inadequate in this example.

In contrast to the conventional gain-scheduling approach, the velocity-based gain-scheduling methodology does not inherently involve a slow variation requirement and includes information about the dynamics at both equilibrium and non-equilibrium operating points. It therefore has the potential to support the design of an incidence inner loop controller which can accommodate rapid variations in the scheduling variable and operation far from equilibrium.

The dynamics of the missile are described, in velocity form, by (29)-(31). Assume that the short period approximation is accurate and Z_δ is sufficiently small that its contribution to the pitch dynamics can be neglected; that is, the pitch dynamics are accurately described by

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{w}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{w} + \begin{bmatrix} M_\delta(\rho) \\ 0 \end{bmatrix}\dot{\delta}, \quad \dot{\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{w} \tag{57}$$

where $\mathbf{w} = \begin{bmatrix} \dot{q} & \dot{\alpha} \end{bmatrix}^{\mathrm{T}}$. The members of the velocity-based linearisation family associated with the nonlinear dynamics are obtained by simply "freezing" the scheduling variable, ρ, in (57). The transfer function, of the velocity-based linearisation corresponding to the value ρ_1 of the scheduling variable, is

$$\dot{\alpha}(s) = \frac{1}{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)}\dot{\delta}(s) \tag{58}$$

The velocity-based linearisations have relative degree two; that is, the difference between the orders of the numerator and denominator of the transfer function is two. Letting

$$\dot{\delta} = \frac{1}{M_\delta(\rho)}\frac{(\dddot{u} + a\ddot{u} + b\dot{u})}{b} \tag{59}$$

where a and b are positive constants, the pitch dynamics may be reformulated, with input $\dot{u}$, as

$$\dot{\omega} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\omega + \begin{bmatrix} \dfrac{M_\alpha(\rho)+b}{b} \\ \dfrac{Z_\alpha(\rho)+a}{b} \end{bmatrix}\dot{u}, \qquad \dot{\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix}\omega + \frac{1}{b}\dot{u} \tag{60}$$

with $\omega = \mathbf{w} - \begin{bmatrix} (\ddot{u} + a\dot{u})/b \\ \dot{u}/b \end{bmatrix}$. The velocity-based linearisation of (60), associated with the value ρ_1 of the scheduling variable, has transfer function

$$\dot{\alpha}(s) = \frac{1}{b}\left(\frac{s^2 + as + b}{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)} \right)\dot{u}(s) \tag{61}$$

and have relative degree zero. Adopting a similar approach to that employed in Appendix A to design the linear family for the conventional gain-scheduled controller, let the members of the controller velocity-based linearisation family have transfer functions

$$\dot{u}(s) = b\left(\frac{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)}{s^2 + as + b} \right)\dot{v}(s) \tag{62}$$

which are simply the reciprocal of the transfer functions, (61), of the augmented plant velocity-based linearisations. The cascade combination of each controller velocity-based linearisation with the corresponding plant velocity-based linearisation has unity transfer function. The velocity-based linearisations of the cascade connection of the plant and controller are simply the cascade connection of the appropriate velocity-based linearisations of the plant and controller (Ref 8) and so have unity transfer function. Owing to the direct relationship between the velocity-form of a nonlinear system and its velocity-based linearisation family (figure 5), the dynamics of the cascade combination of the nonlinear plant and controller have linear dynamics with unity transfer function provided that the rate of variation of the scheduling variable, ρ, is sufficiently slow that the dynamics are insensitive to the choice of controller state.

The slow variation requirement can be relaxed when the controller state is selected appropriately (Ref 9). In the present case, an appropriate choice is

$$\dot{\mathbf{x}}^i = \mathbf{w}^i, \qquad \dot{\mathbf{w}}^i = \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix}\mathbf{w}^i + \begin{bmatrix} -M_\alpha(\rho)-b) \\ -Z_\alpha(\rho)-a \end{bmatrix}\dot{v}, \qquad \dot{u} = \begin{bmatrix} 0 & b \end{bmatrix}\mathbf{w}^i + b\dot{v} \tag{63}$$

The transfer function of the velocity-based linearisations of (63) associated with operating points at which ρ equals ρ_1 is (62) and, when (63) is connected in cascade with (60), the incidence angle satisfies

$$\dot{\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{z} + \dot{v} \tag{64}$$

where $\mathbf{z} = \omega + \mathbf{w}^i$ and

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{z}, \tag{65}$$

In order to ensure internal stability of the cascade combination, assume that the controller dynamics, (63), are stable. Provided the unforced dynamics (65) are also stable, $\mathbf{z}$ decays to zero and the incidence angle is related (after some initial transients) to v by linear dynamics with unity transfer function. This approach to dynamic inversion is a direct generalisation to nonlinear systems of linear pole-zero cancellation (quite distinct from Feedback Linearisation). It is discussed in detail in Ref 9. It should be noted that stability of (63) and (65) reduces, in the purely linear case, to the requirement that the transfer function of the plant and its inverse are both stable thereby avoiding unstable pole-zero cancellation. It should be noted that the stability requirement is not inherent to the velocity-based gain-scheduling approach but rather is a feature of the pole-zero cancellation approach adopted in this example to design the linear controller family on which the gain-scheduled controller is based.

Letting the input to the dynamic inverse controller, (63), be

$$\dot{v}(s) = \frac{K_1}{(s + K_2)}(\alpha_d(s) - \alpha(s)) \tag{66}$$

the closed-loop pitch dynamics are linear with

$$\alpha(s) = \frac{K_1}{s^2 + K_2 s + K_1}\alpha_d(s) \tag{67}$$

as required. Of course, since a measurement of α is not available, the estimated incidence $\hat{\alpha}$ is employed when implementing (66). It follows from (53) that the solution to

$$\dot{\hat{\alpha}} = z_\alpha(\rho)\hat{\alpha} + z_\varepsilon(\rho) \tag{68}$$

is an accurate estimate of α (after some initial transients which decay to zero provided z_α is negative).

Since u is an internal controller state, $\dot{u}$ and $\ddot{u}$ (required in (59)) can be readily derived, without numerical differentiation, from (63) and (66); namely,

$$\dot{u} = \begin{bmatrix} 0 & b \end{bmatrix}\mathbf{w}^i + b\dot{v} \tag{69}$$

$$\ddot{u} = b\begin{bmatrix} 1 & -a \end{bmatrix}\mathbf{w}^i - (Z_\alpha(\rho) + a)\dot{v} + b(-K_2\dot{v} + K_1(\alpha_d(s) - \alpha(s))) \tag{70}$$

However, to derive $\ddot{u}$ requires $\dot{\alpha}_d, \dot{\alpha}$ and $\dfrac{dZ_\alpha(\rho)}{dt}$ which may not be directly available. Alternatively, therefore, assume that the nonlinear mapping, $M(\delta, \rho_{\mathbf{m}})$, relating the aerodynamic moment to the elevator angle δ and scheduling variable $\rho_{\mathbf{m}}$ (which may depend on the airspeed, incidence angle *etc* but not on δ), is invertible in the sense that there exists an elevator angle, $\delta = M^{-1}(M, \rho_{\mathbf{m}})$, corresponding to every pair $(M, \rho_{\mathbf{m}})$. This invertibility condition is rather weak and simply requires that the aerodynamic moment is controllable throughout the flight envelope. Letting

$$\delta = M^{-1}(I_{yy}\left(\hat{M}_\alpha \alpha + \frac{\ddot{u} + a\dot{u} + bu}{b}\right), \rho_{\mathbf{m}}) \tag{71}$$

the pitch dynamics can once again be reformulated as (60), with $M_\alpha(\rho)$ replaced by the constant, $\hat{M}_\alpha$. The input transformation, (71), does not require the calculation of $\ddot{u}$.

It is emphasised that there is no slow variation condition associated with the foregoing inner loop controller (other than that implied by the initial modelling assumption that the short period approximation is accurate). When the velocity-based inner loop controller is employed and the outer acceleration loop controller is as in realisation A, the response obtained in the foregoing missile example is shown in figure 6. In contrast to the conventional gain-scheduled controller, it is evident that the velocity-based controller attains the specified performance of a uniform normal acceleration step response over the flight envelope with rise time of around 0.3 seconds and overshoot less than 5%. This is achieved whilst retaining the divide and conquer approach and continuity with linear design methods which are important features of the conventional gain-scheduling approach. Moreover, although not pursued further here, the velocity-based gain-scheduling approach is not confined to the design of dynamic inversion controllers: since a velocity-based linearisation is associated with every operating point, the controller can instead be designed such that the closed-loop velocity-based linearisations vary, in some appropriate manner, across the operating envelope (Ref 8).

4.4 Short period approximation inaccurate

The foregoing sections employ progressively weaker assumptions about the characteristics of the missile dynamics but the fundamental assumption remaining in section 4.3 is that the short period approximation is accurate. However, this requirement may be readily relaxed by letting

$$\dot{\delta} = \dot{\hat{\delta}} - \frac{\varepsilon}{M_\delta(\rho)} \tag{72}$$

where

$$\varepsilon = \left(\frac{d(M/I_{yy})}{dt} - \frac{1}{I_{yy}}\frac{\partial M}{\partial \alpha}\dot{\alpha} - \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta}\dot{\delta}\right) + \frac{d}{dt}\left(\frac{d(Z/mV)}{dt} - \frac{1}{mV}\frac{\partial Z}{\partial \alpha}\dot{\alpha}\right) \tag{73}$$

It follows from (29)-(31) that the pitch dynamics may be reformulated as

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{w}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{w} + \begin{bmatrix} M_\delta(\rho) \\ 0 \end{bmatrix}\dot{\hat{\delta}}, \quad \dot{\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{w} \tag{74}$$

provided M_δ is not zero. The pitch dynamics, (74), are in the form required by the analysis of section 4.3 which can be re-applied to determine a suitable nonlinear controller.

Alternatively, consider the case when the nonlinear mapping, $M(\delta, \rho_{\mathbf{m}})$, relating the aerodynamic moment to the elevator angle δ and scheduling variable $\rho_{\mathbf{m}}$ (which does not depend on δ), is invertible in the sense that there exists an elevator angle, $\delta = M^{-1}(M, \rho_{\mathbf{m}})$, corresponding to every pair $(M, \rho_{\mathbf{m}})$. Let

$$\delta = M^{-1}(I_{yy}(\hat{M}_\alpha \alpha + \hat{\ddot{\delta}} - \varepsilon_Z), \rho_{\mathbf{m}}) \tag{75}$$

where

$$\varepsilon_Z = \frac{d(Z/mV)}{dt} - \frac{1}{mV}\frac{\partial Z}{\partial \alpha}\dot{\alpha} \tag{76}$$

The missile pitch dynamics may be reformulated as in (74) (with $M_\alpha(\rho)$ replaced by the constant, $\hat{M}_\alpha$, and $M_\delta(\rho)$ unity). The expression for ε_Z is somewhat simpler than that for ε and, in particular, involves only the first time derivative of Z/mV.

It is evident from the foregoing that, in comparison to section 4.3, only a relatively small increase in mathematical complexity is required to relax the short period requirement. However, measurements/estimates of the time derivatives of airspeed, dynamic pressure and mach number are typically required which may, in practice, be difficult to obtain reliably. It should be noted that these measurements/estimates are also necessary in other control approaches, such as Feedback Linearisation (see, for example, Ref 7), and appear to be unavoidable in the general case when the requirement is for uniform dynamics across the flight envelope.

5. Conclusions

Gain-scheduling is widely and successfully employed in flight control applications, where high performance has to be achieved over a broad operating envelope. Nevertheless, conventional theoretical techniques for analysing the dynamics of gain-scheduled systems are poorly developed and provide little support for the gain-scheduling design approach. Moreover, the suitability of the conventional gain-scheduling approach for designing controllers which can accommodate, for example, aggressive manoeuvring far from equilibrium and operation at high angles of attack/post-stall is unclear. In particular, the sensitivity of gain-scheduled controllers to the choice of nonlinear realisation when the scheduling is not sufficiently slowly varying and the difficulties associated with employing incidence angle as a scheduling variable are well known. There is, consequently, interest in the literature in alternative nonlinear control design approaches such as dynamic inversion. However, owing to the substantial body of experience which has been accumulated with gain-scheduling methods both with regard to meeting performance requirements and also such practical issues as safety certification, there is a strong incentive to retain the gain-scheduling approach whilst resolving the foregoing difficulties. The recently developed velocity-based analysis framework associates a linear system with every operating point of a nonlinear system, not just the equilibrium operating points, and so is not confined to near equilibrium operation and does *not* inherently involve a slow variation requirement. Furthermore, it provides a natural and unified framework for gain-scheduling analysis and design which addresses many of the shortcomings of conventional gain-scheduling analysis and design whilst retaining the continuity with linear methods which is the principle feature of the conventional approach.

In this paper, the conventional gain-scheduling design approach to controlling the longitudinal dynamics of a missile is shown to be equivalent to the velocity-based gain-scheduling approach provided

1. the short period approximation is accurate;
2. the aerodynamic moment and force are linear with respect to incidence and elevator angle;
3. the equilibrium operating points parameterise the scheduling variable;
4. the scheduling variable varies sufficiently slowly that the nonlinear controller dynamics are insensitive to the choice of realisation and the velocity-based linearisation family of the controller is directly related to the conventional series expansion linearisation family.

The analysis thereby establishes a rigorous basis for the conventional gain-scheduling design approach. Strong support for the utility of the velocity-based paradigm in the context of the conventional gain-scheduling design approach is provided by its ability to provide an analytic basis for a number of, previously apparently undesirable, aspects of the conventional approach including (i) the practice of neglecting the input, output and state transformations associated with the linearised descriptions used in analysis and design and (ii) the use of frozen scheduling-variable controller linearisations in the design procedure. The latter linearisation neglects the variations in the scheduling variable and differs from the series expansion linearisation of the nonlinear gain-scheduled controller at the relevant equilibrium operating point. In addition, whilst the conventional gain-scheduling design approach is based on the linearisations of the plant about equilibrium operating points, it is shown that, under the foregoing conditions, the resulting gain-scheduled controller is valid, albeit inadvertently, even when operating far from equilibrium. The latter result is quite

encouraging since it indicates that the utility of gain-scheduled controllers is considerably greater than suggested by conventional gain-scheduling analysis.

In addition to providing insight into the conventional gain-scheduling approach, the velocity-based analysis and design framework is employed to design gain-scheduled controllers which progressively relax the restrictions (1)-(4) required by the conventional approach whilst retaining, as far as possible, the continuity with linear design methods of conventional gain-scheduling. In particular, rigorous insight is thereby provided into issues such as the sensitivity of gain-scheduled controllers to the choice of nonlinear realisation and the difficulties associated with employing the incidence and elevator angles as scheduling variables. This insight is exploited to develop gain-scheduled control designs which resolve these issues and the effectiveness of these designs is illustrated by a number of simulation trials.

Whilst illustrated with reference to a simple missile example, the analysis and design techniques developed are, of course, also relevant to other flight control applications.

Acknowledgement

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Appendix A – Missile gain-scheduled controller

The requirement is to design a family of local linear controllers for the family of linear plants, (9). A natural choice of controller configuration is a cascaded inner-outer loop arrangement which employs an attitude inner loop and a normal acceleration outer loop. With this approach, the acceleration outer loop supplies an angle of incidence demand to the attitude inner loop based on the normal acceleration demanded. Firstly, consider the design of the inner loop controller. It follows from the short period dynamics, (9), that, locally to the equilibrium operating point π_0, the transfer function relating incidence angle to elevator angle is

$$\Delta\alpha(s) = \frac{M_\delta(\rho_0)}{s^2 - Z_\alpha(\rho_0)s - M_\alpha(\rho_0)}\Delta\delta(s) \tag{77}$$

where $\Delta\alpha(s)$ and $\Delta\delta(s)$ are, respectively, the Laplace transforms of $\Delta\alpha$ and $\Delta\delta$. Selecting

$$\Delta\delta(s) = \frac{K_1}{M_\delta(\rho_0)}\frac{(s^2 - Z_\alpha(\rho_0)s - M_\alpha(\rho_0))}{s(s+K_2)}(\Delta\alpha_d(s) - \Delta\alpha(s)) \tag{78}$$

where $\Delta\alpha_d$ is the incidence demand from the outer loop acceleration controller, the closed-loop transfer function of the incidence loop is

$$\Delta\alpha(s) = \frac{K_1}{s^2 + K_2 s + K_1}\Delta\alpha_d(s) \tag{79}$$

Of course, an angle of incidence measurement is not available in the present example. However, it may be estimated from the pitch rate measurement using

$$\Delta\dot{\hat{\alpha}} = Z_\alpha(\rho_0)\Delta\hat{\alpha} + Z_\delta(\rho_0)\Delta\delta + \Delta q \tag{80}$$

where $\Delta\hat{\alpha}$ is the estimate of $\Delta\alpha$. The controller parameters, K_1 and K_2 are selected to ensure that the inner loop dynamics have a satisfactory natural frequency and damping whilst respecting the limitations of the actuator and the need to avoid exciting elastic modes of the airframe; in the present example, values for these parameters are selected which correspond to a natural frequency of 50 rad/s and damping factor of 0.7.

Secondly, the design of the outer loop normal acceleration controller is addressed. A normal acceleration step response with rise time (to 95% of final value) of around 0.3 seconds and overshoot of 5% corresponds to a natural frequency of around 10 rad/s and a damping factor of 0.7. Since the inner loop bandwidth of 50 rad/s is large compared to the required bandwidth of the acceleration loop, the design of these loops is effectively decoupled; that is, the dynamics of the inner loop may be neglected when designing the outer loop controller. It should be noted that this time-scale separation is quite natural and, indeed, inherent to the missile configuration since a change in acceleration is initiated by a change in incidence angle and so incidence must always change more rapidly than normal acceleration. Hence, it may be assumed in this context that $\Delta\alpha$ equals $\Delta\alpha_d$. Since the sign of the coefficient of δ in C_L is negative, the normal acceleration in response to a change in the elevator angle, δ, is initially in the opposite direction; that is, the normal acceleration response is non-minimum phase. However, the right-half plane zero associated with the non-minimum phase character of the acceleration response lies well outside the frequency range of interest and may, therefore, also be neglected when designing the outer loop controller. The gain, $Z_\alpha V$, relating changes in the angle of incidence to changes in normal acceleration, varies strongly with the flight condition and, in order to accommodate this variation the gain of the outer loop controller must also be varied. An appropriate outer loop controller transfer function is, therefore,

$$\Delta\alpha_d(s) = \frac{1}{Z_\alpha(\rho_0)V(\rho_0)}\frac{w_n^2}{s(s+2\varsigma w_n)}(\Delta\eta_{Z_d} - \Delta\eta_Z) \tag{81}$$

where $\Delta\eta_{Z_d}$ is the perturbation in the demanded normal acceleration and w_n, $= 10$ rad/s, $\zeta = 0.7$. The corresponding closed-loop dynamics obtained when this outer loop controller is combined with the linearised plant dynamics, (5), (neglecting the inner loop dynamics) are

$$\Delta\eta_Z(s) = \frac{w_n^2}{s^2 + 2\varsigma w_n s + w_n^2}\Delta\eta_{Z_d}(s) + \frac{s^2 + 2\varsigma w_n s}{s^2 + 2\varsigma w_n s + w_n^2}Z_\delta(\rho_o)V(\rho_o)\Delta\delta \tag{82}$$

Evidently, the term, $Z_\delta(\rho_o)V(\rho_o)\Delta\delta$, in (82) is attenuated over the control bandwidth and, consequently, the normal acceleration of the missile, $\Delta\eta_Z$, is essentially related to the demanded acceleration, $\Delta\eta_{Z_d}$, by second order dynamics with natural frequency 10rad/s and damping of 0.7, as required.

A family of linear controller transfer function designs is defined by (78), (80) and (81) which corresponds to the family of plant linearisations, (9), and is parameterised by ρ.

Appendix B – Sensitivity to choice of controller realisation

The nonlinear system

$$\dot{\mathbf{x}} = \mathbf{w} \tag{83}$$
$$\dot{\mathbf{w}} = \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}, \mathbf{r})\dot{\mathbf{r}} \tag{84}$$
$$\dot{\mathbf{y}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x}, \mathbf{r})\dot{\mathbf{r}} \tag{85}$$

has, at the operating point $(\mathbf{x}_1, \mathbf{r}_1)$, the velocity-based linearisation

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{86}$$
$$\dot{\hat{\mathbf{w}}} = \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\hat{\mathbf{w}} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\dot{\mathbf{r}} \tag{87}$$
$$\dot{\hat{\mathbf{y}}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\hat{\mathbf{w}} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\dot{\mathbf{r}} \tag{88}$$

Of course, the dynamics of a linear system are invariant under a non-singular state-transformation. Consider, therefore, the nonlinear system

$$\dot{\chi} = \mathbf{T}(\chi, \mathbf{r})\omega \tag{89}$$

$$\dot{\omega} = \mathbf{T}^{-1}(\chi, \mathbf{r})\nabla_{\mathbf{x}}\mathbf{F}(\chi, \mathbf{r})\mathbf{T}(\chi, \mathbf{r})\omega + \mathbf{T}^{-1}(\chi, \mathbf{r})\nabla_{\mathbf{r}}\mathbf{F}(\chi, \mathbf{r})\,\dot{\mathbf{r}} \tag{90}$$

$$\dot{v} = \nabla_{\mathbf{x}}\mathbf{G}(\chi, \mathbf{r})\mathbf{T}(\chi, \mathbf{r})\omega + \nabla_{\mathbf{r}}\mathbf{G}(\chi, \mathbf{r})\,\dot{\mathbf{r}} \tag{91}$$

for which the velocity-based linearisation, at the operating point $(\mathbf{x}_1, \mathbf{r}_1)$, is

$$\dot{\hat{\chi}} = \mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\omega} \tag{92}$$

$$\dot{\hat{\omega}} = \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\omega} + \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \tag{93}$$

$$\dot{\hat{v}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\omega} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \tag{94}$$

where $\chi, \omega, \hat{\chi}, \hat{\omega} \in \Re^n$ and $\mathbf{T}(\bullet, \bullet)$ is a uniformly bounded non-singular matrix which is differentiable with uniformly bounded derivatives. It can be seen that the velocity-based linearisations, (86)-(88) and (92)-(94), are related by the non-singular transformation

$$\hat{\omega} = \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}}, \quad \hat{\chi} = \hat{\mathbf{x}}, \quad \hat{v} = \hat{\mathbf{y}} \tag{95}$$

and so are dynamically equivalent. Similarly for the velocity-based linearisations at other operating points. However, letting

$$\omega = \mathbf{T}^{-1}(\chi, \mathbf{r})\mathbf{z} \tag{96}$$

the nonlinear system, (89)-(91), may be reformulated as

$$\dot{\chi} = \mathbf{z} \tag{97}$$

$$\dot{\mathbf{z}} = \nabla_{\mathbf{x}}\mathbf{F}(\chi, \mathbf{r})\mathbf{z} + \nabla_{\mathbf{r}}\mathbf{F}(\chi, \mathbf{r})\,\dot{\mathbf{r}} + \varepsilon \tag{98}$$

$$\dot{v} = \nabla_{\mathbf{x}}\mathbf{G}(\chi, \mathbf{r})\mathbf{z} + \nabla_{\mathbf{r}}\mathbf{G}(\chi, \mathbf{r})\,\dot{\mathbf{r}} \tag{99}$$

where $\varepsilon = \dot{\mathbf{T}}(\chi, \mathbf{r})\,\mathbf{T}^{-1}(\chi, \mathbf{r})\mathbf{z}$. Hence, despite the dynamic equivalence of the members of the velocity-based linearisation families, it is evident that the dynamics of the nonlinear systems, (83)-(85) and (89)-(91), are *not* equivalent. The difference between the dynamics is embodied by the perturbation term, ε, and arises from the variation of the state transformation, (95), with the operating point. Nevertheless, the difference between the solutions to the nonlinear systems, (83)-(85), and (97)-(99), over any finite time interval is arbitrarily small provided the magnitude of ε is sufficiently small (see, for example, Ref 19 theorem 2.5).

Figure 1 Non-linear controller realisation A

normal acceleration (g)

Figure 2 Performance of non-linear controllers

Figure 3 Mach number time history.
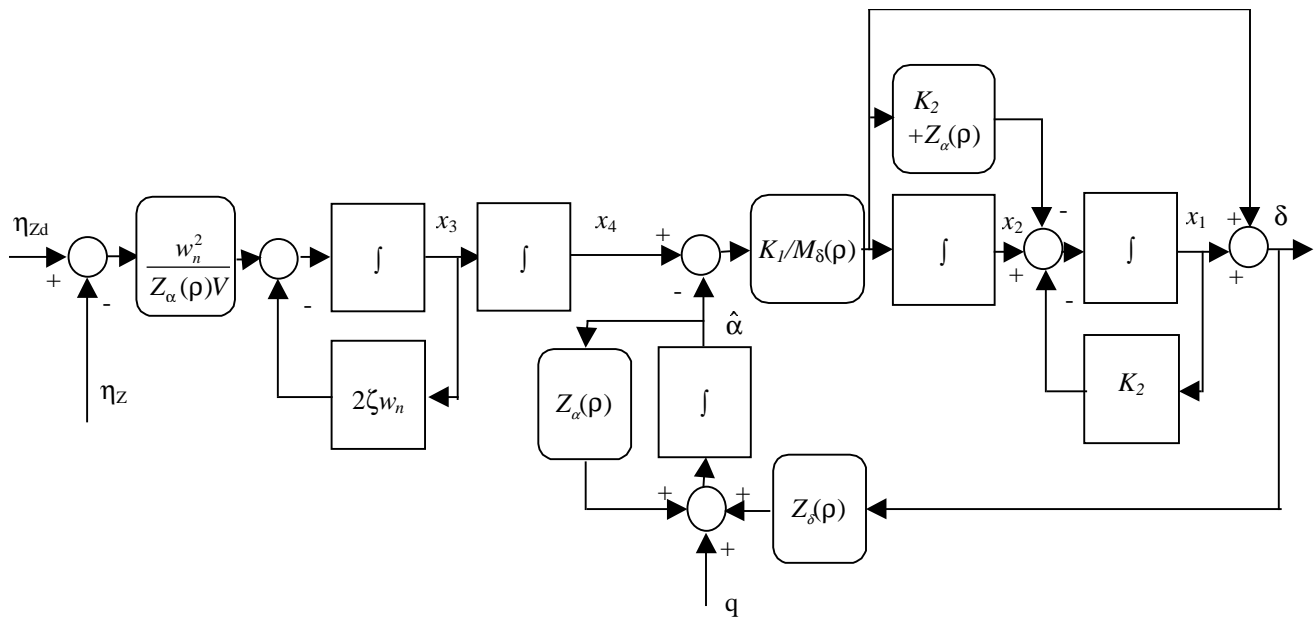
Figure 4 Non-linear controller realisation B

Figure 5 Alternative representations of a nonlinear system

normal acceleration (g)

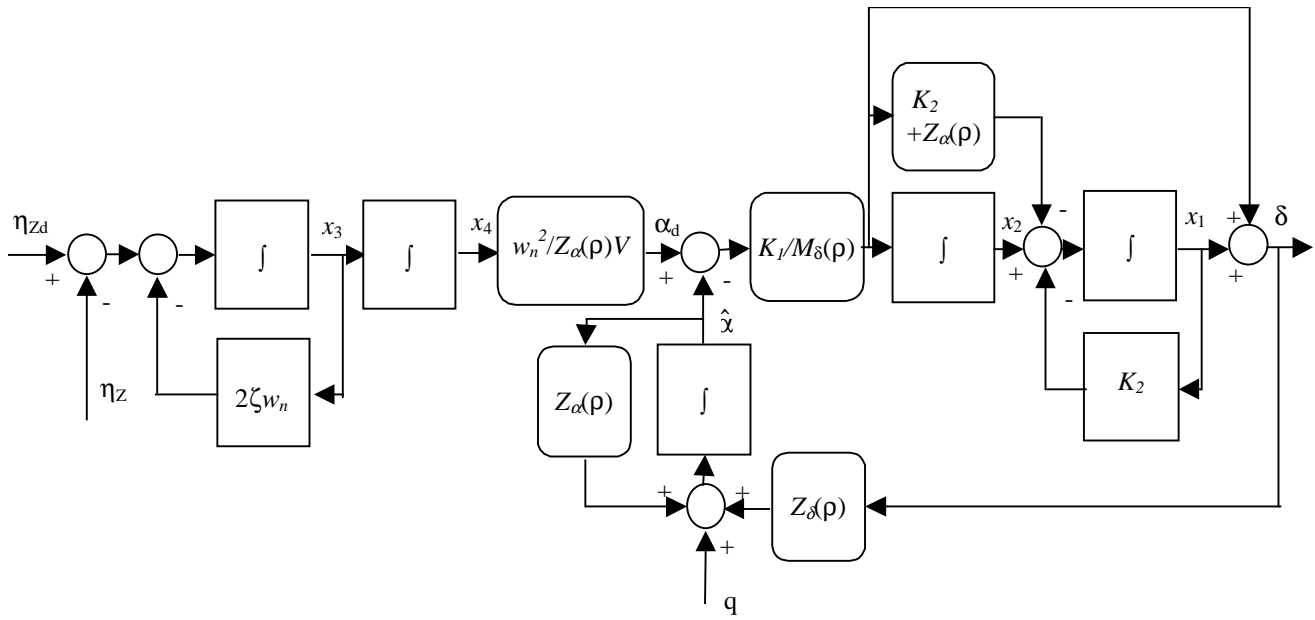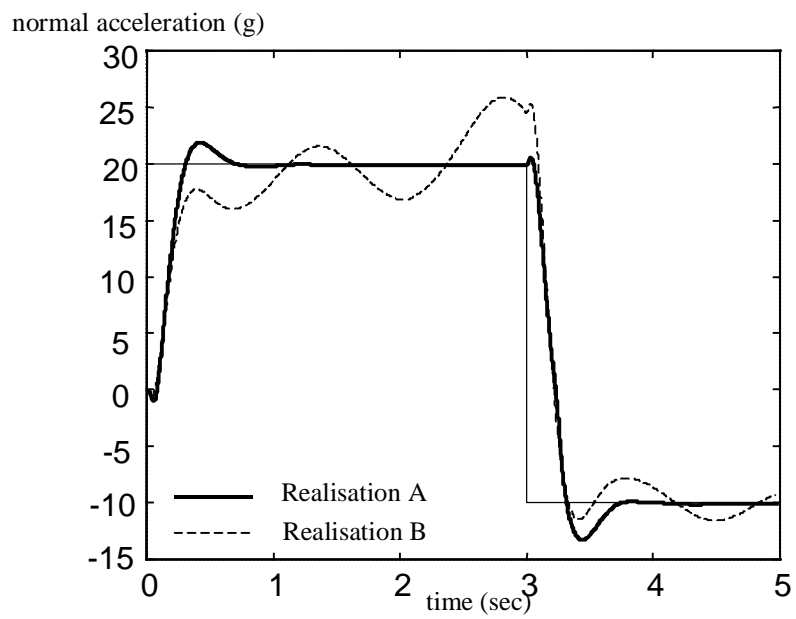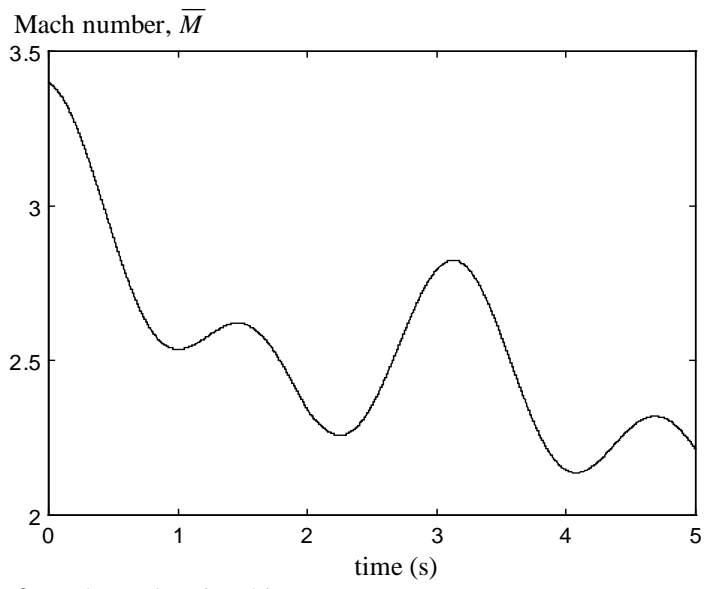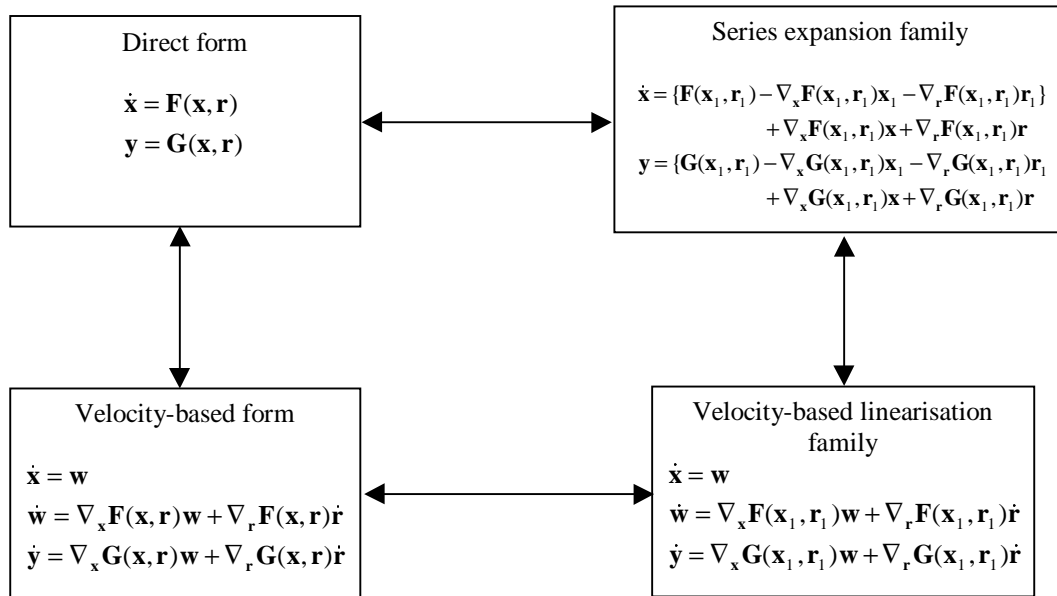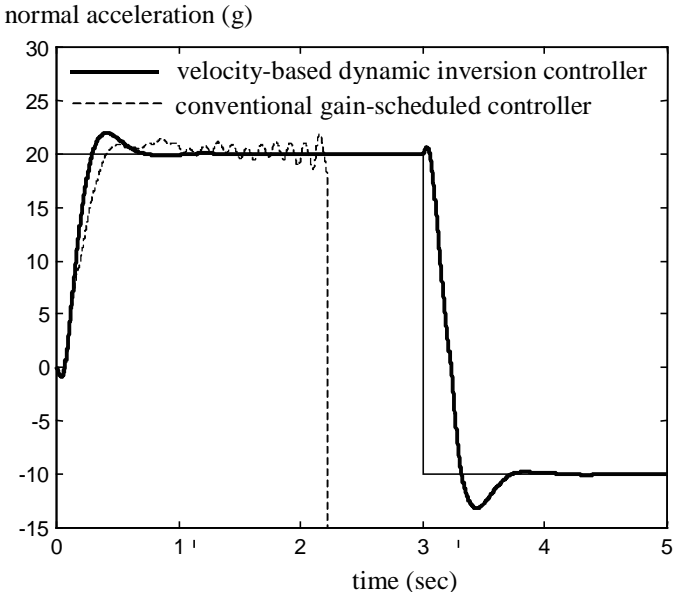**Figure 6** Performance when aerodynamic moment and force depend nonlinearly on incidence and elevator angles.