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On the group generated by C , P and T : $I^2 = T^2 = P^2 = ITP = -1$, with applications to pseudo-scalar mesons

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Abstract

We study faithful representations of the discrete Lorentz symmetry operations of parity P and time reversal T , which involve complex phases when acting on fermions. If the phase of P is a rational multiple of π then $P^{2n} = 1$ for some positive integer n and it is shown that, when this is the case, P and T generate a discrete group, a dicyclic group (also known as a generalised quaternion group) which are generalisations of the dihedral groups familiar from crystallography. Charge conjugation C introduces another complex phase and, again assuming rational multiples of π for complex phases, TC generates a cyclic group of order $2m$ for some positive integer m . There is thus a doubly infinite series of possible finite groups labelled by n and m . Demanding that C commutes with P and T forces $n = m = 2$ and the group generated by P and T is uniquely determined to be the quaternion group. Neutral pseudo-scalar mesons can be simultaneous C and P eigenstates, T commutes with P and C when acting on fermion bilinears so neutral pseudo-scalar mesons can also be T eigenstates. The T -parity should therefore be experimentally observable and the CPT theorem dictates that $T = CP$.

Keywords: Lorentz group, parity, time reversal, fermions, quaternions

1. Introduction

The CPT theorem is a cornerstone of relativistic quantum field theory and of our understanding of matter at the most fundamental level currently accessible to experiment. Parity and time reversal are discrete elements of the Lorentz group acting on the Hilbert space of quantum states and they are intimately related to charge conjugation through the CPT theorem. One

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can ask what is the discrete group generated by the operators \widehat{C} , \widehat{P} and \widehat{T} ? This question was addressed in [1] and, choosing the phase of \widehat{P} so that $\widehat{P}^2 = 1$, three possible groups emerged: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and the dihedral group, D_8 (in [2] a similar discussion for the action of $\widehat{\Sigma} = \widehat{C}\widehat{P}$ and \widehat{T} on physical states gave the same three groups).

When fermionic states are considered there are more possibilities. Racah [3] allowed for $\widehat{P}^2 = -1$ on fermionic states and, with $\widehat{P}^2 = \pm 1$, the list was extended to 8 possible groups in [4], 2 of which are specific to spin-1/2 fermions [5]. In this work these possibilities are explored in more detail, the motivation is to construct all possible faithful representations of the improper Lorentz group.

It will be shown that considering \widehat{P} and \widehat{T} alone acting on fermions, without reference to \widehat{C} , and assuming only that \widehat{P} and \widehat{T} generate a discrete group allows for a doubly infinite series of groups, labelled by two integers n and m . Including \widehat{C} , and assuming it commutes with \widehat{P} and \widehat{T} , reduces this to the two groups in [5], $D_8 \times \mathbb{Z}_2$ and $Q \times \mathbb{Z}_2$, where Q is the quaternion group and \mathbb{Z}_2 is generated by \widehat{C} . The latter group is the only one relevant for Majorana fermions, which are viable candidates for extensions of the standard model that allow for neutrino masses.

The phase ϕ_P in the definition of \widehat{P} does not seem to be physically observable and Weinberg [6] has suggested that that it can be removed completely when there is an internal global $U(1)$ symmetry, such as baryon number or lepton number in the standard model, in which case ϕ_P can be modified by combining \widehat{P} with a global symmetry phase transformation to form a modified parity operator \widehat{P}' satisfying $\widehat{P}'^2 = 1$. The approach here will not assume any such symmetries associated with underlying dynamics, the properties of \widehat{P} will be determined purely in terms of faithful representations of the Lorentz group, without any reference to a specific model.

Another common argument is that the phase of \widehat{P} can be ignored because quantum mechanics only requires projective representations, but this is a dangerous argument. Phases, such as Aharonov–Bohm and Berry phases, can have physical effects—it is safer to work with faithful representations and only project at the end of the calculation. This has the added benefit that it provides a check at the end that the physics does not depend on an overall phase.

The strategy therefore is to construct faithful represents of \widehat{P} and \widehat{T} . If ϕ_P is an irrational multiple of π then \widehat{P} generates a discrete group of infinite order but if the phase is a rational multiple of π the group generated by \widehat{P} is finite. In this paper all possible finite groups generated by acting on spinors with \widehat{C} , \widehat{P} and \widehat{T} are classified.

There are some subtleties in discussing these questions when acting on spinors and the associated action of $SI(2, \mathbb{C})$. Decompose Fock space into invariant sub-spaces of the Lorentz group and let \mathcal{S} denote the sub-space of spinors transforming under the four-dimensional Dirac representation of the proper Lorentz group in four-dimensional Minkowski space–time. Then the definition of \widehat{P} , \widehat{C} and \widehat{T} is such that $\widehat{P} : \mathcal{S} \rightarrow \mathcal{S}$ and $\widehat{C} : \mathcal{S} \rightarrow \mathcal{S}$ but \widehat{T} complex conjugates c -numbers and so maps $\mathcal{S} \rightarrow \mathcal{S}^*$. This means that, while \mathcal{S} is an invariant sub-space of \widehat{P} , it is not invariant under \widehat{T} .

However we can extend \widehat{C} , \widehat{P} and \widehat{T} in an obvious way to operators C , P and T acting on the space $\mathcal{S} \oplus \mathcal{S}^*$ and it is then possible to define C , P and T so that they do indeed generate a discrete group, and it is non-trivial when acting on Dirac fermions.

We can summarise the results quite briefly. Choosing the phase of P to be a rational multiple of π there is a smallest positive integer n for which $P^{2n} = 1$ and P generates a cyclic group of order $2n$. The standard definition of T implies that $T^2 = -1$ and P and T then generate a finite group: for $n = 1$ it is $\mathbb{Z}_4 \times \mathbb{Z}_2$; for $n = 2$ it is the quaternion group Q —denoting space–time inversion $x \rightarrow -x$ by $I = TP$, it is the group defined by

$$I^2 = T^2 = P^2 = ITP = -1. \tag{1}$$

For $n \geq 3$ the group generated by \mathbf{P} and \mathbf{T} is a generalised quaternion group, Q_{4n} when n is even and $Q_{4n} \times \mathbb{Z}_2$ when n is odd.¹ For $n = 2$, Q_8 is the quaternion group Q , of order eight.

When charge conjugation is included there is another undetermined phase, although $\mathbf{C}^2 = 1$ regardless of this phase. If the phase of \mathbf{CT} is also a rational multiple of π then $(\mathbf{CT})^{2m} = 1$, for some positive integer m , and this leads to a doubly infinite set of finite groups, labelled by n and m . This can be reduced to one unique possibility if it is assumed that parity and time reversed transforms of a Majorana spinor are also Majorana, then only $n = m = 2$ is allowed and the group consists of two copies of the quaternion group, $Q \times \mathbb{Z}_2$, where \mathbf{C} generates \mathbb{Z}_2 .

When $n = 2$ the phase of \widehat{P} is $\pm \frac{\pi}{2}$ (with signature $(1, -1, -1, -1)$, and $\det \widehat{P} = 1$). A consequence of demanding that two-point correlators $\Delta(x - y) = \{\psi(x), \bar{\psi}(y)\}$ vanish at space-like separations is that $e^{i\phi_P} = \pm i$ for Majorana fermions [6]. We can turn this argument around and say that demanding that \mathbf{C} commutes with \mathbf{P} implies that $\Delta(x - y)$ vanishes at space-like separations for anti-commuting fields.

Neutral pseudo-scalar mesons, such Kaons and neutral B -mesons, can be simultaneous \mathbf{C} and \mathbf{P} eigenstates, with eigenvalues $CP = \pm 1$ and \mathbf{T} -violation has been observed in the $B^0 - \bar{B}^0$ system. Although \mathbf{T} does not commute with \mathbf{P} on fermions they do commute on fermion bilinears. Neutral pseudo-scalar mesons can therefore be simultaneous eigenstates of \mathbf{C} , \mathbf{P} and \mathbf{T} and the \mathbf{CPT} theorem dictates that their \mathbf{T} parity should be the product of \mathbf{P} and \mathbf{C} , so $T = CP = \pm 1$, which should be experimentally measurable.

The layout of the paper is as follows. In section 2 the algebra of \mathbf{P} and \mathbf{T} is shown to be that of Q_{4n} or $Q_{4n} \times \mathbb{Z}_2$ for even/odd n respectively. In section 3 charge conjugation is included and is shown to restrict n to be 2 if \mathbf{C} commutes with \mathbf{P} and \mathbf{T} . Neutral pseudo-scalar meson systems are discussed in section 5. Finally the results are summarised and discussed in section 6. Conventions and notation are specified in an appendix A but the main text is written so as to be independent of the choice of γ -matrix conventions.

2. The group generated by \mathbf{P} and \mathbf{T}

Consider a massive spin- $\frac{1}{2}$ particle with annihilation operators $a_s(\mathbf{p})$ and the corresponding anti-particle with creation operators $a_{c,s}^\dagger(\mathbf{p})$, for spin $s = \pm \frac{1}{2}$. The field operator is

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \{u_s(\mathbf{p})a_s(\mathbf{p})e^{-ip \cdot x} + v_s(\mathbf{p})a_{c,s}^\dagger(\mathbf{p})e^{ip \cdot x}\}, \quad (2)$$

where $u_s(\mathbf{p})$ is a positive energy spinor and $v_s(\mathbf{p})$ a negative energy spinor associated with the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

In order to make the analysis completely independent of the conventions we shall allow for both $\eta_{\mu\nu} = (-1, +1, +1, +1)$ and $\eta_{\mu\nu} = (+1, -1, -1, -1)$ by defining $\varepsilon = \eta^{00} = \pm 1$.

Under a Lorentz transformation connected to the identity, $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$, this transforms to

$$\psi(x) \rightarrow D(\omega)\psi(\Lambda x), \quad (3)$$

¹ Generalised quaternion groups Q_{4n} (also known as dicyclic groups Dic_n) are generalisations of the dihedral groups D_{2n} , of order $2n$, familiar from crystallography. Dihedral groups are generated by rotations through $\frac{2\pi}{n}$ and reflections \mathbf{R} . Replace \mathbf{R} (with $\mathbf{R}^2 = 1$) by \mathbf{T} (with $\mathbf{T}^4 = 1$) and D_{2n} becomes the generalised quaternion group Q_{4n} .

where $D(\omega)$ is given explicitly in the appendix A (71). We shall denote the space of all such spinor operators that transform this way by \mathcal{S} .

The standard definitions of \hat{P} and \hat{T} in second quantisation are then (see any standard QFT text, e.g. [6], and equations (87) and (88) in the appendix A for our conventions for numerical equalities)

$$\hat{P}\psi(x) = e^{i\phi_P}\gamma^0\psi(x_P) \tag{4}$$

$$\hat{T}\psi(x) = e^{i\phi_T}C^*\beta\gamma_5\gamma^0\psi(x_T), \tag{5}$$

where ϕ_P and ϕ_T are complex phase factors, $x_P = (t, -\mathbf{x})$ denotes the parity transformation on space–time points and $x_T = (-t, \mathbf{x})$ time-reversal. C is the charge conjugation matrix and $C^*\beta$ maps $\mathcal{S} \rightarrow \mathcal{S}^*$.

It is common in quantum field theory textbooks, e.g [6], to write

$$\hat{T}\psi(x) = -e^{i\phi_T}C\gamma_5\psi(x_T),$$

or even

$$\hat{T}\psi(x) = e^{i\phi_T}\gamma^1\gamma^3\psi(x_T),$$

but, while these may be numerically correct in a chosen γ -matrix representation, they obscure the Lorentz transformation properties of the spinors. We wish to write all expressions in a basis independent form and use the notation

$$\hat{T}\psi(x) = e^{i\phi_T}C^*\beta\gamma_5\gamma^0\psi(x_T) \cong e^{i\phi_T}\gamma^1\gamma^3\psi(x_T), \tag{6}$$

where $=$ is reserved for equations involving matrix multiplication that preserve the spinor index structure, it might be called a spinorial equality, while \cong indicates equations that do not preserve the spinor index structure, but are nevertheless numerically correct in a specifically chosen basis (more details are given in the appendix A). If one is confident the index structure is correct \cong is an extremely useful calculational tool, but it should be used with care when discussing the Lorentz transformation properties of spinors.²

The square of the parity operator is proportional to the identity, $\hat{P}^2 = e^{2i\phi_P}(\gamma^0)^2 = \varepsilon e^{2i\phi_P}$, where $\varepsilon = \eta^{00}$. Of course the sign of η^{00} can be absorbed into the phase ϕ_P , so it cannot matter which sign we choose for the metric, but we shall keep ε explicit in order to follow through the mathematical consequences of the different possible conventions.

If $\psi \in \mathcal{S}$ and $\psi' \in \mathcal{S}$ are two Dirac spinors then $\psi^* \in \mathcal{S}^*$ and

$$\bar{\psi} = \psi^\dagger\beta$$

is in the dual vector space to \mathcal{S} , which will be denoted by \mathcal{S}^D . For ordinary complex vector spaces this would mean that $\bar{\psi}\psi' \in \mathbb{C}$, but for spinors

$$\bar{\psi}\psi' = \psi^\dagger\beta\psi'$$

is a Lorentz invariant Fock space operator in second quantisation. β is an hermitian metric on \mathcal{S} , $\beta : \mathcal{S} \rightarrow (\mathcal{S}^D)^\dagger$ and $\beta^\dagger = \beta$ while $C : (\mathcal{S}^D)^T \rightarrow \mathcal{S}$. The important properties of β and C are

$$\beta\gamma^\mu = \varepsilon(\gamma^\mu)^\dagger\beta, \quad \gamma^\mu C = -C(\gamma^\mu)^T, \quad C\beta^*(\gamma^\mu)^* = -\varepsilon\gamma^\mu C\beta^*.$$

²The spinorial equality in equation (5) is obtained by carefully tracking the spinor index structure in the derivation of \hat{T} in the chiral representation, the numerical equality is probably the more familiar one, with signature $(+, -, -, -)$.

which is all that will be used in the ensuing analysis.

Let $\mathbf{1}$ denote the identity matrix acting on \mathcal{S} and $\bar{\mathbf{1}}$ the identity matrix on \mathcal{S}^* then, using a chiral basis in which

$$\beta = \begin{pmatrix} 0 & \bar{\mathbf{1}} \\ \mathbf{1} & 0 \end{pmatrix},$$

gives $\beta \cong \beta^T$ and $\beta^2 \cong 1$, but $\beta \neq \beta^T$ and $\beta^2 \neq 1$. However $\beta^\dagger = \beta$ is correct. With the conventions in appendix A³

$$\gamma^0 \cong \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

so $\beta \cong \gamma^0$ but $\beta \neq \gamma^0$. Similarly $\beta\gamma^0 \cong 1$ and in our conventions C is real, $C^* \cong C$, also $\beta\gamma_5\gamma^0 \cong -\gamma_5\beta\gamma^0 \cong -\gamma_5$ hence equation (6). Spinorial equations involving \cong are valid in any consistent set of conventions for the γ -matrices, numerical equations involving \cong can be convention dependent, though they are always the same in unitarily equivalent representations of the γ -matrices once the sign of η^{00} is chosen, see e.g. appendix A.2 of [7].

The structure of (4) and (5) is perhaps a little clearer if they are re-written as

$$\widehat{P}\psi(x) = i e^{i\phi_P} \gamma_5 \gamma^1 \gamma^2 \gamma^3 \psi(x_P) \tag{7}$$

$$\widehat{T}\psi(x) = e^{i\phi_T} C^* \beta \gamma_5 \gamma^0 \psi(x_T), \tag{8}$$

the index structure is then more logical and $C^*\beta$ is the matrix that maps $D(\omega) \rightarrow D^*(\omega)$. As described in the appendix A $\widehat{P} : \mathcal{S} \rightarrow \mathcal{S}$ and $\widehat{T} : \mathcal{S} \rightarrow \mathcal{S}^*$.

It is straightforward to define

$$\widehat{P}^*\psi^*(x) = \{\widehat{P}\psi(x)\}^* = e^{-i\phi_P} (\gamma^0)^* \psi^*(x_P) \tag{9}$$

$$\widehat{T}^*\psi^*(x) = \{\widehat{T}\psi(x)\}^* = e^{-i\phi_T} C \beta^* \gamma_5^* (\gamma^0)^* \psi^*(x_T) \cong -e^{-i\phi_T} C \gamma_5 \psi^*(x_T). \tag{10}$$

Then $\widehat{P}^* : \mathcal{S}^* \rightarrow \mathcal{S}^*$ and $\widehat{T}^* : \mathcal{S}^* \rightarrow \mathcal{S}$, with $\widehat{T}^* = -\widehat{T}^{-1}$. \mathcal{S} is a sub-space of the full Hilbert space that is an invariant sub-space under \widehat{P} but it is not an invariant sub-space under \widehat{T} .

In the chiral basis in the appendix A

$$\widehat{P}\psi(x) \cong e^{i\phi_P} \beta \psi(x_P) \cong e^{i\phi_P} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(x_P) \tag{11}$$

$$\widehat{P}^*\psi^*(x) \cong e^{-i\phi_P} \beta \psi^*(x_P) \cong e^{-i\phi_P} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi^*(x_P) \tag{12}$$

$$\widehat{T}\psi(x) \cong e^{i\phi_T} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \psi(x_T) \tag{13}$$

$$\widehat{T}^*\psi^*(x) \cong e^{-i\phi_T} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \psi^*(x_T), \tag{14}$$

where $\epsilon \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

³ From now in \cong will refer to the conventions in the chiral basis of appendix A, with $\epsilon = +1$, but ϵ will be kept explicit in spinorial equalities.

The strategy is now to double spinor space to $\mathcal{S} \oplus \mathcal{S}^*$ and construct 8 component spinors

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi^*(x) \end{pmatrix}.$$

This is not a general element of $\mathcal{S} \oplus \mathcal{S}^*$, it is restricted so that the lower component is the complex conjugate of the upper component, and the space of such restricted spinors will be denoted by \mathcal{S} . This idea of doubling the space of states was introduced in [2], though in the present context it is really more of a grading of \mathcal{S} , into one particle states that transform under the Lorentz group under $D(\omega)$ and those that transform under $D^*(\omega)$, rather than a doubling of the space. We really want the action of \hat{P} and \hat{T} on the full Hilbert space and neither \mathcal{S} nor \mathcal{S}^* is the full Hilbert space, which consists of multi-particle states. But it is sufficient to determine the action of P and T on \mathcal{S} in order to determine their action on the full Hilbert space.

\mathcal{S} is an invariant subspace of \hat{P} but not of \hat{T} , it is necessary to go to $\mathcal{S} \oplus \mathcal{S}^*$ to get an invariant sub-space of \hat{T} if we wish to evaluate the full group generated by \hat{P} and \hat{T} .

On \mathcal{S} the action of parity and time-reversal on the spinor indices of $\Psi(x)$ is represented as ordinary multiplication by the 8×8 matrices

$$P = \begin{pmatrix} e^{i\phi_P} \gamma^0 & 0 \\ 0 & e^{-i\phi_P} (\gamma^0)^* \end{pmatrix} \tag{15}$$

$$T = \begin{pmatrix} 0 & e^{-i\phi_T} C \beta^* \gamma_5^* (\gamma^0)^* \\ e^{i\phi_T} C^* \beta \gamma_5 \gamma^0 & 0 \end{pmatrix}. \tag{16}$$

We are now in a position to determine the group generated by P and T using straightforward matrix multiplication. Using the relation (100)

$$C^* \beta \gamma^\mu = -\varepsilon (\gamma^\mu)^* C^* \beta,$$

it is immediate from (15) and (16) that⁴

$$P^{2k} = \varepsilon^k \begin{pmatrix} e^{2ki\phi_P} & 0 \\ 0 & e^{-2ki\phi_P} \end{pmatrix} \tag{17}$$

$$P^{2k+1} = \varepsilon^k \begin{pmatrix} e^{(2k+1)i\phi_P} \gamma^0 & 0 \\ 0 & e^{-(2k+1)i\phi_P} (\gamma^0)^* \end{pmatrix} \tag{18}$$

$$T^2 = -1 \tag{19}$$

$$(PT)^2 = -1, \quad PTP = T. \tag{20}$$

The phase ϕ_P and signature ε drop out of (19) and (20) and we immediately note that

$$PT = TP^{-1}$$

so P and T only commute if $P^2 = 1$.

If ϕ_P is an irrational multiple of π the algebra never closes and the group generated by P is of infinite order. But if ϕ_P is a rational multiple of π then there exists a smallest positive integer n for which

$$P^{2n} = 1 \tag{21}$$

⁴ These matrices are written in 4×4 block form. The 8×8 identity matrix is just denoted by 1.

and P generates the cyclic group \mathbb{Z}_{2n} .

To understand the structure in more detail we analyse $\varepsilon = +1$ and $\varepsilon = -1$ separately.

- $\varepsilon = +1$: first consider specific cases with small n :

- * $n = 1$, $\phi_P = \pm\pi$: $P^{-1} = P$ and (20) implies that $[T, P] = 0$, so T and P generate the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_2$.

- * $n = 2$, $\phi_P = \pm\frac{\pi}{2}$: $P^2 = -1$ and (20) implies that $TP = -PT$. Let $I = TP$ denote space–time inversion, sending space–time points $x \rightarrow -x$, then

$$I^2 = T^2 = P^2 = ITP = -1 \tag{22}$$

generates the quaternion group, Q .⁵

- * $n = 3$, $\phi_P = \pm\frac{\pi}{3}$: in this case (18) implies $P^3 = -\begin{pmatrix} \gamma^0 & 0 \\ 0 & (\gamma^0)^* \end{pmatrix}$. Let

$$V = \begin{pmatrix} \gamma^0 & 0 \\ 0 & (\gamma^0)^* \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} e^{\pm i\pi/3} \mathbf{1} & 0 \\ 0 & e^{\mp i\pi/3} \mathbf{1} \end{pmatrix}.$$

then $U^3 = -1$ and $[P, V] = [T, V] = 0$. From (20)

$$T^{-1}UT = U^{-1}$$

and we have a finite group with presentation

$$\langle U, T \mid U^6 = 1, U^3 = T^2, T^{-1}UT = U^{-1} \rangle, \tag{23}$$

which is the generalised quaternion group Q_{12} of order 12. Including V adds a trivial \mathbb{Z}_2 making the full group $Q_{12} \times \mathbb{Z}_2$. The strategy is to start with P and T , use these to define $V = P^3T^2$ and $U = PV$, then T and U (with the presentation (23)), together with V , give the full group.

The cases $n = 2$ and $n = 3$ generalise to two infinite families:

- * Even n : $P^n = T^2 = -1$ and together P and T generate the generalised quaternion group Q_{4n} of order $4n$ with the presentation

$$Q_{4n} = \langle P, T \mid P^{2n} = 1, P^n = T^2, T^{-1}PT = P^{-1} \rangle. \tag{24}$$

Q_8 is the quaternion algebra.

- * Odd $n > 1$: with

$$V = \begin{pmatrix} \gamma^0 & 0 \\ 0 & (\gamma^0)^* \end{pmatrix}, \tag{25}$$

$P = UV$, U and T generate the generalised quaternion group Q_{4n} with the presentation

$$Q_{4n} = \langle U, T \mid U^{2n} = 1, U^n = T^2, T^{-1}UT = U^{-1} \rangle \tag{26}$$

and, including $V = P^nT^2$, the full group is $Q_{4n} \times \mathbb{Z}_2$.

⁵ We are working with the complex Clifford algebra. The appearance of Q here does not seem to have any obvious relation to the appearance of quaternions in the real Clifford algebra of 2×2 matrices of quaternions.

• $\varepsilon = -1$: the analysis is very similar to $\varepsilon = +1$, but with slight modifications due to the fact that T anti-commutes with $\begin{pmatrix} \gamma^0 & 0 \\ 0 & (\gamma^0)^* \end{pmatrix}$ for this choice.

- * $n = 1, \phi_P = \pm \frac{\pi}{2}$: $P^{-1} = P$ and (20) implies that $TP = PT$. Again T and P simply generate abelian the group $\mathbb{Z}_4 \times \mathbb{Z}_2$.
- * $n = 2, \phi_P = 0, \pi$: so $P^2 = T^2 = -1$ and $TP = -PT$. As for $\varepsilon = +1, I = TP$ and $I^2 = -1$, and the group is again the quaternion group Q .
- * $n = 3, \phi_P = \pm \frac{\pi}{6}$: then

$$U := -P^2 = \begin{pmatrix} e^{\pm\pi i/3} \mathbf{1} & 0 \\ 0 & e^{\mp\pi i/3} \bar{\mathbf{1}} \end{pmatrix}$$

generates \mathbb{Z}_6 and $U^3 = -1$. Also

$$V := P^3 = \mp \begin{pmatrix} i\gamma^0 & 0 \\ 0 & -i(\gamma^0)^* \end{pmatrix} \tag{27}$$

generates \mathbb{Z}_2 . This gives $[T, V] = 0$ and, trivially, $[U, V] = 0$ so V just generates a \mathbb{Z}_2 factor and we only need to worry about T and U . We have $TU = U^{-1}T$ and so T complex conjugates U . Let $W = UT$ then

$$\langle U, T, W \mid U^6 = 1, U^3 = T^2 = W^2 = UTW \rangle$$

is another presentation of Q_{12} . As for $\varepsilon = +1$ the full group is $Q_{12} \times \mathbb{Z}_2$.

- * $n = 4, \phi_P = \pm \frac{\pi}{4}$. $P^4 = -1$ and, from (20), $T^{-1}PT = P^{-1}$. We have the group with presentation

$$\langle P, T \mid P^8 = 1, P^4 = T^2, T^{-1}PT = P^{-1} \rangle. \tag{28}$$

This is the generalised quaternion group Q_{16} .

- The cases $n = 3$ and $n = 4$ generalise to two infinite families

- * Even $n \geq 4$: $P^n = -1$ and the group is again Q_{4n} ,

$$\langle P, T \mid P^{2n} = 1, P^n = T^2, T^{-1}PT = P^{-1} \rangle.$$

- * Odd $n \geq 3$: $\phi_P = \pm \frac{\pi}{2n}$, define $U := -P^2 = \begin{pmatrix} e^{\pm\pi i/n} \mathbf{1} & 0 \\ 0 & e^{\mp\pi i/n} \bar{\mathbf{1}} \end{pmatrix}$ which generates \mathbb{Z}_{2n} , with $U^n = -1$. Let $V := P^n = \mp \begin{pmatrix} i\gamma^0 & 0 \\ 0 & -i(\gamma^0)^* \end{pmatrix}$ generate \mathbb{Z}_2 . With $[T, V] = [U, V] = 0, TU = U^{-1}T$ and $W = UT$ we have

$$\langle U, T, W \mid U^{2n} = 1, U^n = T^2 = W^2 = UTW \rangle$$

and the full group is $Q_{4n} \times \mathbb{Z}_2$.

The group structure is the same with either choice of sign for the signature: $\mathbb{Z}_4 \times \mathbb{Z}_2$ for $n = 1, Q_{4n}$ for even n and $Q_{4n} \times \mathbb{Z}_2$ for odd $n \geq 3$.

This completes the classification of all possible finite groups generated by \mathbf{P} and \mathbf{T} acting on Dirac spinors in \mathcal{S} . We now go on to discuss the unitary properties of \mathbf{P} and \mathbf{T} . Numerically \mathbf{P} and \mathbf{T} are unitary matrices

$$\mathbf{P}^\dagger \cong \mathbf{P}^{-1}, \quad \mathbf{T}^\dagger \cong \mathbf{T}^{-1},$$

but these are not spinorial equalities. The definition of a unitary operator requires defining a bi-linear form on \mathcal{S} and there is a natural Hermitian Lorentz invariant metric,

$$\beta_+ = \frac{1}{2} \begin{pmatrix} \beta & 0 \\ 0 & \beta^* \end{pmatrix}. \quad (29)$$

For two spinors $\Psi \in \mathcal{S}$ and $\Psi' \in \mathcal{S}$

$$\begin{aligned} \widetilde{\Psi}\Psi' &:= \Psi^\dagger \beta_+ \Psi' = \frac{1}{2} (\psi^\dagger \beta \psi' + \psi^T \beta^* \psi'^*) \\ &= \frac{1}{2} (\psi^\dagger \beta \psi' - \psi'^\dagger \beta \psi) = \frac{1}{2} (\overline{\psi} \psi' - \overline{\psi}' \psi) \end{aligned}$$

is Lorentz invariant (the minus sign is due to anti-commutativity of fermions).

This gives

$$(\widetilde{\mathbf{P}\Psi})(\mathbf{P}\Psi') = (\widetilde{\mathbf{T}\Psi})(\mathbf{T}\Psi') = \widetilde{\Psi}\Psi' \quad (30)$$

and both \mathbf{P} and \mathbf{T} are unitary with this bi-linear form.⁶ Unfortunately

$$\widetilde{\Psi}\Psi = 0$$

and this is of little use in constructing a Lagrangian. An alternative is to use

$$\beta_- = \frac{1}{2} \begin{pmatrix} \beta & 0 \\ 0 & -\beta^* \end{pmatrix} \quad (31)$$

for which

$$\begin{aligned} \overline{\Psi}\Psi' &:= \Psi^\dagger \beta_- \Psi' = \frac{1}{2} (\psi^\dagger \beta \psi' - \psi^T \beta^* \psi'^*) = \frac{1}{2} (\psi^\dagger \beta \psi' + \psi'^\dagger \beta \psi) \\ &= \frac{1}{2} (\overline{\psi} \psi' + \overline{\psi}' \psi). \end{aligned} \quad (32)$$

This renders \mathbf{P} unitary but this bi-linear is not invariant under \mathbf{T}

$$\overline{(\mathbf{P}\Psi)}(\mathbf{P}\Psi') = -\overline{(\mathbf{T}\Psi)}(\mathbf{T}\Psi') = \overline{\Psi}\Psi' \quad (33)$$

giving⁷

⁶To avoid cumbersome notation the space-time argument of Ψ is omitted, it should be clear from the context how x is affected. In any case the metric on the full Hilbert space of position dependent spinors includes $\int d^4x$ and the argument of Ψ disappears from the inner product when the integral is evaluated.

⁷Equations (30) and (33) are most easily proven using

$$\mathbf{P}^\dagger \beta_\pm = \beta_\pm \mathbf{P}^{-1} \quad (34)$$

$$\mathbf{T}^\dagger \beta_\pm = \mp \beta_\pm \mathbf{T}. \quad (35)$$

$$\overline{(\widehat{T}\Psi)}(T\Psi) = -\overline{\Psi}\Psi. \tag{36}$$

The standard action of \widehat{T} on four-component spinors in \mathcal{S} gives

$$\widehat{T}(\overline{\psi}\psi) = \overline{(\widehat{T}\psi)}\beta^*(\widehat{T}\psi) = \overline{\psi}\psi$$

and the reason for this difference in sign is the anti-unitary nature of \widehat{T} , which interchanges β and β^* . Alternatively the sign change can be accounted for by demanding that \widehat{T} reverses the order of the spinors (as observed in [16] this is equivalent to $\beta \rightarrow -\beta$ in fermion bi-linears). This property of time reversal can be accommodated in \mathcal{S} by defining

$$\widehat{T}(\overline{\Psi}\Psi') := -\overline{(\widehat{T}\Psi)}(\widehat{T}\Psi') = \overline{\Psi}'\Psi, \tag{37}$$

while

$$\widehat{P}(\overline{\Psi}\Psi') := \overline{(\widehat{P}\Psi)}(\widehat{P}\Psi') = \overline{\Psi}\Psi',$$

remains unchanged.

On \mathcal{S} time reversal \widehat{T} is an anti-linear operator: if ψ is multiplied by a phase, $\psi \rightarrow e^{i\alpha}\psi$, then $\widehat{T}(e^{i\alpha}\psi) = e^{-i\alpha}\widehat{T}\psi$. In \mathcal{S} this is represented by

$$T \begin{pmatrix} e^{i\alpha}\psi \\ e^{-i\alpha}\psi^* \end{pmatrix} = T \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \Psi = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} T\Psi$$

and this is simply matrix multiplication, it does not involve any explicit complex conjugation. Define the 8×8 matrix, in 4×4 block form,

$$\mathcal{I} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

then $\Psi \rightarrow e^{\mathcal{I}\alpha}\Psi$ and matrix multiplication gives

$$T\mathcal{I} = -\mathcal{I}T,$$

with no need to complex conjugate c -numbers explicitly. At the same time it is trivial that

$$P\mathcal{I} = \mathcal{I}P,$$

thus space–time inversion anti-commutes with \mathcal{I}

$$\mathcal{I}\mathcal{I} = -\mathcal{I}\mathcal{I}.$$

Note that multiplication by \mathcal{I} interchanges β_+ and β_- in that

$$\overline{\Psi}\mathcal{I}\Psi' = i\widetilde{\Psi}\Psi'.$$

There is therefore no need to introduce anti-linear operators in \mathcal{S} . The only difference between \widehat{T} and \widehat{P} in this regard is that \widehat{T} reverses the order of spinors whereas \widehat{P} preserves the order.

Now consider the reducible representation of the Clifford algebra

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \varepsilon(\gamma^\mu)^* \end{pmatrix},$$

giving

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu},$$

in terms of which

$$\bar{\Psi} \Gamma^\mu \Psi' = \frac{1}{2} (\bar{\psi} \gamma^\mu \psi' + \bar{\psi}' \gamma^\mu \psi), \quad \tilde{\Psi} \Gamma^\mu \Psi' = \frac{1}{2} (\bar{\psi} \gamma^\mu \psi' - \bar{\psi}' \gamma^\mu \psi).$$

The actions of P and T on these are

$$\hat{P} (\bar{\Psi} \Gamma^\mu \Psi') = \mathcal{P}^\mu{}_\nu (\bar{\Psi} \Gamma^\nu \Psi'), \quad \hat{P} (\tilde{\Psi} \Gamma^\mu \Psi') = \mathcal{P}^\mu{}_\nu (\tilde{\Psi} \Gamma^\nu \Psi') \quad (38)$$

$$\hat{T} (\bar{\Psi} \Gamma^\mu \Psi') = \mathcal{P}^\mu{}_\nu (\bar{\Psi}' \Gamma^\nu \Psi), \quad \hat{T} (\tilde{\Psi} \Gamma^\mu \Psi') = -\mathcal{P}^\mu{}_\nu (\tilde{\Psi}' \Gamma^\nu \Psi) \quad (39)$$

where $\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ implements the parity operation on space-time points,

$x \rightarrow x_P$.

$$\text{With } \Gamma_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5^* \end{pmatrix}$$

$$\bar{\Psi} \Gamma_5 \Psi' = \frac{1}{2} (\bar{\psi} \gamma_5 \psi' - \bar{\psi}' \gamma_5 \psi), \quad \tilde{\Psi} \Gamma_5 \Psi' = \frac{1}{2} (\bar{\psi} \gamma_5 \psi' + \bar{\psi}' \gamma_5 \psi)$$

and

$$\hat{P} (\bar{\Psi} \Gamma_5 \Psi') = \overline{(\hat{P} \Psi)} \Gamma_5 (P \Psi') = -\bar{\Psi} \Gamma_5 \Psi' \quad (40)$$

$$\hat{T} (\bar{\Psi} \Gamma_5 \Psi') = -\overline{(\hat{T} \Psi)} \Gamma_5 (T \Psi') = \bar{\Psi}' \Gamma_5 \Psi \quad (41)$$

$$\hat{P} (\tilde{\Psi} \Gamma_5 \Psi') = \widetilde{(\hat{P} \Psi)} \Gamma_5 (P \Psi') = -\tilde{\Psi} \Gamma_5 \Psi' \quad (42)$$

$$\hat{T} (\tilde{\Psi} \Gamma_5 \Psi') = -\widetilde{(\hat{T} \Psi)} \Gamma_5 (T \Psi') = -\tilde{\Psi}' \Gamma_5 \Psi. \quad (43)$$

3. The groups generated by C , P and T

The charge conjugation operator is defined as

$$\hat{C} \psi(x) = e^{i\phi_C} C \beta^T \psi^*(x) \cong e^{i\phi_C} C \beta \psi^*(x) \quad (44)$$

with

$$C = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \bar{\epsilon}^{-1} \end{pmatrix},$$

$\epsilon \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and ϕ_C another complex phase. The charge conjugation matrix C maps $(S^D)^T \rightarrow S$ and is defined by demanding that $\gamma^\mu C = -(\gamma^\mu)^T C$.

This leads to the natural definition of the 8×8 matrix

$$C = \begin{pmatrix} 0 & e^{i\phi_C} C\beta^* \\ e^{-i\phi_C} C^*\beta & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & e^{i\phi_C} C\beta \\ e^{-i\phi_C} C\beta & 0 \end{pmatrix} \quad (45)$$

acting on S and, using (95),

$$C^2 = 1.$$

From (15) and (16)

$$(CP)^2 = -1 \quad (46)$$

$$(CT)^2 = -\epsilon \begin{pmatrix} e^{2i(\phi_T + \phi_C)} & 0 \\ 0 & e^{-2i(\phi_T + \phi_C)} \end{pmatrix} = -\epsilon e^{2I(\phi_T + \phi_C)}. \quad (47)$$

If $\phi_T + \phi_C$ is an irrational multiple of π the algebra does not close and the group is of infinite order. But if $\phi_T + \phi_C$ is a rational multiple of π then there is a smallest positive integer p for which $e^{ip(\phi_T + \phi_C)} = 1$ and (47) then implies that

$\epsilon = 1$	$\epsilon = -1$	
$(CT)^{4p} = 1,$	$(CT)^{2p} = 1,$	if p is odd,
$(CT)^{2p} = 1,$	$(CT)^p = 1,$	if $p = 2 \pmod 4,$
$(CT)^p = 1,$	$(CT)^p = 1,$	if $p = 0 \pmod 8,$
$(CT)^{\frac{p}{2}} = 1,$	$(CT)^p = 1,$	if $p = 4 \pmod 8,$

and these are all the lowest powers of CT that give 1 on the right-hand side. In general C need not commute with either P or T ,

$$CP = -P^{-1}C \Rightarrow PCP = -C \quad (49)$$

$$CT = \epsilon e^{\frac{2\pi I}{p}} TC. \quad (50)$$

There is a doubly infinite series of possible finite groups generated by P, T and C , characterised by the two integers n and m :

- n even,

$$\langle P, T, C : P^{2n} = T^4 = C^2 = (PT)^4 = (CP)^4 = (CT)^{2m} = 1 \rangle$$

- n odd,

$$\begin{aligned} \langle U, V, T, C : U^{2n} = T^4 = V^2 = C^2 = (UV)^{2n} = (UT)^4 \\ = (CU)^2 = (VT)^2 = (CV)^2 = (CT)^{2m} = 1 \rangle, \end{aligned}$$

where m can be read off from equation (48) for either choice of ϵ and any given p , with V given by (25) for $\epsilon = +1$ and (27) for $\epsilon = -1$.

This doubly infinite series can be reduced to one unique group if we assume that C commutes with P and T on Majorana spinors. Any Dirac spinor Ψ can be written as a sum of two Majorana spinors Ψ_1 and Ψ_2 , with $C\Psi_1 = \pm\Psi_1$ and $C\Psi_2 = \pm\Psi_2$, as

$$\Psi = \Psi_1 + \mathcal{I}\Psi_2.$$

Without loss of generality we can take the plus signs⁸, then

$$\Psi_1 = \frac{1}{2}(\Psi + C\Psi)$$

and, even without any assumption as to the explicit form of the operators when acting on Majorana fermions, we have

$$CP\Psi_1 = \frac{1}{2}(CP + CPC)\Psi$$

$$PC\Psi_1 = \frac{1}{2}(PC + P)\Psi,$$

since $C^2 = 1$. Assuming that C and P commute on Majorana spinors we deduce that

$$CP + CPC = PC + P$$

on Dirac spinors. Now using the explicit form for Dirac spinors P in (15) is block diagonal and C in (45) is block off-diagonal so the only way this can be true is to have

$$CP = PC \quad \text{and} \quad CPC = P.$$

Hence C and P commute on Dirac spinors too. Equation (49) then forces $P = -P^{-1}$ and $P^2 = -1$, which in turn requires $e^{i\phi_P} = -\epsilon e^{-i\phi_P}$. This means that $n = 2$ and $e^{i\phi_P} = \pm i$ for $\epsilon = +1$, $e^{i\phi_P} = -1$ for $\epsilon = -1$.

The possibility of $P^2 = -1$ for fermions was introduced by Racah [3] and it is shown in [6] that it is necessary for Majorana spinors, but this latter argument requires the assumption that two-point functions must vanish at space-like separations. We see here that it can be deduced purely algebraically from the assumption that $[C, P] = 0$.

The allowed parity eigenstates of Dirac fermions are immediate from $P^2 = -1$,

$$P\Psi = \pm\mathcal{I}\Psi.$$

In the particle data tables protons and neutrons are assigned parity $+1$, but this is a modified parity which is discussed in the conclusions.

The same reasoning can be applied to time reversal, assume that the time reversed state of a Majorana spinor Ψ is also Majorana,

$$CT\Psi_1 = T\Psi_1 = TC\Psi_1$$

then

$$[C, T] = 0$$

⁸ If $C\Psi_1 = \pm\Psi_1$ and $C\Psi_2 = \mp\Psi_2$ then Ψ is itself already Majorana. If $C\Psi_i = -\Psi_i$, for $i = 1, 2$, then let $\Psi'_i = \mathcal{I}\Psi_i$ and $C\Psi'_i = +\Psi'_i$.

on Dirac spinors too and $\varepsilon e^{\frac{4\pi I}{p}} = 1$ which requires $p = 1$ or 2 for $\varepsilon = +1$ and $p = 4$ for $\varepsilon = -1$, in all three cases $(CT)^4 = 1$ and $m = 2$. The group is then

$$\langle P, T, C : P^2 = T^2 = (TP)^2 = -1, C^2 = 1, (PC)^2 = (TC)^2 = -1 \rangle, \quad (51)$$

which is a different presentation of $Q \times \mathbb{Z}_2$: the group generated by P and T is the quaternion group (22) and the full group is two copies of the quaternion group, the \mathbb{Z}_2 being generated by C .

Although $P^\dagger \neq -P$ with $e^{i\phi_P} = \pm i$, $P^\dagger \cong -P$ is numerically anti-Hermitian. Indeed

$$P^\dagger \cong -P, \quad T^\dagger \cong -T, \quad I^\dagger \cong -I, \quad C^\dagger \cong -C.$$

Despite the fact that $CI = -IC$ (and $C^\dagger \cong -C$) C is actually unitary with β_- . To see this observe that

$$C^\dagger \beta_\pm = \mp \beta_\pm C,$$

leading to

$$\widehat{C}(\overline{\Psi}\Psi') := \overline{(C\Psi)}C\Psi' = \overline{\Psi}\Psi'.$$

We also have

$$\widehat{C}(\overline{\Psi}\Gamma_5\Psi') := \overline{(C\Psi)}C\Gamma_5\Psi' = -\overline{\Psi}\Gamma_5\Psi', \quad (52)$$

$$\widehat{C}(\overline{\Psi}\Gamma^\mu\Psi') := \overline{(C\Psi)}\Gamma^\mu(C\Psi') = -\overline{\Psi}\Gamma^\mu\Psi', \quad (53)$$

and, with β_+ ,

$$\widehat{C}(\widetilde{\Psi}\Psi') := \widetilde{(C\Psi)}(C\Psi') = -\widetilde{\Psi}\Psi' \quad (54)$$

$$\widehat{C}(\widetilde{\Psi}\Gamma_5\Psi') := \widetilde{(C\Psi)}\Gamma_5(C\Psi') = \widetilde{\Psi}\Gamma_5\Psi' \quad (55)$$

$$\widehat{C}(\widetilde{\Psi}\Gamma^\mu\Psi') := \widetilde{(C\Psi)}\Gamma^\mu(C\Psi') = \widetilde{\Psi}\Gamma^\mu\Psi'. \quad (56)$$

The algebra of P, T and I acting on Weyl fermions is easily deduced in this formalism from the observation that T commutes with Γ_5 while P, I and C anti-commute with it. Hence on Weyl fermions $\Psi_\pm = \frac{1}{2}(1 \pm \Gamma_5)\Psi$ we still have $P^2 = T^2 = I^2 = -1$ and $C^2 = 1$ and $ITP = -1$, although Weyl fermions are of course not eigenstates of C since $C\Psi_\pm = \Psi_\mp$.

4. The CPT theorem

The *CPT* theorem for Dirac fermions follows as usual. We have

$$\Theta = CPT = \begin{pmatrix} e^{i(\phi_T + \phi_C - \phi_P)\gamma_5} & 0 \\ 0 & e^{-i(\phi_T + \phi_C - \phi_P)(\gamma_5)^*} \end{pmatrix}.$$

With $\phi_T + \phi_C = \frac{2\pi}{p}$ and $p = 1, 2, \phi_P = \pm \frac{\pi}{2}$ for $\varepsilon = +1$ while $p = 4, \phi_P = \pi$ for $\varepsilon = -1$, this gives

$$\Theta = CPT = \pm i \Gamma_5.$$

and $\Theta^2 = -1$, $\Theta^\dagger \cong -\Theta$. We also have

$$\Theta^\dagger \beta_\pm = \beta_\pm \Theta,$$

and the choice of ε does not influence these properties of Θ .

Θ changes the sign of both bi-linear forms,

$$\overline{(\Theta\Psi)} (\Theta\Psi') = -\overline{\Psi}\Psi', \quad \widehat{(\Theta\Psi)} (\Theta\Psi') = -\widetilde{\Psi}\Psi'. \quad (57)$$

However $\widehat{\Theta}$ interchanges the order of the fermions, because \widehat{T} does, and so changes the sign

$$\widehat{\Theta} (\overline{\Psi}\Psi') = \overline{\Psi'}\Psi, \quad \widehat{\Theta} (\widetilde{\Psi}\Psi') = \widetilde{\Psi'}\Psi. \quad (58)$$

We have the following bi-linear relations (omitting the argument x)

$$\widehat{\Theta} (\overline{\Psi}\Psi') = \overline{\Psi'}\Psi \quad (59)$$

$$\widehat{\Theta} (\overline{\Psi}\Gamma_5\Psi') = \overline{\Psi'}\Gamma_5\Psi \quad (60)$$

$$\widehat{\Theta} (\overline{\Psi}\Gamma^\mu\Psi') = -\overline{\Psi'}\Gamma^\mu\Psi \quad (61)$$

$$\widehat{\Theta} (\overline{\Psi}\Gamma_5\Gamma^\mu\Psi') = -\overline{\Psi'}\Gamma_5\Gamma^\mu\Psi. \quad (62)$$

Of course $\widehat{\Theta}$ also sends x to $-x$ in $\Psi(x)$ and $\partial_\mu \rightarrow -\partial_\mu$, so

$$\widehat{\Theta} (\overline{\Psi}\Gamma^\mu\partial_\mu\Psi) = \overline{\Psi}\Gamma^\mu\partial_\mu\Psi. \quad (63)$$

and the Dirac Lagrangian is invariant under $\widehat{\Theta}$.

There is another bi-linear invariant that can be constructed on \mathcal{S} , namely

$$\overline{\psi}\psi'_C = e^{i\varphi_C} \psi^T C^{-1} \psi'$$

where $\psi'_C = \widehat{C}\psi'$ is the spinor conjugate to ψ' . For a Majorana spinor $\psi'_C = \psi'$ and $\overline{\psi}\psi'_C = \overline{\psi}\psi'$. The corresponding bi-linear on \mathcal{S} is

$$\overline{\Psi}C\Psi' = e^{i\varphi_C} (\psi^\dagger(C^*)^{-1}\psi'^*) - e^{-i\varphi_C} (\psi^T C^{-1}\psi')$$

since, from (100), $\beta C\beta^* = (C^*)^{-1}$. For a Majorana spinor $\Psi = C\Psi'$ equation (32) shows that

$$\overline{\psi}\psi' + \overline{\psi'}\psi = e^{i\varphi_C} (\psi^\dagger(C^*)^{-1}\psi'^*) - e^{-i\varphi_C} (\psi^T C^{-1}\psi')$$

and for both sides to be real we must have $e^{i\varphi_C} = \pm i$. Since

$$e^{i(\varphi_T + \varphi_C)} = \pm\sqrt{\varepsilon}$$

we conclude that $e^{i\varphi_T} = \pm\sqrt{-\varepsilon}$ also, so this argument determines all three phases factors in P , T and C , up to a sign, $e^{i\varphi_P} = \pm\sqrt{-\varepsilon}$, $e^{i\varphi_T} = \pm\sqrt{-\varepsilon}$ and $e^{i\varphi_C} = \pm i$ for Majorana spinors.

5. Neutral pseudo-scalar mesons

Experimentally the most fruitful place to study the properties of \widehat{C} , \widehat{P} and \widehat{T} is in the physics of pseudo-scalar mesons, particularly B -mesons for \widehat{T} . Pseudo-scalars are represented by operators of the form $\overline{\psi}\gamma_5\psi'$. For example if $\psi_b(x)$ creates a \bar{b} -quark and $\psi_d(x)$ creates a \bar{d} -quark⁹ then

$$B^0 = \overline{\psi}_d\gamma_5\psi_b|0\rangle$$

$$\overline{B}^0 = \overline{\psi}_b\gamma_5\psi_d|0\rangle = (\overline{\psi}_d\gamma_5\psi_b)^\dagger|0\rangle$$

are the neutral B -mesons which are $\widehat{P} = -1$ eigenstates. For a relativistic bound state, like the B^0 , ψ_b here need not be a single particle operator, the \bar{b} -quark will be accompanied by a sea of particles and antiparticles, but for the purposes of this section all that matters is that ψ_b has the same Lorentz transformation properties of a single particle (3), even if it is a composite operator.

Now consider $\overline{\Psi}_d\Gamma_5\Psi_b$ with β_- re-scaled by a factor of $\sqrt{2}$ for convenience,

$$\begin{aligned} \overline{\Psi}_d\Gamma_5\Psi_b &= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_d^\dagger & \psi_d^T \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & -\beta^* \end{pmatrix} \begin{pmatrix} \gamma^5 & 0 \\ 0 & \gamma_5^* \end{pmatrix} \begin{pmatrix} \psi_b \\ \psi_b^* \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \left(\psi_d^\dagger\beta\gamma_5\psi_b - \psi_d^T\beta^*\gamma_5^*\psi_b^* \right) = \frac{1}{\sqrt{2}} \left(\overline{\psi}_d\gamma_5\psi_b + \psi_b^\dagger\gamma_5^\dagger\beta^\dagger\psi_d \right) \\ &= \frac{1}{\sqrt{2}} \left(\overline{\psi}_d\gamma_5\psi_b - \overline{\psi}_b\gamma_5\psi_d \right), \end{aligned}$$

since fermions anti-commute and $\gamma_5^\dagger\beta^\dagger = -\beta\gamma_5$. Acting on the vacuum

$$\overline{\Psi}_d\Gamma_5\Psi_b|0\rangle = \frac{1}{\sqrt{2}}(B^0 - \overline{B}^0) = B_+$$

This is an eigenstate of both \widehat{P} , with $P = -1$ and \widehat{C} , with $C = -1$ so $CP = +1$.

The $CP = -1$ state is constructed using $\frac{1}{\sqrt{2}}\beta_+$,

$$\widetilde{\Psi}_d\Gamma_5\Psi_b = \frac{1}{\sqrt{2}} \left(\overline{\psi}_d\gamma_5\psi_b + \overline{\psi}_b\gamma_5\psi_d \right)$$

and

$$\widetilde{\Psi}_d\Gamma_5\Psi_b|0\rangle = \frac{1}{\sqrt{2}} \left(B^0 + \overline{B}^0 \right) = B_-$$

is an eigenstate of both \widehat{P} , with $P = -1$ and \widehat{C} , with $C = +1$, so this is the other $\widehat{C}\widehat{P}$ eigenstate with $CP = -1$.

In the same way eigenstates of \widehat{T} can be constructed¹⁰ and, while P and T do not commute when acting on fermions, \widehat{T} and \widehat{P} commute when acting on fermion bi-linears so neutral B -mesons can be simultaneous eigenstates of \widehat{C} , \widehat{P} and \widehat{T} . From (33) and (43)

$$\widehat{T} \left(\overline{\Psi}_d\Psi_b \right) = \overline{\Psi}_b\Psi_d$$

⁹ In the Fock space operator in equation (2), a_s annihilates a fermion and $a_{s'}^\dagger$ creates the anti-fermion.

¹⁰ For an eigenvectors of the matrix T the component in S^* is not the complex conjugate of the component in S .

$$\hat{T}(\bar{\Psi}_d \Gamma_5 \Psi_b) = \bar{\Psi}_b \Gamma_5 \Psi_d,$$

while β_+ gives the opposite signs

$$\hat{T}(\tilde{\Psi}_d \Psi_b) = -\tilde{\Psi}_b \Psi_d$$

$$\hat{T}(\tilde{\Psi}_d \Gamma_5 \Psi_b) = -\tilde{\Psi}_b \Gamma_5 \Psi_d.$$

Thus $\bar{\Psi}_d \Gamma_5 \Psi_b$ has $CP = 1, T = 1$ while $\tilde{\Psi}_d \Gamma_5 \Psi_b$ has $CP = -1, T = -1$.

In terms of $\Theta = CPT$

$$\hat{\Theta}(\bar{\Psi}_d \Gamma_5 \Psi_b) = \bar{\Psi}_b \Gamma_5 \Psi_d \tag{64}$$

$$\hat{\Theta}(\tilde{\Psi}_d \Gamma_5 \Psi_b) = \tilde{\Psi}_b \Gamma_5 \Psi_d \tag{65}$$

from which it is immediate that $CPT = 1$ for both B_+ and B_- , as expected from the CPT theorem.

T -violation has been observed in neutral B -meson systems by the BaBar collaboration [8]. It would be interesting if the T -parity could be measured experimentally and compared to the CPT prediction.

6. Discussion

It has been shown that, assuming $P^{2n} = 1$ when acting on Dirac fermions for some integral n , P and T generate one of an infinite series of possible groups, the generalised quaternion groups. Including charge conjugation and assuming that C commutes with P and T singles out the unique option

$$I^2 = P^2 = T^2 = IPT = -1, \quad [C, P] = [C, T] = 0. \tag{66}$$

Although the discussion in sections 2 and 3 involved only single particle states in \mathcal{S} it is easily extended to multiparticle states involving products of \mathcal{S} in the usual way.

If P and T preserve the Majorana property of a Majorana spinor then the inclusion of C eliminates all but one of these possibilities leaving two copies of the quaternion algebra as the only option. While this is perhaps a mathematically pleasing observation it is difficult to think of any experimental consequences.

Before 1956 it was generally believed that P was a symmetry of the fundamental laws of nature, and indeed it is for electromagnetism and the strong nuclear force (the rôle of P in any putative quantum theory of gravity is still open to debate). When Lee and Yang realised in 1956 that P invariance had not been tested in weak interactions [9] they discussed possible experiments to check its status and P -violation was observed in β -decay of Co^{60} very soon afterwards [10], leading to the important conclusion that the Hamiltonian giving rise to β decay does not commute with the parity operator. It was almost immediately suggested by Landau that, if P was replaced by CP , then CP still seemed to be a good symmetry. If so this would imply that C is also violated (but in a manner that exactly cancels P -violation). Then CP violation was discovered in the decay of neutral Kaons to pions in 1964 [12],

$$K_L \rightarrow 2\pi$$

where K_L is CP odd and the 2π state is CP even (the latter is s -wave due to angular momentum conservation, since Kaons and pions are both spin-0). So C -violation does not exactly

cancel P -violation. Famously CP violation is essential to explain the fact that there is more matter than anti-matter in the Universe [11], though the standard model alone does not have large enough CP violation to explain the observed matter/anti-matter asymmetry. Nevertheless the CPT theorem, discovered a few years prior to these events [13–16], suggests that the combination of all three operations C , P and T should be a good symmetry of nature.

The CPT theorem, together with CP violation, implies that T should also be violated, but this is a theoretical prediction based on a particular model of fundamental interactions (assuming a local, Lorentz invariant Lagrangian) and should be tested against observations. Experimental evidence for T -violation in neutral Kaon systems was claimed in [17] and later in [18]. It was confirmed more definitively in a beautiful experiment by the BaBar collaboration [8], following a proposal in [19]. In the chain of $\Upsilon(4S)$ -decays

$$\begin{aligned} \text{I: } \Upsilon &\rightarrow \begin{cases} B^0 & \rightarrow l^+ + X \\ \bar{B}^0 & \rightsquigarrow \begin{cases} B_- & \rightarrow J/\Psi + K_S \\ B_+ & \rightarrow J/\Psi + K_L \end{cases} \end{cases} \\ \text{II: } \Upsilon &\rightarrow \begin{cases} B^0 & \rightsquigarrow \begin{cases} B_- & \rightarrow J/\Psi + K_S \\ B_+ & \rightarrow J/\Psi + K_L \end{cases} \\ \bar{B}^0 & \rightarrow l^- + \bar{X}, \end{cases} \end{aligned}$$

the wavy arrows represent mixing rather than decays. B^0 and \bar{B}^0 are flavour eigenstates while B_{\pm} are CP eigenstates and l^{\pm} are charged leptonic states¹¹. There are 8 possible decay sequences falling into 4 pairs of time reversed oscillations, such as

$$\begin{aligned} &l^+ X \text{ followed by } J/\Psi K_L \\ &J/\Psi K_S \text{ followed by } l^- \bar{X} \end{aligned}$$

for example: in this sequence the semi-leptonic decay $l^+ X$ can only come from a B^0 so the subsequent CP even state, $J/\Psi K_L$, must be due to a B_+ decaying, having oscillated from a \bar{B}^0 , ($\bar{B}^0 \rightsquigarrow B_+$). The $l^- \bar{X}$ can only come from a \bar{B}^0 which has oscillated from a B_+ ($B_+ \rightsquigarrow \bar{B}^0$) because the earlier CP odd $J/\Psi K_S$ state must have come from a B_- . Experimentally the time of the two oscillations is different and, since the decays involved are CP conserving, this asymmetry can only be due to T -violation. Up to date information on the CPT theorem in B -meson systems can be found in [20, 21] and for K -mesons in [22].

CPT invariance, which has a strong theoretical foundation, replaces the now discredited ideas of P invariance and CP invariance. It predicts that it is four-dimensional inversion $x \rightarrow -x$, combined with the interchange of particles with anti-particles, that is a symmetry of nature. This is very natural: the definitions of P and T require choosing a reference frame and, in a relativistic setting, they should be combined into $I = TP$. A relativistic statement of the CPT theorem would be the CI theorem: that C -violation is exactly cancelled by I violation. Landau would have got it right if only he had thought relativistically, as Bell did in [16] where I was emphasised over P and T separately.

¹¹ It is essential for the experiment that, in decay sequence I, B^0 and \bar{B}^0 form an entangled state: \bar{B}^0 is inferred from $B^0 \rightarrow l^+ + X$ and it is not observed directly. If it were observed it could not oscillate to the CP eigenstate B_- . Similarly, in sequence II, B_+ and B_- are an entangled state and B_- is not observed directly.

Assuming that \mathbf{P} and \mathbf{T} commute with \mathbf{C} leads to the conclusion that that a fermionic eigenstate of parity must have $\mathbf{P}^2 = -1$, so its parity is $\pm i$. Particle data tables however give the parity of protons and neutrons as $+1$. This is because when there is a global $U(1)$ symmetry, such as baryon number \mathbf{B} , a new parity operator

$$\mathbf{P}' = e^{i\vartheta_B \mathbf{B}} \mathbf{P}$$

can be defined and ϑ_B chosen to ensure that $\mathbf{P}'^2 = 1$, [6]. If \mathbf{P} is conserved so is \mathbf{P}' , since the $U(1)$ is a symmetry, and by convention $\mathbf{P}' = +1$ is chosen for baryons. If \mathbf{P} is replaced with \mathbf{P}' the quaternion group is replaced by the dihedral group D_8 , realising the second group identified in [5].

Some insight into the possible phase of \mathbf{P} (rather than \mathbf{P}') can be obtained by a short calculation along the lines of [23], page 41, which shows that reflecting a plane-wave solution of the Dirac equation $\psi_{\text{inc}}(t, z)$, with wavevector $\mathbf{k} = k\hat{z}$, off an infinitely high potential barrier ($V_0 \rightarrow \infty$ in [23]), followed by a rotation of π about the z -axis, produces a reflected wave $\psi_{\text{ref}} = i\gamma_{\text{Dirac}}^0 \psi_{\text{inc}}(t, -z)$, suggesting that indeed $e^{i\phi_P} = i$ and $n = 2$.

One might try to measure ϕ_P in a two slit experiment in which one path is subjected to reflection in a mirror followed by a rotation of π about an axis perpendicular to the mirror while the other path is left alone (the rotation could be induced by applying an appropriate magnetic field). This is equivalent to \mathbf{P} acting on one path but not the other and a non-zero phase ϕ_P might then be expected to show up as a shift in the interference pattern. However this is probably not possible as reflecting the fermion off a mirror will inevitably impart a momentum to the mirror, no matter how small, which will make it possible, at least in principle, to determine which slit the fermion has gone through and presumably this will destroy the interference pattern, even if the recoil of the mirror is not measured. Certainly, if $\mathbf{P}^2 = -1$, then \mathbf{P} is not Hermitian¹² and it cannot be an observable. \mathbf{P} acting on fermions is not a physical observable, though acting on fermion bi-linears it is. Indeed the whole quaternion algebra (51), while mathematically precise and elegant, does not seem to be physically observable. When acting on fermion bi-linears it reduces to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

There are however physical consequences of $[\mathbf{C}, \mathbf{P}] = [\mathbf{C}, \mathbf{T}] = 0$. Weinberg [6] argues that $e^{i\phi_P} = \pm i$ for Majorana fermions. Starting from (2), with $a_s = a_{c,s}$, he concludes that $e^{-i\phi_P} = -e^{i\phi_{c,P}}$, where ϕ_P is the phase for ψ and $\phi_{c,P}$ is the phase for the conjugate spinor ψ_c . For a Majorana spinor $\phi_{c,P} = \phi_P$ so necessarily $\phi_P = \pm \frac{\pi}{2}$. But the explicit form (2) is derived from the assumption that the two-point correlator $\Delta(x - y) = \{\psi(x), \bar{\psi}(y)\}$ vanishes for space-like separations—this assumption fixes the relative phases of the positive and negative energy terms in (2), or equivalently, the relative phases of a_s and $a_{c,s}$. We can turn this argument around and say that assuming \mathbf{C} commutes with \mathbf{P} and \mathbf{T} implies that the two-point correlator of anti-commuting fields vanishes for space-like separations, implying an intimate correlation between complex conjugation of wave-functions and causal structure. This is perhaps not too surprising when one considers that the chiral representations $D_{\pm}(\omega)$ of the Lorentz group acting on Dirac spinors are faithful representations of $Sl(2, \mathbb{C})$ with a complex parameter $\theta + i\alpha$ (see (71)), clearly the appearance of i here is related to the causal structure. If \mathbf{C} did not commute with the improper Lorentz group the causal structure could be violated.

One final remark on time-reversal. In the Υ decays described above \mathbf{T} -violation was deduced as a consequence of two processes, $\bar{B}^0 \rightsquigarrow B_+$ and $B_+ \rightsquigarrow \bar{B}^0$, happening in opposite orders. For

¹² Explicitly

$$(\mathbf{P}\bar{\Psi}) \Psi = \bar{\Psi} \mathbf{P}^{-1} \Psi$$

and \mathbf{P} is only hermitian if $\mathbf{P}^{-1} = \mathbf{P}$. It is obviously anti-Hermitian if $\mathbf{P}^2 = -1$.

P and T the order also matters, not only do P and T not commute with the Hamiltonian, but in general they do not commute with each other. However while P and T do not commute when acting on fermions they do commute when acting on fermion bi-linears so it is probably not possible to see this non-commutativity experimentally. However we can hope to measure T on fermion bi-linears. Neutral pseudo-scalar mesons, such as Kaons and neutral B -mesons can be eigenstates of PC with $CP = \pm 1$. Since \hat{T} maps \mathcal{S} to \mathcal{S}^* it is not possible for a fermion in \mathcal{S} to be an eigenstate of \hat{T} , but a state in \mathcal{S} can be an eigenstate of T . Since all three of C , P and T are mutually commuting on fermion bi-linears, it should be possible to define the T parity of such states, or equivalently their Θ parity. From the CPT theorem their T -parity should be equal to their CP -parity.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Conventions

To be specific we give conventions with metric signature

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{67}$$

and

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{68}$$

($\mu, \nu = 0, 1, 2, 3$). The opposite signature can be accommodated by sending $\gamma^\mu \rightarrow i\gamma^\mu$, keeping γ_5 and β fixed.

In the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \bar{\sigma}^i & 0 \end{pmatrix} \tag{69}$$

with $i = 1, 2, 3$ and σ^i the Pauli matrices (numerically $\bar{\sigma}^i = -\sigma^i$). The chirality matrix is

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\bar{\mathbf{1}} \end{pmatrix}. \tag{70}$$

It is standard to write $\sigma^0 = \bar{\sigma}^0$ and $\bar{\sigma}^i = -\sigma^i$, but this can be confusing as they act on different spaces. When convenient we shall write

$$\sigma^0 \cong \bar{\sigma}^0, \quad \bar{\sigma}^i \cong -\sigma^i$$

meaning the matrices are numerically equal but have different spinor index structure. Thus the identity matrices acting on the different Weyl sectors, denoted $\mathbf{1}$ and $\bar{\mathbf{1}}$, satisfy $\mathbf{1} \cong \bar{\mathbf{1}}$ but $\mathbf{1} \neq \bar{\mathbf{1}}$. In the text, when it is deemed necessary for clarity, ‘=’ will be referred to as spinorial

equality and ‘ \cong ’ as numerical equality (for definiteness numerical equalities are given for signature (67)). In particular

$$(\sigma^\mu)^\dagger = \sigma^\mu, \quad (\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu, \quad \bar{\mathbf{1}} = \mathbf{1}^\dagger.$$

This may seem like an overly pedantic distinction but it proves to be an extremely useful notation to keep track of the Lorentz transformation properties of spinors.

The generators of the Lorentz group in the spinor representation are

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

and a general Lorentz transformation on spinors is

$$D(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}},$$

where $\omega_{0i} = \alpha n_i$ is a boost in the \mathbf{n} direction with rapidity α and $\omega_{ij} = \epsilon_{ijk}\theta^k$ a rotation in the i - j plane. In the chiral basis $J^{\mu\nu}$ are block diagonal

$$J^{0i} = \frac{i}{2} \begin{pmatrix} -\tau^i & 0 \\ 0 & \bar{\tau}^i \end{pmatrix} := \begin{pmatrix} J_+^{0i} & 0 \\ 0 & J_-^{0i} \end{pmatrix}$$

$$J^{ij} = \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \tau^k & 0 \\ 0 & \bar{\tau}^k \end{pmatrix} := \begin{pmatrix} J_+^{ij} & 0 \\ 0 & J_-^{ij} \end{pmatrix}$$

with

$$\tau^i = -\frac{1}{2}(\sigma^0\bar{\sigma}^i - \sigma^i\bar{\sigma}^0) = \frac{i}{2}\epsilon^i{}_{jk}\sigma^j\bar{\sigma}^k$$

$$\bar{\tau}^i = \frac{1}{2}(\bar{\sigma}^0\sigma^i - \bar{\sigma}^i\sigma^0) = \frac{i}{2}\epsilon^i{}_{jk}\bar{\sigma}^j\sigma^k,$$

and

$$D(\omega) = \begin{pmatrix} e^{-\frac{i}{2}(\theta-i\alpha).\tau} & 0 \\ 0 & e^{-\frac{i}{2}(\theta+i\alpha).\bar{\tau}} \end{pmatrix}.$$

As is well known the four-dimensional Dirac representation $J^{\mu\nu}$ is a reducible representation of the Lorentz group and decomposes into two inequivalent Weyl representations $J_+^{\mu\nu}$ and $J_-^{\mu\nu}$, with $J_\pm^i = \frac{1}{2}(\epsilon^i{}_{jk}J^{jk} \pm iJ^{0i})$ corresponding to τ^i and $\bar{\tau}^i$ respectively, and

$$D_+(\omega) = e^{-\frac{i}{2}(\theta-i\alpha).\tau} \quad \text{and} \quad D_-(\omega) = e^{-\frac{i}{2}(\theta+i\alpha).\bar{\tau}}. \quad (71)$$

Denoting the space of Dirac spinors by \mathcal{S} then $\gamma^\mu : \mathcal{S} \rightarrow \mathcal{S}$, $D(\omega) : \mathcal{S} \rightarrow \mathcal{S}$ and \mathcal{S} decomposes into the two eigenspaces of γ_5 , \mathcal{S}_\pm , with

$$\gamma_5 : \mathcal{S}_\pm = \pm\mathcal{S}_\pm, \quad D_+(\omega) : \mathcal{S}_+ \rightarrow \mathcal{S}_+ \quad \text{and} \quad D_-(\omega) : \mathcal{S}_- \rightarrow \mathcal{S}_-.$$

Complex conjugation interchanges these two inequivalent representations, up to an equivalence $D_\pm^*(\omega) \sim D_\mp(\omega)$,

$$D_+^*(\omega) = e^{\frac{i}{2}(\theta+i\alpha).\tau^*} = e^{-\frac{i}{2}(\theta+i\alpha).(-\tau^*)} = e^{-\frac{i}{2}(\theta+i\alpha).(\bar{\tau}\bar{\tau}^{-1})} = \bar{\tau}D_-(\omega)\bar{\tau}^{-1}$$

$$D_-^*(\omega) = e^{\frac{i}{2}(\theta-i\alpha).\bar{\tau}^*} = e^{-\frac{i}{2}(\theta-i\alpha).(-\bar{\tau}^*)} = e^{-\frac{i}{2}(\theta-i\alpha).(\tau\tau^{-1})} = \tau D_+(\omega)\tau^{-1},$$

with $\epsilon = i\tau^2$ and $\bar{\epsilon} = -i\bar{\tau}^2$. Thus the reducible representations $D(\omega)$ and $\{D(\omega)\}^*$ are also equivalent,

$$\{D(\omega)\}^* = \mathcal{C}^{-1}D(\omega)\mathcal{C} \quad \text{with } \mathcal{C} = \begin{pmatrix} 0 & \epsilon^{-1} \\ \bar{\epsilon}^{-1} & 0 \end{pmatrix}. \quad (72)$$

If $\psi_+ \in \mathcal{S}_+$ and $\psi_- \in \mathcal{S}_-$, then a Dirac spinor ψ decomposes into two Weyl spinors as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (73)$$

In the dreaded dotted and undotted spinor notation, with spinor indices $a = 1, 2$ and $\dot{a} = 1, 2$, this is

$$\begin{pmatrix} (\psi_+)_a \\ (\psi_-)^{\dot{a}} \end{pmatrix}, \quad (74)$$

and the components of $\sigma^0, \bar{\sigma}^0, \sigma^i, \bar{\sigma}^i, \tau^i$ and $\bar{\tau}^i$ are

$$\delta_{ab}, \quad \delta^{\dot{a}\dot{b}}, \quad (\sigma^i)_{ab}, \quad -(\sigma^i)^{\dot{a}\dot{b}}, \quad (\tau^i)_a{}^{\dot{b}}, \quad \text{and} \quad (\bar{\tau}^i)^{\dot{a}}{}_b,$$

respectively. Numerically

$$\sigma^i \cong -\bar{\sigma}^i \cong \tau^i \cong \bar{\tau}^i$$

while $\bar{\tau}^i = (\tau^i)^\dagger$ is a spinorial equality. When multiplying matrices together in a manner that preserves Lorentz transformation properties a lower dotted index can only be contracted with an upper dotted index and a lower undotted index can only be contracted with an upper undotted index. Thus $\sigma^\mu \bar{\sigma}^\nu$ is allowed but $\sigma^\mu \sigma^\nu$ is not, though $\sigma^\mu \bar{\sigma}^\nu \cong \sigma^\mu \sigma^\nu$. The identity matrices $\mathbf{1}$ and $\bar{\mathbf{1}}$, acting on \mathcal{S}_+ and \mathcal{S}_- respectively, have components

$$\delta_a{}^b \quad \text{and} \quad \delta^{\dot{a}}{}_{\dot{b}}.$$

The product $\bar{\mathbf{1}}\mathbf{1}$ is not allowed and strictly speaking the transpose matrix $\mathbf{1}^T \neq \mathbf{1}$. Also $\sigma^0 \cong \bar{\sigma}^0 \cong \mathbf{1} \cong \bar{\mathbf{1}}$, but they should not be identified.

Denote the vector space dual to \mathcal{S} by \mathcal{S}^D , then an inner product on the space of Dirac spinors is defined between two spinors $\psi \in \mathcal{S}$ and $\psi' \in \mathcal{S}$ as

$$\bar{\psi}\psi' = \psi^\dagger \beta \psi',$$

where $\psi^\dagger \beta \in \mathcal{S}^D$. Then $\beta: \mathcal{S} \rightarrow (\mathcal{S}^D)^\dagger$ is an Hermitian Lorentz invariant metric, $\beta = \beta^\dagger$, satisfying

$$D^\dagger(\omega)\beta D(\omega) = \beta$$

which acts on γ -matrices as

$$\beta\gamma^\mu = (\gamma^\mu)^\dagger\beta. \quad (75)$$

Explicitly

$$\beta = \begin{pmatrix} 0 & \bar{\mathbf{1}} \\ \mathbf{1} & 0 \end{pmatrix} \quad \text{and} \quad \beta^{-1} = \begin{pmatrix} 0 & \mathbf{1} \\ \bar{\mathbf{1}} & 0 \end{pmatrix},$$

so

$$\bar{\psi}\psi' = \psi_+^\dagger\psi'_- + \psi_-^\dagger\psi'_+ = (\psi_+^*)_{\dot{a}}(\psi'_-)^{\dot{a}} + (\psi_-^*)^a(\psi'_+)_a. \tag{76}$$

and, when $\psi' = \psi$,

$$\bar{\psi}\psi = \psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+. \tag{77}$$

The matrices β , β^{-1} , and β^* are not spinorially equal, though numerically

$$\beta \cong \beta^{-1} \cong \beta^*, \quad \beta^2 \cong 1 \quad \text{and} \quad \beta \cong \gamma^0 \cong (\gamma^0)^\dagger.$$

However

$$\beta = \beta^\dagger \tag{78}$$

is correct.

There are also spaces \mathcal{S}_+^D dual to \mathcal{S}_+ and \mathcal{S}_-^D dual to \mathcal{S}_- with $(\psi_-^{\dot{a}})^* = (\psi_-^*)^a$, $(\psi_+^a)^* = (\psi_+^*)_{\dot{a}}$ and $\psi_-^* \in \mathcal{S}_+^D$ and $\psi_+^* \in \mathcal{S}_-^D$. So

$$* : \mathcal{S}_- \rightarrow \mathcal{S}_+^D, \quad * : \mathcal{S}_+ \rightarrow \mathcal{S}_-^D.$$

Lorentz invariant metrics can be defined on \mathcal{S}_+ and \mathcal{S}_- separately. Let

$$(\psi_+^D)^a = \epsilon^{ab}(\psi_+)_b \quad \text{and} \quad (\psi_-^D)_{\dot{a}} = \bar{\epsilon}_{\dot{a}b}\psi_-^{\dot{b}}$$

with

$$\epsilon \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{components } \epsilon^{ab})$$

and

$$\bar{\epsilon} \cong \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{components } \bar{\epsilon}_{\dot{a}b}).$$

Then ϵ is a metric on \mathcal{S}_+ and $\bar{\epsilon}$ a metric on \mathcal{S}_- . These are Lorentz invariant metrics in that

$$D_+^T(\omega) \epsilon D_+(\omega) = \epsilon, \quad D_-^T(\omega) \bar{\epsilon} D_-(\omega) = \bar{\epsilon}.$$

The inner product of ψ_+ with itself is

$$(\psi_+)_a \epsilon^{ab} (\psi_+)_b = \psi_+ \psi_+^D = \psi_+^D \psi_+,$$

which would vanish if $(\psi_+)_a$ were commuting objects, but if they anti-commute (either as second quantised single particle fields or as Grassmann variables in first quantisation) then the above definitions are consistent.

The inverse metrics are

$$\epsilon^{-1} \cong \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{components } \epsilon_{ab})$$

and

$$\bar{\epsilon}^{-1} \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{components } \bar{\epsilon}^{\dot{a}b}).$$

This follows general relativistic conventions where it is standard to write the components of the inverse of the metric $\bar{\epsilon}_{\dot{a}\dot{b}}$ as $\bar{\epsilon}^{\dot{a}\dot{b}}$, so $\bar{\epsilon}_{\dot{a}\dot{c}}\bar{\epsilon}^{\dot{c}\dot{b}} = \delta_{\dot{a}}^{\dot{b}}$, and $\epsilon^{ac}\epsilon_{cb} = \delta^a_b$. Just as in general relativity, spinor indices are raised and lowered by using these metrics (multiplying from the left, if multiplying from the right their transposes must be used). In index free notation

$$\epsilon = -\epsilon^T \tag{79}$$

$$\bar{\epsilon} = -\bar{\epsilon}^T \tag{80}$$

$$\epsilon^{-1}\epsilon = \mathbf{1} \tag{81}$$

$$\epsilon\epsilon^{-1} = \mathbf{1}^T \tag{82}$$

$$\bar{\epsilon}\bar{\epsilon}^{-1} = \bar{\mathbf{1}} \tag{83}$$

$$\bar{\epsilon}^{-1}\bar{\epsilon} = \bar{\mathbf{1}}^T \tag{84}$$

$$\epsilon^* = \bar{\epsilon}^{-1} \tag{85}$$

$$\bar{\epsilon}^* = \epsilon^{-1}. \tag{86}$$

While numerically

$$\begin{aligned} \epsilon &\cong \epsilon^* \cong -\epsilon^{-1} \cong -\bar{\epsilon} \cong -\bar{\epsilon}^* \cong \bar{\epsilon}^{-1} \\ \epsilon^2 &\cong \bar{\epsilon}^2 \cong -\mathbf{1}. \end{aligned}$$

The parity, time reversal and charge conjugation operators, \hat{P} , \hat{T} and \hat{C} respectively, are defined in the usual way in second quantisation (see e.g. [6] with some minor changes due to different conventions)

$$\hat{P}\psi(x) = e^{i\phi_P}\gamma^0\psi(x_P) \cong e^{i\phi_P}\beta\psi(x_P) \tag{87}$$

$$\hat{T}\psi(x) = e^{i\phi_T}C^*\beta\gamma_5\gamma^0\psi(x_T) \cong -e^{i\phi_T}C\gamma_5\psi(x_T) \tag{88}$$

$$\hat{C}\psi(x) = e^{i\phi_C}C\beta^T\psi^*(x) \cong e^{i\phi_C}C\beta\psi^*(x) \tag{89}$$

where the charge conjugation matrix transposes the γ -matrices,

$$\gamma^\mu C = -C(\gamma^\mu)^T, \tag{90}$$

$e^{i\phi_P}$, $e^{i\phi_T}$, $e^{i\phi_C}$ are arbitrary complex phases, $x_P = (t, -\mathbf{x})$ and $x_T = (-t, \mathbf{x})$. Explicitly

$$C = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \bar{\epsilon}^{-1} \end{pmatrix} \cong \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \tag{91}$$

and

$$\mathcal{C} = C\beta^T$$

in (72), so

$$\hat{C}\{D(\omega)\psi\} = e^{i\phi_C}\mathcal{C}\{D(\omega)\}^*\psi^* = e^{i\phi_C}D(\omega)\mathcal{C}\psi^*.$$

From (85) and (86),

$$C^* = \begin{pmatrix} (\epsilon^{-1})^* & 0 \\ 0 & (\bar{\epsilon}^{-1})^* \end{pmatrix} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \epsilon \end{pmatrix},$$

with

$$C^T = -C, \quad C^\dagger = -C^*, \quad C^* \cong C \cong -C^{-1}, \quad C^2 \cong -1 \quad \text{and} \quad C^\dagger C \cong \mathbf{1}.$$

It is natural that γ^0 should appear in the definitions of \widehat{P} and \widehat{T} as these require choosing a time-like direction and planar space-like foliations perpendicular to the chosen time direction: there is no such information in β , which is Lorentz invariant.

Clearly $\widehat{P} : \mathcal{S} \rightarrow \mathcal{S}$ and $\widehat{C} : \mathcal{S} \rightarrow \mathcal{S}$ interchange \mathcal{S}_+ and \mathcal{S}_- , while the charge conjugation matrix $C : (\mathcal{S}^D)^T \rightarrow \mathcal{S}$. Also $\widehat{T} : \mathcal{S}_\pm \rightarrow \mathcal{S}_\pm^*$.

In practice \widehat{C} , \widehat{P} and $\widehat{C}\widehat{T}$ are easily implemented using the numerical equalities in (87)–(89),

$$\widehat{P}\psi(x) \cong e^{i\phi_P} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(\mathcal{P}x) \tag{92}$$

$$\widehat{T}\psi(x) \cong e^{i\phi_T} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \psi(\mathcal{T}x) \tag{93}$$

$$\widehat{C}\psi(x) \cong e^{i\phi_C} \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \psi^*(x). \tag{94}$$

Since $\widehat{P} : \mathcal{S}_\pm \rightarrow \mathcal{S}_\mp$, $\widehat{T} : \mathcal{S}_\pm \rightarrow \mathcal{S}_\pm^* \sim \mathcal{S}_\mp$ and $\widehat{C} : \mathcal{S}_\pm \rightarrow \mathcal{S}_\mp$, none of these discrete symmetry operators can be represented on an irreducible representation of the proper Lorentz group.

The anti-particle of an anti-particle is the particle,

$$\widehat{C}(\widehat{C}\psi) = \psi,$$

so $\widehat{C}^2 = 1$.

If ψ and ψ' are two fermions in \mathcal{S} then

$$\overline{(\widehat{C}\psi)}(\widehat{C}\psi') = \psi^T \beta^* C^\dagger \beta C \beta^* \psi'^* = -\psi^T \beta^* \psi'^* = \overline{\psi'}\psi,$$

since spinors anti-commute and $\beta^\dagger = \beta$ so $\beta^* = \beta^T$. This uses the important identity

$$C\beta^* C^* \beta = 1 \tag{95}$$

which is easily proven by using (76) and (91) to show that

$$C\beta^* C^* \beta \gamma^\mu = \gamma^\mu C\beta^* C^* \beta$$

and the only matrix that commutes with all four γ^μ is a multiple of the identity matrix, so

$$(C\beta^*)(C^*\beta) = \lambda \mathbf{1}.$$

Sine the eigenvalues of $C\beta^*$ and $C^*\beta$ have complex conjugate eigenvalues λ must be a positive real number and, since from (76) and (91) the eigenvalues of β and C have unit modulus, $\lambda = 1$. In particular

$$\overline{(\widehat{C}\psi)}(\widehat{C}\psi) = \overline{\psi}\psi,$$

with the physical interpretation that anti-particles have the same mass as particles.

For \widehat{T} there is no explicit complex conjugation in (88) yet $\widehat{T}\psi(x) \in \mathcal{S}^*$, it is the matrix $C^*\beta$ that maps \mathcal{S} to \mathcal{S}^* . The complex conjugate of (88) is

$$\left\{ \widehat{T}\psi(x) \right\}^* = e^{-i\phi_T} C\beta^* \gamma_5^* (\gamma^0)^* \psi^*(x_T) \in \mathcal{S} \tag{96}$$

$$\Rightarrow \widehat{T} \left\{ \widehat{T}\psi(x) \right\}^* = C^* \beta \gamma_5 \gamma_0 C \beta^* \gamma_5^* (\gamma^0)^* \psi^*(x) = -\psi^*(x). \tag{97}$$

using (95) and (90). The metric on \mathcal{S}^* is β^* so we have the inner product

$$\begin{aligned} (\widehat{T}\psi)^\dagger \beta^* (\widehat{T}\psi') &= \psi^\dagger (\gamma^0)^\dagger \gamma_5^\dagger \beta^\dagger C^T \beta^* C^* \beta \gamma_5 \gamma^0 \psi' \\ &= -\psi^\dagger (\gamma^0)^\dagger \gamma_5^\dagger \beta \gamma_5 \gamma^0 \psi' \\ &= \overline{\psi} \psi', \end{aligned} \tag{98}$$

because $\gamma_5^\dagger \beta = -\beta \gamma_5$.

Since \widehat{T} maps $\mathcal{S} \rightarrow \mathcal{S}^*$ we cannot apply \widehat{C} in (89) to $\widehat{T}\psi$ directly, but it can be applied to $\left\{ \widehat{T}\psi \right\}^*$ in (96),

$$\begin{aligned} \widehat{C} \left\{ \widehat{T}\psi \right\}^* &= e^{i\phi_C} C\beta^T \left\{ e^{-i\phi_T} C\beta^* \gamma_5^* (\gamma^0)^* \psi^* \right\}^* \\ &= e^{i(\phi_C + \phi_T)} C\beta^T C^* \beta \gamma_5 \gamma^0 \psi \\ &= e^{i(\phi_C + \phi_T)} \gamma_5 \gamma^0 \psi. \end{aligned}$$

This is in \mathcal{S} and so we can apply \widehat{P} directly to obtain

$$\widehat{P}\widehat{C} \left\{ \widehat{T}\psi \right\}^* = e^{i(\phi_P + \phi_C + \phi_T)} \gamma^0 \gamma_5 \gamma^0 \psi = -e^{i(\phi_P + \phi_C + \phi_T)} \gamma_5 \psi.$$

In the text any specific choice of γ -matrix representation is avoided as much as possible—no physical result can depend on the choice of representation. The definitions

$$\beta \gamma^\mu = \varepsilon (\gamma^\mu)^\dagger \beta, \quad \gamma^\mu C = -C (\gamma^\mu)^T, \tag{99}$$

implying

$$C^* \beta \gamma^\mu = -\varepsilon (\gamma^\mu)^* C^* \beta, \quad C\beta^* C^* \beta = 1, \tag{100}$$

are sufficient to derive all the formulae in the text without choosing a representation for the γ -matrices.

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