Non-locality of non-Abelian anyons

2009 New J. Phys. 11 103023

(http://iopscience.iop.org/1367-2630/11/10/103023)

New Journal of Physics
The open-access journal for physics

Non-locality of non-Abelian anyons

G K Brennen[1], **S Iblisdir**[2], **J K Pachos**[3] **and J K Slingerland**[4,5]
[1] Centre for Quantum Information Science and Security, Macquarie University, 2109 NSW, Australia
[2] Departament Estructura i Constituents de la Matèria, Universitat de Barcelona, 08028 Barcelona, Spain
[3] School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, UK
[4] Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Road, Dublin, Ireland
[5] Department of Mathematical Physics, National University of Ireland, Maynooth, Ireland
E-mail: gbrennen@ics.mq.edu.au

New Journal of Physics **11** (2009) 103023 (18pp)
Received 22 January 2009
Published 12 October 2009
Online at http://www.njp.org/
doi:10.1088/1367-2630/11/10/103023

Abstract. Entangled states of quantum systems can give rise to measurement correlations of separated observers that cannot be described by local hidden variable theories. Usually, it is assumed that entanglement between particles is generated due to some distance-dependent interaction. Yet anyonic particles in two dimensions have a nontrivial interaction that is purely topological in nature. In other words, it does not depend on the distance between two particles, but rather on their exchange history. The information encoded in anyons is inherently non-local even in the single subsystem level making the treatment of anyons non-conventional. We describe a protocol to reveal the non-locality of anyons in terms of correlations in the outcomes of measurements in two separated regions. This gives a clear operational measure of non-locality for anyonic states and it opens up the possibility to test Bell inequalities in quantum Hall liquids or spin lattices.

Contents

1. Introduction 2
2. Anyons and their Hilbert space 3
3. Bell inequalities and spin-$\frac{1}{2}$ particles 4
4. Bell inequalities and $SU(2)_k$ anyons 7
5. Explicit construction of Bell violating states for $k = 2$ and 3 9
5.1. The $SU(2)_2$ case . . . 9
5.2. The Fibonacci case . . . 10
6. Bell inequalities and quantum doubles 11
7. Conclusions 14
Acknowledgments 14
Appendix A. The Clauser–Horne–Shimony–Holt (CHSH) inequality and Ising anyons 14
Appendix B. The quantum double model 16
References 17

1. Introduction

Quantum mechanics is a non-local theory: it allows for correlations between distant systems that cannot be explained in terms of a local preparation. Early in the history of quantum mechanics many believed that non-locality was due to incompleteness of quantum theory. Einstein, Podolsky and Rosen (EPR) in their seminal work [1] made this debate explicit by introducing a model with *local hidden* variables (LHVs). Their values would complement the information supplied by quantum mechanics, thus restoring locality. Bell inequalities aim at validating or rejecting this view from experimental data [2]. To date, unlike LHV theories, the predictions of quantum mechanics have been consistent with all Bell tests.

Particle exchange gives a striking example of non-locality in quantum mechanics. Bosons and fermions are governed by the trivial and signed one-dimensional representations of the symmetric group, respectively, and this introduces a non-local constraint on the form of the wave function for those particles. More generally, the many-particle wave function can transform in a nontrivial representation of the fundamental group of configuration space when particles are adiabatically exchanged [3]–[5]. For planar systems, this group is the braid group and particles transforming in nontrivial braid group representations have been dubbed *anyons*. Anyonic exchange interactions are topological in nature and do not change on variation of the distance between the particles, or the metric of the spacetime manifold. A well-known example where such interactions appear is the Aharonov–Bohm effect [6].

One may ask if the non-locality, which results from the effect of particle exchange is measurable as a violation of Bell inequalities. The issue is subtle for bosons and fermions since states that appear entangled in a first quantized picture may no longer appear so in second quantization. Under the generalized entanglement approach of [7] one quantifies entanglement of a state with respect to the purity of a subalgebra. If one imposes a superselection rule on particle number, then states that appear entangled with respect to a mode subalgebra may no longer be entangled with respect to a Lie subalgebra of bosonic or fermionic operators.

This becomes evident in a Bell measurement because the local parties can presumably do not measure along directions which mix particle number superselection sectors (see also [8, 9] for a discussion of these issues). A study of non-locality of anyons actually avoids most of these subtleties because there is a strong restriction on the set of observables in the physical theory. The only relevant observables are topological charges so any test of non-locality would have only to focus on correlations of charge measurements obtained by separated observers. In this paper we provide a Bell-like test which delineates when such correlations cannot be reproduced by any LHV model. Ultimately, for our test the existence of states with non-local correlations depends on the underlying physical theory which gives rise to the anyonic particle excitations. However, the results in this paper are independent of the detailed microscopic physics in those theories.

To date, most Bell tests have been performed on entangled light beams, but there is certainly an interest in showing that material media can be used to demonstrate non-locality. Experiments involving a photon and an atom, two atoms or even kaons have been proposed or even carried out [10]. Some of these schemes could in principle be implemented in a fractional quantum Hall liquid [11], whereas others can be associated with arrays of Josephson junctions [12] or atoms in optical lattices [13].

2. Anyons and their Hilbert space

Anyons can be split into two main categories: they can be Abelian or non-Abelian. Given labels $\{a_j\}$ for the different types of anyons we assign fusion rules that determine the outcome of bringing two anyons together,

$$a_i \times a_j = \sum_k N_{a_i a_j}^{a_k} a_k. \tag{1}$$

Here $N_{ab}^c \in \mathbb{N}$ counts the number of ways of combining a and b to obtain c. Non-Abelian anyons have $\sum_c N_{ab}^c \geqslant 2$ for some pair a, b, whereas in the Abelian case, the labels of the fused anyons determine a unique outcome.

In a physical system with anyons, the low-energy part of the Hilbert space can be thought of as a tensor product $\mathcal{H} = \mathcal{H}_{\text{local}} \otimes \mathcal{H}_{\text{non-local}}$, where the first factor describes local degrees of freedom, which we will ignore, and the second describes topological degrees of freedom associated with the anyons. These topological degrees of freedom may arise as a result of nontrivial topology of the space supporting the anyons. For Abelian anyons, this is in fact the only possibility; in the Abelian toric code models [14] for instance, the non-local degrees of freedom are described by elements of the first homology groups of the surface with finite group coefficients. In principle, one can probe non-local correlations in these topological degrees of freedom, but the observables involved would need to be non-local themselves[6].

For non-Abelian anyons, even on a contractible surface, there are non-local degrees of freedom associated with the different fusion outcomes. A number of proposals have been made on how the associated quantum numbers, or topological charges, might be measured by interferometry (see [11, 16] for further references and [17] for an overview of the measurement theory). We will not go into the details of interferometric measurements here, but rather just assume that we can do projective measurements onto the various fusion channels. The

[6] For example, in the toric code, $\mathcal{H}_{\text{non-local}}$ is isomorphic to the space of two qubits and single qubit observables correspond to non-contractible string operators, see e.g. [15].

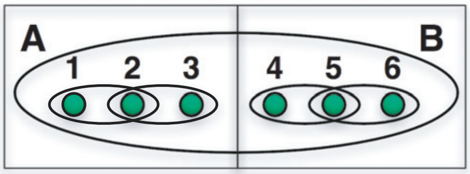

Figure 1. A Bell-type measurement on six particles. First a joint measurement (large oval) of total topological charge (or spin) on a pure state of all six particles is made and the result kept if the result is zero. Particles 1–3 are then sent to Alice, whereas particles 4–6 are sent to Bob. Alice performs measurements of total charge on pairs 1, 2 and 2, 3 and Bob performs measurements on pairs 4, 5 and 5, 6. For some quantum states the correlator $\langle W \rangle$ exceeds the bound set by LHV theories.

non-local Hilbert state space of n anyons $(a_1, a_2, \dots, a_n)$ with total charge c has $\dim(\mathcal{H}_{\text{non-local}}) = \sum_{b_1, b_2, \dots, b_{n-2}} N^{b_1}_{a_1 a_2} N^{b_2}_{b_1 a_3} N^{b_3}_{b_2 a_4} \dots N^{c}_{b_{n-2} a_n}$. This Hilbert space usually does not admit a tensor product structure, e.g. the dimension could be prime, and thus does not obviously fit the usual paradigm for tests of non-locality. Nevertheless, we show that topological interactions can indeed be used to demonstrate non-locality in the EPR sense. In order to do this, we consider two classes of anyonic theories: the $\mathrm{SU}(2)_k$ models, including the Fibonacci model [11] and a model based on discrete gauge theory [18, 19]. These models are important both for their potential to process quantum information fault tolerantly [14, 20] and for their viability for experimental realization [11]. For these cases the fusion spaces are at most one dimensional, i.e. $N^c_{ab} < 2$ for all (a, b, c). Non-commuting measurements project onto different ways of combining particles a, b, c to yield d. Measurement bases are labelled by the intermediate products x and x' obtained by fusing a, b, c. The unitary transformation that describes the change from one of these bases to the other is given by the so-called F matrices, and the recoupling formula is $|(ab)c \to d; x\rangle = \sum_{x'} (F^d_{abc})^{x'}_{x} |(a(bc) \to d; x'\rangle$.

3. Bell inequalities and spin-$\frac{1}{2}$ particles

In order to build intuition for the anyonic case we describe the general framework by employing distinguishable spin-$\frac{1}{2}$ particles with nontrivial fusion properties. The fusion rules are given by the angular momentum decomposition of tensor products of vector spaces. For example, the combination of two spin-$\frac{1}{2}$ particles gives $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ that closely resembles the anyonic fusion rules (1). We will show that it is possible to violate a Bell inequality by performing only *total spin* measurements, i.e. without measuring *projections* of the total spin along specified directions. Consider a system divided into two spatially non-overlapping subsystems A and B, conveniently labelled as Alice and Bob, each one possessing three spin-$\frac{1}{2}$ particles, as seen in figure 1. First, we perform a joint measurement on the total spin $\vec{S}_{\text{tot}} = \sum_{j=1}^{6} \vec{s}_j$ and post-select the $S_{\text{tot}} = 0$ outcome that has state space dimension five. Second, we define a set of measurement operators $\{\Upsilon^{\mathrm{A}}_{1,2}, \Upsilon^{\mathrm{A}}_{2,3}, \Upsilon^{\mathrm{B}}_{4,5}, \Upsilon^{\mathrm{B}}_{5,6}\}$, where $\Upsilon_{i,j} = (\vec{s}_i + \vec{s}_j)^2 - \mathbf{1}$. The eigenvalues of $\Upsilon_{i,j}$ are +1 in the triplet space and -1 for the singlet and the operators $\Upsilon^{\mathrm{A(B)}}$ act on the subsystems A(B). The operator pair within A or B is non-commuting but $[\Upsilon^{\mathrm{A}}_{i,j}, \Upsilon^{\mathrm{B}}_{k,l}] = 0$. Consider the expectation value of the operator

$$W = \Upsilon^{A}_{1,2}\Upsilon^{B}_{4,5} + \Upsilon^{A}_{1,2}\Upsilon^{B}_{5,6} - \Upsilon^{A}_{2,3}\Upsilon^{B}_{5,6} + \Upsilon^{A}_{2,3}\Upsilon^{B}_{4,5}. \tag{2}$$

For a classical theory, even in the presence of LHVs (see [21] and references therein), the Bell inequality for W is $|\langle W\rangle_{\mathrm{LHV}}| \leqslant 2$. This can be derived straightforwardly as follows [22]. Assume independence of the two subsystems (locality) so that the joint probabilities for pairs of outcomes is just the product of the individual probabilities which could depend on a hidden variable λ, drawn from a fixed distribution $p(\lambda)$. For the above quorum of observables with outcomes $\{m^{A}_{1,2}, m^{A}_{2,3}, m^{B}_{4,5}, m^{B}_{5,6}\} \in \pm 1$ we have

$$(m^{A}_{2,3} + m^{A}_{1,2})m^{B}_{4,5} - (m^{A}_{2,3} - m^{A}_{1,2})m^{B}_{5,6} = \pm 2.$$

Hence, in the LHV model, the outcomes must satisfy

$$\begin{aligned} |W_{\mathrm{LHV}}| &= |\int \mathrm{d}\lambda p(\lambda)\langle(\Upsilon^{A}_{1,2}(\lambda)\Upsilon^{B}_{4,5}(\lambda) + \Upsilon^{A}_{1,2}(\lambda)\Upsilon^{B}_{5,6}(\lambda) - \Upsilon^{A}_{2,3}(\lambda)\Upsilon^{B}_{5,6}(\lambda) + \Upsilon^{A}_{2,3}(\lambda)\Upsilon^{B}_{4,5}(\lambda))\rangle| \\ &= |\int \mathrm{d}\lambda p(\lambda)(m^{A}_{2,3}(\lambda) + m^{A}_{1,2}(\lambda))m^{B}_{4,5}(\lambda) - (m^{A}_{2,3}(\lambda) - m^{A}_{1,2}(\lambda))m^{B}_{5,6}(\lambda)| \\ &\leqslant 2. \end{aligned}$$

Quantum mechanically, the maximum value of $|\langle W\rangle|$ is obtained for eigenstates of W with maximum eigenvalue, i.e. $|\langle W\rangle| \leqslant \sqrt{7}$. Our aim is to find a violation of the classical upper bound in the subspace of states with $S_{\mathrm{tot}} = 0$. Note that our protocol allows for measuring correlations without the need of a shared reference frame between Alice and Bob [23] thus giving a simple and unambiguous test of Bell inequalities. In the anyonic case treated below the operators $\Upsilon_{i,j}$ also have eigenvalue -1 when the fusion outcome is the vacuum and +1 otherwise.

At this point we pause briefly to mention that while we will be operating on Hilbert spaces that do not admit a tensor product structure, Tsirelson's inequality [24] applies nonetheless. That is, for arbitrary operators in equation (2) that have the same commutation structure and square to $\mathbf{1}$, quantum mechanics demands $|\langle W\rangle| \leqslant \sqrt{8}$. For example, it may be that the Hilbert space is a direct sum of tensor products, e.g. $d_1 \times d_1 + d_2 \times d_2$. However, that space may be embedded into a tensor product space of dimension $(d_1 + d_2) \times (d_1 + d_2)$, and hence the measurement outcomes cannot violate the quantum Bell-like inequialities proved for tensor products [25, 26].

There are two natural orthonormal bases for a three-particle system based on the two different orders of fusing the three particles. These are graphically represented by fusion trees in figure 2(b). The basis change for three particles with charges (a, b, c) fusing to d is given by the matrices $F^{d}_{a,b,c}$. For the case of SU(2) these matrices just describe angular momentum recoupling and their matrix elements are the Wigner 6-j symbols:

$$(F^{d}_{abc})^{x}_{f} = \sqrt{(2x+1)(2f+1)} \begin{Bmatrix} a & b & f \\ c & d & x \end{Bmatrix} \tag{3}$$

and satisfy the orthogonality relation: $\sum_x (F^{d}_{abc})^{f}_{x}(F^{d}_{abc})^{f'}_{x} = \delta_{f,f'}$. For six particles with total spin 0, we get four natural product bases from the two pairs of bases for each triple. The

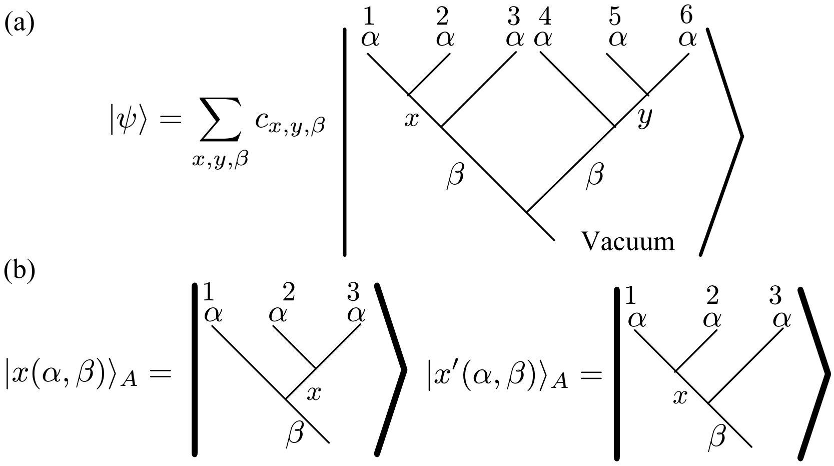

Figure 2. The state space of anyons in our protocol represented as fusion trees. (a) An arbitrary state of six α type anyons with trivial total charge expanded in terms of fusion outcomes local to A and B. In the models considered here, β is its own antiparticle, but it is straightforward to generalize. (b) A 'local' fusion basis satisfying $\Upsilon^{\mathrm{A}}_{2,3}|x(\alpha,\beta)\rangle_{\mathrm{A}} = \pm|x(\alpha,\beta)\rangle_{\mathrm{A}}$ and $\Upsilon^{\mathrm{A}}_{1,2}|x'(\alpha,\beta)\rangle_{\mathrm{A}} = \pm|x'(\alpha,\beta)\rangle_{\mathrm{A}}$ for x a vacuum state or a particle and similarly for B. For each subsystem the bases are related by an F move: $|x'(\alpha,\beta)\rangle = \sum_x (F^{\beta}_{\alpha\alpha\alpha})^x_{x'}|x(\alpha,\beta)\rangle$.

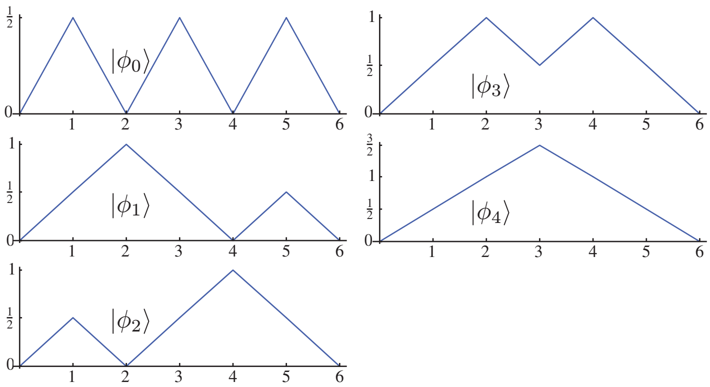

Figure 3. Bratteli diagram for SU(2) describing the various paths for combining six spin-$\frac{1}{2}$ particles with total spin zero. Each path defines a distinguishable state in the Hilbert space. In the $\mathrm{SU}(2)_2$ case the Bratteli diagram that corresponds to $|\phi_4\rangle$ is absent due to the charge truncation condition equation (7).

contributing particle labels are the spin values $\{0, \frac{1}{2}, 1, \frac{3}{2}\}$ and the only relevant F-matrix for changing between bases is $F^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$, which, for the SU(2) case is given by

$$F \equiv F^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

in the basis given by fusion trees with intermediate spins 0 and 1. The distinguishable states in the Hilbert space can be represented by Bratteli diagrams (see figure 3) which label the different paths for addition of angular momentum. For the group SU(2) with six spin-$\frac{1}{2}$ particles in the total spin zero sector, the dimensionality of the fusion space is five. We can then re-express those states using the local fusion basis defined in figure 2:

$$\begin{aligned} &|\phi_0\rangle = |0'\rangle_{\mathrm{A}}|0\rangle_{\mathrm{B}}, \quad |\phi_1\rangle = |1'\rangle_{\mathrm{A}}|0\rangle_{\mathrm{B}}, \quad |\phi_2\rangle = |0'\rangle_{\mathrm{A}}|1\rangle_{\mathrm{B}}, \\ &|\phi_3\rangle = |1'\rangle_{\mathrm{A}}|1\rangle_{\mathrm{B}}, \quad |\phi_4\rangle = \left|1\left(\tfrac{1}{2},\tfrac{3}{2}\right)\right\rangle_{\mathrm{A}}\left|1\left(\tfrac{1}{2},\tfrac{3}{2}\right)\right\rangle_{\mathrm{B}}, \end{aligned} \tag{4}$$

where $|x\rangle \equiv |x(\frac{1}{2},\frac{1}{2})\rangle$ and $|x'\rangle = \sum_x F^x_{x'}|x\rangle$. In spin components we have $|\phi_0\rangle = |\Psi^-\rangle_{1,2} \otimes |\Psi^-\rangle_{3,4} \otimes |\Psi^-\rangle_{5,6}$, where $|\Psi^-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, so the state $|\phi_0\rangle$ has three adjacent singlet pairs. Notice that in order to have trivial total spin the local bases occur in pairs that share the same label β.

In the basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\} \sqcup |\phi_4\rangle$, we have

$$W = (F^\dagger\sigma^z F \otimes F^\dagger\sigma^z F + F^\dagger\sigma^z F \otimes \sigma^z + \sigma^z \otimes F^\dagger\sigma^z F - \sigma^z \otimes \sigma^z) \oplus 2|\phi_4\rangle\langle\phi_4|. \tag{5}$$

The Hilbert space splits into different sectors labelled by β, which are conserved by the action of W. The expression for the witness in equation (5) is easily obtained by noting that since our measurement operators have two outcomes ± 1 that $\Upsilon^{\mathrm{A}}_{1,2} = \sigma^z$ and $\Upsilon^{\mathrm{A}}_{2,3} = F^\dagger\sigma^z F$, and similarly for the operators on B. There is a four-dimensional sector with $\beta = \frac{1}{2}$ and a one-dimensional sector with $\beta = \frac{3}{2}$, containing $|\phi_4\rangle$. No Bell violation can occur in the $\beta = \frac{3}{2}$ sector, since the measurement operators commute in that sector. Maximally, Bell violating states are thus orthogonal to $|\phi_4\rangle$. We consider a one parameter family of states that gives a good representation of the full Hilbert space in terms of the Bell inequalities. Indeed, for the states

$$|r(a)\rangle = \frac{a}{\sqrt{2}}(|\phi_0\rangle + |\phi_3\rangle) + \frac{\sqrt{1-a^2}}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle), \tag{6}$$

with $-1 \leqslant a \leqslant 1$ we plot the expectation value $\langle W\rangle$ seen in figure 4. The maximal violation ($\langle W\rangle_{\max,\min} = \pm\sqrt{7} \approx \pm 2.6458$) is obtained for $a_\pm = \mp\sqrt{(7 \pm 2\sqrt{7})/14}$, while non-violating regions of the parameter a can also be identified.

4. Bell inequalities and SU(2)$_k$ anyons

Consider now a two-dimensional system with quasiparticle excitations described by SU(2)$_k$ Chern–Simons–Witten theories. The corresponding fusion rules satisfy the addition of angular momentum with the constraints

$$j_1 \times j_2 \to j; \quad \text{only if } j_1, j_2, j \leqslant k/2 \quad \text{and} \quad j_1 + j_2 + j \leqslant k. \tag{7}$$

It is quickly verified that for $k \geqslant 3$, the total charge *zero* sector of six particles labelled by spin-$\frac{1}{2}$ again has five states, labelled by the same fusion trees as in the SU(2) case (see figure 3). The F matrices will differ, but for our purposes, the only relevant recoupling is still $\tilde{F}^{1/2}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$. Computing the quantum $6-j$ symbols, we find (see for instance [27])

$$F \equiv \tilde{F}^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} = \frac{1}{[2]_q}\begin{pmatrix} 1 & \sqrt{[3]_q} \\ \sqrt{[3]_q} & -1 \end{pmatrix},$$

where the quantum integers are defined as $[m]_q = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$ for m integer. For the SU(2)$_k$ theories, $q = \mathrm{e}^{2\pi\mathrm{i}/(k+2)}$. In the limit $k \to \infty$, then $[m]_q \to m$. As before, we can

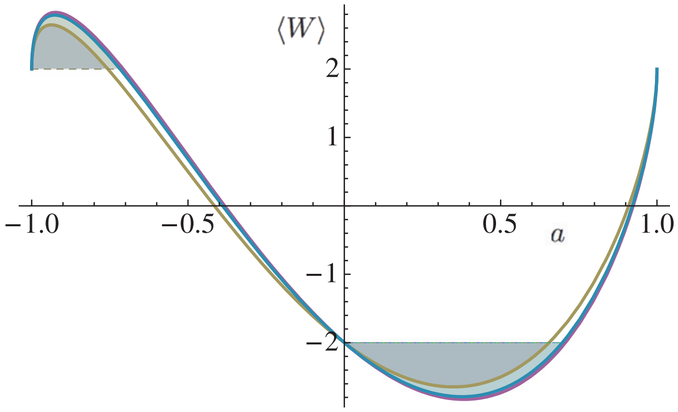

Figure 4. The expectation value of the Bell witness W as a function of the amplitude of mixing for the total charge zero states $|r(a)\rangle$ in equations (6). The yellow, blue and red lines correspond to an SU(2), $SO(3)_3$, $SU(2)_2$ theory with six spin-$\frac{1}{2}$, τ and σ particles, respectively. The shaded region corresponds to states which violate the inequality derived for LHV models.

label the states in the local fusion basis, as in equation (4), and still use equation (5) for W which is derived in the same way but with the appropriate F matrix.

The state $|\phi_0\rangle$ is obtained by creating spin-$\frac{1}{2}$ particle anti-particle pairs at positions (1, 2), (3, 4) and (5, 6) out of the vacuum. For the one parameter family of states $|r(a)\rangle$ we find maximal violation at

$$a_+ = -\frac{1}{\sqrt{8\cos(2\pi/(k+2)) + \cos(4\pi/(k+2)) + 5}}\left[\cos^2\left(\frac{2\pi}{k+2}\right) + 4\cos\left(\frac{2\pi}{k+2}\right) + \sqrt{2\cos^4\left(\frac{\pi}{k+2}\right)\left(8\cos\left(\frac{2\pi}{k+2}\right) + \cos\left(\frac{4\pi}{k+2}\right) + 5\right)} + 2\right]^{1/2} \tag{8}$$

and at $a_- = \sqrt{1 - a_+^2}$, where

$$\langle W\rangle = \pm\sec^2\left(\frac{\pi}{k+2}\right)\sqrt{4\cos\left(\frac{2\pi}{k+2}\right) + \frac{1}{2}\cos\left(\frac{4\pi}{k+2}\right) + \frac{5}{2}}.$$

It is easy to verify that $k \to \infty$ corresponds to SU(2). A qualitative difference between anyonic systems and the spin systems discussed before is that, while the z-components of the spins of all particles are in principle measurable, there are not necessarily any observables associated with the z-components of the 'q-spins' of the anyons. Only $SU(2)_q$ invariant quantities, such as the total q-spins of groups of anyons, can be observables, or at any rate topologically protected observables. This can be traced back to the superselection rule that says that the total q-spin of all anyons together must be trivial. If it were possible to measure the z-components of every anyons' q-spin, then the state obtained would no longer be invariant under $SU(2)_q$. In fact, a similar rule would hold for confined particles in gauge theory and so for a better analogy, one may think of the $SU(2)_q$ invariance as being closer to an SU(2) gauge symmetry rather than spin.

5. Explicit construction of Bell violating states for $k = 2$ and 3

5.1. The $SU(2)_2$ case

It was shown by Freedman *et al* [28] that the anyonic theories with $k \geqslant 3$, $k \neq 4$ are universal for quantum computation. Hence, for those theories, the Bell violating states can be obtained by topological braiding operations alone acting, for example, on the fiducial state $|\phi_0\rangle$. We now check to see if it is possible to generate a state that violates the inequality for the $k = 2$ case. Up to charge factors that affect the Abelian part of braiding, the $SU(2)_2$ anyons are believed to exist in the $\nu = \frac{5}{2}$ plateau of the fractional quantum Hall effect [29, 31, 32] and $p_x + \mathrm{i}p_y$ superconductors and possibly in systems involving topological insulators. They come in three varieties, the vacuum, 1, the fermion, ψ, and the non-Abelian anyon, σ, that satisfy the nontrivial fusion rules,

$$\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi. \tag{9}$$

The counterclockwise exchange of two σ particles, which fuse to either 1 or ψ, results in the matrix evolution $R = 1 \oplus \mathrm{i}$ expressed in the basis labelled by the fusion channels $\{1, \psi\}$. The state evolution produced by the exchange of particles with no immediate fusion channel is found by employing the recoupling matrix F. Expressed in the basis $\{|x(\sigma, \sigma)\rangle_A |y(\sigma, \sigma)\rangle_B; x, y \in \{1, \psi\}\}$, we have the following representation of the nearest-neighbor left–right exchanges that are generators of the braid group $\mathcal{B}_6$:

$$\begin{aligned} B_1 &= \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma^x} \otimes \mathbf{1}_2, \quad B_2 = \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma^z} \otimes \mathbf{1}_2, \quad B_3 = \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma^x \otimes \sigma^z}, \\ B_4 &= \mathbf{1}_2 \otimes \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma^x}, \quad B_5 = \mathbf{1}_2 \otimes \mathrm{e}^{-\mathrm{i}(\pi/4)\sigma^z}, \end{aligned} \tag{10}$$

where B_j results from the exchange of j and $j+1$ particles in a counterclockwise manner. As a simple initial state we can consider $|\phi_0\rangle$ that is produced from (1, 2), (3, 4) and (5, 6) pairs created from the vacuum.

The braid group generators, B_j, are in the Clifford group, so we cannot generate a dense set in SU(4) by braiding alone[7]. But can we still obtain Bell violating states? In [33, 34], an LHV model was introduced for a pair of qubits that exactly reproduces the set of allowed operations in the present model. There are two distinct configurations of shared vacuum pairs (up to relabelling of particles by Alice or Bob) both of which can be obtained from $|\phi_0\rangle$ by braiding. Hence, it is not possible to build Bell violating states starting out from three shared vacuum pairs using topologically protected operations alone.

Despite the impossibility of producing a Bell violating state from $|\phi_0\rangle$ by topological gates, one can in fact straightforwardly obtain a maximally Bell violating state using non-topological gates[8]. Let us employ the non-Clifford gate $D = \mathrm{e}^{-\mathrm{i}(\pi/8)\sigma^z} \otimes \mathbf{1}$. This can be implemented by bringing the two and three σ anyons nearby, thus shifting the energy of the fermionic fusion channel [36] such that a relative phase $\mathrm{e}^{\mathrm{i}\pi/4}$ is accumulated on that channel. From these operations one can build the controlled phase gate in the following way $\mathrm{CP} = \mathrm{e}^{\mathrm{i}\pi/4} B_2 B_1 B_2 B_3^{-1} B_2^{-1} B_1^{-1} B_5$. A simple searching algorithm provides us with the sequence that produces $|r(a_-)\rangle = -CPB_3B_4DB_2B_3|\phi_0\rangle$, with $\langle W\rangle = -2\sqrt{2}$, thus saturating the Tsirelson bound.

[7] Complementing topological operations by some noisy non-topological operations, one can achieve universality [30].

[8] For an alternative construction of this kind, but using the more standard tensor product structure of four-anyon qubits, giving a total of eight anyons, see [35].

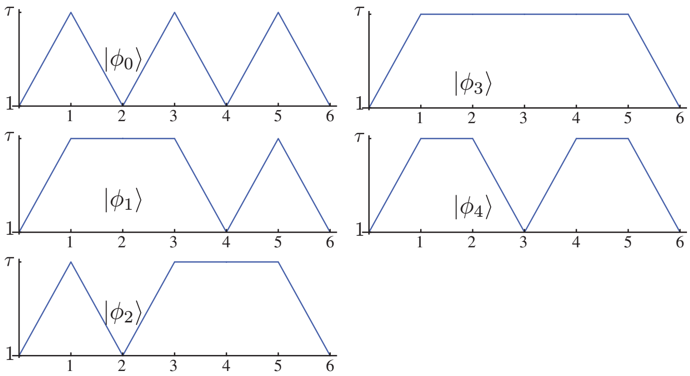

Figure 5. Bratteli diagram for $SO(3)_3$ describing the various paths for combining 6 charge τ particles with total charge zero. Each path defines a distinguishable state in the Hilbert space.

In fact, it is indeed possible to realize a maximally violating state without braiding at all. Consider a state $|\phi_0'\rangle$ given by a distribution of singlet pairs on (1, 6), (2, 5) and (3, 4). This state $|\phi_0'\rangle$ is related to the fiducial distribution of pairs by the following braid word $|\phi_0'\rangle = \frac{1}{\sqrt{2}}(|0'\rangle_A|0\rangle_B + |1'\rangle_A|1\rangle_B) = B_2^{-1}B_3^{-1}B_5B_4B_3B_2|\phi_0\rangle$. This state would be maximally violating if we could measure in arbitrary local bases. For our fixed measurement quorum $|\phi_0'\rangle$ is related to the maximally violating state by $e^{-i(\pi/8)\sigma^y}\otimes \mathbf{1}|\phi_0'\rangle = |r(a_+)\rangle$ and hence it suffices to implement the local unitary $e^{-i(\pi/8)\sigma^y} = e^{-i(\pi/4)\sigma^z}e^{-i(\pi/8)\sigma^x}e^{i(\pi/4)\sigma^z}$ on Alice's side to obtain a Bell violation. The z rotation is simply achieved by bringing anyons two and three near each other as above, and similarly the x rotation is performed by pushing one and two together. This scheme is re-described in appendix A in a way that emphasizes its analogy with current photon experiments used to demonstrate non-locality. Note that the maximal Bell violation in these two constructions actually saturates the Tsirelson inequality, making the $SU(2)_2$ case at the same time 'maximally quantum mechanical' and 'topologically classical'.

5.2. The Fibonacci case

Let us turn now to Fibonacci anyons from the $SO(3)_3$ theory. This is the theory obtained from $SU(2)_3$ but using only integer spin particles: the vacuum 1 and the non-Abelian anyon τ, with nontrivial fusion rule $\tau\times\tau = 1+\tau$. The relevant recoupling matrix is

$$F = F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

expressed in the basis of 1 and τ. The dimension of the topological Hilbert space of $m+1$ type τ anyons with total charge zero is f_m, the mth Fibonacci number, hence there are five states in the fusion space as depicted in figure 5. These states can be decomposed into superpositions of products of local basis states as in equation (4) where $|0\rangle = |1(\tau,\tau)\rangle$ and $|1\rangle = |\tau(\tau,\tau)\rangle$ and $|\phi_4\rangle = |\tau(\tau,1)\rangle_A|\tau(\tau,1)\rangle_B$. The state $|\phi_0\rangle$ is the state obtained by creating type τ particle–anti-particle pairs on (1, 2), (3, 4), (5, 6) out of the vacuum. For the one parameter family of states $|r(a)\rangle$ we find the same maximal violation as in equation (8) for $SU(2)_3$: $\langle W\rangle = \pm 2\sqrt{-7+4\sqrt{5}} \approx \pm 2.7887$. This is not very surprising, considering that the

$SU(2)_3$ theory is equivalent to the product of the Fibonacci theory and an Abelian theory with $\mathbb{Z}_2$ fusion rules (see for instance [37]).

The action under braiding is represented by the matrix $R_{\tau\tau} = e^{i4\pi/5} \oplus e^{i7\pi/5}$ expressed in the basis $\{1, \tau\}$. We obtain the following representation of the generators of the braid group $\mathcal{B}_6$ expressed in the basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\} \sqcup |\phi_4\rangle$:

$$\begin{aligned} B_1 &= [F R_{\tau\tau} F^{-1} \otimes \mathbf{1}_2] \oplus (e^{i7\pi/5}), \quad B_2 = [R_{\tau\tau} \otimes \mathbf{1}_2] \oplus (e^{i7\pi/5}), \\ B_3 &= O^\dagger \left[e^{i4\pi/5} \oplus e^{i7\pi/5} \oplus e^{i7\pi/5} \oplus \begin{pmatrix} M_1^1 & M_0^1 \\ M_1^0 & M_0^0 \end{pmatrix} \right] O, \\ B_4 &= [\mathbf{1}_2 \otimes F R_{\tau\tau} F^{-1}] \oplus (e^{i7\pi/5}), \quad B_5 = [\mathbf{1}_2 \otimes R_{\tau\tau}] \oplus (e^{i7\pi/5}), \end{aligned} \tag{11}$$

where O maps the product basis to the basis $\{|\phi_j\rangle\}_{j=0}^4$, and $M = (F_{\tau\tau\tau}^\tau)^{-1} R_{\tau\tau} F_{\tau\tau\tau}^\tau$. A length 25 braid word produces a Bell violation: $|\Psi\rangle = [B_3 B_4^{-1} B_1^{-1} B_3^{-1} B_2^{-1}]^5 |\phi_0\rangle$ with $\langle\Psi|W|\Psi\rangle = 2.5310$. With longer braid words it is possible to get arbitrarily close to maximal violation [20].

6. Bell inequalities and quantum doubles

In the models above, measurements by Alice and Bob had two outcomes for the two fusion products of the anyons. To accommodate more outcomes we can use higher dimensional Bell witnesses [38]. We demonstrate how this works in another anyonic model with excitations in one-to-one correspondence with irreducible representations of a Hopf algebra, $D(G)$, the quantum double of a finite group G [18, 19]. We focus on the simplest non-Abelian finite group, S_3, the group of permutations on three objects (see appendix B). A further simplification can be achieved by restricting to a particular fusion subalgebra of $D(S_3)$, $\{1, \Lambda, \Phi\}$, with nontrivial fusion rules

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi.$$

The magnetic charge Φ with quantum dimension two carries non-Abelian statistics and the fusion of n such particles gives $\Phi^{\times n} = \frac{1}{3}(2^{n-1} + (-1)^n)(1 + \Lambda) + \frac{1}{3}(2^n + (-1)^{n-1})\Phi$. As before, we will work in the superselection sector with total trivial charge. The smallest number of particles in this sector that could hope to violate a Bell inequality should have fusion space dimension $\geqslant 4$. If we are to pick measurement operators for Alice and Bob that measure total charge on pairs of Φ particles and we want two non-commuting operators on each side then we require at least six particles in total. Exactly six particles suffices, giving Hilbert space dimension 11 for the vacuum sector.

Either by using the representation theory of $D(S_3)$, or by solving the pentagon and hexagon equations directly (see appendix B), we find the following recoupling and braid matrices, expressed in the basis $\{1, \Lambda, \Phi\}$:

$$F \equiv F_{\Phi\Phi\Phi}^\Phi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad R \equiv R_{\Phi\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

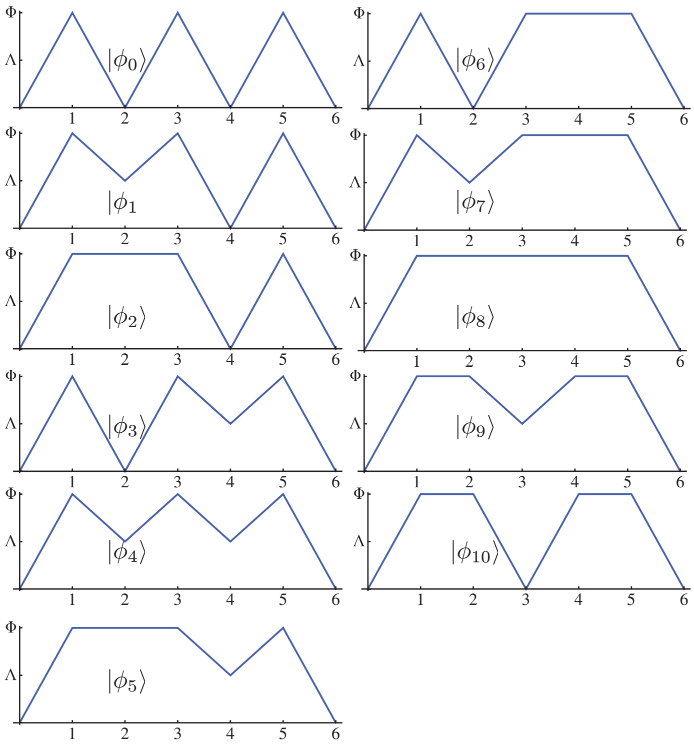

Figure 6. Bratteli diagram for a subalgebra of $D(S_3)$ describing the various paths for combining six Φ particles with total charge zero. Each path defines a distinguishable state in the Hilbert space.

A basis of the 11-dimensional vacuum sector of the six anyon Hilbert space, depicted in figure 6, is re-expressed in the fusion bases for A and B as

$$\begin{aligned}\{|\phi_j\rangle\}_{j=0}^{10} = \Big\{ &\sum_y F_1^y |y(\Phi,\Phi)\rangle_A |1(\Phi,\Phi)\rangle_B, \sum_y F_\Lambda^y |y(\Phi,\Phi)\rangle_A |1(\Phi,\Phi)\rangle_B,\\ &\sum_y F_\Phi^y |y(\Phi,\Phi)\rangle_A |1(\Phi,\Phi)\rangle_B, \sum_y F_1^y |y(\Phi,\Phi)\rangle_A |\Lambda(\Phi,\Phi)\rangle_B,\\ &\sum_y F_\Lambda^y |y(\Phi,\Phi)\rangle_A |\Lambda(\Phi,\Phi)\rangle_B, \sum_y F_\Phi^y |y(\Phi,\Phi)\rangle_A |\Lambda(\Phi,\Phi)\rangle_B\\ &\sum_y F_1^y |y(\Phi,\Phi)\rangle_A |\Phi(\Phi,\Phi)\rangle_B, \sum_y F_\Lambda^y |y(\Phi,\Phi)\rangle_A |\Phi(\Phi,\Phi)\rangle_B,\\ &\sum_y F_\Phi^y |y(\Phi,\Phi)\rangle_A |\Phi(\Phi,\Phi)\rangle_B, |\Phi(\Phi,\Lambda)\rangle_A |\Phi(\Phi,\Lambda)\rangle_B, |\Phi(\Phi,1)\rangle_A |\Phi(\Phi,1)\rangle_B \Big\}.\end{aligned} \tag{12}$$

The state $|\phi_0\rangle$ is obtained by creating type Φ particle–anti-particle pairs on (1, 2), (3, 4), (5, 6) out of the vacuum.

Now in analogy to the cases studied for $\mathrm{SU}(2)_k$, we could look for a Bell-like inequality but using measurement operators with three outcomes. Let Alice have one operator $\Upsilon_{1,2}^{A}$ which measures the outcome of total charge for particles 1 and 2 with outcomes $\{1, \Lambda, \Phi\}$ and another, non-commuting operator, $\Upsilon^{\mathrm{A}}_{2,3}$ that measures total charge for particles 2 and 3 with outcomes $\{1, \Lambda, \Phi\}$. In other words, $\Upsilon^{\mathrm{A}}_{1,2}$ is a measurement in the basis $\{(F^{\beta}_{\alpha\alpha\alpha})^{\dagger}|y(\alpha, \beta)\rangle_{\mathrm{A}}\}$ with outcome $m^{\mathrm{A}}_{1,2} = y \in \{1, \Lambda, \Phi\}$ and $\Upsilon^{\mathrm{A}}_{2,3}$ is a measurement in the basis $\{|y(\alpha, \beta)\rangle_{\mathrm{A}}\}$ with outcome $m^{\mathrm{A}}_{2,3} = y$. Similarly, let Bob have two measurement operators, $\Upsilon^{\mathrm{B}}_{4,5}$ that measures in the basis $\{(F^{\beta}_{\alpha\alpha\alpha})^{\dagger}|y(\alpha, \beta)\rangle_{\mathrm{B}}\}$ with outcome $m^{B}_{4,5} = y$, and $\Upsilon^{\mathrm{B}}_{5,6}$ which measures onto the basis $\{|y(\alpha, \beta)\rangle_{\mathrm{B}}\}$ with outcome $m^{\mathrm{B}}_{5,6} = y$. Now $(F^{1}_{\Phi\Phi\Phi})^{y}_{x} = \delta_{x,\Phi}\delta_{y,\Phi} = (F^{\Lambda}_{\Phi\Phi\Phi})^{y}_{x}$, so in the subspace of $\{|\phi_9\rangle, |\phi_{10}\rangle\}$, the measurement operators all commute. These states cannot yield a Bell violation and we can focus on the states in the nine-dimensional orthogonal subspace which is isomorphic to the Hilbert space of two three-dimensional particles (qutrits).

In [38] it was shown how to construct Bell inequalities for bipartite systems of equal but arbitrary finite dimension. In particular for two qutrits the authors introduce the witness I_3 which for all LHV theories satisfies $|\langle I_3\rangle| \leqslant 2$, whereas for quantum mechanical systems $|\langle I_3\rangle| \leqslant 4$. To simplify notation let us introduce the projectors $\pi_y \equiv |y(\Phi, \Phi)\rangle\langle y(\Phi, \Phi)|$ and $\tilde{\pi}_y \equiv F^{\dagger}|y(\Phi, \Phi)\rangle\langle y(\Phi, \Phi)|F$. For the quorum of observables above, the Bell witness I_3 is

$$\begin{aligned} I_3 = {} & \tilde{\pi}_1 \otimes \tilde{\pi}_1 + \tilde{\pi}_\Lambda \otimes \tilde{\pi}_\Lambda + \tilde{\pi}_\Phi \otimes \tilde{\pi}_\Phi + \pi_\Phi \otimes \tilde{\pi}_1 + \pi_1 \otimes \tilde{\pi}_\Lambda + \pi_\Lambda \otimes \tilde{\pi}_\Phi + \pi_1 \otimes \pi_1 + \pi_\Lambda \otimes \pi_\Lambda \\ & + \pi_\Phi \otimes \pi_\Phi + \tilde{\pi}_1 \otimes \pi_1 + \tilde{\pi}_\Lambda \otimes \pi_\Lambda + \tilde{\pi}_\Phi \otimes \pi_\Phi - \tilde{\pi}_1 \otimes \tilde{\pi}_\Lambda + \tilde{\pi}_\Lambda \otimes \tilde{\pi}_\Phi + \tilde{\pi}_\Phi \otimes \tilde{\pi}_1 - \pi_1 \otimes \tilde{\pi}_1 \\ & + \pi_\Lambda \otimes \tilde{\pi}_\Lambda + \pi_\Phi \otimes \tilde{\pi}_\Phi - \pi_1 \otimes \pi_\Lambda + \pi_\Lambda \otimes \pi_\Phi + \pi_\Phi \otimes \pi_1 - \tilde{\pi}_\Lambda \otimes \pi_1 + \tilde{\pi}_\Phi \otimes \pi_\Lambda + \tilde{\pi}_1 \otimes \pi_\Phi \\ & + 2|\Phi(\Phi, \Lambda)\rangle\langle\Phi(\Phi, \Lambda)| \otimes |\Phi(\Phi, \Lambda)\rangle\langle\Phi(\Phi, \Lambda)| \\ & + 2|\Phi(\Phi, 1)\rangle\langle\Phi(\Phi, 1)| \otimes |\Phi(\Phi, 1)\rangle\langle\Phi(\Phi, 1)|. \end{aligned} \tag{13}$$

The state with the largest violation has $\langle I_3\rangle = -2.5216$.

In the basis $\{|x(\Phi, \Phi)\rangle_{\mathrm{A}}|y(\Phi, \Phi)\rangle_{\mathrm{B}}; x, y \in \{1, \Lambda, \Phi\}\} \sqcup |\Phi(\Phi, \Lambda)\rangle_{\mathrm{A}}|\Phi(\Phi, \Lambda)\rangle_{\mathrm{B}} \sqcup |\Phi(\Phi, 1)\rangle_{\mathrm{A}}|\Phi(\Phi, 1)\rangle_{\mathrm{B}}\}$ the representation of the generators for $\mathcal{B}_6$ is given by

$$\begin{aligned} & B_1 = [FRF^{-1} \otimes \mathbf{1}_3] \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi}, \quad B_2 = [R \otimes \mathbf{1}_3] \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi}, \\ & B_3 = O^{\dagger}\left[R^{1}_{\Phi\Phi} \oplus R^{\Lambda}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Lambda}_{\Phi\Phi} \oplus R^{1}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi} \oplus \begin{pmatrix} M^{\Phi}_{\Phi} & M^{\Phi}_{\Lambda} & M^{\Phi}_{1} \\ M^{\Lambda}_{\Phi} & M^{\Lambda}_{\Lambda} & M^{\Lambda}_{1} \\ M^{1}_{\Phi} & M^{1}_{\Lambda} & M^{1}_{1} \end{pmatrix} \right] O, \\ & B_4 = [\mathbf{1}_3 \otimes FRF^{-1}] \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi}, \quad B_5 = [\mathbf{1}_3 \otimes R] \oplus R^{\Phi}_{\Phi\Phi} \oplus R^{\Phi}_{\Phi\Phi}, \end{aligned} \tag{14}$$

where O maps the product basis to the basis $\{|\phi_j\rangle\}_{j=0}^{10}$, and $M = F^{-1}RF$. Here $B_j^2 = \mathbf{1}_{11}\forall j$, so we have the permutation group S_6, as mentioned before. We compute the action on states consisting of vacuum magnetic charge pairs. The state $|\phi_0\rangle = |\Phi, \Phi; (1, 2)\rangle|\Phi, \Phi; (3, 4)\rangle|\Phi, \Phi; (5, 6)\rangle$ is the fiducial state, and the other distinct configuration of vacuum magnetic charge pairs is $|\Phi, \Phi; (1, 4)\rangle|\Phi, \Phi; (2, 5)\rangle|\Phi, \Phi; (3, 6)\rangle = B_3B_4B_2B_1|\phi_0\rangle$, hence it suffices to consider the orbit of $|\phi_0\rangle$. An exhaustive search through $6! = 720$ braid words corresponding to all distinct permutations in S_6 finds that, while $\langle I_3\rangle$ is not constant under braiding, we do find that in all cases $|\langle I_3\rangle| \leqslant 2$. Hence we require some operation beyond braiding to produce a violation of LHV under our protocol. Even if we restrict to non-topologically protected operations that just involve interacting pairs of particles, we can indeed produce a Bell violating state. Consider the family of states $|\phi'\rangle = D_{3,4}(\alpha_1, \alpha_2)D_{1,2}(\alpha_3, \alpha_4)D_{2,3}(\alpha_5, \alpha_6)B_1B_5B_3B_2B_3B_4|\phi_0\rangle$, where $D_{i,j}(\alpha, \beta)$ is the non-topologically protected gate obtained by bringing anyons i and j of type Φ nearby each other and allowing them to interact for a time such that the fusion channel $\Phi \times \Phi \to \Lambda$ accumulates a phase $e^{i\alpha}$ and the fusion channel $\Phi \times \Phi \to \Phi$ accumulates a phase $e^{i\beta}$. Optimizing $|\langle I_3 \rangle|$ over the interaction phases, we find a violation $\langle \phi' | I_3 | \phi' \rangle = 2.0512$ for the angles: $\alpha_1 = 0.7943$, $\alpha_2 = 0.3989$, $\alpha_3 = 3.5531$, $\alpha_4 = 0.9257$, $\alpha_5 = -0.8525$ and $\alpha_6 = 0.1036$. No systematic attempt was made to optimize the violation over other braid words and it is likely stronger violations could be found.

7. Conclusions

We have described a protocol to reveal non-locality in several classes of non-Abelian anyonic theories. The need for at least six anyons shared between two parties arises because each party needs three anyons in order to have two non-commuting topologically protected observables. It is possible if this could be reduced using a shared resource which fixes a common gauge, akin to using a shared reference frames to reveal non-locality in mode entanglement with bosons [23]. The size of the maximum violation depends on the recoupling matrices F and the ability to generate Bell violating states beginning from three vacuum charge pairs depend on the power of the braiding operations. It is intriguing to ask whether one could find intermediate anyonic theories which have the power to generate Bell violating states by topologically protected gates, but are not universal for topological quantum computation.

Acknowledgments

SI thanks Belén Paredes for stimulating discussions on representations of anyonic statistics by spin-$\frac{1}{2}$ particles [39], as well as support from the Generlaitat de Catalunya, MEC (Spain), and the European project QAP. GKB thanks Boris Tsirelson for helpful correspondence regarding the quantum bound for Bell like inequalities. JS received support through the Science Foundation of Ireland through Principal Investigator Award 08/IN.1/I1961. SI acknowledges support from the Generalitat de Catalunya, MEC (Spain), and the European project QAP. This work was supported in parts by the EU grants EMALI and SCALA, EPSRC and the Royal Society.

Appendix A. The Clauser–Horne–Shimony–Holt (CHSH) inequality and Ising anyons

In this appendix, we show the equivalence between one of our schemes and a scheme where the CHSH inequality [22] is violated by a maximally entangled two-qubit state. Let us start by briefly recalling this inequality. Let us consider two spatially separated parties, Alice and Bob, sharing many copies of a bipartite system. Each of them can perform either of two measurement on each copy; A_1 or A_2 for Alice, and B_1 or B_2 for Bob. The outcomes of A_1, A_2, B_1, B_2 are ± 1. The CHSH inequality reads

$$\langle A_1 B_1 + A_2 B_2 + A_2 B_1 - A_1 B_2 \rangle \leqslant 2. \tag{A.1}$$

When Alice and Bob perform the measurements

$$A_1 = \sigma^z, \quad A_2 = \sigma^x, \tag{A.2}$$

$$B_1 = \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z), \quad B_2 = \frac{1}{\sqrt{2}}(\sigma^x - \sigma^z), \tag{A.3}$$

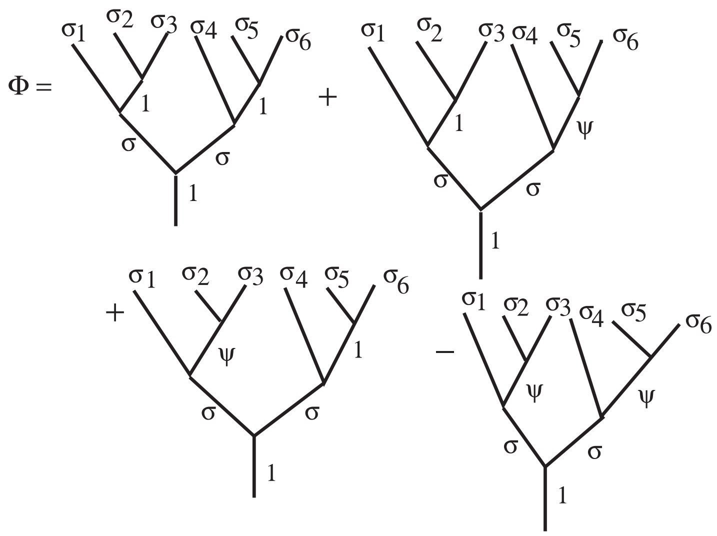

Figure A.1. State of three pairs of Ising anyons express in a 'right–right' basis.

on the state

$$|\phi^+\rangle = |0,0\rangle + |0,1\rangle + |1,0\rangle - |1,1\rangle, \tag{A.4}$$

then the l.h.s. of (A.1) is equal to $\sqrt{8}$, which is the maximal value attainable by quantum theory.

Let us now consider an $SU(2)_2$ theory and suppose that three pairs of σ anyons (spin-$\frac{1}{2}$ irrep particles) are created from the vacuum: (σ_1, σ_6), (σ_2, σ_5), (σ_3, σ_4). Quasi-particles $\sigma_1, \sigma_2, \sigma_3$ go to Alice whereas $\sigma_4, \sigma_5, \sigma_6$ go to Bob. Using a sequence of F-moves, one can write the corresponding state shared by Alice and Bob as drawn in figure A.1.

On Alice's side (left part of each tree), we formally define a $|0\rangle$ state of a computational basis as a state where her anyons 1 and 2 have fused to yield a charge-'1' quasi-particle before fusing with her anyon 3. The $|1\rangle$ state is defined as a state where her anyons 1 and 2 fuse to ψ. A similar computational basis is defined on Bob's side. Then, it is obvious that the state Alice and Bob share is of the form (A.4). Therefore, it should allow for a maximal violation of the CHSH inequality by operations only on Alice's side and on Bob's side. Of course, the state shared by Alice and Bob is not truly (A.4). Alice and Bob do not have two qubits, and their Hilbert space is not a tensor product. Rather they have excitations defined over an entangled vacuum, and their Hilbert space cannot be factorized. But as far as the CHSH inequality is concerned, all that matters is to reproduce the mean values $\langle A_i B_j\rangle$ as if their state were (A.4). For that, we only need to implement the measurements (A.2) and (A.3).

The matter is simple on Alice's side. A_1 is a measurement where she fuses her anyons 2 and 3, and then the result with her anyon 1 (outcome '0' if the intermediate particle is trivial, and '1' if the intermediate particle is ψ). A_2, in turn, can be implemented by fusing her anyons 1 and 2, and then the result with the anyon 1, as can be seen from the expressions for the elements of the F-matrix.

In order to implement the necessary measurement's on Bob's side, we use the D-gate. Bringing anyons 5 and 6 close to each other for a given amount of time allows to implement operations of the form $D^z(\beta) = \mathrm{e}^{\mathrm{i}\beta\sigma^z}$, whereas bringing anyons 4 and 5 for a given amount of time allows to implement $D^x(\alpha) = \mathrm{e}^{\mathrm{i}\alpha\sigma^x}$. From the identity $\mathrm{e}^{\mathrm{i}\alpha\sigma^x}\mathrm{e}^{\mathrm{i}\beta\sigma^z}\sigma^x\mathrm{e}^{-\mathrm{i}\beta\sigma^z}\mathrm{e}^{-\mathrm{i}\alpha\sigma^x}$

$= \sin 2\alpha \sin 2\beta\sigma^z + \cos 2\beta\sigma^x - \cos 2\alpha \sin 2\beta\sigma^y$, we see that choosing the 'interaction times' $(\alpha, \beta) = (\pi/4, \pi/8)$ implements B_1, whereas the choice $(\alpha, \beta) = (\pi/4, 7\pi/8)$ implements B_2.

Appendix B. The quantum double model

Here we give a brief account of anyons governed by the irreducible representations of a Hopf algebra, $D(G)$, the quantum double of a finite group G [18, 19]. The particles can carry electric and magnetic charge and are labelled by $\Pi_\alpha^{[a]}$ where $[a]$ denotes a conjugacy class of G which labels the magnetic charge, and α, which labels the electric charge, denotes a unitary irreducible representation of the centralizer of an element in the conjugacy class $[a]$. The dimension of the carrier space for each irreducible representation, which equals the quantum dimension of the particle $\Pi_\alpha^{[a]}$ is $d_\alpha^{[a]} = |[a]||\alpha|$. We focus on the simplest non-Abelian finite group, S_3, the group of permutations on three objects. Elements of S_3 are organized into three conjugacy classes: $[e] = \{e\}$ the identity element, $[t] = \{t_0, t_1, t_2\}$ the transpositions and $[c] = \{c_+, c_-\}$ the cyclic permutations. The eight irreducible representations for $D(S_3)$ are

$$\begin{aligned}
&\Pi_{\delta_+}^{[e]} \quad d = 1 \quad \text{(vacuum)},\\
&\Pi_{\beta_0}^{[c]}, \Pi_{\gamma_0}^{[t]} \quad d = 2, 3 \quad \text{(pure magnetic charges)},\\
&\Pi_{\delta_-}^{[e]}, \Pi_{\delta_2}^{[e]} \quad d = 1, 2 \quad \text{(pure electric charges)},\\
&\Pi_{\beta_1}^{[c]}, \Pi_{\beta_2}^{[c]}, \Pi_{\gamma_1}^{[t]} \quad d = 2, 2, 3 \quad \text{(dyonic combinations)}.
\end{aligned} \tag{B.1}$$

A complete derivation of the fusion rules for this model is given in [40]. In the toric code realization of these anyon models, the quantum dimensions actually count local degrees of freedom associated with the anyon. In the discrete gauge theory context, these degrees of freedom are also present in the description of the system, but some of them are gauge. A single particle's electric charge and magnetic charge can always be measured locally (or at least within a region of size characteristic of the particles), by braiding with other locally prepared charge pairs and measuring the outcome of fusion of the pairs. Truly non-local properties are contained in the fusion space. To explore this we pick a fusion subalgebra of $D(S_3)$: $\{\Pi_{\delta_+}^{[e]}, \Pi_{\delta_-}^{[e]}, \Pi_{\beta_0}^{[c]}\}$ which we label for convenience $\{1, \Lambda, \Phi\}$. The nontrivial fusion rules are

$$\Lambda \times \Lambda = 1, \quad \Lambda \times \Phi = \Phi, \quad \Phi \times \Phi = 1 + \Lambda + \Phi.$$

These fusion rules are the same as the fusion rules for the representations of S_3 itself and also the same as the fusion rules of the integer spin sectors of $\mathrm{SU}(2)_4$. The particles are their own anti-particles. The magnetic charge Φ with quantum dimension two carries non-Abelian statistics and the fusion of n such particles gives: $\Phi^{\times n} = \frac{1}{3}(2^{n-1} + (-1)^n)(1 + \Lambda) + \frac{1}{3}(2^n + (-1)^{n-1})\Phi$. As before, we will work in the superselection sector with total trivial charge. The smallest number of particles in this sector that could hope to violate a Bell inequality should have fusion space dimension $\geqslant 4$. If we are to pick measurement operators for Alice and Bob that measure total charge on pairs of Φ particles and we want two non-commuting operators on each side then we require at least six particles in total. Exactly six particles suffices giving Hilbert space dimension 11 for the vacuum sector.

Either by using the representation theory of $D(S_3)$, or by solving the pentagon and hexagon equations directly, we find the following recoupling and braid matrices, expressed in the basis

$\{1, \Lambda, \Phi\}$,

$$F \equiv F^{\Phi}_{\Phi\Phi\Phi} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad R \equiv R_{\Phi\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We notice immediately that R has eigenvalues ± 1, so that we will end up with a representation of the permutation group when 'braiding' the anyons. Nevertheless, these anyons are not bosons or fermions, since this representation is non-Abelian. The fact that we have a permutation group representation does signal the fact that braiding in this theory is not universal for quantum computation. This is in fact a general property of braiding in discrete gauge theories.

References

[1] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[2] Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[3] Leinaas J M and Myrheim J 1977 *Nuovo Cimento* B **37** 1
[4] Goldin G A, Menikoff R and Sharp D H 1980 *J. Math. Phys.* **21** 650
[5] Wilzcek F 1982 *Phys. Rev. Lett.* **48** 1144
[6] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[7] Barnum H, Knill E, Ortiz G, Somma R and Viola L 2004 *Phys. Rev. Lett.* **92** 107902
[8] Eckert K, Schliemann J, Bruss D and Lewenstein M 2002 *Ann. Phys.* **299** 88
[9] Wiseman H and Vaccaro J A 2003 *Phys. Rev. Lett.* **91** 097902
[10] Moehring D L, Madsen M J, Blinov B B and Monroe C 2004 *Phys. Rev. Lett.* **93** 090410
Matsukevich D N, Maunz P, Moehring D L, Olmschenk S and Monroe C 2008 *Phys. Rev. Lett.* **100** 150404
Bertlmann R A and Hiesmayr B C 2001 *Phys. Rev.* A **63** 062112
[11] Nayak C, Simon S H, Stern A, Freedman M and Sarma S D 2008 *Rev. Mod. Phys.* **80** 1083
[12] Douçot B, Ioffe L B and Vidal J 2004 *Phys. Rev.* B **69** 214501
[13] Aguado M, Brennen G K, Verstraete F and Cirac J I 2008 *Phys. Rev. Lett.* **101** 260501
[14] Kitaev A 2003 *Ann. Phys.* **303** 2
[15] Hu M-G, Deng D-L and Chen J-L 2008 arXiv:0810.1157
[16] Das Sarma S, Freedman M and Nayak C 2005 *Phys. Rev. Lett.* **94** 166802
[17] Bonderson P, Shtengel K and Slingerland J K 2008 *Ann. Phys.* **323** 2709–55
[18] Bais F A, van Driel P and de Wild Propitius M 1992 *Phys. Lett.* B **280** 63
[19] de Wild Propitius M and Bais F A 1998 *Particles and Fields (CRM Series in Mathematical Physics)* ed G Semenoff and L Vinet (New York: Springer) pp 353–439 arXiv:hep-th/9511201
[20] Freedman M H, Larsen M J and Wang Z 2002 *Commun. Math. Phys.* **227** 605–22
[21] Gisin N 2007 arXiv:quant-ph/0702021
[22] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880–4
[23] Bartlett S D, Rudolph T and Spekkens R 2007 *Rev. Mod. Phys.* **79** 555
[24] Tsirelson B S 1980 *Lett. Math. Phys.* **4** 93
[25] Tsirelson B S 1993 *Hadronic J.* (Suppl.) **8** 329
[26] Scholz V B and Werner R F 2008 arXiv:0812.4305
[27] Slingerland J K and Bais F A 2001 *Nucl. Phys.* B **612** 229–90
[28] Freedman M, Larsen M and Wang Z 2002 *Commun. Math. Phys.* **227** 605
Freedman M, Larsen M and Wang Z 2002 *Commun. Math. Phys.* **228** 177

[29] Moore G and Read N 1991 *Nucl. Phys.* B **360** 362
[30] Bravyi S 2006 *Phys. Rev.* A **73** 042313
[31] Willett R L *et al* 1987 *Phys. Rev. Lett.* **59** 1776
[32] Willett R L, Pfeiffer L N and West K W 2009 *Proc. Natl Acad. Sci. USA* **106** 8853
[33] Spekkens R W 2007 *Phys. Rev.* A **75** 032110
[34] Gibbons K S, Hoffman M J and Wootters W K 2004 *Phys. Rev.* A **70** 062101
[35] Zhang C, Tewari S and das Sarma S 2007 *Phys. Rev. Lett.* **99** 220502
[36] Lahtinen V *et al* 2008 *Ann. Phys.* **323** 2286
[37] Bonderson P 2007 Non-Abelian anyons and interferometry *PhD Thesis* California Institute of Technology, Pasadena
[38] Collins D *et al* 2002 *Phys. Rev. Lett.* **88** 040404
[39] Paredes B unpublished
[40] de Wild Propitius M 1995 Topological interactions in broken gauge theories *PhD Thesis* University of Amsterdam