On the simultaneous diagonal stability of a pair of positive linear systems

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Abstract

In this paper we derive a necessary and sufficient condition for the existence of a common diagonal quadratic Lyapunov function for a pair of positive linear time-invariant (LTI) systems.

1 Introduction

The theory of positive LTI systems has historically assumed a position of great importance in systems theory and has been applied in the study of a wide variety of dynamic systems [1, 2, 3, 4]. Recently, new application studies in communication systems [5], formation flying [6], and several other areas, have highlighted the importance of switched (hybrid) positive linear systems (PLS). While positive LTI systems are well understood, many basic properties of switched PLS remain to be ascertained. The most important of these concerns their stability. In this paper we report initial results on the stability of switched positive linear systems. While we are ultimately interested in the general stability properties of such systems, our focus in this paper is on the existence of a diagonal common quadratic Lyapunov function for a pair of stable positive linear systems. Diagonal quadratic Lyapunov functions play a central role in the study of time-invariant positive linear systems; this is one of the few system classes that is known a-priori to have a diagonal Lyapunov function [3, 2].

A natural question in our context is therefore to ask whether necessary and sufficient conditions for the existence of a common diagonal Lyapunov function can be obtained

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for a pair of positive linear systems. In this paper we provide an affirmative answer to this question and in Theorem 4.2 below derive, in the spirit of the arguments presented in [7], algebraic conditions that are both necessary and sufficient for the existence of a such a function. Roughly speaking our Theorem says that a diagonal common quadratic Lyapunov function exists if and only if a multi-dimensional matrix pencil is non-singular. While the obtained condition is not simple, it does provide considerable insight into the existence of such functions and can provide a basis for simple design laws for switched positive linear systems.

Before proceeding we note that the existence of diagonal Lyapunov functions is an important research area in its own right (several papers have appeared on this topic [8, 9, 10]). For example, these functions arise in the study of decentralised and interconnected systems,[11, 12] (as well as in the study of neural networks [13], and asynchronised systems). The existence question for common diagonal Lyapunov systems arises naturally as the study of such systems extends to interconnections of time-varying and switched systems.

This paper is organised as follows. Definitions and preliminary results are presented in Sections' 2 and 3 and the main result of the paper is given in Section 4. Some applications of the main result are briefly discussed in Section 5 before the conclusions are presented in section 6.

2 Mathematical preliminaries

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. Similarly, for a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} denotes the element in the (i, j) position of A, and $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0(a_{ij} \geq 0)$ for $1 \leq i, j \leq n$. $A \succ B$ ($A \succeq B$) means that $A - B \succ 0$ ($A - B \succeq 0$). We write A^T for the transpose of A. For P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ means that the matrix P is positive definite.

The spectral radius of a matrix A is the maximum modulus of the eigenvalues of A and is denoted by $\rho(A)$. Also we shall denote the maximal real part of any eigenvalue of A by $\mu(A)$. If $\mu(A) < 0$ (all the eigenvalues of A are in the open left half plane) A is said to be *Hurwitz*

Finally for two matrices $A, B \in \mathbb{R}^{n\times k}$ we denote the entrywise or *Hadamard* product [14] of A and B by $A \circ B$. The notion of an irreducible matrix, which we introduce now, shall be important in the sequel.

Irreducible matrices:

A permutation matrix $P \in \mathbb{R}^{n\times n}$ is a matrix with exactly one entry in each row and column equal to one and all other entries zero. For such matrices $P^T = P^{-1}$ and the similarity transformation $A \to PAP^T$ permutes the rows and columns of A in the same way. In particular, diagonal elements are permuted among themselves. (Thus if D is diagonal, so is PDP^T.)

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *reducible* if there exists a permutation matrix P and some k with $1 \leq k \leq n-1$ such that PAP^T has the form

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \tag{1}$$

where $A_{11} \in \mathbb{R}^{k\times k}, A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}, A_{21} \in \mathbb{R}^{(n-k)\times k}$ and 0 is the zero matrix in $\mathbb{R}^{k\times(n-k)}$. If a matrix is not reducible, then it is *irreducible*.

Perron eigenvalues and eigenvectors:

The following classical result forms the starting point for the general theory of irreducible non-negative matrices.

Theorem 2.1 *Suppose $A \in \mathbb{R}^{n\times n}$ is an irreducible non-negative matrix. Then*

(i) $\rho(A) > 0$ is an eigenvalue of A with algebraic multiplicity one;

(ii) There is some vector $x \succ 0$ in $\mathbb{R}^n$ such that $Ax = \rho(A)x$.

This theorem guarantees that the eigenspace of A corresponding to $\rho(A)$ is one-dimensional. For background on irreducible and non-negative matrices consult [2].

Positive LTI systems and Metzler matrices:

The LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$. Basically, if the system starts in the non-negative orthant of $\mathbb{R}^n$, it remain there for all time. See [3] for a description of the basic theory and several applications of positive linear systems.

It is well-known that the system Σ_A is positive if and only if the off-diagonal entries of the matrix A are non-negative. Matrices of this form are known as Metzler matrices. If A is Metzler we can write $A = N - \alpha I$ for some non-negative N and a scalar $\alpha \geq 0$. Note that if the eigenvalues of N are $\lambda_1, \ldots \lambda_n$, then the eigenvalues of $N - \alpha I$ are $\lambda_1 - \alpha, \ldots \lambda_n - \alpha$. Thus the Metzler matrix $N - \alpha I$ is Hurwitz if and only if $\alpha > \rho(N)$.

We now record some fundamental properties of Metzler matrices corresponding to those established by Theorem 2.1 for non-negative matrices. First of all, note that $A = N - \alpha I$ is irreducible if and only if N is irreducible.

Theorem 2.2 *Let $A = N - \alpha I \in \mathbb{R}^{n\times n}$ be Metzler and irreducible. Then*

(i) $\mu(A) = \rho(N) - \alpha$ is an eigenvalue of A of algebraic (and geometric) multiplicity one;

(ii) There is an eigenvector $x \succ 0$ with $Ax = \mu(A)x$.

The next result concerning positive combinations of Metzler Hurwitz matrices was pointed out in [14].

Lemma 2.1 *Let A_1, A_2 be Metzler and Hurwitz. Then $A_1 + \gamma A_2$ is Hurwitz for all $\gamma > 0$ if and only if $A_1 + \gamma A_2$ is non-singular for all $\gamma > 0$.*

In [7], the notions of strong and weak common quadratic Lyapunov functions were defined. We now specialize one of these notions to diagonal functions.

Diagonal common quadratic Lyapunov functions:

Given a set of stable LTI systems $\Sigma_{A_i} : \dot{x}(t) = A_i x(t)$, $A_i \in \mathbb{R}^{n\times n}, 1 \leq i \leq k$, if there exists a diagonal matrix $D > 0$ in $\mathbb{R}^{n\times n}$ such that

$$A_i^T D + DA_i < 0 \quad 1 \leq i \leq k, \tag{2}$$

then we say that $V(x) = x^T Dx$ is a *strong diagonal common quadratic Lyapunov function* (or strong diagonal CQLF) for the family of systems Σ_{A_i}.

Of course, for a family of LTI systems to possess a strong diagonal CQLF, each individual system must have a strong diagonal quadratic Lyapunov function. If each of the systems Σ_{A_i} is positive, this is indeed the case [3, 2]. Finally we note the well-known fact that for a Hurwitz matrix A and a diagonal $D > 0$, $A^T D + DA < 0$ if and only if $A^{-T} D + DA^{-1} < 0$.

3 Some results on Metzler matrices and diagonal Lyapunov functions

In this section we present some technical results that are useful in deriving the principal contribution of this paper. The next lemma is easily checked by direct calculation.

Lemma 3.1 *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix and D a diagonal matrix with non-negative entries. Then $A^T D + DA$ is also a Metzler matrix.*

We now show that if the Metzler matrix A has no zero entries, then $A^T D + DA$ is irreducible for any non-zero diagonal matrix $D \geq 0$.

Lemma 3.2 *Suppose $A \in \mathbb{R}^{n \times n}$ is a Metzler matrix such that $a_{ij} \neq 0$ for $1 \leq i, j \leq n$, and D is a non-zero diagonal matrix with non-negative entries. Then the Metzler matrix $A^T D + DA$ is irreducible.*

Proof: Write $Q = A^T D + DA$. Then if Q is reducible, there is some permutation matrix P such that PQP^T has the form (1) for some $k, 1 \leq k \leq n-1$. It now follows that $(PAP^T)^T(PDP^T) + (PDP^T)(PAP^T) = PQP^T$. Writing $\bar{A} = PAP^T, \bar{D} = PDP^T, \bar{Q} = PQP^T$, we have that $\bar{A}^T \bar{D} + \bar{D}\bar{A}$ has the form (1).

If we now consider the $k \times (n-k)$ zero block in $\bar{Q}$, we see that

$$\begin{array}{rcl} \bar{a}_{1(k+1)}\bar{d}_1 + \bar{a}_{(k+1)1}\bar{d}_{k+1} & = & 0, \bar{a}_{1(k+2)}\bar{d}_1 + \bar{a}_{(k+2)1}\bar{d}_{k+2} = 0, \ldots, \bar{a}_{1n}\bar{d}_1 + \bar{a}_{n1}\bar{d}_n = 0 \\ \bar{a}_{2(k+1)}\bar{d}_2 + \bar{a}_{(k+1)2}\bar{d}_{k+1} & = & 0, \bar{a}_{2(k+2)}\bar{d}_2 + \bar{a}_{(k+2)2}\bar{d}_{k+2} = 0, \ldots, \bar{a}_{2n}\bar{d}_2 + \bar{a}_{n1}\bar{d}_n = 0 \\ \vdots & \vdots & \quad\quad \vdots \quad\quad\quad\quad \vdots \\ \bar{a}_{k(k+1)}\bar{d}_k + \bar{a}_{(k+1)k}\bar{d}_{k+1} & = & 0, \bar{a}_{k(k+2)}\bar{d}_k + \bar{a}_{(k+2)k}\bar{d}_{k+2} = 0, \ldots, \bar{a}_{kn}\bar{d}_k + \bar{a}_{nk}\bar{d}_n = 0. \end{array} \quad (3)$$

Given that all terms of the form $\bar{a}_{ij}\bar{d}_i$ or $\bar{a}_{ij}\bar{d}_j$ where $i \neq j$ have the same sign, it follows that each of these terms in (3) above is zero. As D is not the zero matrix, this implies that $\bar{a}_{ij} = 0$ for some i, j. But this then contradicts the assumption that the original A has no zero entries. Thus if A has no zero entries, $A^T D + DA$ must be irreducible as claimed.

The following result describes a simple necessary condition for two stable LTI systems to have a strong diagonal CQLF. Note the similarity to the necessary conditions previously established for the general CQLF problem (see for instance [7]).

Lemma 3.3 *Let Σ_{A_1}, Σ_{A_2} be two stable LTI systems and assume that they have a strong diagonal CQLF. Then for all non-singular diagonal matrices D in $\mathbb{R}^{n \times n}$ $A_1 + DA_2D$ and $A_1^{-1} + DA_2D$ are Hurwitz and hence non-singular.*

Proof: Firstly, note that if $V(x) = x^T \bar{D} x$ is a strong diagonal CQLF for Σ_{A_1}, Σ_{A_2}, then it is also a strong diagonal CQLF for $\Sigma_{A_1^{-1}}, \Sigma_{A_2}$. Furthermore for any non-singular diagonal D,

$$(DA_2D)^T \bar{D} + \bar{D}(DA_2D) = D(A_2^T \bar{D} + \bar{D}A_2)D < 0.$$

Thus, by congruence $V(x) = x^T \bar{D} x$ would also define a strong diagonal CQLF for Σ_{A_1}, Σ_{DA_2D}, and a strong diagonal quadratic Lyapunov function for $\Sigma_{A_1 + DA_2D}$. This implies that $A_1 + DA_2D$ is Hurwitz and hence non-singular. The identical argument shows that $A_1^{-1} + DA_2D$ is also non-singular.

Finally for this section, as Σ_A has a strong diagonal Lyapunov function if and only if Σ_A, $\Sigma_{A^{-1}}$ have a strong diagonal CQLF, the preceding lemma can be used to derive the following simple necessary conditions for a single stable LTI system to have a diagonal Lyapunov function.

Corollary 3.1 *Let Σ_A be a stable LTI system with $A \in \mathbb{R}^{n\times n}$. Then a necessary condition for Σ_A to have a diagonal quadratic Lyapunov function is that $A + DA^{-1}D$ and $A + DAD$ are Hurwitz for all non-singular diagonal matrices $D \in \mathbb{R}^{n\times n}$.*

4 The main result

In this section we shall consider a pair of stable positive LTI systems Σ_{A_1}, Σ_{A_2}, where A_1, A_2 are Hurwitz Metzler matrices in $\mathbb{R}^{n\times n}$ all of whose entries are non-zero. We shall derive a necessary and sufficient condition for Σ_{A_1}, Σ_{A_2} to have a strong diagonal CQLF. This condition is closely related to the matrix pencil conditions derived for the general CQLF existence question in [7].

The next theorem considers the marginal situation of two systems $\Sigma_{A_1}, \Sigma_{A_2}$ for which there is no strong diagonal CQLF but for which there is a non-zero diagonal $D \geq 0$ satisfying

$$\begin{aligned} A_1^T D + DA_1 &\leq 0 \\ A_2^T D + DA_2 &\leq 0. \end{aligned} \tag{4}$$

A similar situation is considered for the general CQLF problem in [7]. We shall see that in this situation, the necessary conditions of Lemma 3.3 are violated.

Theorem 4.1 *Let Σ_{A_1}, Σ_{A_2} be stable positive LTI systems, where A_1, A_2 are Hurwitz Metzler matrices in $\mathbb{R}^{n\times n}$ with no zero entries. Assume that there is no strong diagonal CQLF for Σ_{A_1}, Σ_{A_2}. Furthermore suppose that there is some non-zero diagonal $D \geq 0$ satisfying (4). Then there is some diagonal matrix $D_0 > 0$ such that $A_1 + D_0A_2D_0$ is singular.*

Proof: First of all, note that from Lemma 3.1 and Lemma 3.2 it follows that $Q_1 = A_1^T D + DA_1$, and $Q_2 = A_2^T D + DA_2$ are irreducible and Metzler. It now follows that $\mu(Q_i) = 0$ is an eigenvalue of algebraic multiplicity one for $i = 1, 2$. Thus the rank of Q_i is $n - 1$ for $i = 1, 2$. Furthermore we can choose vectors $x_1 \succ 0$, $x_2 \succ 0$ such that $Q_1x_1 = 0, Q_2x_2 = 0$.

The next stage in the proof is to show that there can be no diagonal matrix D' with

$$x_1^T D' A_1 x_1 < 0 \tag{5}$$

$$x_1^T D' A_2 x_2 < 0. \tag{6}$$

We shall prove this by contradiction. First of all suppose that there is some D' satisfying (5). We shall show that by choosing $\delta_1 > 0$ sufficiently small, it is possible to guarantee that $A_1^T(D + \delta_1 D') + (D + \delta_1 D')A_1$ is negative definite. Firstly, consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : \|x\| = 1 \text{ and } x^T D' A_1 x \geq 0\}.$$

Note that if the set Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $A_1^T(D + \delta_1 D') + (D + \delta_1 D')A_1$ negative definite. Hence, we assume that Ω_1 is non-empty.

The function that takes x to $x^T D' A_1 x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T D A_1 x$ is strictly negative on Ω_1.

Let M_1 be the maximum value of $x^T D' A_1 x$ on Ω_1, and let M_2 be the maximum value of $x^T D A_1 x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the diagonal matrix

$$D + \delta_1 D'.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $\|x\| = 1$, it follows that for all non-zero vectors x of norm 1

$$x^T(A_1^T(D + \delta_1 D') + (D + \delta_1 D')A_1)x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Since the above inequality is unchanged if we scale x by any non-zero real number, it follows that $A_1^T(D+\delta_1 D')+(D+\delta_1 D')A_1$ is negative definite. As A_1 is Hurwitz, this implies that the matrix $D + \delta_1 D'$ is positive definite.

The same argument can be used to show that there is some $C_2 > 0$ such that

$$x^T(A_2^T(D + \delta_1 D') + (D + \delta_1 D')A_2)x < 0$$

for all non-zero x, for $0 < \delta_1 < C_2$. So, if we choose δ less than the minimum of C_1, C_2, we would have a positive definite diagonal matrix

$$D_1 = D + \delta D'$$

which defined a strong diagonal CQLF for Σ_{A_1} and Σ_{A_2}.

As there is no diagonal solution to (5), (6) it follows that

$$x_1^T D' A_1 x_1 < 0 \iff x_2^T D' A_2 x_2 > 0 \tag{7}$$

for diagonal D'. It follows from this that

$$x_1^T D' A_1 x_1 = 0 \iff x_2^T D' A_2 x_2 = 0.$$

The expressions $x_1^T D' A_1 x_1$, $x_2^T D' A_2 x_2$, viewed as functions of D', define linear functionals on the space of diagonal matrices in $\mathbb{R}^{n\times n}$. Moreover, we have seen that the null sets of these functionals are identical. So they must be scalar multiples of each other. Furthermore, (7) implies that they are negative multiples of each other. So there is some constant $k > 0$ such that

$$x_1^T D' A_1 x_1 = -k x_2^T D' A_2 x_2 \tag{8}$$

for all diagonal D'. In fact, we can take $k = 1$ as we may replace x_2 with $x_2/\sqrt{k}$ if necessary.

On expanding out equation (8) (with $k = 1$) and equating coefficients, it follows that

$$x_1 \circ A_1 x_1 = -x_2 \circ A_2 x_2. \tag{9}$$

Now as $x_1 \succ 0$, $x_2 \succ 0$, there is some diagonal matrix $D_0 > 0$ such that $x_2 = D_0 x_1$. But then, it follows from (9) that $A_2 x_2 = -D_0^{-1} A_1 x_1$ and hence that $(D_0^{-1}A_1 + A_2 D_0)x_1 = 0$. This means that

$$det(A_1 + D_0 A_2 D_0) = det(D_0) det(D_0^{-1} A_1 + A_2 D_0) = 0$$

as claimed.

We can now apply Lemma 3.3 and Theorem 4.1 to derive the main result of the paper.

Theorem 4.2 *Let Σ_{A_1}, Σ_{A_2} be stable positive LTI systems, where A_1, A_2 are Hurwitz, Metzler matrices in $\mathbb{R}^{n\times n}$ with no zero entries. Then a necessary and sufficient condition for $\Sigma_{A_1}, \Sigma_{A_2}$ to have a strong diagonal CQLF is that $A_1 + DA_2D$ is non-singular for all diagonal $D > 0$.*

Proof: The necessity was proven in Lemma 3.3. Now suppose that there is no diagonal CQLF for Σ_{A_1}, Σ_{A_2}. Then,

(i) For $\alpha > 0$ sufficiently large, $\Sigma_{A_1 - \alpha I}$, Σ_{A_2} will have a strong diagonal CQLF.

(ii) If we define $\alpha_0 = inf\{\alpha > 0 : \Sigma_{A_1-\alpha I}, \Sigma_{A_2}$ have a strong diagonal CQLF $\}$, then $\Sigma_{A_1-\alpha_0 I}$, Σ_{A_2} satisfy the conditions of Theorem 4.1.

(iii) It follows that there is some diagonal $D > 0$ such that $A_1 - \alpha_0 I + DA_2D$ is singular.

From item (iii), it follows that $A_1 + DA_2D$ has a positive real eigenvalue. However both A_1 and DA_2D are Hurwitz Metzler matrices, and it follows from Lemma 2.1 that there is some positive $\gamma > 0$ such that $A_1 + \gamma DA_2D$ is singular. Hence, defining $\bar{D} = \sqrt{\gamma}D$, we have that $A_1 + \bar{D}A_2\bar{D}$ is singular. This completes the proof of the theorem.

5 Applications

In this section we present two simple applications of Theorem 4.2. Throughout the section we assume that $\Sigma_{A_1}, \Sigma_{A_2}$ are two stable positive LTI systems such that $A_1, A_2 \in \mathbb{R}^{n \times n}$ have no zero entries.

We note the following easily verifiable facts.

(i) It follows from Corollary 3.1 that $DA_1D + A_1$ is Hurwitz and Metzler for all diagonal $D > 0$.

(ii) If B is any Metzler matrix with $A_1 \succeq B$, then B is also Hurwitz [14].

(iii) If $A_1 \succeq B$, then for any diagonal $D > 0$, $DA_1D + A_1 \succeq DA_1D + B$.

Thus if $A_1 \succeq A_2$, it follows from item (iii) that for all positive diagonal D, $DA_1D+A_1 \succeq DA_1D+A_2$. Hence from (i) and (ii) it follows that DA_1D+A_2 is Hurwitz for all diagonal $D > 0$. Thus applying Theorem 4.2 we have the following known result [15, 16].

Theorem 5.1 *Let Σ_{A_1}, Σ_{A_2} be two positive LTI systems where $A_1, A_2 \in \mathbb{R}^{n \times n}$ have no zero entries and $A_1 \succeq A_2$. Then $\Sigma_{A_1}, \Sigma_{A_2}$ have a strong diagonal CQLF.*

It is in fact possible to slightly strengthen Theorem 5.1 by noting that, for a fixed diagonal $D_1 > 0$, as D ranges over all positive diagonal matrices, so too does $DD_1 = D_1D$. So if we know that $DD_1A_1D_1D + A_2$ is non-singular for all positive diagonal D, then $DA_1D + A_2$ is non-singular for all positive diagonal D. This gives us the following result.

Corollary 5.1 *Let Σ_{A_1}, Σ_{A_2} be two positive LTI systems where $A_1, A_2 \in \mathbb{R}^{n \times n}$ have no zero entries. Suppose that for some diagonal $D_1 > 0$ $D_1A_1D_1 \succeq A_2$. Then $\Sigma_{A_1}, \Sigma_{A_2}$ have a strong diagonal CQLF.*

6 Conclusions

In this paper we have considered the problem of finding a common diagonal Lyapunov function for a pair of linear time-invariant systems. In general this problem is considerably more difficult than the standard common quadratic Lyapunov function existence problem for a pair of LTI systems. This follows from the fact that the problem of determining whether a diagonal Lyapunov function exists for a single system is still open (and is the subject of much research activity). We have shown for pairs of positive LTI systems, which are individually guaranteed to have a diagonal Lyapunov function, it is possible to apply a simple geometric argument to determine necessary and sufficient conditions for a common diagonal Lyapunov function to exist. While the obtained condition is difficult (if not impossible) to verify, examples have been given to show that the result provides a basis for design criteria for switched positive LTI systems. Future research will consider the more general problem of the existence of diagonal Lyapunov functions.

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References

[1] D. Luenberger, *Introduction to Dynamic Systems: Theory, Models, and Applications.* Wiley, 1979.

[2] A. Berman and R. Plemmons, *Non-negative matrices in the mathematical sciences.* SIAM Classics in Applied Mathematics, 1994.

[3] L. Farina and S. Rinaldi, *Positive linear systems.* Wiley Interscience Series, 2000.

[4] T. Haveliwala and S. Kamvar, "The second eigenvalue of the Google matrix," tech. rep., Stanford University, March 2003.

[5] R. Shorten, D. Leith, J. Foy, and R. Kilduff, "'towards an analysis and design framework for congestion control in communication networks," in *Proceedings of the 12th Yale workshop on adaptive and learning systems*, 2003.

[6] A. Jadbabaie, J. Lin, and A. S. Morse, "Co-ordination of groups of mobile autonomous agents using nearest neighbour rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[7] R. N. Shorten, K. S. Narendra, and O. Mason, "On common quadratic Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 110–113, 2003.

[8] N. Oleng and K. Narendra, "On the existence of diagonal solutions to the Lyapunov equation for a third order system," in *Proceedings of American Control Conference*, 2003.

[9] J. Kraaijevanger, "A characterization of Lyapunov diagonal stability using Hadamard products," *Linear Algebra and its Applications*, vol. 151, pp. 245–254, 1991.

[10] G. P. Barker, A. Berman, and R. J. Plemmons, "Positive Diagonal Solutions to the Lyapunov Equations," *Linear and Multilinear Algebra*, vol. 5, no. 3, pp. 249–256, 1978.

[11] D. Siljak, *Decentralized control of complex systems.* Academic Press, San Diego, 1991.

[12] M. Vidyasagar, "New directions of research in nonlinear system theory," *Proceedings of the IEEE*, vol. 74, no. 8, pp. 1060 – 1091, 1986.

[13] E. Kaszkurewicz and A. Bhaya, *Matrix diagonal stability in systems and computation.* Birkhauser, 1999.

[14] R. Horn and C. Johnson, *Topics in matrix analysis.* Cambridge University Press, 1991.

[15] O. Mason and R. Shorten, "A conjecture on the existence of common quadratic Lyapunov functions for positive linear systems," in *Proceedings of American Control Conference*, 2003.

[16] Y. Mori, T. Mori, and Y. Kuroe, "On a class of linear constant systems which have a common quadratic Lyapunov function," in *Proceedings of the IEEE Conference on Decision and Control*, 1998.