COMPONENTS OF THE INVOLUTION MODULE IN BLOCKS WITH CYCLIC OR KLEIN-FOUR DEFECT GROUP

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Abstract. Each p-block of a finite group has an associated defect group, which is a p-subgroup of the group. Each real 2-block has, in addition, an associated *extended defect group*, which is is a 2-subgroup of the group that contains a defect group as a subgroup of index ≤ 2. We consider the possible extended defect groups of a real 2-block that has a cyclic or a Klein-four defect group. In each case we describe the modules in the block that are components in the permutation module of the group acting by conjugation on its involutions. We also determine the Frobenius-Schur indicators of the irreducible characters in the block.

1. Introduction

Let G be a finite group and let $\mathcal{O}$ be a complete discrete valuation ring that has a fraction field F of characteristic 0 and residue field $k := \mathcal{O}/J(\mathcal{O})$ of characteristic $p > 0$. We assume that F and k are splitting fields for all subgroups of G. These are standard assumptions in [15]. Mostly, but not exclusively, $p = 2$. We use $\mathcal{R}$ to denote either of the rings $\mathcal{O}$ or k. Now G acts by conjugation on

$$\Omega := \{g \in G \mid g^2 = 1_G\}.$$

We call the resulting permutation module $\mathcal{R}\Omega$ the *involution module* of G. In this paper we classify the components (indecomposable dirrect summands) of $\mathcal{R}\Omega$ that belong to real 2-blocks of G whose defect groups are cyclic or Klein-four groups. This enables us to determine the Frobenius-Schur indicators of the irreducible characters in such blocks.

As usual G is a right $G \times G$-set via $x\cdot(g_1, g_2) := g_1^{-1}xg_2$, for all $x \in G$ and $g_1, g_2 \in G$. If H is a subgroup of G, then $\Delta H := \{(h, h) \mid h \in H\}$ is the diagonal of H in $G \times G$, and if $h \in H$, then $\mathcal{C}l_H(h)$ is the H-conjugacy class that contains h. We simplify this to $\mathcal{C}l(h)$, if $H = G$.

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The wreath product $G \wr \Sigma$ is a split extension of $G \times G$ by the cyclic group $\Sigma = \langle \sigma \rangle$ of order 2. Here $(g_1, g_2)\sigma = \sigma(g_2, g_1)$, for all $g_1, g_2 \in G$. We extend G to a right $G \wr \Sigma$-set by defining $x \cdot \sigma := x^{-1}$, for all $x \in G$ (see [13]). The map $g \to (g, g^{-1})\sigma$, establishes a $G \wr \Sigma$-set isomorphism between G and the $G \wr \Sigma$-conjugacy class $\mathcal{C}l_{G\wr\Sigma}(\sigma)$ of σ. So $\mathcal{R}G$ is isomorphic to the induced module $(\mathcal{R}_{\Delta G\times\Sigma})\uparrow^{G\wr\Sigma}$, as $\mathrm{C}_{G\wr\Sigma}(\sigma) = \Delta G \times \Sigma$.

Throughout this paper B is a p-block of G. This means that B is a $G \times G$-component of $\mathcal{R}G$. According to J. A. Green, there is a p-subgroup D of G such that ΔD is a vertex of B, as $G \times G$-module. This D is also a *defect group* of B, in the sense of R. Brauer. There is a set $\mathrm{Irr}(B)$ of irreducible complex G-characters and a set of $\mathcal{R}G$-modules attached to B. If M is a $\mathcal{R}G$-module, we let MB denote the sum of all submodules of M that belong to B. In particular

$$\mathcal{R}\Omega B \text{ is the sum of all submodules of } \mathcal{R}\Omega \text{ that belong to } B.$$

The *contragradient block* B^o is the p-block of G that is the image of B under σ. Thus $B = B + B^o$ is a $G \wr \Sigma$-component of $\mathcal{R}G$. We let $\hat{B}$ be the p-block of $G \wr \Sigma$ that contains this component. The block B^o contains the complex conjugates of the irreducible characters in B and the duals of the $\mathcal{R}G$-modules in B. We say that B is *real* if $B = B^o$.

For the rest of this section $p = 2$ and B is a real 2-block of G. So B is a $G \wr \Sigma$-component of $\mathcal{R}G$. By [14, Lemma 10], there exists $e \in \mathrm{N}_G(D)$, with $e^2 \in D$, such that $\Delta D\langle \Delta e\, \sigma\rangle$ is a vertex of B as $G \wr \Sigma$-module. We call the 2-group $E := D\langle e\rangle$ an *extended defect group* of B. If B is the principal 2-block of G then $E = D$; otherwise $[E : D] = 2$.

We justify the adjective 'extended' as follows: There is a conjugacy class of G which occurs in the support of the block idempotent of B and on which the central character of B does not vanish. Any such class is called a *defect class of* B. It is known that there exists $g \in G$ in a defect class of B such that the defect group D is a Sylow 2-subgroup of the centralizer $\mathrm{C}_G(g)$ of g in G. According to [10], B has a real defect class. By [14] there exists $g \in G$ in this class such that E is a Sylow 2-subgroup of the *extended centralizer* $\mathrm{C}^*_G(g)$ of g in G. Here $\mathrm{C}^*_G(g) := \{x \in G \mid x^{-1}gx = g^{\pm 1}\}$.

The Frobenius-Schur indicator of a generalized character χ of G is the integer $\epsilon(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$. Suppose that χ is irreducible. Then $\epsilon(\chi) \in \{0, \pm 1\}$. Moreover $\epsilon(\chi) = 0$, if χ is not real-valued, and $\epsilon(\chi) = +1$ if χ is the character of a real representation. Otherwise $\epsilon(\chi) = -1$. The structure of E affects the Frobenius-Schur indicators of the irreducible characters in B, as the following two lemmas indicate.

Lemma 1.1. *B has an irreducible character with Frobenius-Schur indicator -1 and height zero if and only if E/D' does not split over D/D'.*

Proof. This is a result of R. Gow. See Theorem 5.6 in [9]. □

Lemma 1.2. *The following are equivalent:*

(i) *E splits over D;*
(ii) *$\mathcal{R}\Omega B \neq 0$;*
(iii) $\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(1) \neq 0$.

Proof. This is Theorem 2 in the author's paper [14]. □

We call B a *strongly real* 2-block of G, if it satisfies these conditions.
We shall prove a number of general results in Section 2, in particular the following two theorems about real 2-blocks:

Theorem 1.3. *Suppose that B is strongly real. Then there is a component of $\mathcal{R}\Omega B$ that has vertex $\mathrm{C}_D(x)$, for some $x \in \Omega$ such that $E = D\langle x\rangle$. Moreover, each component of $\mathcal{R}\Omega B$ has a vertex contained in $\mathrm{C}_D(y)$, for some $y \in \Omega$ such that $E = D\langle y\rangle$.*

In the terminology of Brauer, a *root* of B (in $D\,\mathrm{C}_G(D)$) is a 2-block β of $D\,\mathrm{C}_G(D)$ such that $\beta^G = B$. Each root of B has defect group D, and the roots of B form a single $\mathrm{N}_G(D)$-orbit under conjugation.

Theorem 1.4. *B has a real root if and only if $E \leq D\,\mathrm{C}_G(D)$.*

The remainder of the paper concentrates on real 2-blocks that have a cyclic or Klein-four defect group. For such blocks we have:

Lemma 1.5. *Suppose that D is cyclic or a Klein-four group. Then B has an irreducible character with Frobenius-Schur indicator -1 if and only if E does not split over D.*

Proof. This follows from Lemma 1.1 and the fact that all irreducible characters in B have height zero, as detailed below. □

E. C. Dade described the decomposition matrices, and much of the character theory of blocks with cyclic defect group in [6]. In particular, if D is cyclic, then there is a unique irreducible kG-module S in B, and there are $|D|$ ordinary irreducible G-characters in B. It is also known that for each $i = 1, 2, \dots, |D|$, there is a unique indecomposable $\mathcal{R}G$-module S_i in G such that S_i has i composition factors. The subgroup of index $\gcd(|D|, i)$ in D is a vertex of S_i.
Here are names for 2-groups with maximal subgroups that are cyclic:

- $D_{2^n} := \langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{-1}\rangle$ is a dihedral group;

- $Q_{2^n} := \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1}\rangle$ is a generalized quaternion group;
- $SD_{2^n} := \langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1}\rangle$ is a semidihedral group;
- $M_{2^n} := \langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}+1}\rangle$ is a modular group.

The term 'modular 2-group' is used occasionally in finite group theory. A recent example can be found in [8].

Here is our main result about real 2-blocks with a cyclic defect group:

Theorem 1.6. *If D is cyclic, then one of the following is true:*

(i) *E is cyclic. Two irreducible characters in B are real-valued; one of these has Frobenius-Schur indicator -1.*
(ii) *$E \cong D \times \mathbb{Z}_2$. Two irreducible characters in B are real-valued and $k\Omega B \cong S \oplus S$.*
(iii) *E is a dihedral group. All irreducible characters in B are real-valued and $k\Omega B \cong S_{|D|/2} \oplus S_{|D|/2}$.*
(iv) *E is a semi-dihedral group. Half of the irreducible characters in B are real-valued and $k\Omega B \cong S_{|D|/2}$.*
(v) *E is a modular group. Two irreducible characters in B are real-valued and $k\Omega B \cong S_2$.*
(vi) *E is a generalized quaternion group. All irreducible characters in B are real-valued; half of these have Frobenius-Schur indicator -1.*

We use $\mathrm{rad}(M)$ to denote the radical and $\mathrm{soc}(M)$ to denote the socle of a kG-module M. The *Loewy series* of M is $M \supset \mathrm{rad}(M) \supset \mathrm{rad}^2(M) \supset \dots$. We use P_M to denote the projective cover of M. If $A, B, \dots$ are irreducible modules, then $U(A, B, \dots)$ is a *uniserial module* with successive Loewy factors $A, B, \dots$.

Now suppose that D is a Klein-four group. Then $D = \langle s, t\rangle$, where s, t and st are involutions. R. Brauer showed in [4, Section 7] that B has four ordinary irreducible characters and either one or three irreducible modules. K. Erdmann [7, Theorem 4] Green correspondence and Auslander-Reiten theory to describe the possible decomposition matrices of B.

We list the irreducible characters in B as $\chi_1, \chi_2, \chi_3, \chi_4$. Let β be a root of B and let $I = \{n \in \mathrm{N}_G(D) \mid \beta^n = \beta\}$ be the inertial group of β in $\mathrm{N}_G(D)$. Then $[I : \mathrm{C}_G(D)]$ is an odd integer, called the *inertial index* of B. If the inertial index is 1 then B has a single irreducible module S; if the inertial index is 3 then B has three irreducible modules S, X and Y. In the latter case s, t and st are conjugate in G and $\mathrm{C}_G(s)$ has a unique 2-block b_1 such that $b_1^G = B$. Moreover, b_1 has inertial index

1. Let ψ_1 denote the unique irreducible Brauer character in b_1. We obtain the following by combining results from [4] and [7].

Lemma 1.7. *If D is a Klein-four group, there are three block types:*

(I) *B has inertial index 1 and $\mathrm{rad}(P_S)/S \cong S \oplus S$. All decomposition numbers are 1;*

(II) *B has inertial index 3 and*

$$\mathrm{rad}(P_S)/S \cong X \oplus Y,\ \mathrm{rad}(P_X)/X \cong Y \oplus S,\ \mathrm{rad}(P_Y)/Y \cong S \oplus X.$$

Thus $D_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

There is a sign $\delta = \pm 1$ such that for all 2-regular elements $g \in \mathrm{C}_G(s)$, we have $\chi_1(gs) = \chi_3(gs) = \chi_4(gs) = \delta\psi_1(g)$ and $\chi_2(gs) = -\delta\psi_1(g)$;

(III) *B has inertial index 3 and*

$$\mathrm{rad}(P_S)/S \cong U(X,S,Y) \oplus U(Y,S,X),$$
$$\mathrm{rad}(P_X)/X \cong U(S,Y,S), \quad \mathrm{rad}(P_Y)/Y \cong U(S,X,S).$$

Thus $D_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

There is a sign $\delta = \pm 1$ such that for all 2-regular elements $g \in \mathrm{C}_G(s)$, we have $\chi_1(gs) = \chi_2(gs) = \delta\psi_1(g)$ and $\chi_3(gs) = \chi_4(gs) = -\delta\psi_1(g)$.

We give our main results for real 2-blocks with Klein-four defect groups in three theorems:

Theorem 1.8. *Let B be of type (I). Then one of the following holds:*

(i) *$E \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. All four irreducible characters in B are real-valued; two of these have Frobenius-Schur indicator -1.*

(ii) *$E \cong \mathbb{Z}_2^3$. All four irreducible characters in B are real-valued and $k\Omega B \cong S^4$;*

(iii) *$E \cong D_8$. Two irreducible characters in B are real-valued and $k\Omega B \cong U(S,S)$ is indecomposable with a vertex of order 2.*

Theorem 1.9. *Let B be of type (II). Then one of the following holds:*

(i) *$E \cong \mathbb{Z}_2^3$. Two irreducible characters in B are real-valued and $k\Omega B \cong S^2 \oplus X \oplus Y$;*

(ii) $E \cong D_8$. *All four irreducible characters in B are real-valued and $k\Omega B$ is indecomposable with a vertex of order* 2. *Moreover* $\mathrm{hd}(k\Omega B) \cong \mathrm{rad}(k\Omega B) \cong S \oplus X \oplus Y$.

Theorem 1.10. *Let B be of type (III). Then one of the following holds:*

(i) $E \cong \mathbb{Z}_2^3$. *All four irreducible characters in B are real-valued and $k\Omega B \cong S^2 \oplus U(X, S, Y) \oplus U(Y, S, X)$;*

(ii) $E \cong D_8$. *Two irreducible characters in B are real-valued and $k\Omega B$ is indecomposable with a vertex of order* 2. *Moreover* $\mathrm{hd}(k\Omega B) \cong \mathrm{soc}(k\Omega B) \cong S$ *and* $\mathrm{rad}(k\Omega B)/\mathrm{soc}(k\Omega B) \cong X \oplus Y$.

2. Preliminary results

In this section we describe some results about blocks that do not depend on the isomorphism type of their defect groups. Generally B is a p-block of G and D is a defect group of B. If $p = 2$, and B is a real 2-block, then E is an extended defect group of B that contains D. Each $g \in G$ can be written uniquely as $g = g_p g_{p'} = g_{p'} g_p$, for g_p a p-element in G and $g_{p'}$ a p'-element in G. In fact, $g_p, g_{p'} \in \langle g \rangle$.

Lemma 2.1. *Suppose that $D \trianglelefteq G$ and that B has a real root and inertial index* 1. *Then the number of real irreducible characters in B equals the number of real irreducible characters of D.*

Proof. Let β be a real root of B. By [15, 5.8.14], there is a unique irreducible Brauer character ψ in β, and the irreducible characters in β are $\{X_\lambda \mid \lambda \in \mathrm{Irr}(D)\}$. Here if $\lambda \in \mathrm{Irr}(D)$,

$$X_\lambda(g) = \begin{cases} \psi(g_{p'})\lambda(g_p), & \text{if } g_p \in D, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } g \in D\,\mathrm{C}_G(D).$$

Since ψ is real, the character X_λ is real if and only if λ is real.

Let $g \in G \backslash D\mathrm{C}_G(D)$. Then by hypothesis $\beta^g \neq \beta$. So $X_\lambda^g \neq X_\lambda$. It follows that $X_\lambda{\uparrow}^G$ is an irreducible character in B. Moreover, the set of irreducible characters in B is $\{X_\lambda{\uparrow}^G \mid \lambda \in \mathrm{Irr}(D)\}$. Clearly $X_\lambda{\uparrow}^G$ is real if X_λ is real. Suppose that X_λ is not real. Then X_λ and its complex conjugate $\overline{X_\lambda}$ both belong to β. It follows that they are not G-conjugate. So $X_\lambda{\uparrow}^G$ is not real. The Lemma follows from this. □

We state without proof the next two results.

Lemma 2.2. *Let $N \trianglelefteq G$, let M be a kN-module and let L be a kG/N-module. Then $M{\uparrow}^G \otimes L \cong \dim(L)\, M{\uparrow}^G$.*

Lemma 2.3. *Let $p = 2$ and let $I \leq H \leq G$ with $[H : I] = 2$. Then there is a short exact sequence of $\mathcal{R}G$-modules*

$$0 \to \mathcal{R}_H{\uparrow}^G \to \mathcal{R}_I{\uparrow}^G \to \mathcal{R}_H{\uparrow}^G \to 0.$$

Thus the composition multiplicity of an irreducible kG-module in $k_I{\uparrow}^G$ is twice its multiplicity in $k_H{\uparrow}^G$.

A *p-permutation* $\mathcal{R}G$-module is an $\mathcal{R}G$-module whose restriction to a p-subgroup of G is a permutation module. An indecomposable $\mathcal{R}G$-module is a p-permutation module if and only if it has a source module that is trivial. By [15, 4.8.9] if M is a p-permutation kG-module then there is a unique p-permutation $\mathcal{O}G$-module $\tilde{M}$ such that $M = \tilde{M} \otimes_{\mathcal{O}} k$. We refer to the character of $\tilde{M}$ as the *$\mathcal{O}$-character* of M.

Lemma 2.4. *Let $\mathcal{X}$ be a G-set and let V be a p-subgroup of G. Green correspondence establishes a multiplicity preserving correspondence:*

$$\{G\text{-components of } \mathcal{R}\mathcal{X} \text{ that have vertex } V\}$$
$$\updownarrow$$
$$\{\mathrm{N}_G(V)/V\text{-components of } \mathcal{R}\,\mathrm{C}_{\mathcal{X}}(V) \text{ that are projective}\}.$$

If $M \leftrightarrow N$ under this correspondence, and M belongs to B, then N belongs to a p-block b of $\mathrm{N}_G(V)$ such that $b^G = B$.

Proof. This is Theorem 3.2 of [5]. □

Lemma 2.5. *Let M be an indecomposable p-permutation $\mathcal{R}G$-module let χ_M be the $\mathcal{O}$-character of M, and let g be a p-element in G. Then*

$$\chi_M(g) \begin{cases} \in \mathbb{Z}_+, & \text{if } g \text{ belongs to some vertex of } M; \\ = 0, & \text{otherwise.} \end{cases}$$

Proof. The character value $\chi_M(g)$ is the number of trivial components of $M{\downarrow}_{\langle g \rangle}$. See [12] for details. □

The following is a variant of Lemma 9.7 of [1].

Lemma 2.6. *Let M be an indecomposable p-permutation $\mathcal{R}G$-module and let H be a subgroup of G. Let V be a vertex of M such that $V \cap H$ is not a proper subgroup of $U \cap H$, for any vertex U of M. Then some component of $M{\downarrow}_H$ has vertex $V \cap H$.*

Proof. By hypothesis $\mathcal{R}_V$ is a component of $M{\downarrow}_V$. So $\mathcal{R}_{V\cap H}$ is a component of $M{\downarrow}_{V\cap H}$. Choose a component S of $M{\downarrow}_H$ such that $\mathcal{R}_{V\cap H}$ is a component of $S{\downarrow}_{V\cap H}$. Now $V\cap H$ is a vertex of $\mathcal{R}_{V\cap H}$. So $V \cap H \leq W$, for some vertex W of S, by Lemma 4.3.4 of [15]. Similarly $W \leq U$, for some vertex U of M. Thus $V\cap H \leq W \leq U \cap H$. The assumption on $V \cap H$ forces $V \cap H = U \cap H$. We conclude that $V \cap H = W$. □

The Frobenius automorphism of k is given by $\mathrm{Fr}(\lambda) = \lambda^p$, for all $\lambda \in k$. If M is a kG-module, then we can apply Fr to the entries in the matrices representing the elements of G in $\mathrm{GL}(M)$, to get a

kG-module M^{Fr} called the *Frobenius twist* of M. If M has Brauer character ϕ, then M^{Fr} has Brauer character ϕ^{Fr} where $\phi^{\mathrm{Fr}}(g) = \phi(g^p)$, for all p-regular $g \in G$. Clearly M is indecomposable/irreducible if and only if M^{Fr} is indecomposable/irreducible. Moreover, as Fr commutes with induction, if M is indecomposable then M and M^{Fr} share the same vertices. Let B^{Fr} be the p-block of G that contains the Frobenius twists of all the kG-modules in B. Then B and B^{Fr} have the same defect classes and defect groups. Moreover $\mathrm{Irr}(B^{\mathrm{Fr}}) = \{\chi^{\mathrm{Fr}} \mid \chi \in \mathrm{Irr}(B)\}$, where $\chi^{\mathrm{Fr}}(g) := \chi(g_p g_{p'}^p)$, for all $g \in G$.

Lemma 2.7. *There is a multiplicity and vertex preserving correspondence between the components of $\mathcal{R}\Omega B$ and the components of $\mathcal{R}\Omega B^{\mathrm{Fr}}$.*

Proof. This follows from the discussion above and the fact that $k\Omega = (k\Omega)^{\mathrm{Fr}}$. □

Lemma 2.8. *Suppose that B is a real 2-block of G. Then*

$$B{\downarrow_{\Delta G\times\Sigma}} = \mathcal{R}\Omega B^{\mathrm{Fr}} \oplus X,$$

where no component of X has a vertex that contains Σ.

Proof. Here we are identifying the 2-blocks of G and $\Delta G \times \Sigma$. Now $kG{\downarrow_{\Delta G\times\Sigma}} = k\Omega \oplus k(G\backslash\Omega)$, where $k\Omega$ is centralized by Σ, and no component of $k(G\backslash\Omega)$ has a vertex that contains Σ. Suppose that M is a component of the inflation of $k\Omega B^{\mathrm{Fr}}$ to $\Delta G \times \Sigma$. Then M has a vertex that contains Σ. But $\mathrm{C}_{G\wr\Sigma}(\Sigma) = \Delta G \times \Sigma$, and $(B^{\mathrm{Fr}})^{G\wr\Sigma} = \hat{B}$, by Lemma 16 of [14]. It then follows from Lemma 3.7a of [11] that M is a direct summand of $kG\hat{B} = B$. □

Proof of Theorem 1.3. Choose $e \in E$ so that $E = D\langle e\rangle$ and set $H := \Delta G\times\Sigma$. Each vertex of B is ΔG-conjugate to $(\Delta D\langle\Delta e\sigma\rangle)^{(1,y)}$, for some $y \in G$. Suppose that such a vertex contains σ. Then there exists $d_0 \in D$ such that $\sigma = (\Delta(d_0e)\sigma)^{(1,y)}$. But $(\Delta(d_0e)\sigma)^{(1,y)} = (d_0ey, y^{-1}d_0e)\sigma$. So $y = d_0e$, whence $E = D\langle y\rangle$, and $y \in \Omega$. Moreover

$$(1) \qquad (\Delta D\langle\Delta e\sigma\rangle)^{(1,y)} \cap H = \Delta\,\mathrm{C}_D(y) \times \Sigma.$$

Let $x \in \Omega$ be such that $E = D\langle x\rangle$ and $\mathrm{C}_D(x) \not< \mathrm{C}_D(y)$, for all $y \in \Omega$ such that $E = D\langle y\rangle$. Set $V := (\Delta D\langle\Delta e\sigma\rangle)^{(1,x)}$. Then V is a vertex of B, and (1) implies that $V \cap H \not< U \cap H$, for any vertex U of B. Lemma 2.6 shows that there is a component of $B{\downarrow_{\Delta G\times\Sigma}}$ that has vertex $V \cap H = \Delta\,\mathrm{C}_D(x) \times \Sigma$. We conclude from Lemma 2.8 that $\mathcal{R}\Omega B^{\mathrm{Fr}}$ has a component that has vertex $\mathrm{C}_D(x)$. The first statement now follows from Lemma 2.7.

Let M be a component of $\mathcal{R}\Omega B$, as $\mathcal{R}G$-module, and let $W \leq D$ be a vertex of M. Then M has vertex $\Delta W \times \Sigma$, as $\mathcal{R}\Delta G \times \Sigma$-module.

Lemma 2.8 implies that M^{Fr} is a component of $B{\downarrow_{\Delta G\times\Sigma}}$. Using (1), there exists $y\in E$ such that $E=D\langle y\rangle$ and $W\leq \mathrm{C}_D(y)$. This proves the second statement. □

The vertices of components of $\mathcal{R}\Omega$ also have a lower bound:

Lemma 2.9. *Suppose that $p=2$. Then each projective component of $\mathcal{R}\Omega$ belongs to a real 2-block with a trivial defect group.*

Proof. This is the main result of [13]. □

We will make use of the following special result in Sections 4 and 5:

Lemma 2.10. *Suppose that B is a strongly real 2-block such that $D\trianglelefteq G$ and $E\leq D\,\mathrm{C}_G(D)$. Set $Z:=\Omega(\mathrm{Z}(D))$. Consider kZ as $Z\rtimes G/D$-module, with Z acting regularly and G/D acting by conjugation. Then there is a self-dual irreducible kG-module T in B such that the module*

$$\sum_{i\geq 0} T\otimes \mathrm{rad}^i(kZ)/\mathrm{rad^{i+1}(kZ)}$$

is a direct summand of $k\Omega B$, and all its components have vertex D.

Proof. We identify the 2-blocks of G, ΔG and $\Delta G\times\Sigma$. The group G acts by conjugation on Z, and $D\leq \mathrm{C}_G(Z)$. So there is a well-defined action of G/D on Z. The group Z acts regularly on kZ. In this way kZ is a module for the semi-direct product group $Z\rtimes G/D$.

Choose $e\in\Omega_{\mathrm{C}_G(D)}$ such that $E=D\times\langle e\rangle$. Then $(\Delta e\sigma)^{(1,e)}=\sigma$, and $\Delta D^{(1,e)}=\Delta D$. So $V:=\Delta D\times\Sigma$ is a vertex of B. Now

$$\Delta G\times\Sigma\;\leq\;\mathrm{N}_{G\wr\Sigma}(V)\;\leq\;\mathrm{N}_{G\wr\Sigma}(\Delta D)\;=\;\Delta G(\mathrm{C}_G(D)\wr\Sigma).$$

Let $c\in\mathrm{C}_G(D)$ be such that $(1,c)$ normalizes V. Then $\sigma^{(1,c)}=(c,c^{-1})\sigma$ belongs to V. This forces $c\in Z$. So $\mathrm{N}_{G\wr\Sigma}(V)=\Delta G(Z\wr\Sigma)$.

By Lemma 2.4, there is a component fB of $B{\downarrow_{\mathrm{N}_{G\wr\Sigma}(V)}}$ that has vertex V and kernel containing V. In particular all components of $fB{\downarrow_{\Delta G\times\Sigma}}$ have vertex V. So by Lemma 2.8, the module $fB{\downarrow_{\Delta G}}$ is a direct summand of $k\Omega B^{\mathrm{Fr}}$, and all of its components have vertex D.

Now $\mathrm{N}_{G\wr\Sigma}(V)/V\cong Z\rtimes G/D$. So we can regard fB as a projective indecomposable $Z\rtimes G/D$-module. Let S be the irreducible socle of fB. Then S is self-dual, as B and hence fB are self-dual. Proposition 18.4 of [1] implies that

$$fB{\downarrow_{\Delta G/\Delta D}}\;\cong\;\sum_{i\geq 0} S\otimes_k \mathrm{rad}^i(kZ)/\mathrm{rad}^{i+1}(kZ).$$

The result now follows from Lemma 2.7, and the fact that $kZ=(kZ)^{\mathrm{Fr}}$. □

The following result is similar to Lemma 13.7 in [1].

Lemma 2.11. *Let P be a p-subgroup of G. Then*

$$B{\downarrow}_{P\,\mathrm{C}_G(P)\wr\Sigma} \quad = \quad \sum(b+b^o)\ \oplus\ X,$$

where b ranges over the p-blocks of $P\,\mathrm{C}_G(P)$ such that $b^G = B$ and no component of X has a vertex that contains ΔP.

Proof. Set $H := P\,\mathrm{C}_G(P)$. Then $kG{\downarrow}_{H\wr\Sigma} = kH \oplus k(G\backslash H)$, as $kH \wr \Sigma$-modules. Now ΔP does not centralize any element of $G\backslash H$. So by Lemma 2.4, no component of $k(G\backslash H)$ has a vertex that contains ΔP.

The $kH \wr \Sigma$-components of kH have the form $b + b^o$, where b is a p-block of H. Let b be such a block and let $\hat{b}$ be the p-block of $H \wr \Sigma$ that contains $b + b^o$. As P is a normal subgroup of H, it is contained in a defect group of b, by [1, 13.6]. So ΔP is contained in a vertex of $b + b^o$, as $kH \wr \Sigma$-module. Now $\mathrm{C}_{G\wr\Sigma}(\Delta P) = \mathrm{C}_G(P) \wr \Sigma \leq H \wr \Sigma$. So $\hat{b}^{G\wr\Sigma} = \hat{B}$, by Lemma 3.7a of [11]. It follows from this that $b^G = B$. □

The $\mathcal{O}$-character χ_B of B is computed in Lemma 8 of [14] as:

$$(2)\qquad \begin{aligned} \chi_B\left((g_1,g_2)\right) &= \sum_{\chi\in\mathrm{Irr}(B)} \chi(g_1^{-1})\chi(g_2), \\ \chi_B\left((g_1,g_2)\sigma\right) &= \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(g_1g_2), \end{aligned} \qquad \text{for all } g_1,g_2\in G.$$

Our next result appears as Theorem (4B) in R. Brauer's paper [3]:

Corollary 2.12. *Let $g \in G$. Then*

$$(3)\qquad \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(g) \quad = \quad \sum_{b}\sum_{\hat{\chi}\in\mathrm{Irr}(b)} \epsilon(\hat{\chi})\hat{\chi}(g),$$

where b ranges over the p-blocks of $\mathrm{C}_G(g_p)$ such that $b^G = B$.

Proof. We claim that the p-part of $(g,1)\sigma \in G\wr\Sigma$ generates a group that contains Δg_p. Suppose first that p is odd. There is a unique $s \in \langle g_p\rangle$ such that $s^2 = g_p$. Then Δs is a p-element and $\langle \Delta g_p\rangle = \langle \Delta s\rangle$. Also $(\Delta s)^{-1}(g,1)\sigma$ is a p'-element, because its square is $(g_{p'}, g_{p'})\sigma$. So Δs is the p-part of $(g,1)\sigma$. This proves the claim when p is odd. Conversely, suppose that $p = 2$. Then $(g,1)\sigma$ has 2-part $(g_2,1)\sigma$. So the claim follows from the fact that $\Delta g_2 = ((g_2,1)\sigma)^2$.

With $P = \langle g_p\rangle$, Lemma 2.11 gives

$$B{\downarrow}_{\mathrm{C}_G(g_2)\wr\Sigma} = \sum_{b^G=B}(b+b^o)\ \oplus\ X,$$

where no component of X has a vertex that contains Δg_p. It then follows from the previous paragraph, and Theorem 4.7.4 of [15], that $\chi_B((tu,1)\sigma)) = \sum_{b^G=B}\chi_b((g,1)\sigma))$. Equation (3) is a consequence of this and (2). □

The following is a variant of [14, Corollary 15]:

Lemma 2.13. *Suppose that B is a real 2-block of G. Let t be a 2-element in G. Then $\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)\chi(t)$ is a nonnegative integer, which is positive if and only if t is G-conjugate to e^2, for some $e \in E$ such that $E = D\langle e\rangle$.*

Proof. The element $(t,1)\sigma \in G \wr \Sigma$ has 2-power order, as its square is the 2-element $\Delta t \in G \times G$. By (2), $\chi_B((t,1)\sigma) = \sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)\chi(t)$. So Lemma 2.5 implies that $\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)\chi(t)$ is a non-negative integer.

Suppose that $(t,1)\sigma$ belongs to a vertex of B. Then there exist $g_1, g_2 \in G$ and $e \in E$ such that $E = D\langle e\rangle$ and $((t,1)\sigma)^{(g_1,g_2)} = (e,e)\sigma$. Thus $g_1^{-1}tg_2 = g_2^{-1}g_1 = e$, whence $e^2 = t^{g_1}$ is G-conjugate to t.

Conversely, suppose that $e^2 = t^{g_1}$, where $g_1 \in G$ and $E = D\langle e\rangle$. Then $((t,1)\sigma)^{(g_1,g_1e^{-1})} = (e,e)\sigma$. So $(t,1)\sigma$ belongs to a vertex of B. The result now follows from Lemma 2.5. □

Lemma 2.14. *Let $p = 2$, let P be a principal indecomposable kG-module and let Φ be the $\mathcal{O}$-character of P. Then the multiplicity of the irreducible module $P/\mathrm{rad}(P)$ as a composition factor of $k\Omega$ is $\epsilon(\Phi)$.*

Proof. See [14, Lemma 3] or [16, Lemma 1]. □

A *subgroup pair* of G is a pair (H,I) such that $H \leq I \leq G$ and $[I:H] = 2$. We say that a pair (J,K) is a *subpair* of (H,I) if $J \leq H$ and $KH = I$. More generally, (J,K) is G-conjugate to a subpair of (H,I) if there exists $g \in G$ such that (J^g,K^g) is a subpair of (H,I). Clearly G-conjugacy is a transitive relation on subgroup pairs.

If B is a real non-principal 2-block of G then (D,E) is a subgroup pair of G. The G-conjugates of (D,E) are called the *defect pairs* of B.

Lemma 2.15. *Suppose that B is a real 2-block of G. Let $H \leq G$ and let b be an admissible real non-principal 2-block of H such that $B = b^G$. Then each defect pair of b is a subpair of a defect pair of B. In particular, if b is strongly real then B is strongly real, while if D is a defect group of b, then each defect pair of b is a defect pair of B.*

Proof. If $X \subseteq G$, set $X^+ := \sum_{x\in X}x$, in kG. Let ω_B, ω_b be the central character of B, respectively b. Let C be a real defect class of B. Then $\omega_b((C\cap H)^+) = \omega_B(C^+) \neq 0_k$. We pair the contribution of each H-conjugacy class in $C\cap H$ to $\omega_b((C\cap H)^+)$ with that of its inverse class in $C \cap H$. In this way we see that there is a real H-conjugacy class $C_1 \subseteq C \cap H$ such that $\omega_b(C_1^+) \neq 0_k$. Let $g \in C_1$, let D_H be a Sylow 2-subgroup of $\mathrm{C}_H(g)$ and let E_H be a Sylow 2-subgroup of $\mathrm{C}_H^*(g)$ that contains D_H. Theorem 2.1 in [10] implies that some defect pair (D_b, E_b) of b is a subpair of (D_H, E_H). But $g \in C$. So (D_H, E_H) is G-conjugate to a subpair of (D, E). It follows that (D_b, E_b) is G-conjugate to a subpair of (D, E).

Suppose that b is strongly real. Then C_1 is a strongly real class of H. So $C \supseteq C_1$ is a strongly real class of G. Thus B is strongly real.

Finally, suppose that D is a defect group of b. Then D_b is a Sylow 2-subgroup of $\mathrm{C}_G(g)$ and E_b is a Sylow 2-subgroup of $\mathrm{C}_G^*(g)$. It follows that (D_b, E_b) is a defect pair of B. □

Proof of Theorem 1.4. Suppose that B has a real root β. Let (D, E_1) be a defect pair of β. Lemma 2.15 implies that (D, E_1) is a defect pair of B. So there exists an element $g \in G$ in a defect class of B such that D is a Sylow 2-subgroup of $\mathrm{C}_G(g)$ and E_1 is a Sylow 2-subgroup of $\mathrm{C}_G^*(g)$. But then g is a real element in $\mathrm{C}_G(D)$.

Conversely, suppose that $E \leq D\,\mathrm{C}_G(D)$. Lemma 2.11 gives

$$B{\downarrow}_{D\,\mathrm{C}_G(D)\wr\Sigma} = \sum(\beta_1 + \beta_1^o) \oplus X,$$

where β_1 ranges over the roots of B, and no component of X has a vertex that contains ΔD. Suppose that $E = D\langle e\rangle$. Then $\Delta D\langle\Delta e\sigma\rangle$ is a vertex of B, as $G \wr \Sigma$-module. The hypothesis is that $\Delta D\langle\Delta e\sigma\rangle \leq D\,\mathrm{C}_G(D) \wr \Sigma$. It then follows from Lemma 2.6 that there is a root β_1 of B such that $\beta_1 + \beta_1^o$ has vertex $\Delta D\langle\Delta e\sigma\rangle$. But then β_1 is a real 2-block of $D\,\mathrm{C}_G(D)$. We deduce that so too is its $\mathrm{N}_G(D)$-conjugate β. □

3. Cyclic Defect group

In this section B is a real 2-block of G that has a cyclic defect group. We follow Dade's original approach from [6]. Recall the notation introduced in Section 1. Let d be a generator for the defect group D and suppose that $|D| = 2^n$. Then $t := d^{2^{n-1}}$ is the unique involution in D. Let χ be the Brauer character of the irreducible B-module S. So χ is real-valued. We note that $P := S_{2^n}$ is the unique projective indecomposable B-module and $S_{2^{n-1}}$ is the unique indecomposable B-module that has $\langle t\rangle$ as a vertex.

Set $H := \mathrm{C}_G(t)$. As $H \geq \mathrm{N}_G(D)$, Brauer's first main theorem implies that there is a unique block b of H such that $b^G = B$, and D is a defect group of b. As b is unique it is also real, and Lemma 2.15 implies that each of its defect pairs is a defect pair of B. Let T be the unique irreducible kH-module in b. We may label the indecomposable

kH-modules in b as $T := T_1, T_2, \dots, T_{2^n}$, where T_i is uniserial with i composition factors, each isomorphic to T.

Lemma 3.1. *Let V be a nontrivial subgroup of D and let f be the Green correspondence with respect to (G, V, H). Then $f(S_i) = T_i$ or $f(S_i) = T_{2^n-i}$, for each $i = 1, 2, \dots, 2^n$.*

Proof. As usual $\overline{\mathrm{End}}_{kG}(M)$ is the quotient of $\mathrm{End}_{kG}(M)$ by the subspace of projective endomorphisms, for any module M. We make use of [1, 21.3, 21.4], and the fact that B and b have inertial index 1.

Let $1 \leq i \leq 2^{n-1}$ and let θ be a non-zero G-endomorphism of S_i. As $i + i - 2^n \leq 0$, and as S_i has composition length i, [1, 21.3] implies that θ is not a projective homomorphism. Thus $\mathrm{End}_{kG}(S_i) \cong \overline{\mathrm{End}}_{kG}(S_i)$. We deduce from this that $\overline{\mathrm{End}}_{kG}(S_i)$ has dimension i as k-vector space. The Heller translate of S_i is S_{2^n-i}. So $\overline{\mathrm{End}}_{kG}(S_{2^n-i}) \cong \overline{\mathrm{End}}_{kG}(S_i)$, by [1, 20.6]. Thus $\overline{\mathrm{End}}_{kG}(S_{2^n-i})$ also has dimension i as k-vector space. The same results hold for $\overline{\mathrm{End}}_{kG}(T_i)$ and $\overline{\mathrm{End}}_{kG}(T_{2^n-i})$.

Now let $1 \leq i \leq 2^n$. Then $\overline{\mathrm{End}}_{kG}(S_i) \cong \overline{\mathrm{End}}_{kG}(fS_i)$ by [1, Corollary 4]. The result follows from this and the previous paragraph. □

We can label the irreducible characters in B as X_λ, where λ ranges over $\mathrm{Irr}(D)$. The characters X_λ with $\lambda \neq 1$ are called the *exceptional characters* in B. The trivial character X_1 is the unique *non-exceptional character* in B. If $d_{\lambda,\chi}$ is the multiplicity of χ in X_λ, then $d_{\lambda,\chi} = 1$, for all λ. We use a dash, as in χ', X'_λ, for other objects associated to b.

Lemma 3.2. *If $\lambda \in \mathrm{Irr}(D)$, then X_λ is real if and only if X'_λ is real.*

Proof. The automorphism group of D is a 2-group. But B has odd inertial index. So $\mathrm{C}_G(D)$ is the inertial subgroup of a root of B in $\mathrm{N}_G(D)$.

If $g \in G$ is 2-regular then $X_\lambda(g) = \chi(g)$ and $X'_\lambda(g) = \chi'(g)$ are both real-valued. If $g \in G$ is such that g_2 is G-conjugate to an element of $D \backslash \{1\}$ then Corollary 1.9 of [6] implies that $X_\lambda(g) = \pm X'_\lambda(g)$ (the symbols $\mathrm{C}_i, \mathrm{N}_i$ and ϕ_i in the statement of that Corollary are the same for B and b, while the symbols ε_i and γ_i are signs). If $g \in G$ is such that g_2 is not G-conjugate to an element of D then by a theorem of J. A. Green, $X_\lambda(g) = 0$ and $X'_\lambda(g) = 0$. The lemma follows easily from these facts. □

Let X be a component of $k\Omega B$ and let V be a vertex of X that is contained in D. As $D \neq \langle 1 \rangle$, Lemma 2.9 implies that $\langle t \rangle \leq V$. Thus $\mathrm{N}_G(V) \leq H$. Let f be the Green correspondance with respect to (G, V, H). Then by Lemma 2.4, fX is a component of $k\Omega(H)$ that belongs to b and that has kernel containing $\langle t \rangle$.

By the results of section 5.8.3 of [15], there is a unique 2-block $\overline{b}$ of $\overline{H} := H/\langle t\rangle$ that is dominated by b. Now $\overline{b}$ is real. Let $g \in H$ belong to a real defect class of b. Then g is 2-regular. The proof of Lemma 5.8.9 of [15] can be adapted to show that $\mathrm{C}^*_{\overline{H}}(\overline{g}) = \overline{\mathrm{C}^*_H(g)}$. So we may assume that $(\overline{D}, \overline{E})$ is a defect pair of $\overline{b}$.

Lemma 3.3. *Suppose that B is not the principal block. Let $s \in \Omega$ be such that $k\mathcal{C}l(s)B \neq 0$. Then some G-conjugate of s lies in $E\backslash D$.*

Proof. The proof is by induction on $|E|$. The base case is $D = \langle 1\rangle$ and E is cyclic of order 2. In that case the conclusion follows from Theorem 19 of [14].

By Lemma 2.4 we may assume that $s \in H$, and that $k_{\mathrm{C}_H(s)}\!\uparrow^H$ has a componext X that belongs to b. Then X can be regarded as a component of $k_{\overline{\mathrm{C}_H(s)}}\!\uparrow^{\overline{H}}$ that belongs to $\overline{b}$. Now $\overline{\mathrm{C}_H(s)}$ is a subgroup of index ≤ 2 in $\mathrm{C}_{\overline{H}}(\overline{s})$. Thus $k\,\mathrm{C}_{\overline{H}}(\overline{s})\!\uparrow^{\overline{H}}$ has a composition factor of $\overline{b}$, using Lemma 2.3. Now $k\,\mathrm{C}_{\overline{H}}(\overline{s})\!\uparrow^{\overline{H}}$ is a direct summand of $k\Omega(\overline{H})$, as $\overline{H}$-modules. Moreover, $\overline{b}$ is a real 2-block with cyclic defect group $\overline{D}$ of smaller order than D. By our inductive assumption, $\overline{s}$ is $\overline{H}$-conjugate to an element of $\overline{E}\backslash\overline{D}$. It follows from this that s is H-conjugate to an element of $E\backslash D$. □

Abelian extended defect group

We prove parts (i) and (ii) of Theorem 1.6. So assume that $E \leq \mathrm{C}_G(D)$. Theorem 1.4 implies that b has a real root β. As D has two real-valued linear characters, Lemma 2.1 implies that b has two real-valued irreducible characters. We deduce from Lemma 3.2 that B has two real-valued irreducible characters. As all decomposition numbers in B are one, both of these characters have the same degree.

Suppose that E is a cyclic group. Lemma 1.2 implies that $k\Omega B = 0$ and $\sum_{\lambda\in\Lambda} \epsilon(X_\lambda)X_\lambda(1) = 0$. So one of the real-valued irreducible characters in B has Frobenius-Schur indicator $+1$ and the other has indicator -1.

Now suppose that $E \cong D \times \mathbb{Z}_2$. Lemma 2.14 implies that $k\Omega B$ has two composition factors that are isomorphic to S. But Theorem 1.3 implies that $k\Omega B$ has at least one component with vertex D. The only possibility is that $k\Omega B = S \oplus S$. The $\mathcal{O}$-character of $k\Omega B$ is the sum of the two real irreducible characters in B.

Note that $k\Omega(H)b = T \oplus T$. This can be used to show that T is the Green correspondent of S with respect to (G, D, H).

To prove the other parts of Theorem 1.6, we need some information on the elements $s \in \Omega$ such that $k\mathcal{C}l(s)B \neq 0$. In case B is the principal 2-block of G, we have $E = D$, $T = k_H$, and $k\mathcal{C}l(1_H)B = k_H$ and $k\mathcal{C}l(t)B = k_H$.

Suppose that B is not the principal 2-block of G. Then $E = D \times \langle e\rangle$, where e is an involution. Moreover, e and te are the only involutions in $E\backslash D$. We claim that S occurs twice as a component of $k\mathcal{C}l(e)$, if e and te are conjugate in G. Otherwise it occurs once as a component of each of $k\mathcal{C}l(e)$ and $k\mathcal{C}l(te)$. For, b covers a unique real 2-block of H/D that has defect pair $(D/D, E/D)$. By Theorem 19 of [14], S occurs once as a component of $k_{\mathrm{C}_{H/D}(eD)}{\uparrow^{H/D}}$. Now e and te are the only involutions in $E\backslash D$. So $\mathrm{N}_H\langle t, e\rangle$ is the inverse image of $\mathrm{C}_{H/D}(eD)$ in H. We deduce that S occurs once as a component of $k_{\mathrm{N}_H\langle t,e\rangle}{\uparrow^H}$.

Say that e is H-conjugate to te. Then $[\mathrm{N}_H\langle t, e\rangle : \mathrm{C}_H(e)] = 2$. Lemma 2.3 implies that T occurs twice as a composition factor of $k\mathcal{C}l_H(e)$. But then $k\mathcal{C}l_H(e)b = T \oplus T$, by our work above. It then follows from Lemma 2.4 that $k\mathcal{C}l(e)B = S \oplus S$,

Say that e is not H-conjugate to te. Then $\mathrm{N}_H\langle t, e\rangle = \mathrm{C}_H(e)$. So $k\mathcal{C}l(e)b = T$, and by the same argument, $k\mathcal{C}l(te)b = T$. If e is G-conjugate to te, then Lemma 2.4 implies that $k\mathcal{C}l(e)B = S \oplus S$. Otherwise, Lemma 2.4 implies that $k\mathcal{C}l(e)B = S$ and $k\mathcal{C}l(de)B = S$.

Dihedral or semi-dihedral extended defect group

We prove parts (iii) and (iv) of Theorem 1.6. So assume that E is dihedral or semi-dihedral. As E splits over D, all real irreducible characters in B have Frobenius-Schur indicator $+1$. Let $E = D\langle e\rangle$, where $e^2 = 1$. Then $E\backslash D$ is a union of two conjugacy classes of E:

$$\{d^{2m}e \mid m = 0, 1, \ldots\} \quad \text{and} \quad \{d^{2m+1}e \mid m = 0, 1, \ldots\}.$$

Now $\mathrm{C}_D(xe) = \langle t\rangle$, for each $x \in D$. As $\langle t\rangle$ is a minimal nontrivial subgroup of D, Corollary 18 of [14] and Lemma 2.9 imply that each B-component of $k\Omega$ has vertex $\langle t\rangle$, and hence that each such component is isomorphic to $S_{2^{n-1}}$. Similarly, each b-component of $k\Omega_H$ is isomorphic to $T_{2^{n-1}}$. It then follows from Lemma 2.4 that $T_{2^{n-1}}$ is the Green correspondent of $S_{2^{n-1}}$ with respect to (G, D, H).

We use induction on $|E|$ to prove that $S_{2^{n-1}}$ occurs twice as a component of $k\mathcal{C}l(e)$, if e and de are conjugate in G. Otherwise it occurs once as a component of each of $k\mathcal{C}l(e)$ and $k\mathcal{C}l(de)$. The base case $|E| = 4$ is covered by the abelian extended defect group case.

We can apply our inductive hypothesis to $\overline{b}$, as $(\overline{D}, \overline{E})$ is a defect pair of $\overline{b}$, $\overline{E}$ is a dihedral group, and $|\overline{E}| < |E|$. Note that e and te are conjugate in E, since t is an even power of d. So the inverse image $\mathrm{N}_H\langle t, e\rangle$ of $\mathrm{C}_{\overline{H}}(\overline{e})$ in H contains $\mathrm{C}_H(e)$ as a subgroup of index 2.

Suppose first that e is H-conjugate to de. Then $\overline{e}$ is $\overline{H}$-conjugate to $\overline{de}$. By our inductive hypothesis, $T_{2^{n-2}}$ occurs twice as a component of $k_{\mathrm{C}_{\overline{H}}(\overline{e})}{\uparrow}^{\overline{H}}$. The inflation of the latter module to H is $k_{\mathrm{N}_H\langle t,e\rangle}{\uparrow}^{H}$. Thus T has multiplicity $2^{n-1} = 2\times 2^{n-2}$ as a composition factor of $k_{\mathrm{N}_H\langle t,e\rangle}{\uparrow}^{H}$. Lemma 2.3 then implies that T has multiplicity $2^n = 2\times 2^{n-1}$ as a composition factor of $k_{\mathrm{C}_H(e)}{\uparrow}^{H} = k\mathcal{C}l_H(e)$. It follows that $k\mathcal{C}l_H(e)$ has two components that are isomorphic to $T_{2^{n-1}}$. We conclude from Lemma 2.4 that $S_{2^{n-1}}$ occurs twice as a component of $k\mathcal{C}l(e)$.

Suppose that e is not H-conjugate to de. Then $\overline{e}$ is not $\overline{H}$-conjugate to $\overline{de}$. By our inductive hypothesis, $T_{2^{n-2}}$ occurs once as a component of each of $k_{\mathrm{C}_{\overline{H}}(\overline{e})}{\uparrow}^{\overline{H}}$ and $k_{\mathrm{C}_{\overline{H}}(\overline{de})}{\uparrow}^{\overline{H}}$. Repeating the argument above, $T_{2^{n-1}}$ occurs once as a component of each of $k\mathcal{C}l_H(e)$ and $k\mathcal{C}l_H(de)$. Suppose that e is G-conjugate to de. Then Lemma 2.4 implies that $S_{2^{n-1}}$ occurs twice as a component $k\mathcal{C}l(e)$. Otherwise, Lemma 2.4 implies that $S_{2^{n-1}}$ occurs once as a component of each of $k\mathcal{C}l(e)$ and $k\mathcal{C}l(de)$.

If E is dihedral, both e and te are involutions. Then $k\Omega$ has two components that belong to B. Both are isomorphic to $S_{2^{n-1}}$. It then follows from Lemma 2.14 that all irreducible characters in B are real. On the other hand, if E is semi-dihedral, only e is an involution. Then $k\Omega$ has one component that belongs to B. This component is isomorphic to $S_{2^{n-1}}$. It then follows from Lemma 2.14 that exactly half the irreducible characters in B are real.

Modular extended defect group

We prove part (v) of Theorem 1.6. So assume that E is a modular 2-group i.e. that $E = D\langle e\rangle$ where $e^2 = 1$ and $d^e = td$. Then e and te are conjugate in E and are the only involutions in $E\backslash D$. As $\mathrm{C}_D(e) = \langle d^2\rangle$, Theorem 1.3 implies that some B-component of $k\Omega$ has vertex $\langle d^2\rangle$.

Let s be one of the two elements of order 4 in D. Then se and tse both have order 4. So $\overline{e}$ and $\overline{se}$ are involutions in $\overline{E}$ that are not conjugate in $\overline{H}$. As $\overline{b}$ has abelian extended defect group $\overline{E}$s, our proof of part (i) of Theorem 1.6 implies that T occurs once as a component of $k_{\mathrm{C}_{\overline{H}}(\overline{e})}{\uparrow}^{\overline{H}}$.

The inverse-image of $\mathrm{C}_{\overline{H}}(\overline{e})$ in H is $\mathrm{N}_H\langle t,e\rangle$. So T occurs once as a component of $k_{\mathrm{N}_H\langle t,e\rangle}{\uparrow}^{H}$. As $[\mathrm{N}_H\langle t,e\rangle : \mathrm{C}_H(e)] = 2$, Lemma 2.3 implies that T occurs twice as a composition factor of $k\mathcal{C}l_H(e)$. In view of Lemma 3.3, this accounts for all b-composition factors in $k\Omega_H$. Now T_2 is the only indecomposable b-module that has at most two composition factors and vertex $\langle d^2\rangle$. So by the first paragraph, T_2 occurs once as a component of $k\mathcal{C}l_H(e)$. Lemma 2.4 then implies that fT_2 occurs once as a component of $k\mathcal{C}l(e)$. Here fT_2 is the Green correspondent of T_2 with respect to (G, D, H).

The previous paragraph and Lemma 2.14 imply that b has two real irreducible characters. So by Lemma 3.2 B also has two real irreducible characters. So S occurs with multiplicity 2 as a composition factor of $k\Omega$. In particular the B-component fT_2 of $k\mathcal{C}l(e)$ has at most 2 composition factors. We deduce from this and Lemma 3.1 that $fT_2 = S_2$. Thus S_2 is the unique component of $k\Omega$ that belongs to B.

Generalized quaternion extended defect group

We prove part (vi) of Theorem 1.6. So assume that E is a generalized quaternion group i.e. $E = D\langle e\rangle$ where $e^2 = t$ and $d^e = d^{-1}$. Since E does not split over D, Lemma 1.2 implies that $k\Omega B = 0$ and $\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(1) = 0$. Since all irreducible characters in B have the same degree, it follows that B has an equal number of irreducible characters with Frobenius-Schur indicator $+1$ and -1. The same result holds for b.

Now $\overline{b}$ has defect pair $(\overline{D}, \overline{E})$ and $\overline{E}$ is a dihedral group. We deduce from part (iii) of Theorem 1.6 that all 2^{n-1} irreducible characters in $\overline{b}$ are real with Frobenius-Schur indicator $+1$. So the remaining 2^{n-1} irreducible characters in b are real with Frobenius-Schur indicator -1.

Lemma 3.2 and the previous paragraph show that all irreducible characters in B are real. We conclude from the first paragraph that half the irreducible characters in B have Frobenius-Schur indicator $+1$ and the other half have Frobenius-Schur indicator -1.

4. Type (I) blocks with Klein-four defect group

We will prove Theorem 1.8 in this section. So assume that B is a real 2-block of G with Klein-four defect group D that is of type (I), and E is an extended defect group of B that contains D.

Recall the notation of Section 1. From the decomposition matrix of B, all four irreducible characters in B have the same degree. Set $b := \beta^{\mathrm{N}_G(D)}$. Then b is the Brauer correspondent of B. There is a unique irreducible $k\,\mathrm{C}_G(D)$-module T_0 in β. As $\mathrm{C}_G(D)$ is the inertial group of T_0 in $\mathrm{N}_G(D)$, the induced module $T := T_0{\uparrow}^{\mathrm{N}_G(D)}$ is the unique irreducible $k\,\mathrm{N}_G(D)$-module in b.

According to [4, VII], the subsections of B are $(\pi_1, b_1), (\pi_2, b_2), (\pi_3, b_3)$ and $(1, B)$. Here π_i is a 2-element and b_i is a 2-block of $\mathrm{C}_G(\pi_i)$ such that $b_i^G = B$. Moreover, for each i there exists an element $n_i \in G$ such that $b_i = (\beta^{n_i})^{\mathrm{C}_G(\pi_i)}$. Each b_i has a unique irreducible Brauer character

ψ_i, and the generalized decomposition number of each χ_j with respect to (π_i, ψ_i) is ± 1.

Abelian extended defect group

We prove part (i) of Theorem 1.8. So assume that $E \leq \mathrm{C}_G(D)$. Then β is a real block, by Theorem 1.4. Thus b_1, b_2 and b_3 are real blocks. We deduce from this and the fact that the generalized decomposition numbers of B are ± 1 that all four irreducible characters in B are real-valued.

Suppose that E does not split over D. Then Lemma 1.2 implies that $k\Omega B = 0$ and $\sum_{i=1}^4 \epsilon(\chi_i)\chi_i(1) = 0$. It follows that two of the irreducible characters in B have Frobenius-Schur indicator $+1$, while the other two have Frobenius-Schur indicator -1.

Suppose then that E splits over D. We apply Lemma 2.10 to $\mathrm{N}_G(D)$ and its 2-block b. Thus the $k\,\mathrm{N}_G(D)$-module

$$M := \sum_{i \geq 0} T \otimes_k \mathrm{rad}^i(kD)/\mathrm{rad}^{i+1}(kD)$$

is a direct summand of $k\Omega_{\mathrm{N}_G(D)}b$ and each of its components has vertex D. The factors $\mathrm{rad}^i(kD)/\mathrm{rad}^{i+1}(kD)$ are inflated from $\mathrm{C}_G(D)$ and T is induced from a $k\,\mathrm{C}_G(D)$-module. It then follows from Lemma 2.2 that $M \cong T^4$.

Let fT be the kG-module that is the Green correspondent of T with respect to $(G, D, \mathrm{N}_G(D))$. Then fT^4 is a direct summand of $k\Omega B$, by Lemma 2.4. But according to Lemma 2.14, the multiplicity of S as a composition factor of $k\Omega$ is $\sum_{i=1}^4 \epsilon(\chi_i) \leq 4$. It follows that $\epsilon(\chi_i) = +1$, for $i = 1, 2, 3, 4$, and also that $fT = S$ and $k\Omega B = S^4$. Finally, the $\mathcal{O}$-character of $k\Omega B$ is 4χ, where χ is the unique irreducible character in B which takes a positive integer value on each element of D. This completes the proof of part (i) of Theorem 1.8.

D_8 extended defect group

We prove part (ii) of Theorem 1.8. So assume that $E \cong D_8$. Write $E = \langle s, t, e\rangle$ and $D = \langle s, t\rangle$, where s, t and e are involutions, and $s^e = s$ and $t^e = st$. Then $\mathrm{C}_D(xe) = \langle s\rangle$, for all $x \in D$. So by Theorem 1.3 and Lemma 2.9, every component of $k\Omega B$ has vertex $\langle s\rangle$.

Recall that χ_B is the $G \wr \Sigma$-character of the lift of B to characteristic zero. Since $(te)^2 = s$, Lemma 2.13 that

$$\sum_{i=1}^{4} \epsilon(\chi_i)\chi_i(s) \qquad \text{is a positive integer.} \tag{4}$$

Suppose we are in Case (I)1 of Lemma 1.7. Then s, t and st are conjugate in G and the subsections of B are $(s, b_1), (s, b_2), (s, b_3)$ and $(1, B)$. Equation 7.2_1 of [4] shows that

$$\begin{aligned}\sum_{i=1}^{4} \epsilon(\chi_i)\chi_i(s) \quad &= \quad \psi_1(s)(\epsilon(\chi_1) + \epsilon(\chi_2) - \epsilon(\chi_3) - \epsilon(\chi_4)) \\ &\quad + \psi_2(s)(\epsilon(\chi_1) - \epsilon(\chi_2) + \epsilon(\chi_3) - \epsilon(\chi_4)) \\ &\quad + \psi_3(s)(\epsilon(\chi_1) - \epsilon(\chi_2) - \epsilon(\chi_3) + \epsilon(\chi_4)).\end{aligned}$$

It then follows from (4) that at least two of $\chi_1, \chi_2, \chi_3, \chi_4$ are not real-valued. But every real 2-block has at least one real-valued irreducible character. So we may assume that χ_1 and χ_2 are real, while $\chi_3 = \overline{\chi_4}$.

Suppose we are in Case (I)2 of Lemma 1.7. Then s and t are not conjugate in G and the subsections of B are $(s, b_1), (t, b_2), (t, b_3)$ and $(1, B)$. Equation 7.2_1 of [4] shows that

$$\sum_{i=1}^{4} \epsilon(\chi_i)\chi_i(s) = \psi_1(s)(\epsilon(\chi_1) + \epsilon(\chi_2) - \epsilon(\chi_3) - \epsilon(\chi_4))$$

It then follows from (4) that at least two of the irreducible characters in B are not real-valued. So just as before we may assume that χ_1 and χ_2 are real-valued, while $\chi_3 = \overline{\chi_4}$.

In either of the above two cases B has two real-valued irreducible characters and the decomposition matrix of B is a 4×1-column of 1's. It then follows from Lemma 2.14 that S occurs twice as a composition factor of $k\Omega$. Now S is not a component of $k\Omega$, as it has vertex D. Thus $k\Omega B$ is an indecomposable uniserial module $U(S, S)$ whose vertex is $\langle s \rangle$. The $\mathcal{O}$-character of $k\Omega B$ can be taken to be $\chi_1 + \chi_2$.

Similar arguments show that exactly one of b_1, b_2, b_3 is a real block.

Case (I)3 of Lemma 1.7 does not occur, as t and st are conjugate in E. This completes the proof of part (ii) of Theorem 1.8.

5. Type (II) and (III) blocks with Klein-four defect group

We will prove Theorems 1.9 and 1.10 in this section. So assume that B is a real 2-block of G with Klein-four defect group D that is of type (II) or (III), and E is an extended defect group of B that contains D. Recall the notation of Section 1. Lemma 1.7 shows that B has three irreducible modules S, X and Y. In both types we may assume that S is a self-dual module. It is then clear from the decomposition matrices given in Lemma 1.7 that χ_1 and χ_2 are real-valued irreducible characters. Then $\epsilon(\chi_1) = \epsilon(\chi_2) = +1$, and $\epsilon(\chi_i) \geq 0$, for $i = 3, 4$, by Lemma 1.5.

Set $b = \beta^{\mathrm{N}_G(D)}$. Then b is the Brauer correspondent of B in $\mathrm{N}_G(D)$. Now β has a unique irreducible module T_0. The inertial group I of

β in $\mathrm{N}_G(D)$ equals that of T_0, and $I/\mathrm{C}_G(D)$ is cyclic of order 3. It follows that T_0 has three extensions to I, which we label as S_I, X_I and Y_I. Then $S_1 := S_I{\uparrow}^{\mathrm{N}_G(D)}$, $X_1 := X_I{\uparrow}^{N_G(D)}$ and $Y_1 := Y_I{\uparrow}^{\mathrm{N}_G(D)}$ are the three irreducible $k\,\mathrm{N}_G(D)$-modules that belong to b. In particular b is of type (II) or (III). We may assume that S_I and S_1 are self-dual.

Abelian extended defect group

We prove part (i) of Theorems 1.9 and 1.10. Assume that $E \leq \mathrm{C}_G(D)$. We claim that in this case E splits over D. Theorem 1.4 implies that β is a real 2-block of $\mathrm{C}_G(D)$. So T_0 is a self-dual $k\,\mathrm{C}_G(D)$-module. We may choose notation so that S_I is a self-dual kI-module. Then $S_I \otimes U = X_I$ and $S_I \otimes U^* = Y_I$, where U is a non-trivial 1-dimensional $kI/\mathrm{C}_G(D)$-module. In particular $Y_1 = X_1^*$. So S_1 is the only self-dual irreducible b-module.

Now b is a real block of type (II) or (III), with defect pair (D, E). Let $\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3, \hat{\chi}_4$ be the irreducible characters in b, with the notation chosen so that S_1, X_1 and Y_1 appear in the modular decomposition of $\hat{\chi}_2$, while X appears in the modular decomposition of $\hat{\chi}_3$ and Y appears in the modular decomposition of $\hat{\chi}_4$. It is clear that $\hat{\chi}_1$ and $\hat{\chi}_2$ are real-valued. As Y_1 is the dual of X_1, the characters $\hat{\chi}_3$ and $\hat{\chi}_4$ form a complex conjugate pair.

Let Φ_X be the principal indecomposable character corresponding to X_1. Then

$$\epsilon(\Phi_X) = \epsilon(\hat{\chi}_2) + \epsilon(\hat{\chi}_3) = \epsilon(\hat{\chi}_2) \neq 0.$$

So $\epsilon(\hat{\chi}_2) = +1$. Similarly $\epsilon(\Phi_Y) = 1$ and $\epsilon(\Phi_S) = 2$ and $\epsilon(\hat{\chi}_1) = +1$. It then follows from Lemma 2.14 that S_1 occurs twice and each of X_1 and Y_1 occurs once, as a composition factor of $k\Omega_{\mathrm{N}_G(D)}$. We then deduce from Lemma 1.2 that E splits over D, which proves our claim.

Regard kD as a $k\,\mathrm{N}_G(D)/\mathrm{C}_G(D)$-module, via the conjugation action of $\mathrm{N}_G(D)$ on D. Then

(5) $kD/\mathrm{rad}(kD)$ and $\mathrm{rad}(kD)/\mathrm{rad}^2(kD)$ are trivial $\mathrm{N}_G(D)$-modules.

Now by direct calculation, $\mathrm{rad}(kD)/\mathrm{rad}^2(kD) \cong U \oplus U^*$, as $I/\mathrm{C}_G(D)$-modules. Moreover, if $\mathrm{N}_G(D)/\mathrm{C}_G(D) \cong \Sigma_3$, then $\mathrm{rad}(kD)/\mathrm{rad}^2(kD) \cong U{\uparrow}^{\mathrm{N}_G(D)}$, as $k\,\mathrm{N}_G(D)/\mathrm{C}_G(D)$-modules. We note also that

$$\begin{aligned} S_I{\uparrow}^{\mathrm{N}_G(D)} \otimes U{\uparrow}^{\mathrm{N}_G(D)} &\cong (S_I \otimes U){\uparrow}^{\mathrm{N}_G(D)} \oplus (S_I \otimes U^*){\uparrow}^{\mathrm{N}_G(D)} \\ &\cong X_I{\uparrow}^{\mathrm{N}_G(D)} \oplus Y_I{\uparrow}^{\mathrm{N}_G(D)}. \end{aligned}$$

So whether $\mathrm{N}_G(D)/\mathrm{C}_G(D) \cong \mathbb{Z}_3$ or Σ_3, we have

$$S_1 \otimes \mathrm{rad}(kD)/\mathrm{rad}^2(kD) \cong X_1 \oplus Y_1. \qquad (6)$$

In the hypothesis of Lemma 2.10, take the group to be $\mathrm{N}_G(D)$ and the block to be b. As S_1 is the only self-dual irreducible b-module, we conclude from (5), (6) and Lemma 2.10 that

$$S_1 \oplus S_1 \oplus X_1 \oplus Y_1$$

is a direct summand of $k\Omega_{\mathrm{N}_G(D)}b$.

Let f be the Green correspondence with respect to $(G, D, \mathrm{N}_G(D))$. Lemma 2.4 and the previous paragraph imply that

(7) $\quad fS_1 \oplus fS_1 \oplus fX_1 \oplus fY_1$ is a direct summand of $k\Omega B$.

Now s is not a square in E, as E is an elementary abelian 2-group. It then follows from Lemma 2.13 that

$$\sum_{i=1}^{4} \epsilon(\chi_i)\chi_i(s) = 0. \tag{8}$$

Suppose that B is of type (II). Then (8) and Lemma 1.7 imply that

$$\delta\psi_1(1)\left(\epsilon(\chi_1) - \epsilon(\chi_2) + \epsilon(\chi_3) + \epsilon(\chi_4)\right) = 0.$$

Thus $\overline{\chi}_3 = \chi_4$. Considering the decomposition matrix of B, Lemma 2.14 implies that S occurs twice, and both X and Y occur once, as composition factors of $k\Omega$. It then follows from (7) that $fS_1 = S$, and fX_1 and fY_1 are distinct irreducible B-modules, and

$$k\Omega B = S \oplus S \oplus X \oplus Y.$$

The character of the lift of $k\Omega B$ to characteristic zero is $2\chi_1 + \chi_3 + \chi_4$. Notice that χ_3 and χ_4 are non-real constituents of the involution module.

Suppose that B is of type (III). Then (8) and Lemma 1.7 imply that

$$\delta\psi_1(1)\left(\epsilon(\chi_1) + \epsilon(\chi_2) - \epsilon(\chi_3) - \epsilon(\chi_4)\right) = 0.$$

Thus $\epsilon(\chi_3) = \epsilon(\chi_4) = +1$, and hence X and Y are self-dual modules. Considering the decomposition matrix of B, Lemma 2.14 implies that S occurs four times, and both X and Y occur twice, as composition factors of $k\Omega$.

From the structure of the projective indecomposable B-modules, each indecomposable B-module that is not irreducible contains S as a composition factor. Now $fX_1^* = fY_1$, as $X_1^* = Y_1$. In particular neither fX_1 nor fY_1 is irreducible. So both fX_1 and fY_1 contain S as a composition factor. It follows from this that S occurs at most two times as a composition factor of $fS_1 \oplus fS_1$.

Now X is not a direct summand of a permutation module, as it is not liftable to an $\mathcal{R}G$-module. In particular $fS_1 \ncong X$. We claim that X is not a submodule of fS_1. Suppose otherwise. Then X is also a quotient module of fS_1, as fS_1 and X are self-dual. It follows that X occurs at least twice as a composition factor of fS_1. This is impossible, as X occurs at most two times as a composition factor of $fS_1 \oplus fS_1$. Our claim follows. In the same way, Y is not a submodule of fS_1.

The previous paragraph implies that S is a submodule of fS_1. We claim that $fS_1 = S$. Suppose otherwise. Then by duality S occurs at least twice as a composition factor of fS_1. This is impossible, as S occurs at most two times as a composition factor of $fS_1 \oplus fS_1$. Our claim follows.

We claim that S is not a submodule of fX_1. For otherwise there is a non-projective map in $\mathrm{Hom}(fS_1, fX_1)$. It follows from the Green correspondence theorem that there is a nonzero homomorphism $S_1 \to X_1$. But this is nonsense, as S_1 and X_1 are non-isomorphic irreducible $\mathrm{N}_G(D)$-modules. This proves our claim. By a similar argument, S is not a factor module of fX_1.

The previous paragraph means that we may choose notation so that X is a submodule of fY_1. Then by duality, X is a factor module of fX_1. This accounts for all occurrences of X as a composition factor of $k\Omega$. By exhaustion, Y is a submodule of fX_1 and a factor module of fY_1. This accounts for all occurrences of Y as a composition factor of $k\Omega$. We now know that $X = \mathrm{hd}(fX_1) = \mathrm{soc}(fY_1)$ and $Y = \mathrm{hd}(fY_1) = \mathrm{soc}(fX_1)$. We deduce from this that $fX_1 = U(X, S, Y)$ and $fY_1 = U(Y, S, X)$.

We have shown that

$$k\Omega B = S \oplus S \oplus U(X, S, Y) \oplus U(Y, S, X).$$

The $\mathcal{O}$-character of S is χ_1. The $\mathcal{O}$-character of $U(X, S, Y)$ takes a positive integer value at s. So it must equal χ_2. Similarly $U(Y, S, X)$ has $\mathcal{O}$-character χ_2.

D_8 extended defect group

We prove part (ii) of Theorems 1.9 and 1.10. So suppose that $E \cong D_8$. Then $E = D : \langle e \rangle$, where e has order 2, $s^e = s$ and $t^e = st$. Clearly $\mathrm{C}_D(xe) = \langle s \rangle$, for all $x \in D$. Theorem 1.3 then implies that every component of $k\Omega B$ has vertex $\langle s \rangle$. Moreover, as $s = (te)^2$, Lemma 2.13 implies that

$$(9) \qquad \sum_{i=1}^{4} \epsilon(\chi_i)\chi_i(s) \qquad \text{is a positive integer.}$$

According to [4], there is a unique block b_1 of $\mathrm{C}_G(s)$ such that $b_1^G = B$. Moreover, b_1 has defect group D and extended defect group E. As s and t are not conjugate in $\mathrm{C}_G(s)$, the block b_1 is of type (I). Let T

be the unique irreducible b_1-module and let ψ_1 be the Brauer character of T. Then $k\Omega_{C_G(s)}b_1 = U(T,T)$, by Theorem 1.8. Lemma 2.4 now implies that $k\Omega B$ is the Green correspondent of $U(T,T)$. In particular $k\Omega B$ is indecomposable.

By [2, 6.6.3] the module $U(T,T)$ is its own Heller translate. As Green correspondence commutes with Heller translation, it follows that $k\Omega B$ is its own Heller translate. So we have a short exact sequence

$$0 \to k\Omega B \to P \to k\Omega B \to 0, \tag{10}$$

where P is the projective cover of $k\Omega B$.

Suppose that B is of type (II). Then (9) and Lemma 1.7 imply that

$$\delta\psi_1(1)(\epsilon(\chi_1) - \epsilon(\chi_2) + \epsilon(\chi_3) + \epsilon(\chi_4)) \quad \text{is a positive integer.}$$

Thus χ_3 and χ_4 are real-valued. From Lemma 2.14, and the decomposition matrix of B given in Lemma 1.7, we see that $k\Omega B$ has two composition factors isomorphic to each of S, X and Y. Considering (10), this shows that P has four composition factors isomorphic to each of S, X and Y. It follows from this that P is the direct sum of the projective covers of S, X and Y, and hence $\mathrm{hd}(k\Omega B) \cong \mathrm{rad}(k\Omega B) \cong S \oplus X \oplus Y$.

Now $\mathrm{End}_{kG}(k\Omega B)$ is 4-dimensional; the identity map, together with the projections onto each of the submodules S, X and Y, constitute a basis. Let $\chi = \sum_{i=1}^4 n_i\chi_i$ be the $\mathcal{O}$-character of $k\Omega B$. Then $\chi(s) = \psi_1(n_1-n_2+n_3+n_4)$ is a positive integer, while $n_1+n_2 = n_2+n_3 = n_2+n_4 = 2$. So either $n_1 = n_2 = n_3 = n_4 = 1$ or $n_1 = n_3 = n_4 = 2, n_2 = 0$. The latter case would imply that $\mathrm{End}_{kG}(k\Omega B)$ is 12-dimensional. We conclude that $\chi = \chi_1 + \chi_2 + \chi_3 + \chi_4$. This completes the proof of part (ii) of Theorem 1.9.

Suppose that B is of type (III). Then (9) and Lemma 1.7 imply that

$$\delta\psi_1(1)(\epsilon(\chi_1) + \epsilon(\chi_2) - \epsilon(\chi_3) - \epsilon(\chi_4)) \quad \text{is a positive integer.}$$

Thus $\overline{\chi_3} = \chi_4$ and $\delta = +1$. From Lemma 2.14, and the decomposition matrix of B given in Lemma 1.7, we see that $k\Omega B$ has two composition factors isomorphic to S, and one to each of X and Y. Considering (10), this shows that P has four composition factors isomorphic to S and two to each of X and Y. It follows from this that P is the projective cover of S. Thus $\mathrm{soc}(k\Omega B) \cong \mathrm{hd}(k\Omega B) \cong S$, and $\mathrm{rad}(k\Omega B)/\mathrm{soc}(k\Omega B) \cong X \oplus Y$. The character of the lift of $k\Omega B$ to characteristic zero is $\chi_1+\chi_2$. This completes the proof of part (ii) of Theorem 1.10.

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References

[1] J. Alperin, *Local Representation Theory,* Cambridge Studies Advanced Math. 11, Cambridge Univ. Press, 1986.
[2] D. Benson, *Representations and cohomology I: basic representation theory of finite groups and associative algebras,* Camb. Stud. Adv. Math. 30, Cambridge Univ. Press, 1991.
[3] R. Brauer, Some applications of the theory of blocks of characters of finite groups III, *J. Algebra* **3** (1966), 225–255.
[4] R. Brauer, Some applications of the theory of blocks of characters of finite groups IV, *J. Algebra* (4) **17** (1971), 489–521.
[5] Michel Broué, On Scott Modules and p-Permutation Modules: An approach through the Brauer Morphism, *Proc. Amer. Math. Soc.* (3) **93** (1985), 401–408.
[6] E. C. Dade, Blocks with cyclic defect groups, *Ann. Math.* (1) **84** (1966), 20–48.
[7] K. Erdmann, Blocks whose defect groups are Klein four groups: a correction, *J. Algebra* (2) **76** (1982), 505–518.
[8] R. Foote, L. Jones, Homogeneously generated groups, *J. Group Theory* **6** (2003), 321–330.
[9] R. Gow, Real-valued and 2-rational group characters, *J. Algebra* (2) **61** (1979), 388–413.
[10] R. Gow, Real 2-blocks of characters of finite groups, *Osaka J. Math.* **25** (1988), 135–147.
[11] J. A. Green, On the Brauer homomorphism, *J. Lond. Math. Soc.* (2) **17** (1978), 58–66.
[12] P. Landrock, *Finite Group Algebras and their modules,* Cambridge University Press, Cambridge, 1983.
[13] J. Murray, Projective modules and involutions, *J. Algebra* (2) **299** (2006), 616-622.
[14] J. Murray, Strongly real 2-blocks and the Frobenius-Schur indicator, *Osaka J. Math.* (1) **43** (2006), 201-213.
[15] H. Nagao, Y. Tsushima, *Representations of finite groups*, Academic Press, 1987.
[16] G. R. Robinson, The Frobenius-Schur indicator and projective modules, *J. Algebra* (1) **126** (1989), 252–257.

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