# BRANCHING RULES FOR SPECHT MODULES

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ABSTRACT. Let $\Sigma_n$ be the symmetric group of degree $n$, and let $F$ be a field of characteristic distinct from 2. Let $S_F^\lambda$ be the Specht module over $F\Sigma_n$ corresponding to the partition $\lambda$ of $n$. We find the indecomposable components of the restricted module $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and the induced module $S_F^\lambda \uparrow^{\Sigma_{n+1}}$. Namely, if $b$ and $B$ are block idempotents of $F\Sigma_{n-1}$ and $F\Sigma_{n+1}$ respectively, then the modules $S_F^\lambda \downarrow_{\Sigma_{n-1}} b$ and $S_F^\lambda \uparrow^{\Sigma_{n+1}} B$ are 0 or indecomposable. We give examples to show that the assumption char $F \neq 2$ cannot be dropped.

## 1. INTRODUCTION

Let $n$ be a positive integer and let $\Sigma_n$ be the symmetric group of degree $n$. For any field $F$ and any partition $\lambda$ of $n$, the Specht module $S_F^\lambda$ is defined to be the submodule of the permutation module $F_{\Sigma_\lambda} \uparrow^{\Sigma_n}$ spanned by certain elements called polytabloids, where $\Sigma_\lambda$ is the Young subgroup associated to $\lambda$ and $F_{\Sigma_\lambda}$ is the principal $F\Sigma_\lambda$-module. (See [1] for definitions.) Specht modules play a central role in the representation theory of the symmetric group, because in characteristic 0 the Specht modules are the simple $F\Sigma_n$-modules, while in characteristic $p$ the heads of the Specht modules $S_F^\lambda$ such that $\lambda$ is $p$-regular are the simple $F\Sigma_n$-modules. When the field $F$ has charactersitic 0, the structure of the restriction of $S_F^\lambda$ to $\Sigma_{n-1}$ is given by the Classical Branching Rule: the module $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ is a direct sum $\bigoplus_\mu S_F^\mu$, where $\mu$ runs through all partitions of $n-1$ obtained from $\lambda$ by removing a node from its Young diagram. In 1971, Peel [4] gave the first characteristic $p$ version of the branching rule. He showed that there is a series of submodules such that the successive quotients are the Specht modules $S_F^\mu$, where $\mu$ runs through the same set. Nevertheless, the structure of the restriction $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ is not well understood. For example, the problem of finding a composition series is open and very difficult, and the socle is not known. See Kleshchev [2] for an introduction to recent work on $S_F^\lambda \downarrow_{\Sigma_{n-1}}$.

In this paper, we find the indecomposable components of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$, when the characteristic of $F$ is not 2. These are given by Theorem 3.4: if $b$ is a block idempotent of $F\Sigma_{n-1}$, then $S_F^\lambda \downarrow_{\Sigma_{n-1}} b$ is 0 or indecomposable. Thus there is a bijection between the set of indecomposable components of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and the set of $p$-cores that can be obtained from $\lambda$ by removing first one node and then a sequence of rim $p$-hooks. We also prove the analogous theorem for the induced module $S_F^\lambda \uparrow^{\Sigma_{n+1}}$. The two proofs are almost identical. We give examples to show that the assumption char $F \neq 2$ cannot be dropped.

The combinatorial part of the proof is in section 2. Here we find the minimal polynomials for the actions of $E_{n-1}$ on $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and $E_{n+1}$ on $S_F^\lambda \uparrow^{\Sigma_{n+1}}$, where

$E_k$ is the sum of all the transpositions in $\Sigma_k$. These polynomials have degrees $m$ and $m + 1$ respectively, where $m$ is the number of distinct parts of $\lambda$. The results of section 2 are valid for all fields, not just those of odd characteristic.

In section 3, we investigate the algebras $\mathcal{E} = \mathrm{End}_{F\Sigma_{n-1}}(S_F^\lambda \downarrow_{\Sigma_{n-1}})$ and $\mathcal{F} = \mathrm{End}_{F\Sigma_{n+1}}(S_F^\lambda \uparrow^{\Sigma_{n+1}})$. Under the assumption that char $F \neq 2$, we use the results from section 2 to show that the natural maps $Z(F\Sigma_{n-1}) \to \mathcal{E}/J(\mathcal{E})$ and $Z(F\Sigma_{n+1}) \to \mathcal{F}/J(\mathcal{F})$ are surjective, where $J(\mathcal{E})$ and $J(\mathcal{F})$ are the Jacobson radicals of $\mathcal{E}$ and $\mathcal{F}$. The main theorem follows easily.

## 2. The minimal polynomials of the sum of all transpositions acting on the restriction and induction of a Specht module

Throughout this paper $n$ is a fixed positive integer and $\lambda$ is a fixed partition of $n$. We orient the Young diagram $[\lambda]$ left to right and top to bottom. This means that the first row is the one at the top and the first column is the one at the left. The $(i, j)$ node is in the $i$th row and the $j$th column. We will use $\widehat{n}$ to denote the set $\{1, \ldots, n\}$ and let $\Sigma_n$ denote the group of permutations of $\widehat{n}$. Permutations and homomorphisms will generally act on the right. The *Murphy element* $L_n$ is the sum of all transpositions in $\Sigma_n$ that are not in $\Sigma_{n-1}$ (with $L_1 := 0$). We use $E_n$ to denote the sum of all transpositions in $\Sigma_n$. So $E_n$ is the 1-st elementary symmetric function in the Murphy elements.

Let $F$ be any field and let $S^\lambda$ denote the Specht module, defined over $F$, corresponding to $\lambda$. We use the notation

$$\mathcal{R} \quad \text{for the restriction of } S^\lambda \text{ to } \Sigma_{n-1} \text{ and}$$

$$\mathcal{I} \quad \text{for the induction of } S^\lambda \text{ to } \Sigma_{n+1}.$$

The purpose of this section is to compute the minimal polynomial of $E_{n-1}$ acting on $\mathcal{R}$ and the minimal polynomial of $E_{n+1}$ acting on $\mathcal{I}$.

We consider a $\lambda$-*tableau* to be a bijective map $t : [\lambda] \to \widehat{n}$. The value of $t$ at a node $(r, c)$ is denoted by $t_{rc}$. The group $\Sigma_n$ acts on the set of all $\lambda$-tableaux by functional composition; $(t\pi)_{rc} = t_{rc}\pi$, for each $\pi \in \Sigma_n$.

Suppose that $\lambda$ has $l$ nonzero parts $[\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l]$. We regard a $\lambda$-*tabloid* as an ordered partition $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ of $\widehat{n}$ such that the cardinality of $\mathcal{P}_u$ is $\lambda_u$, for $u = 1, \ldots, l$. Each $\lambda$-tableau $t$ determines the $\lambda$-tabloid $\{t\}$ whose $u$-th part is the set of entries in the $u$-th row of $t$. If $s$ is a $\lambda$-tableau, then $\{t\} = \{s\}$ if and only if $s = t\pi$, for some $\pi$ in the row stabilizer $R_t$ of $t$. We denote the column stabilizer of $t$ by $C_t$. We denote by $M^\lambda$ the $F\Sigma_n$-module consisting of all formal $F$-linear combinations of $\lambda$-tabloids.

Adapting the notation of James [1], let $(r_1, c_1), \ldots, (r_m, c_m)$ be the removable nodes of $[\lambda]$, ordered so that $r_1 < \ldots < r_m$ and $c_1 > \ldots > c_m$. Set $r_0 = 0 = c_{m+1}$. The addable nodes of $[\lambda]$ are the $(m + 1)$ nodes $(r_u + 1, c_{u+1} + 1)$, for $u = 0, \ldots, m$. We use $\lambda\!\downarrow_u$ to denote the partition of $n - 1$ obtained by decrementing the $r_u$-th part of $\lambda$ by 1, for $u \in \widehat{m}$. In addition, we use $\lambda\!\uparrow^u$ to denote the partition of $n + 1$ obtained by incrementing the $(r_u + 1)$-th part of $\lambda$ by 1, for $u \in \widehat{m+1}$.

We need special notation for certain subsets of a $\lambda$-tableau $t$. For the rest of the paper, suppose that $\lambda$ has parts of $m$ different nonzero lengths. For any $u \in \widehat{m}$, let $H_u(t)$ be the set of entries in the union of the top $r_u$ rows of $t$, and let $V_u(t)$ be the set of entries in the union of columns of $t$ numbered from $c_{u+1} + 1$ to $c_u$ (inclusive). Clearly $H_1(t) \subset \ldots \subset H_m(t)$, while $V_m(t), \ldots, V_1(t)$ forms a partition of $t$. Also

$V_u(t) \subseteq H_v(t)$ if and only if $u \leq v$. As $H_u(t)$ depends only on the rows of $t$, we may define $H_u(\{t\}) := H(t)$.

By Theorem 9.3 in [1], $\mathcal{R}$ has a Specht series

$$0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_m = \mathcal{R},$$

with $\mathcal{R}_u / \mathcal{R}_{u-1} \cong S^{\lambda \downarrow u}$, for $u \in \widehat{m}$. Also, by 17.14 in [1], $\mathcal{I}$ has a Specht series

$$\mathcal{I} = \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots \supset \mathcal{I}_{m+1} \supset \mathcal{I}_{m+2} = 0,$$

with $\mathcal{I}_u / \mathcal{I}_{u+1} \cong S^{\lambda \uparrow u}$, for $u \in \widehat{m+1}$. Each factor $\mathcal{I}/\mathcal{I}_{u+1}$ is isomorphic to a submodule of the permutation module $M^{\lambda \uparrow u}$.

**Lemma 2.1.** *Suppose that the $F\Sigma_n$-module $M$ has a Specht series $0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$. Let $z \in Z(F\Sigma_n)$ and let $u \in \widehat{m}$. Then there is a scalar $z_u$ in $F$ such that the map $M_u/M_{u-1} \to M_u/M_{u-1}$ given by multiplication by $z$ is equal to $z_u$ times the identity map.*

*Proof.* If char $F = 0$, then $M_u/M_{u-1}$ is an irreducible $F\Sigma_n$-module (a Specht module), and the conclusion is obvious. If char $F = p$ is positive, then $M_u/M_{u-1}$ is the $p$-modular reduction of an irreducible module defined over a suitable discrete valuation ring of characteristic 0. The conclusion follows in this case from the characteristic zero case. $\qquad\square$

This lemma allows us to give the following upper bound on the degrees of the minimal polynomials of $E_{n-1}$ and $E_{n+1}$.

**Corollary 2.2.** *The minimal polynomial of $E_{n-1}$ acting on $\mathcal{R}$ has degree at most $m$, while the minimal polynomial of $E_{n+1}$ acting on $\mathcal{I}$ has degree at most $m+1$.*

*Proof.* Let $u \in \widehat{m}$. Lemma 2.1 shows that $\mathcal{R}_u(E_{n-1} - z_u) \subseteq \mathcal{R}_{u-1}$, for some scalar $z_u$. It follows from a simple inductive argument that $\mathcal{R} \prod_{u=1}^{m}(E_{n-1} - z_u) = 0$. A similar argument deals with the action of $E_{n+1}$ on $\mathcal{I}$. $\qquad\square$

It will turn out that the polynomials given in the proof of Corollary 2.2 are minimal. Before we prove this, we will identify the scalars $z_u$ in terms of Young diagrams.

The residue of a node $(r, c)$ is the scalar $(c - r)1_F$. We set $E(\lambda)$ as the sum of the residues of all nodes in $[\lambda]$. So $E(\lambda)$ is the 1-st elementary symmetric function in the residues. An easy calculation shows that $E(\lambda) = \sum_{i=1}^{l} \frac{1}{2}\lambda_i(\lambda_i + 1 - 2i)1_F$. The next lemma is a special case of a more general result proved by G. E. Murphy [3]: 1-st elementary symmetric function can be replaced by any symmetric function in $n$ variables.

**Lemma 2.3.** *$E_n$ acts as the scalar $E(\lambda)$ on $S^\lambda$.*

*Proof.* Let $t$ be a $\lambda$-tableau, let $(r, c) \in [\lambda]$ and let $i = t_{rc}$. Fix $1 \leq c_r < c$. Then by a simple Garnir relation (section 7 of [1]), $e_t \sum_j (i, j) = e_t$, where $j$ runs over all entries in the $c_r$-th column of $t$. Also $e_t(i, j) = -e_t$, for each entry $j$ above $i$ in column $c$ of $t$. It follows that

$$e_t \sum_j (i, j) = (c - r)e_t,$$

where $j$ runs over those elements of $\widehat{n}$ that lie in $t$ in columns strictly left of $i$ or in the same column as $i$ but strictly above $i$. If we sum over all $(r, c) \in [\lambda]$, each

transposition $(i, j)$ occurs exactly once on the left hand side, while the coefficient of $e_t$ on the right hand side is $E(\lambda)$. $\qquad\square$

If $t$ is a $\lambda$-tableau, the polytabloid $e_t$ is the following element of $M^\lambda$:

$$e_t := \sum_{\pi \in C_t} \mathrm{sgn}\,\pi\{t\pi\}.$$

It is well known that the polytabloids span the Specht module $S^\lambda$. James' description of $\mathcal{R}$, and the Garnir relations, show that $e_t$ lies in $\mathcal{R}_u \backslash \mathcal{R}_{u-1}$ if $n \in V_u(t) \backslash H_{u-1}(t)$ (although we do not use this fact).

We next describe the induced module $\mathcal{I}$. Suppose that $u \in \widehat{m+1}$. Let $T$ be a $\lambda\uparrow^u$-tableau, and let $t$ denote the restriction of $T$ to $[\lambda]$. Then the $(\lambda, T)$-polytabloid $e_T^\lambda$ is the following element of $M^{\lambda\uparrow^u}$:

$$e_T^\lambda := \sum_{\pi \in C_t} \mathrm{sgn}\,\pi\{T\pi\}.$$

In Section 17 of [1], James has shown that when $u = m + 1$, the corresponding $(\lambda, T)$-polytabloids span an $F\Sigma_{n+1}$-submodule of $M^{\lambda\uparrow^{m+1}}$, which is isomorphic to the induced module $\mathcal{I}$. We will always work with this copy of $\mathcal{I}$.

When we are showing that the polynomials given in the proof of 2.2 are minimal, it will be convenient to look at the action of the Murphy elements $L_n$ and $L_{n+1}$ rather than $E_{n-1}$ and $E_{n+1}$. The following lemma provides a link between these actions. If $t$ is a $\lambda$-tableau, its *extension to* $[\lambda\uparrow^{m+1}]$ is the $\lambda\uparrow^{m+1}$-tableau that is obtained from $t$ by appending $n + 1$ to the bottom of the first column.

**Lemma 2.4.** *Let $t$ be a $\lambda$-tableau and let $T$ be its extension to $[\lambda\uparrow^{m+1}]$. Suppose that $f(x) \in F[x]$. Then*

$$
\begin{aligned}
e_t\, f(E_{n-1}) &= e_t\, f(E(\lambda) - L_n); \\
e_T^\lambda\, f(E_{n+1}) &= e_T^\lambda\, f(E(\lambda) + L_{n+1}).
\end{aligned}
$$

*Proof.* Lemma 2.3 shows that $E_n$ acts as the scalar $E(\lambda)$ on $\mathcal{R}$. The first statement then follows from $E_{n-1} = E_n - L_n$.

Consider the subspace $V$ of $M^{\lambda\uparrow^{m+1}}$ spanned by all $e_U^\lambda$ such that $U$ is a $\lambda\uparrow^{m+1}$-tableau with $n + 1$ in the unique entry of its last row. The subspace $V$ is a direct summand of the restriction of $\mathcal{I}$ to $\Sigma_n$, and as an $F\Sigma_n$-submodule, $V$ is clearly isomorphic to $S^\lambda$. Thus $e_T^\lambda$ lies in a direct summand of the restriction of $\mathcal{I}$ to $\Sigma_n$ that is isomorphic to $S^\lambda$. So Lemma 2.3 shows that $e_T^\lambda E_n = E(\lambda)e_T^\lambda$. The second statement now follows from $E_{n+1} = E_n + L_{n+1}$, and the fact that $E_n L_{n+1} = L_{n+1}E_n$. $\qquad\square$

When we are showing that the polynomials given in the proof of 2.2 are minimal, we will want to show that there is a $\lambda$-tableau $t$ such that the set of vectors $\{e_t(L_n)^i \mid 0 \le i \le m-1\}$ is linearly independent. This will be accomplished using the following technical lemma concerning the action of $L_n$ on $\mathcal{R}$.

**Lemma 2.5.** *Let $t$ be a $\lambda$-tableau such that $n \in V_m(t) \backslash H_{m-1}(t)$. For each $u \in \widehat{m-1}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Set $s = t\,(n, x_{m-1}, x_{m-2}, \ldots, x_1)$. Let $i$ be a positive integer with $i \le m - 1$. Then the coefficient of $\{s\}$ in the expansion of*

$e_t(L_n)^i$ *into tabloids is*

$$0, \quad \text{when } 0 \le i \le m - 2;$$
$$1, \quad \text{when } i = m - 1.$$

*Proof.* Clearly $(L_n)^i = \sum (w_i, n)(w_{i-1}, n) \ldots (w_1, n)$, where $(w_1, \ldots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n-1}$. Let $(y_1, \ldots, y_i)$ be a function $\widehat{i} \to \widehat{n-1}$, let $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$, and assume that $\{s\}$ appears with nonzero coefficient in the expansion of $e_t \theta$. We have two goals: (a) to show that $i = m - 1$, and (b) to show that when $i = m - 1$, the sequence $(y_1, \ldots, y_{m-1})$ is equal to the sequence $(x_1, \ldots, x_{m-1})$. The second part of the lemma follows easily from this second goal, as we now show. In the sum $\sum e_t(w_i n) \ldots (w_1 n)$, $\{s\}$ can appear in only one term, namely $e_t(x_{m-1}, n) \ldots (x_1, n)$. Since this term is equal to $e_t(n, x_{m-1}, x_{m-2}, \ldots, x_1) = e_s$, $\{s\}$ appears with coefficient 1.

Since $e_t \theta = e_{t\theta}$, there exists $\pi$ in the column stabilizer of $t\theta$ such that $\{s\} = \{t\,\theta\,\pi\}$. Let $u \in \widehat{m-1}$. Then by construction $x_u \in V_{u+1}(s) \backslash H_u(s)$; since $\{s\} = \{t\theta\pi\}$, it follows that $x_u \notin H_u(t\,\theta\,\pi)$. As $\pi^{-1}$ is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \ldots \cup V_m(t\theta)$. Thus

$$(1) \qquad \forall u \in \widehat{m-1}, \quad x_u \theta^{-1} \in V_{u+1}(t) \cup \ldots \cup V_m(t).$$

In particular, $\theta$ does not fix any of the $m-1$ distinct symbols $x_1, \ldots, x_{m-1} \in \widehat{n-1}$.

In this paragraph, we will show that $\theta$ does not fix $n$. Assume that $\theta$ does fix $n$. If the symbols in the list $y_1, \ldots, y_i$ were distinct, $\theta$ would be the cycle $(y_i, y_{i-1}, \ldots, y_1, n)$; since $\theta$ fixes $n$, it follows that there is some repetition in the list $y_1, \ldots, y_i$. Since $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$ and $\theta$ fixes $n$, the only symbols potentially moved by $\theta$ are on the list $y_1, \ldots, y_i$. Since this list contains a repeat, $\theta$ moves at most $i - 1$ symbols. The previous paragraph shows that $\theta$ moves at least $m - 1$ symbols. Therefore $m \le i$. But by hypothesis $i \le m - 1$. This contradiction shows that $\theta$ moves $n$.

We now know that $\theta$ moves all the $m$ symbols in $\{x_1, \ldots, x_{m-1}, n\}$. Since $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$, $\theta$ can only move symbols on the list $y_1, y_2, \ldots, y_i, n$. By hypothesis, $i \le m - 1$. It follows that $i = m - 1$, which is part (a) of our goal. It also follows that the sets $\{x_1, \ldots, x_{m-1}\}$ and $\{y_1, \ldots, y_{m-1}\}$ coincide and that the elements on the list $y_1, y_2, \ldots, y_{m-1}$ are distinct. Hence $\theta$ is equal to the $m$-cycle $(y_{m-1}, y_{m-2}, \ldots, y_1, n)$. In particular, $y_{m-1}\theta^{-1} = n$. From (1) applied with $u = m - 1$, $x_{m-1}\theta^{-1} = n$. (This is because $n$ is the only symbol moved by $\theta$ that is in $V_m(t)$.) Hence $y_{m-1} = x_{m-1}$. From this fact and (1) applied with $u = m - 2$, it follows that $x_{m-2}\theta^{-1} = x_{m-1}$. Hence $y_{m-2} = x_{m-2}$. Continuing in this way, by reverse induction on $u$, it follows that for all $u \in \widehat{m-1}$, $y_u = x_u$. This gives goal (b) above, and completes the proof. $\square$

The corresponding result for the action of $L_{n+1}$ on $\mathcal{I}$ is:

**Lemma 2.6.** *Let $t$ be a $\lambda$-tableau and let $T$ be its extension to $[\lambda \uparrow^{m+1}]$. For each $u \in \widehat{m}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Set $S = T(n + 1, x_m, x_{m-1}, \ldots, x_1)$. Let $i$ be a positive integer with $i \le m$. Then the multiplicity of $\{S\}$ in the expansion of $e_T^\lambda (L_{n+1})^i$ into tabloids is*

$$0, \quad \text{when } 0 \le i \le m - 1;$$
$$1, \quad \text{when } i = m.$$

*Proof.* Clearly we have $(L_{n+1})^i = \sum (w_i, n+1)(w_{i-1}, n+1)\ldots(w_1, n+1)$, where $(w_1, \ldots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n}$. Let $(y_1, \ldots, y_i)$ be a function $\widehat{i} \to \widehat{n}$, let $\theta = (y_i, n+1)(y_{i-1}, n+1)\ldots(y_1, n+1)$, and assume that $\{S\}$ appears with nonzero multiplicity in the expansion of $e_T^\lambda \theta$ as a linear combination of tabloids. Then there exists $\pi$ in the column stabilizer of $t\theta$ such that $\{S\} = \{T\,\theta\,\pi\}$.

As $\pi$ fixes the single entry in the last row of $T\theta$, and $x_m$ occupies this node in $S$, it follows that $(n+1)\theta = x_m$. Let $u \in \widehat{m-1}$ and let $s$ denote the restriction of $S$ to $\lambda$. Then $x_u \in V_{u+1}(s)\backslash H_u(s)$, whence $x_u \notin H_u(t\,\theta\,\pi)$. As $\pi^{-1}$ is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \ldots \cup V_m(t\theta)$. Thus

$$(2) \qquad x_u\theta^{-1} \in V_{u+1}(t) \cup \ldots \cup V_m(t).$$

In particular, $\theta$ does not fix $x_u$.

From its definition, $\theta$ moves at most $i+1$ elements of $\widehat{n+1}$. But $\theta$ does not fix any of the $m+1$ distinct symbols $n+1, x_m, \ldots, x_1$, and $i \leq m$. So we must have $i = m$. Together with (2), this implies that $x_u\theta^{-1} \in \{x_{u+1}, \ldots, x_m\}$. Reverse induction on $u$ shows that $x_u\theta^{-1} = x_{u+1}$. Thus $\theta$ coincides with the $(m+1)$-cycle $(n+1, x_m, x_{m-1}, \ldots, x_2, x_1)$. We conclude that $x_u = y_u$, for $u \in \widehat{m}$. This shows that $\theta$ occurs with multiplicity 1 in the expansion of $(L_{n+1})^m$ as a linear combination of group elements, whence $\{S\}$ appears with multiplicity 1 in the expansion of $e_T^\lambda(L_{n+1})^m$ as a linear combination of tabloids in $M^{\lambda\uparrow^{m+1}}$. $\qquad\square$

We can now prove the main result of this section.

**Theorem 2.7.** *The minimal polynomial of $E_{n-1}$ acting on $\mathcal{R}$ is*

$$\prod_{u=1}^{m} (x - E(\lambda\downarrow_u)),$$

*while the minimal polynomial of $E_{n+1}$ acting on $\mathcal{I}$ is*

$$\prod_{u=1}^{m+1} (x - E(\lambda\uparrow^u)).$$

*Proof.* First, we will prove the result on $\mathcal{R}$. Let $t$ be as in Lemma 2.5. Then Lemma 2.5 implies that the set of vectors $\{e_t(L_n)^i \mid 0 \leq i \leq m-1\}$ is linearly independent. It follows from Lemma 2.4 that the set $\{e_t(E_{n-1})^i \mid 0 \leq i \leq m-1\}$ is linearly independent. So the minimal polynomial of $E_{n-1}$ has degree at least $m$. But Lemma 2.3 and the proof of Corollary 2.2 show that $\mathcal{R}\prod_{u=1}^{m}(E_{n-1} - E(\lambda\downarrow_u)) = 0$.

The result on $\mathcal{I}$ follows from an identical argument using Lemma 2.6 in place of Lemma 2.5. $\qquad\square$

## 3. The indecomposable components of the restriction and induction of a Specht module

The purpose of this section is to compute the indecomposable components of $\mathcal{R}$ and $\mathcal{I}$, when the characteristic of $F$ is not 2. It is convenient to consider an $F\Sigma_n$-module $M$ that shares the following properties in common with $\mathcal{R}$ and $\mathcal{I}$:

   (1) $M$ has a Specht series

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M,$$

       such that $M_u/M_{u-1} \cong S^{\lambda_u}$, where $\lambda_u$ is a partition of $n$, for each $u \in \widehat{m}$.
   (2) The labelling partitions satisfy $\lambda_1 \lhd \ldots \lhd \lambda_m$.

(3) There exists $z \in Z(F\Sigma_n)$ such that the minimal polynomial of $z$ acting on $M$ has degree $m$.

Looking at the proof of Corollary 2.2, we see that $z$ has minimal polynomial $\prod_{u=1}^{m}(x - z_u)$, where $z$ acts as the scalar $z_u$ on the Specht factor $M_u/M_{u-1}$.

**Lemma 3.1.** *There exists $\tau \in M$ such that for all $u \in \widehat{m}$, $\tau \prod_{i=u+1}^{m}(z - z_i)$ lies in $M_u \backslash M_{u-1}$.*

*Proof.* The hypothesis on the degree of the minimal polynomial of $z$ implies that there exists $\tau \in M$ such that $\tau z^{m-1}$ does not lie in the span of the vectors $\{\tau, \tau z, \ldots, \tau z^{m-2}\}$. Set $\tau_u = \tau \prod_{i=u+1}^{m}(z - z_i)$. Repeated application of Lemma 2.1 shows that $\tau_u \in M_u$. We claim that $\tau_u \notin M_{u-1}$. Suppose otherwise. Then $\tau_u \prod_{i=1}^{u-1}(z - z_u) \subseteq M_{u-1} \prod_{i=1}^{u-1}(z - z_u) = 0$, again using Lemma 2.1. Thus $\tau \prod_{i=1, i \neq u}^{m}(z - z_i) = 0$. This contradicts our choice of $\tau$. $\square$

We now consider the endomorphism ring of a module that has properties (1) and (2) in common with $M$.

**Lemma 3.2.** *Suppose that char $F \neq 2$. Let $\theta$ be a $F\Sigma_n$-endomorphism of $M$. Then*
  (1) *for all $u \in \widehat{m}$, $M_u\theta \subseteq M_u$;*
  (2) *for all $u \in \widehat{m}$, there is a well-defined $\Sigma_n$-endomorphism $\theta_u : M_u/M_{u-1} \to M_u/M_{u-1}$ given by $(v + M_{u-1})\theta_u = v\theta + M_{u-1}$;*
  (3) *the map $\Phi : \mathrm{End}_{F\Sigma_n}(M) \to \bigoplus_u \mathrm{End}_{F\Sigma_n}(M_u/M_{u-1})$ given by $(\theta)\Phi = (\theta_1, \ldots, \theta_m)$ is an algebra homomorphism;*
  (4) *the kernel of $\Phi$ is the Jacobson radical of $\mathrm{End}_{F\Sigma_n}(M)$.*

*Proof.* First, we prove (i). By induction, we may assume that $M_{u-1}\theta \subseteq M_{u-1}$. Suppose that $M_u\theta \nsubseteq M_u$. Choose $v$ so that $m \geq v > u$ and $v$ is maximal so that $M_u\theta \nsubseteq M_{v-1}$. Then $M_u\theta \subseteq M_v$, and applying $\theta$ to elements of $M_u$ induces a well-defined nonzero $\Sigma_n$-homomorphism

$$M_u/M_{u-1} \to M_v/M_{u-1} \twoheadrightarrow M_v/M_{v-1}.$$

But $\lambda_u \triangleleft \lambda_v$. This, together with the fact that char $F \neq 2$, contradicts 13.17 of [1], proving (i). Part (ii) follows easily from part (i).

It is immediate from the definition of $\theta_u$ that $\Phi$ is an algebra homomorphism. As char $F \neq 2$, the only $\Sigma_n$-endomorphisms of $M_u/M_{u-1}$ are scalar multiples of the identity, by 13.17 of [1]. It follows that the codomain of $\Phi$ is commutative and semisimple. Any element of the kernel must send $M_u$ to $M_{u-1}$ for all $u$; therefore the kernel is nilpotent. $\square$

We now compute the indecomposable summands of $M$.

**Proposition 3.3.** *Assume that char $F \neq 2$. Let $B$ be a block idempotent of $F\Sigma_n$. Then the $F\Sigma_n$-module $MB$ is $0$ or indecomposable.*

*Proof.* Assume that $MB \neq 0$. Let $A$ be the algebra $\mathrm{End}_{F\Sigma_n}(MB)$. Identify the algebra $A$ in the natural way with a direct summand of the algebra $\mathrm{End}_{F\Sigma_n}(M)$. We will use the notation and results from Lemma 3.2 throughout this proof. Our goal is to show that $A/J(A)$ has dimension 1 over $F$.

Suppose then that $\theta \in A$. Let $w$ be maximal such that the Specht module $M_w/M_{w-1}$ belongs to $B$. Our task is to show that if $\theta_w = 0$, then $\theta_u = 0$ for all $u$ such that $M_u/M_{u-1}$ belongs to $B$. (The proposition follows easily from this. Let $\phi$

be in $A$. Then there is a scalar $c$ such that the map $\phi_w$ is $c$ times the identity. Let $\theta = \phi - c1_A$. Then $\theta_w = 0$. Since $\theta_u$ is also 0 for all $u$ with $M_u/M_{u-1}$ belonging to $B$, it follows from the last part of Lemma 3.2 that $\theta \in J(A)$. Hence $A/J(A)$ has dimension 1.)

Now assume that $\theta_w = 0$, and let $u$ be an integer such that $M_u/M_{u-1}$ belongs to $B$. Let $\tau \in M$ be as in Lemma 3.1, set $\tau_u := \tau \prod_{i=u+1}^{m}(z - z_i)$, and set $\tau_w := \tau \prod_{i=w+1}^{m}(z - z_i)$. The lemma states that $\tau_u \in M_u \backslash M_{u-1}$ and $\tau_w \in M_w \backslash M_{w-1}$. Since $u \le w$, we have

$$
\begin{aligned}
\tau_u \theta &= \left( \tau_w \prod_{i=u+1}^{w} (z - z_i) \right) \theta \\
&= \tau_w \theta \prod_{i=u+1}^{w} (z - z_i), \quad \text{as } \theta \text{ is in } \mathrm{End}_{F\Sigma_n}(M), \\
&\in M_{w-1} \prod_{i=u+1}^{w} (z - z_i), \quad \text{as } \theta_w = 0 \text{ implies that } \tau_w \theta \in M_{w-1}, \\
&= \left( M_{w-1} \prod_{i=u+1}^{w-1} (z - z_i) \right) (z - z_w) \\
&\subseteq M_u(z - z_w), \quad \text{using Lemma 2.1 repeatedly.}
\end{aligned}
$$

Now $M_u/M_{u-1}$ and $M_w/M_{w-1}$ both belong to $B$. So $z_u = z_w$, since both scalars are equal to the image of $z$ under the central character of $B$. Lemma 2.1 and the last inclusion displayed above then show that $\tau_u\theta \in M_{u-1}$. But $\tau_u \notin M_{u-1}$, as proved in Lemma 3.1, and $\mathrm{End}_{F\Sigma_n}(M_u/M_{u-1})$ is one-dimensional, by 13.17 of [1]. We conclude that $\theta_u = 0$, as required.                     $\square$

We have now done all the work to prove the main result of this paper.

**Theorem 3.4.** *Assume that* $\mathrm{char}\, F \neq 2$. *Let $b$ be a block idempotent of $F\Sigma_{n-1}$. Then the $F\Sigma_{n-1}$-module $(S^\lambda\!\downarrow_{S_{n-1}})b$ is $0$ or indecomposable. Let $B$ be a block idempotent of $F\Sigma_{n+1}$. Then the $F\Sigma_{n+1}$-module $(S^\lambda\!\uparrow^{S_{n+1}})B$ is $0$ or indecomposable.*

*Proof.* We know that $\mathcal{R}$ and $\mathcal{I}$ satisfy properties (1) and (2) of $M$. That they also satisfy property (3) is a consequence of Theorem 2.7. The result now follows from Proposition 3.3.                     $\square$

We will finish by giving examples to show that the assumption $\mathrm{char}\, F \neq 2$ cannot be dropped in Theorem 3.4.

Assume that $\mathrm{char}\, F = 2$. Consider the Specht module $S^{(6,1,1,1)}$. The decomposition matrix for $\Sigma_9$ given in [1] shows that $S^{(8,1)}$ and $S^{(6,3)}$ are simple and that $S^{(6,1,1,1)}$ has a composition series with factors $S^{(8,1)}$ and $S^{(6,3)}$. By 23.8 in [1], $S^{(6,1,1,1)}$ is self-dual, so there is another composition series in which the factors appear in the other order. It follows that $S^{(6,1,1,1)} \cong S^{(8,1)} \oplus S^{(6,3)}$.

Now consider the restriction of $S^{(6,1,1,1)}$ to $\Sigma_8$. All components of the restriction belong to the principal 2-block of $\Sigma_8$, which is the block with empty core. Since $S^{(6,1,1,1)}$ is decomposable, so is its restriction to $\Sigma_8$.

For the other counterexample, let $M = S^{(6,1,1)} \uparrow^{\Sigma_9}$. The module $M$ has a Specht series with factors $S^{(7,1,1)}$, $S^{(6,2,1)}$, and $S^{(6,1,1,1)}$. These factors belong to 2-blocks with cores $(1)$, $(1)$, and $(2,1)$ respectively. It follows that if $B$ is the

block idempotent corresponding to 2-core $(2, 1)$, then $MB \cong S^{(6,1,1,1)}$; thus $MB$ is decomposable.

## 4. Acknowledgement

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