# REAL 2-REGULAR CLASSES AND 2-BLOCKS

RODERICK GOW
JOHN MURRAY*

ABSTRACT. Suppose that $G$ is a finite group. We show that every 2-block of $G$ has a defect class which is real.

As a partial converse, we show that if $G$ has a real 2-regular class with defect group $D$ and if $\mathbf{N}(D)/D$ has no dihedral subgroup of order 8, then $G$ has a real 2-block with defect group $D$.

More generally, we show that every 2-block of $G$ which is weakly regular relative to some normal subgroup $N$ has a defect class which is real and contained in $N$. We give several applications of these results and also investigate some consequences of the existence of non-real 2-blocks.

## 1. INTRODUCTION

Let $G$ be a finite group and let $p$ be a prime divisor of $|G|$. If $B$ is a $p$-block of $G$, then the complex conjugates of the irreducible characters in $B$ form another $p$-block, $B^o$, of $G$. We say that $B$ is a real $p$-block if $B = B^o$. In the case that the prime $p$ is 2, it is easy to prove that a real 2-block has at least one real defect class. One of the main results of this paper, Theorem 3.5, shows that in fact an arbitrary 2-block has a real defect class. As a consequence, in order for $G$ to have a 2-block with defect group $D$, it must have at least one real 2-regular class with defect group $D$.

The converse of this last statement is false. For instance, the symmetric group $S_4$ has a real 2-regular class of defect zero, yet it has no 2-blocks of defect zero. However, Theorem 4.8 establishes the following partial converse: Let $D$ be a 2-subgroup of $G$ and let $\mathbf{N}(D)$ denote its normalizer in $G$. Suppose that $\mathbf{N}(D)/D$ has no dihedral subgroups of order 8. Then $G$ has a 2-block with defect group $D$ if and only if $G$ has a real 2-regular class with defect group $D$. We conclude Section 4 with a number of examples illustrating the use of this theorem.

In Section 5, we investigate blocks in relation to a normal subgroup $N$. Our main result is that each 2-block of $G$ which is weakly regular relative to $N$ has a real defect class which is contained in $N$. If $N$ has odd order, this gives the following purely group-theoretic consequence: $N$ contains a conjugacy class of $G$ with a given defect group if and only if $N$ contains a real conjugacy class of $G$ with the same defect group.

## 2. PRELIMINARIES

Much of our notation comes from [NT89]. In particular, we fix a $p$-modular system $(K, \mathcal{O}, F)$ for $G$ and assume that $K$ contains a primitive $|G|$-th root of unity. (For

*Date*: September 29, 1999.
1991 *Mathematics Subject Classification.* 20C20.
*The second author was supported by a Forbairt Postdoctoral Grant while writing this paper.

much of this paper, the prime $p$ will be 2, although a number of our results hold for any prime $p$.) We use $\mathcal{P}$ to denote the unique maximal ideal in $\mathcal{O}$. If $V$ is an $\mathcal{O}$-module, and $v$ is an element of $V$, then $v^*$ will denote the image of $v$ in the $F$-module $V/\mathcal{P}V$.

If $\mathcal{K}$ is a conjugacy class of $G$, then $\mathcal{K}^o$ will denote the conjugacy class whose members are the inverses of the elements of $\mathcal{K}$. We say that $\mathcal{K}$ is a *real* class if $\mathcal{K} = \mathcal{K}^o$. We use $\mathbf{C}(g)$ to denote the centralizer, and $\mathbf{C}^*(g)$ the extended centralizer, of an element $g$ of $G$. In particular $\mathbf{C}^*(g)$ is the stabilizer of the set $\{g, g^{-1}\}$ in $G$. The Sylow $p$-subgroups of the centralizers of the elements of a conjugacy class $\mathcal{K}$ are called the *defect groups* of $\mathcal{K}$. When $p = 2$, and $\mathcal{K}$ is a non-identity real class of $G$, we call the Sylow 2-subgroups of the extended centralizers of the elements of $\mathcal{K}$ the *extended defect groups* of $\mathcal{K}$.

The elements of $G$ form a basis for the group algebra $FG$. So every element of $FG$ is of the form $x = \sum_{g\in G} \beta(x,g)g$, where the $\beta(x,g)$ are elements of $F$. We define the element $x^o \in FG$ by $x^o = \sum \beta(x, g^{-1})g$. Let $S^+$ denote the sum of the elements of a subset $S$ of $G$ in $FG$. The elements $\mathcal{K}^+$, where $\mathcal{K}$ is a conjugacy class of $G$, form a basis for the centre $ZFG$ of $FG$. If $x \in ZFG$, we use $\beta(x, \mathcal{K}^+)$ to denote the coefficient of $\mathcal{K}^+$ in $x$.

The $p$-blocks of $G$ correspond in a one-to-one manner with the primitive idempotents of $ZFG$ and also with the $F$-algebra epimorphisms $ZFG \to F$. If $B$ is a $p$-block of $G$ with associated primitive idempotent $e$ and $F$-algebra epimorphism $\omega$, then we will express this association by $B \leftrightarrow e \leftrightarrow \omega$.

Let $\chi$ be an irreducible $K$-character of $G$. The central character of $\chi$ is the map $\omega_\chi : Z\mathcal{O}G \to \mathcal{O}$, given by $\omega_\chi(x) := \chi(x)/\chi(1_G)$, for each $x \in Z\mathcal{O}G$. The irreducible $K$-characters of $G$ are partitioned into blocks, with $\chi$ belonging to the block determined by $B \leftrightarrow e \leftrightarrow \omega$ if and only if

$$\omega(\mathcal{K}^+) = \omega_\chi(\mathcal{K}^+)^*,$$

for each conjugacy class $\mathcal{K}$ of $G$. Set $\omega^o(z) := \omega(z^o)$, for $z \in ZFG$. Then $\omega^o$ is an $F$-algebra epimorphism, and it is clear that $B^o \leftrightarrow e^o \leftrightarrow \omega^o$ is the associated $p$-block. Moreover, the complex conjugate $\chi^o$ of $\chi$ belongs to $B^o$.

The defect group of a block is defined in Section 3.6 of [NT89]. We collect a number of well-known results about $p$-blocks in the following lemma.

**Lemma 2.1.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a $p$-block of $G$, with defect group $D$. Let $\mathcal{K}$ be a conjugacy class of $G$. Then $\omega(\mathcal{K}^+) = 0_F$, unless some defect group of $\mathcal{K}$ contains $D$, and $\beta(e, \mathcal{K}^+) = 0_F$, unless $\mathcal{K}$ is $p$-regular and some defect group of $\mathcal{K}$ is contained in $D$.*

We note that, given a $p$-block $B \leftrightarrow e \leftrightarrow \omega$, there exists at least one class $\mathcal{L}$ such that $\beta(e, \mathcal{L}^+) \neq 0_F$ and $\omega(\mathcal{L}^+) \neq 0_F$. Any such $\mathcal{L}$ is called a *defect class* for $B$.

In later sections, we will be concerned with showing the existence of real 2-blocks with a given defect group. For this purpose, the proof of Lemma 1.2 of [G88] may be adapted to show the following result.

**Lemma 2.2.** *Suppose that $G$ has a 2-block with defect group $D$ and let $e_D$ denote the sum of the 2-block idempotents of $FG$ with defect group $D$. Then $G$ has a real 2-block with defect group $D$ if and only if there exists a real 2-regular class $\mathcal{K}$ with defect group $D$ for which $\beta(e_D, \mathcal{K}^+) \neq 0_F$.*

## 3. Real defect classes

*Brauer Characters* and *Principal Indecomposable Characters* are defined in Section 6.3 of [NT89]. If $B$ is a $p$-block of $G$, with defect group $D$, and if $\chi$ is an irreducible $K$-character in $B$, then $\chi(1)/|G:D|$ is an element of $\mathcal{O}$. Moreover, there exists at least one irreducible $K$-character $\theta$ in $B$ for which $\theta(1)/|G:D|$ is a unit in $\mathcal{O}$. We say that any such $\theta$ has *height zero*. The following result is collected from [O66]:

**Proposition 3.1.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a $p$-block of $G$ with defect group $D$. Let $\mathcal{K}$ be a $p$-regular class with defect group $D$. Then*

$$\beta(e, \mathcal{K}^+) = \big(\dim(B)/|G||\mathcal{K}|\big)^* \, \omega(\mathcal{K}^{o+}).$$

*Proof.* Let $\theta$ be an irreducible $K$-character in $B$ which has height zero. Then $\chi(1)/|\mathcal{K}|$ and $\chi(1)/\theta(1)$ are elements of $\mathcal{O}$, for each irreducible $K$-character $\chi$ in $B$. Also $\omega_\chi(\mathcal{K}^+) \equiv \omega_\theta(\mathcal{K}^+) \bmod \mathcal{P}$. Thus, given any element $k$ in $\mathcal{K}$, we have

$$\begin{aligned} \chi(k) &\equiv \frac{\chi(1)}{|\mathcal{K}|} \frac{|\mathcal{K}|}{\theta(1)} \theta(k) \mod \mathcal{P} \\ &\equiv \frac{\chi(1)}{\theta(1)} \theta(k) \mod \mathcal{P} \end{aligned}$$

If $\psi$ is an irreducible Brauer character of $G$ in $B$ then $\Psi$ will denote the corresponding principal indecomposable character. Now $\psi$ is a $\mathbb{Z}$-linear combination of the restrictions of the ordinary irreducible $K$-characters in $B$ to the $p$-regular elements of $G$. So there exist integers $r_{\chi,\psi}$ such that $\psi = \sum r_{\chi,\psi}\, \chi$ on $p$-regular elements, where $\chi$ ranges over the irreducible $K$-characters in $B$. Thus

$$\psi(k)^* = \left( \sum_{\chi \in B} r_{\chi,\psi} \frac{\chi(1)}{\theta(1)} \right)^* \theta(k)^* = \left( \frac{\psi(1)}{\theta(1)} \right)^* \theta(k)^*.$$

Now $\sum_{\chi\in B} \chi(1)\chi(k^{-1}) = \sum_{\psi \in B} \Psi(1)\psi(k^{-1})$. So

$$\beta(e, \mathcal{K}^+) = \left( |G|^{-1} \sum_{\chi \in B} \chi(1)\chi(k^{-1}) \right)^* = \sum_{\psi \in B} \left( \frac{\Psi(1)}{|G|} \right)^* \psi(k^{-1})^*,$$

taking into account the fact that $\Psi(1)_p \geq |G|_p$ for each $\Psi$. Thus

$$\beta(e, \mathcal{K}^+) = \left( \sum_{\psi \in B} \frac{\Psi(1)\psi(1)}{|G| \; \theta(1)} \right)^* \theta(k^{-1})^* = \left( \frac{\dim(B)}{|G| \, \theta(1)} \right)^* \theta(k^{-1})^*$$

The Proposition now follows from the fact that $\theta(k^{-1})^* = \omega(\mathcal{K}^{o+})\,(\theta(1)/|\mathcal{K}|)^*$. □

**Corollary 3.2.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of $G$ with defect group $D$ and that $\mathcal{K}$ is a 2-regular class with defect group $D$. Then $\beta(e, \mathcal{K}^+) = \omega(\mathcal{K}^{o+})$.*

*Proof.* This follows from Proposition 3.1 and the fact that $1_F$ is the only non-zero element in the prime field of $F$. □

We note in passing that Proposition 3.1 implies the following result of R. Brauer, [B76].

**Corollary 3.3.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a $p$-block of $G$ with defect group $D$. Set $|G|_p = p^a$ and $|D| = p^d$. Then the $p$-part of* $\dim(B)$ *equals $p^{2a-d}$.*

*Proof.* It follows from the existence of a defect class (see the note after Lemma 2.1) and Proposition 3.1 that $\dim(B)/|G|\,|G:D|$ is a unit in $\mathcal{O}$. □

**Proposition 3.4.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of $G$ with defect group $D$ and let $\mathcal{R}_D$ denote the union of the real 2-regular classes of $G$ which have defect group $D$ Then $\omega(\mathcal{R}_D^+) = 1_F$ and $\sum_{\mathcal{K}\subseteq\mathcal{R}_D} \beta(e,\mathcal{K}^+) = 1_F$.*

*Proof.* Let $\mathcal{K}_1, \ldots, \mathcal{K}_s$, denote all the real 2-regular classes of $G$ with defect group $D$ and let $\mathcal{L}_1, \mathcal{L}_1^o, \ldots, \mathcal{L}_t, \mathcal{L}_t^o$ denote all the non-real 2-regular classes of $G$ with defect group $D$. Taking into account Lemma 2.1, we have

$$\omega(e) = \sum_{i=1}^{s} \beta(e,\mathcal{K}_i^+)\omega(\mathcal{K}_i^+) + \sum_{j=1}^{t} \left(\beta(e,\mathcal{L}_j^+)\,\omega(\mathcal{L}_j^+) + \beta(e,\mathcal{L}_j^{o+})\,\omega(\mathcal{L}_j^{o+})\right).$$

Then using Corollary 3.2, we get

$$\begin{aligned}\omega(e) &= \sum_{i=1}^{s} \omega(\mathcal{K}_i^+)^2 + \sum_{j=1}^{t} \left(\omega(\mathcal{L}_j^{o+})\,\omega(\mathcal{L}_j{}^+) + \omega(\mathcal{L}_j^+)\,\omega(\mathcal{L}_j^{o+})\right), \\ &= \Big(\sum_{i=1}^{s} \omega(\mathcal{K}_i^+)\Big)^2, \quad \text{as } F \text{ has characteristic } 2, \\ &= \omega(\mathcal{R}_D^+)^2.\end{aligned}$$

Thus $1_F = \omega(e) = \omega(\mathcal{R}_D^+)^2$, which in turn implies that $\omega(\mathcal{R}_D^+) = 1_F$. The second equation now follows from Corollary 3.2. □

We now present the main result of this section.

**Theorem 3.5.** *Every 2-block has a real defect class.*

*Proof.* Let $B \leftrightarrow e \leftrightarrow \omega$ be a 2-block of $G$ which has defect group $D$. It follows from the previous theorem that there exists a real 2-regular class $\mathcal{K}$ of $G$, with defect group $D$, such that $\omega(\mathcal{K}^+) \neq 0_F$. But $\beta(e,\mathcal{K}^+) \neq 0_F$, using Proposition 3.1. So $\mathcal{K}$ is a real defect class for $B$. The result follows. □

Our corollary furnishes a necessary condition for the existence of 2-blocks which seems to have been overlooked until now.

**Corollary 3.6.** *Suppose that $G$ has a 2-block with defect group $D$. Then $G$ has a real 2-regular conjugacy class with defect group $D$.*

**Example 3.7.** Let $n > 1$ be an odd integer and let $q$ be a power of an odd prime $p$. Suppose that $n$ is relatively prime to $q-1$. Consideration of rational canonical forms shows that the only real 2-regular class of defect zero in the simple group $\mathrm{SL}_n(q)$ is that containing a regular unipotent element, whose minimal polynomial is $(x-1)^n$. It is known that $\mathrm{SL}_n(q)$ has 2-blocks of defect 0, so the class of regular unipotent elements is the only real defect class for such blocks. Taking $n = 3$ and $\gcd(3, q-1) = 1$, we find that the number of 2-blocks of defect 0 is $q(q+1)/3$. Each of these blocks contains a unique irreducible character of degree $(q-1)^2(q+1)$, which is not real-valued, and they all have the same real defect class, consisting of elements of order $p$.

Let $B \leftrightarrow e \leftrightarrow \omega$ be a real 2-block of $G$ with defect group $D$. Suppose that $\omega(\mathcal{K}^+) \neq 0_F$ and $\omega(\mathcal{L}^+) \neq 0_F$, where $\mathcal{K}$ and $\mathcal{L}$ are real classes of $G$ and $\mathcal{K}$ has defect group $D$. The first author showed [G88, 2.1] that each extended defect group of $\mathcal{K}$ is contained in some extended defect group of $\mathcal{L}$, thus refining Lemma 2.1. This need not hold when $B$ is non-real, as the following example shows:

**Example 3.8.** Let $\mathcal{S}_n$ ($\mathcal{A}_n$) denote the symmetric (alternating) group of degree $n$. Suppose that $n = m(m+1)/2$ is a triangular number, where $m > 0$. Then $\mathcal{S}_n$ has a 2-block of defect zero, containing the irreducible character corresponding to the triangular partition $[m, \dots, 3, 2, 1]$. Let $\chi$ be an irreducible constituent of the restriction of this character to $\mathcal{A}_n$. Then $\chi$ lies in a 2-block $B$ of defect zero. Now suppose that $m$ is even and greater than 2. The classes of cycle type $[(2m-3)^2, 2m-9, 2m-13, 2m-17, \dots]$, and $[2m-1, 2m-3, 2m-5, \dots, 7, 1^3]$, if $m \equiv 2 \mod 4$, of $\mathcal{S}_n$ form single classes $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{A}_n$. Moreover both have defect 0, and are real in $\mathcal{A}_n$, elements being inverted by involutions of cycle type

$$[2^{(m/2)^2}], \quad \text{for } \mathcal{K}, \text{ if } m \equiv 0 \mod 4;$$
$$[2^{(m/2)^2+1}], \quad \text{for } \mathcal{K}, \text{ if } m \equiv 2 \mod 4;$$
$$[2^{(m/2)^2-1}], \quad \text{for } \mathcal{L}, \text{ if } m \equiv 2 \mod 4.$$

By repeated applications of the Murnaghan-Nakayama formula, it can be shown that $\chi$ takes the value $\pm 1$ on elements of $\mathcal{K}$ or $\mathcal{L}$. So $\mathcal{K}$, and $\mathcal{L}$ if $m \equiv 2 \mod 4$, are real defect classes for $B$. However, $\mathcal{K}$ and $\mathcal{L}$ are inverted by non-conjugate involutions.

## 4. Existence of real 2-blocks

Given a $p$-subgroup $D$ of $G$, we will write $e_D = e_D(G)$ for the sum of the block idempotents of $FG$ with defect group $D$. In the case that $D$ is the trivial subgroup of $G$, we will write $e_0 = e_0(G)$ for the sum of the block idempotents of defect 0. Similar notation will be used for subgroups of $G$. We will also write $\mathcal{R}_0 = \mathcal{R}_0(G)$ for the union of the real $p$-regular classes which have trivial defect group. Set $W := \mathbf{N}(D)/D$, and let $\pi : F\mathbf{N}(D) \to FW$ be the $F$-algebra epimorphism induced by the natural group epimorphism $\pi : \mathbf{N}(D) \to W$.

**Lemma 4.1.** *Suppose that $k$ is a $p$-regular element of $G$ with $p$-defect group $D$. Then $\beta(e_D(G), k) = \beta(e_0(W), \pi(k))$.*

*Proof.* Let $B \leftrightarrow e \leftrightarrow \omega$ be a $p$-block of $G$ with defect group $D$. Brauer's First Main Theorem establishes a bijection between the $p$-blocks of $G$ with defect group $D$ and the $p$-blocks of $\mathbf{N}(D)$ with defect group $D$. Let $\tilde{B} \leftrightarrow \tilde{e} \leftrightarrow \tilde{\omega}$ correspond to $B \leftrightarrow e \leftrightarrow \omega$ under this bijection. It follows from Theorem 5.2.15 of [NT89] that

$$\beta(e, k) = \beta(\tilde{e}, k). \tag{4.2}$$

Suppose that $k$ lies in the class $\mathcal{K}$ of $\mathbf{N}(D)$ and that $\pi(k)$ lies in the class $\mathcal{L}$ of $W$. Then $\mathcal{K}$ is $p$-regular with $p$-defect group $D$. So by Lemma 5.8.9 of [NT89], we have $\pi(\mathcal{K}^+) = \mathcal{L}^+$. It follows that

$$\beta(\tilde{e}, \mathcal{K}^+) = \beta(\pi(\tilde{e}), \mathcal{L}^+). \tag{4.3}$$

Theorem 5.8.7.(ii) of [NT89] implies that $\pi(\tilde{e})$ is a sum of $p$-blocks of $W$ with trivial defect group. Hence

$$\pi(\tilde{e}) = \pi(\tilde{e})e_0(W). \tag{4.4}$$

Suppose that $B_W \leftrightarrow e_W \leftrightarrow \omega_W$ is a $p$-block of $W$, with trivial defect group, such that $\beta(e_W, \mathcal{L}^+) \neq 0_F$. Then $\omega_W \circ \pi : ZF\mathbf{N}(D) \to F$ is an $F$-algebra epimorphism. Let $B_N \leftrightarrow e_N \leftrightarrow \omega_N$ be the associated $p$-block of $\mathbf{N}(D)$. Then

$$\begin{aligned} \omega_N(\mathcal{K}^{o+}) &= \omega_W \circ \pi(\mathcal{K}^{o+}) = \omega_W(\mathcal{L}^{o+}) \\ &= \lambda \cdot \beta(e_W, \mathcal{L}^+), \text{ where } \lambda \neq 0_F, \text{ by Proposition 3.1 and Corollary 3.3} \\ &\neq 0_F, \quad \text{as } \beta(e_W, \mathcal{L}^+) \neq 0_F. \end{aligned}$$

It follows from Lemma 2.1 that some defect group of $B_N$ is contained in $D$, the unique $p$-defect group of $\mathcal{K}^o$. But $D$ is a normal $p$-subgroup of $\mathbf{N}(D)$. So every defect group of $B_N$ contains $D$. We deduce that $B_N$ is a $p$-block with defect group $D$. Also $\omega_N(\pi(e_W)) = \omega_W(e_W) = 1_F$. So $e_W = e_W\,\pi(e_N)$. The result now follows from (4.2), (4.3) and (4.4). □

We take $p$ to be 2 for the rest of this section and for each positive integer $n$, we set $\mathcal{Q}_n = \mathcal{Q}_n(G) := \{g \in G \mid g \text{ has order } n\}$.

**Lemma 4.5.** $e_0 = \mathcal{R}_0^+ + \mathcal{Q}_4^+ + \mathcal{Q}_4^+\mathcal{Q}_2^+$.

*Proof.* This follows from the following equalities:

$$\begin{aligned} (1_G + \mathcal{Q}_2^+)^2 &= \mathcal{R}_0^+, && \text{by Proposition 4.1 of [M99]} \\ (1_G + \mathcal{Q}_2^+ + \mathcal{Q}_4^+)(1_G + \mathcal{Q}_2^+) &= e_0\ , && \text{by Corollary 5.9 of [M99].} \end{aligned}$$

□

If $g$ is a real element of $G$ of defect zero, then it is well known that there exists an involution $t$ which inverts $g$. So $s := gt$ is also an involution which inverts $g$. Moreover, if $g = uv$, where $u$ and $v$ are involutions, then both $u$ and $v$ invert $g$. Any two involutions which invert $g$ are conjugate in $C^*(g)$, and hence come from a single conjugacy class of involutions in $G$.

We will use $D_8$ to denote a dihedral group of order 8 and $K_4$ to denote an elementary abelian group of order 4.

**Proposition 4.6.** *Suppose that $g$ is a real element of $G$ which has trivial defect group. Let $s$ and $t$ be involutions such that $g = st$. Then*

$$\beta(e_0, g) = 1_F + |\{u \in \mathcal{Q}_2 \mid \langle s, u\rangle \cong D_8, \langle u, t\rangle \cong K_4\}|1_F.$$

*Proof.* It follows from Lemma 4.5 that

$$\beta(e_0, g) = 1_F + |\Phi_1(g)|1_F,$$

where $\Phi_1(g) := \{(x, y) \in \mathcal{Q}_4 \times \mathcal{Q}_2 \mid xy = g\}$.

Define $(x, y)^t := (x^{-yt}, y^t)$, for $(x, y) \in \Phi_1(g)$. We check that $(x, y)^t \in \Phi_1(g)$ and that $\left((x, y)^t\right)^t = (x, y)$. In this way we get an action of the 2-group $\langle t\rangle$ on $\Phi_1(g)$.

Each orbit of $\langle t\rangle$ on $\Phi_1(g)$ has size 1 or 2. So

$$\beta(e_0, g) = 1_F + |\Phi_2(g)|1_F, \tag{4.7}$$

where $\Phi_2(g) := \{(x, y) \in \mathcal{Q}_4 \times \mathcal{Q}_2 \mid xy = g, x = x^{-yt}, y = y^t\}$.

Set $\Phi_3(g) := \{u \in \mathcal{Q}_2 \mid <s,u> \cong D_8, <u,t> \cong K_4\}$. Suppose that $(x,y) \in \Phi_2(g)$ and $u \in \Phi_3(g)$. We check that $yt \in \Phi_3(g)$ and $(su, ut) \in \Phi_2(g)$. Moreover the maps $\Phi_2(g) \to \Phi_3(g)$, $(x,y) \to yt$ and $\Phi_3(g) \to \Phi_2(g)$, $u \to (su, ut)$ are inverses of each other. Thus $|\Phi_2(g)| = |\Phi_3(g)|$. The proposition now follows from (4.7).

□

In a similar fashion we can show, with the notation above, that

$$\begin{aligned}\beta(e_0,g) &= 1_F + |\{\, u \in \mathcal{Q}_2 \mid <s,u> \cong <u,t> \cong D_8 \,\}| 1_F \\ &= 1_F + |\{\, (u,v) \in \mathcal{Q}_2 \times \mathcal{Q}_2 \mid <s,u> \cong <u,v> \cong <v,t> \cong K_4 \,\}| 1_F.\end{aligned}$$

The next result, which uses Proposition 4.6 in a crucial way, gives a partial converse to Corollary 3.6 and also generalizes Theorem 2 of [T71].

**Theorem 4.8.** *Suppose that no subgroup of $W := \mathbf{N}(D)/D$ is isomorphic to $D_8$. Then $\beta(e_D(G), g) = 1_F$, for each real 2-regular element $g$ with defect group $D$. In particular, the following are equivalent:*

(a). *$G$ has a real 2-regular element with defect group $D$;*
(b). *$G$ has a 2-block with defect group $D$;*
(c). *$G$ has a real 2-block with defect group $D$.*

*Proof.* We have

$$\begin{aligned}\beta(e_D(G), g) &= \beta(e_0(W), \pi(g)), \quad \text{by Lemma 4.1} \\ &= 1_F, \quad \text{using the hypothesis and Proposition 4.6.}\end{aligned}$$

This proves the first assertion.

Suppose that $G$ has a real 2-regular element $g$ with defect group $D$. Then $\beta(e_D(G), g) = 1_F$. So $G$ has a real 2-block with defect group $D$, by Lemma 2.2. Thus (a) $\implies$ (c).

The implication (c) $\implies$ (b) is trivial.

The implication (b) $\implies$ (a) follows from Corollary 3.6. This completes the proof. □

In the situation of Theorem 4.8, we can sometimes guarantee the existence of more than one block with defect group $D$.

**Corollary 4.9.** *Suppose that no subgroup of $W := \mathbf{N}(D)/D$ is isomorphic to $D_8$. Suppose further that $G$ has $r$ real 2-regular classes, $\mathcal{K}_1, \ldots, \mathcal{K}_r$, with defect group $D$, which have non-conjugate extended defect groups. Then $G$ has at least $r$ distinct real 2-blocks with defect group $D$.*

*Proof.* It follows from Theorem 4.8 that $\beta(e_D(G), \mathcal{K}_i^+) = 1_F$, for $i = 1, \ldots, r$. Corollary 3.2 implies that if $1 \le i \le r$, then $G$ has a 2-block $B_i \leftrightarrow e_i \leftrightarrow \omega_i$ with defect group $D$ such that $\omega_i(\mathcal{K}_i^+) \neq 0_F$. Moreover, as in the proof of Lemma 2.2, we may choose each $B_i$ to be real.

As the principal 2-block is the only real 2-block with maximal defect, and the trivial class is the only real class with maximal defect, we may assume that $D$ is not a Sylow 2-subgroup of $G$. It then follows from Corollary 2.2 of [G88] that $B_i \neq B_j$, for all distinct $i, j$ from $\{1, \ldots, r\}$. This completes the proof.

□

**Note 4.10.** The condition that no subgroup of $W$ is isomorphic to $D_8$ is equivalent to the condition that all the involutions in a Sylow 2-subgroup of $W$ commute with each other and thus generate an elementary abelian 2-subgroup of $W$. Corollary

1 of [GL74], taken in conjunction with later work of Bombieri and Thompson on the classification of groups of Ree type, implies that if $G$ is a simple group with no subgroup isomorphic to $D_8$, $G$ is isomorphic either to a group of Lie type of rank 1 over a field of even characteristic, or to $\mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \bmod 8$ and $q > 3$, or to the Janko group $J_1$, or to a Ree group ${}^2G_2(3^{2n+1})$. Excluding the group $\mathrm{PSL}_2(2^2)$ which is isomorphic to $\mathrm{PSL}_2(5)$, the groups of Lie type of rank 1 over a field of even characteristic are $\mathrm{PSL}_2(2^n)$, $n \geq 3$, $\mathrm{Sz}(2^{2n+1})$, $n \geq 1$, and $\mathrm{PSU}_3(2^n)$, $n \geq 2$, and each of these groups has a unique 2-block of defect 0, which contains the Steinberg character.

We say that $D$ is a *maximal Sylow 2-intersection* in $G$ if $D = S \cap T$, where $S \neq T$ are Sylow 2-subgroups of $G$, and if $D \leq P \cap Q$, where $P \neq Q$ are Sylow 2-subgroups of $G$, then $D = P \cap Q$. The following is a special case of Theorem 4.8.

**Corollary 4.11.** *Suppose that $D$ is a maximal Sylow 2-intersection in $G$. Then the conditions (a), (b) and (c) of Theorem 4.8 are equivalent for $G$.*

*Proof.* The hypothesis implies that $W = \mathbf{N}(D)/D$ has a trivial intersection Sylow 2-subgroup. An elementary argument (due to M. Suzuki) shows that $W$ has no subgroups isomorphic to $D_8$. The result now follows from Theorem 4.8.

□

Our next corollary can also be proved using the Brauer-Suzuki theorem, but the methods developed in this section provide a self-contained approach.

**Corollary 4.12.** *Suppose that a Sylow 2-subgroup of $G$ is generalized quaternion or cyclic. Then either $G$ has a unique involution or it has a real 2-block of defect 0.*

*Proof.* Suppose that $G$ has two different involutions $s$ and $t$. Consider their product $st$. This is not the identity. We claim that $st$ has odd order. For if this is not the case, there exists an involution $u$ which commutes with both $s$ and $t$. This contradicts the fact that $G$ contains no elementary abelian subgroup of order 4. Thus $st$ has odd order. Also $st$ is real, since it is inverted by $t$.

Let $S$ be a Sylow 2-subgroup of the extended centralizer $\mathbf{C}^*(st)$ of $st$ which contains $t$. Then $t$ is the unique involution in $S$, but $t \notin \mathbf{C}(st)$. Since $\mathbf{C}(st) \cap S$ is a Sylow 2-subgroup of $\mathbf{C}(st)$, it follows that this latter group has odd order. In particular $st$ has defect zero.

Finally, as a Sylow 2-subgroup contains no dihedral subgroup of order 8, it follows from Theorem 4.8 that $G$ has a real 2-block of defect 0.

□

The Brauer-Suzuki theorem gives the more precise information that any real defect zero element of $G$ lies in $O_{2'}(G)$. This strengthens Proposition 5 of [T74].

**Corollary 4.13.** *Suppose that $G$ does not possess subgroups $H$ and $K$ with $H \triangleleft K$ and $K/H \cong D_8$. Then all 2-blocks of $G$ have maximal defect if and only if $G$ has a normal Sylow 2-subgroup.*

*Proof.* Let $S$ be a Sylow 2-subgroup of $G$. We may suppose that $|S| > 1$. Theorem 4.8 implies that all 2-blocks of $G$ have maximal defect if and only if $G$ has no non-identity real 2-regular elements. Now if $S$ is normal in $G$, it is elementary to check that $G$ has no non-identity real 2-regular elements. Conversely, suppose that

$G$ has no non-identity real 2-regular elements. Then $\bar{G} := G/O_2(G)$ also has no non-identity real 2-regular elements. We claim that $\bar{G}$ has odd order. For let $s$ an $t$ be involutions in $\bar{G}$. Then $s$ inverts every element in $< st >$ and hence $st$ has 2-power order. It follows from the hypothesis on the subgroups of $G$ that $(st)^2 = 1$. In particular $s$ and $t$ commute. So the involutions in $\bar{G}$ form a normal 2-subgroup. We conclude that $\bar{G}$ has no involutions, which proves our claim.

□

The following result, which generalizes [H68], is a special case of the previous corollary.

**Theorem 4.14.** *Suppose that a Sylow 2-subgroup $S$ of $G$ is abelian, or is a direct product of a quaternion group of order 8 with an elementary abelian 2-group. Then all 2-blocks of $G$ have maximal defect if and only if $S$ is normal in $G$.*

The groups $S$ in the theorem above are precisely the Dedekind 2-groups, i.e. those 2-groups all of whose subgroups are normal.

## 5. Blocks and Normal Subgroups

Let $N$ be a normal subgroup of $G$. We say that a $p$-block $B \leftrightarrow e \leftrightarrow \omega$ of $G$ *covers* a $p$-block $b \leftrightarrow e_b \leftrightarrow \omega_b$ of $N$ if the restriction of some irreducible character in $B$ has an irreducible constituent in $b$. It can be shown that each block of $G$ covers a $G$-orbit of blocks of $N$. More precisely $B$ covers $b$ if and only if $e = e\, e_b^G$, where $e_b^G$ denotes the sum of the distinct $G$-conjugates of $e_b$ in $G$. We will use $\mathrm{Bl}(G|b)$ to denote the set of blocks of $G$ which cover $b$.

Suppose now that $B$ covers $b$. We say that $B$ is *weakly regular* (relative to $N$) if it has maximal defect among the blocks of $G$ which cover $b$. It follows from Theorem 5.5.16 of [NT89] that the weakly regular blocks in $\mathrm{Bl}(G|b)$ have a common defect group, $D$ say. Moreover, $D \cap N$ is a defect group of $b$ and $DN/N$ is a Sylow $p$-subgroup of $I(b)/N$, where $I(b)$ is the inertial subgroup of $b$ in $G$. We call $D$ a defect group of $b$ in $G$. Those blocks in $\mathrm{Bl}(G|b)$ which are not weakly regular have defect groups strictly contained in $D$. We now specialize to $p = 2$.

**Lemma 5.1.** *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of $N$. Then the number of weakly regular blocks which cover $b$ is odd and*

$$\beta(e_b^G, \mathcal{K}^+) = \omega(\mathcal{K}^{o+}),$$

*for each 2-block $B \leftrightarrow e \leftrightarrow \omega$ of $G$ which covers $b$ and each 2-regular class $\mathcal{K}$ of $G$ which is contained in $N$ and which has defect group $D$.*

*Proof.* Let $B_i \leftrightarrow e_i \leftrightarrow \omega_i$, $1 \leq i \leq s$, be a complete list of the blocks of $G$ which cover $b$, ordered so that $B_1, \ldots, B_t$ are all the weakly regular blocks. It follows from Theorem 5.5.5 of [NT89] that

$$\omega_i(\mathcal{K}^+) = \omega_j(\mathcal{K}^+), \tag{5.2}$$

for each class $\mathcal{K}$ of $G$ which is contained in $N$, and each pair $i, j \in \{1, \ldots, s\}$. So we can assume, and we do, that $B$ is weakly regular.

We may write

$$e_1 + \cdots + e_s = e_b^G = \sum \beta(e_b^G, \mathcal{K}^+)\mathcal{K}^+, \tag{5.3}$$

where $\mathcal{K}$ ranges over the 2-regular classes of $G$ which are contained in $N$. Since $B$ covers $b$, we have

$$1_F = \omega(e) = \omega(e_b^G) = \sum \beta(e_b^G, \mathcal{K}^+)\omega(\mathcal{K}^+). \tag{5.4}$$

It follows that there is at least one 2-regular class $\mathcal{L}$ contained in $N$ with

$$\beta(e_b^G, \mathcal{L}^+) \neq 0_F \quad \text{and} \quad \omega(\mathcal{L}^+) \neq 0_F.$$

As $B$ has defect group $D$, we deduce from Lemma 2.1 that $D$ is contained in a defect group of $\mathcal{L}$. Furthermore, it follows from (5.3) that there is an index $j$, with $1 \leq j \leq s$, such that $\beta(e_j, \mathcal{L}^+) \neq 0_F$. Since $B_j$ has a defect group contained in $D$, Lemma 2.1 implies that so too does $\mathcal{L}$. We conclude that $\mathcal{L}$ has defect group $D$.

Now $B_i$ has defect group $D$, for $1 \leq i \leq t$, and $B_i$ has a defect group strictly contained in $D$, for $t < i \leq s$. So

$$\beta(e_i, \mathcal{L}^+) = \begin{cases} \omega_i(\mathcal{L}^{o+}), & \text{for } 1 \leq i \leq t, \quad \text{by Proposition 3.1,} \\ 0_F, & \text{for } t < i \leq s, \quad \text{by Lemma 2.1.} \end{cases}$$

It follows that

$$\beta(e_b^G, \mathcal{L}^+) = \sum_{i=1}^{t} \omega_i(\mathcal{L}^{o+}) = t^*\omega(\mathcal{L}^{o+}), \quad \text{using (5.3) and (5.2).}$$

In particular, $t$ is odd. Finally, if $\mathcal{K}$ is any 2-regular class of $G$ with defect group $D$ that is contained in $N$, identical arguments now show that $\beta(e_b^G, \mathcal{K}^+) = \omega(\mathcal{K}^{o+})$. □

Easy examples show that the following corollary is false when $p \neq 2$.

**Corollary 5.5.** *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of $N$. Then $b$ is $G$-conjugate to $b^o$ if and only if some real weakly regular 2-block of $G$ covers $b$.*

*Proof.* The 'if' part of this statement is straightforward.

Suppose that $b$ is $G$-conjugate to $b^o$, and that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of $G$ which covers $b$. Then $B$ covers $b^o$. So

$$\omega^o(\mathcal{K}^+) = \omega(\mathcal{K}^{o+}) = \omega_b^o(\mathcal{K}^{o+}) = \omega_b(\mathcal{K}^+),$$

for each class $\mathcal{K}$ of $G$ which is contained in $N$, using Theorem 5.5.5 of [NT89]. It follows that $B^o$ covers $b$, again using Theorem 5.5.5 of [NT89]. Now $B$ is weakly regular if and only if $B^o$ is weakly regular. So the number of non-real weakly regular blocks in $\mathrm{Bl}(G|b)$ is even. We deduce from Lemma 5.1 that there are an odd number of real weakly regular blocks in $\mathrm{Bl}(G|b)$, proving the 'only if' part. □

We let $\mathcal{R}_D^N$ denote the union of the real 2-regular classes of $G$ which have defect group $D$ and which are contained in $N$. The following result generalizes Proposition 3.4:

**Proposition 5.6.** *Suppose that $b \leftrightarrow e_b \leftrightarrow \omega_b$ is a 2-block of $N$ with defect group $D$ in $G$. Then $\sum_{\mathcal{K} \subseteq \mathcal{R}_D^N} \beta(e_b^G, \mathcal{K}^+) = 1_F$. Also $\omega(\mathcal{R}_D^{N+}) = 1_F$, for each 2-block $B \leftrightarrow e \leftrightarrow \omega$ of $G$ which covers $b$.*

*Proof.* We have

$$\begin{aligned} 1_F &= \sum \beta(e_b^G, \mathcal{K}^+)\,\omega(\mathcal{K}^+), \quad \text{by (5.4)},\\ &= \sum \omega(\mathcal{K}^{o+})\,\omega(\mathcal{K}^+), \quad \text{by Lemma 5.1}, \end{aligned}$$

where $\mathcal{K}$ ranges over the 2-regular classes of $G$ which have defect group $D$ and which are contained in $N$. The proof now proceeds along the same lines as that of Proposition 3.4.

□

We can now prove the following generalization of Theorem 3.5:

**Theorem 5.7.** *Every 2-block of $G$ which is weakly regular relative to $N$ has a real defect class which is contained in $N$.*

*Proof.* This follows immediately from Propositions 3.1 and 5.6.

□

The aim of the remainder of this section is to prove Theorem 5.10. This result, which is concerned with the reality of elements in a non-trivial normal subgroup of odd order, does not appear to have a purely group-theoretic proof. The basic idea for the next lemma goes back to Theorem 5E of [BF55] and was also used in our context by [W77].

**Lemma 5.8.** *Let $N$ be a normal subgroup of odd order in $G$. Every conjugacy class of $G$ which is contained in $N$ is a defect class for some real weakly regular 2-block of $G$.*

*Proof.* Let $\mathcal{K}$ be a conjugacy class of $G$ which is contained in $N$, and let $k$ be an element of $\mathcal{K}$. We define a class function $\theta$ of $G$ by

$$\theta := \sum_{\psi} \omega_\psi(\mathcal{K}^{o+})\psi(k)\psi,$$

where $\psi$ ranges over the $K$-irreducible characters of $G$. Let $P$ be a Sylow 2-subgroup of $G$ and let $g$ be a non-identity element of $P$. Since no conjugate of $g$ can be expressed as a product of elements of $N$, it follows that $\theta(g) = 0$. See, for example, Problem 3.9 of [I94]. Also $\theta(1) = |G|$. So

$$|G : P| = \frac{\theta(1)}{|P|} = (\theta_P, 1_P) = \sum_{\psi} \omega_\psi(\mathcal{K}^{o+})\psi(k)(\psi_P, 1_P),$$

where $\psi_P$ denotes the restriction of $\psi$ to $P$, and $1_P$ denotes the principal character of $P$. Also $\omega_\psi(\mathcal{K}^{o+})\psi(k) = \omega_{\psi^o}(\mathcal{K}^{o+})\psi^o(k)$ and $(\psi_P, 1_P) = (\psi_P^o, 1_P)$, for each $\psi$, where $\psi^o$ denotes the complex conjugate of $\psi$. It follows that

$$|G : P| \equiv \sum \omega_\psi(\mathcal{K}^{o+})\psi(k)(\psi_P, 1_P) \bmod \mathcal{P},$$

where $\psi$ ranges over the real $K$-irreducible characters of $G$. Since $|G : P|$ is odd, there must be a real-valued irreducible $K$-character $\chi$ with $\chi(k) \not\equiv 0 \bmod \mathcal{P}$, $\omega_\chi(\mathcal{K}^{o+}) \not\equiv 0 \bmod \mathcal{P}$, and $(\chi, 1_P)_P$ an odd integer.

Let $B \leftrightarrow e \leftrightarrow \omega$ be the, necessarily real, 2-block of $G$ which contains $\chi$. Since $\chi(k) \not\equiv 0 \bmod \mathcal{P}$ and $\omega_\chi(\mathcal{K}^{o+}) \not\equiv 0 \bmod \mathcal{P}$, it follows that $\mathcal{K}$ is a defect class for $B$. So $B$ is weakly regular, using the definition of weak regularity given in Section 5.5 of [NT89].

□

The following corollary is a straightforward application of a result of P. Fong.

**Corollary 5.9.** *Suppose that $G$ is solvable. Then $G$ has a 2-block of non-maximal defect if and only if $G$ has a real non-principal 2-block.*

*Proof.* Any real non-principal 2-block necessarily has non-maximal defect.

On the other hand, suppose that $G$ has a 2-block of non-maximal defect. Then the 2-regular core $O_{2'}(G)$ of $G$ contains a class $\mathcal{K}$ of non-maximal defect, by Theorem (1G)(ii) of [F62]. Thus $G$ has a real 2-block of non-maximal defect, using the previous lemma.

□

We note that Corollary 5.9 need not hold for a group that is not solvable. For example, the Mathieu group $M_{11}$ has exactly three 2-blocks, namely, the principal block and two non-real blocks of defect 0. Theorem 6.6 of this paper shows that this phenomenon can only occur in a group whose Sylow 2-subgroup has order at least 16 (the Sylow 2-subgroup of $M_{11}$ has order 16).

We now prove our main result on the existence of real elements in normal subgroups of odd order.

**Theorem 5.10.** *Let $N$ be a normal subgroup of $G$ which has odd order, and let $D$ be a 2-subgroup of $G$. Then $G$ has a conjugacy class with defect group $D$ which is contained in $N$ if and only if $G$ has a real conjugacy class with defect group $D$ which is contained in $N$.*

*Proof.* Suppose that $G$ has a conjugacy class which is contained in $N$ and which has defect group $D$. Then $G$ has a real 2-block $B$ which is weakly regular and which also has defect group $D$, using Lemma 5.8. So $N$ contains a real defect class for $B$, by Theorem 5.7. This class has defect group $D$. The result follows.

□

## 6. Non-Real 2-Blocks

Let $G_D$ denote the union of the 2-regular classes of $G$ which have defect group $D$. The following result is similar to Proposition 3.4:

**Proposition 6.1.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a 2-block of $G$ with defect group $D$. Then* $\omega(G_D^+) = \begin{cases} 1_F, & \text{if } \omega = \omega^o; \\ 0_F, & \text{if } \omega \neq \omega^o. \end{cases}$

*Proof.* We have

$$\omega(e^o) = \sum \beta(e^o, \mathcal{K}^+)\, \omega(\mathcal{K}^+),$$

where $\mathcal{K}$ ranges over the 2-regular classes with defect group $D$. Using Corollary 3.2, we have

$$\omega(e^o) = \sum \omega(\mathcal{K})^2 = \left(\sum \omega(\mathcal{K}^+)\right)^2 = \omega(G_D^+)^2.$$

The proposition follows from the fact that $\omega(e^o) = \begin{cases} 1_F, & \text{if } \omega = \omega^o; \\ 0_F, & \text{if } \omega \neq \omega^o. \end{cases}$

□

This result can be used to gain additional group-theoretic information when $G$ has a non-real 2-block whose defect group is not a Sylow 2-subgroup of $G$. Let $G_{2'}$ denote the set of 2-regular elements of $G$. We begin with the proof of a result that is quite well known.

**Lemma 6.2.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a non-principal 2-block of $G$. Then $\omega(G_{2'}^+) = 0_F$.*

*Proof.* Let $\chi$ be an irreducible $K$-character in $B$, and let $d$ be the defect of $B$. We define a class function $\theta$ on $G$ by setting

$$\theta(g) := \begin{cases} 2^d\chi(g), & \text{if } g \in G_{2'}; \\ 0, & \text{if } g \in G \setminus G_{2'}. \end{cases}$$

It is known that $\theta$ is an integral combination of the irreducible $K$-characters in $B$. See, for example Lemma 3.6.33 (i) of [NT89]. Since $B$ is not the principal block, $(\theta, 1_G) = 0$, where $1_G$ is the principal character of $G$. It follows that $\chi(G_{2'}^+) = 0$, and hence that $\omega_\chi(G_{2'}^+) = 0$. The result follows. □

We proceed to the main result of this section.

**Theorem 6.3.** *Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a non-real 2-block of $G$ whose defect group $D$ is not a Sylow 2-subgroup of $G$. Then there exists a non-identity 2-regular class $\mathcal{K}$ of $G$, whose defect group strictly contains $D$, such that $\omega(\mathcal{K}^+) \neq 0_F$.*

*Proof.* Certainly $B$ is not the principal block of $G$ (as $B \neq B^o$). So $\omega(G_{2'}^+) = 0_F$, by the previous lemma. Now $\omega(\mathcal{K}^+) = 0_F$, if a defect group of a class $\mathcal{K}$ does not contain $D$, using Lemma 2.1. Moreover, $\omega(G_D^+) = 0_F$, by Proposition 6.1, and $\omega(1_G) = 1_F$. Since $D$ is not a Sylow 2-subgroup of $G$, it follows that $\sum \omega(\mathcal{K}^+) = 1_F$, where $\mathcal{K}$ ranges over the non-identity 2-regular conjugacy classes of $G$ that have a defect group which strictly contains $D$. The result follows. □

Theorem 4.8 shows that, in certain circumstances, the existence of a non-real 2-block with defect group $D$ implies the existence of a real 2-block with the same defect group. We end this section by considering an extension of this idea.

The proof of the following result relies on transfer techniques. We omit the details, which are well known.

**Lemma 6.4.** *Suppose that the finite group $G$ has a dihedral Sylow 2-subgroup. Let $t$ be an involution in $G$. Then $\mathbf{C}(t)$ has a normal 2-complement.*

While we are primarily interested in the prime 2, the next result, giving a sufficient condition for the existence of a non-principal $p$-block, holds for any prime $p$.

**Proposition 6.5.** *Let $D$ be a $p$-subgroup of $G$ and let $E$ be a $p$-subgroup of $\mathbf{N}(D)$ which is not a Sylow $p$-subgroup of $\mathbf{N}(D)$. Set $W := \mathbf{N}(D)/D$. Suppose that $G$ has a $p$-regular class with defect group $E$ and that $\mathbf{N}_W(E/D)$ has a normal $p$-complement. Then $G$ has a non-principal $p$-block $B_G$ which has a defect group containing $E$.*

*Proof.* Recall that $\pi : F\mathbf{N}(D) \to FW$ is the $F$-algebra epimorphism induced by the natural group epimorphism $\pi : \mathbf{N}(D) \to W$. Let $k$ be a $p$-regular element of $G$ which has defect group $E$, let $\mathcal{K}$ be the conjugacy class of $\mathbf{N}(D)$ which contains

$k$, and let $\mathcal{L}$ be the conjugacy class of $W$ which contains $\pi(k)$. Then, as in Lemma 4.1, the class $\mathcal{L}$ has defect group $J := E/D$ in $W$ and $\pi(\mathcal{K}^+) = \mathcal{L}^+$.

Set $H = \mathbf{N}_W(J)$. Now $\mathcal{L} \cap \mathbf{C}_W(J)$ is a $p$-regular class of $H$ which has defect group $J$, and by hypothesis $H$ has a normal $p$-complement. Using Theorem 1. of [T77], we see that $H$ has a $p$-block $B_H \leftrightarrow \omega_H$, with defect group $J$, such that $\omega_H((\mathcal{L} \cap \mathbf{C}_W(J))^+) \neq 0_F$. Brauer's first main theorem then implies that $W$ has a $p$-block $B_W \leftrightarrow \omega_W$, with defect group $J$, such that $\omega_W(\mathcal{L}^+) \neq 0_F$.

Let $B_N \leftrightarrow \omega_N$ be the unique $p$-block of $\mathbf{N}(D)$ which dominates $B_W$, and let $R$ be a defect group of $B_N$. Then $R$ contains $D$, by a theorem of Brauer, and $J$ is conjugate in $W$ to a subgroup of $R/D$, by Theorem 5.8.7 (ii) of [NT89]. Since $E/D = J$, this implies that $E$ is conjugate in $\mathbf{N}(D)$ to a subgroup of $R$. Also $\omega_N(\mathcal{K}^+) = \omega_W \circ \pi(\mathcal{K}^+) = \omega_W(\mathcal{L}^+) \neq 0_F$. So some conjugate of $R$ is contained in $E$, the defect group of $\mathcal{L}$. It follows that $B_N$ has defect group $E$.

Since $E\mathbf{C}(E) \leq \mathbf{N}(D)$, Corollary 5.3.7 of [NT89] guarantees that the induced block $B_G := B_N^G$ is defined. But $B_N$ is not the principal $p$-block of $\mathbf{N}(D)$, since $E$ is not a Sylow $p$-subgroup of $\mathbf{N}(D)$. So $B_G$ is not the principal $p$-block of $G$, by Brauer's third main theorem (see Theorem 5.6.1 of [NT89]). Finally, it is a standard fact that some defect group of $B_G$ contains $E$.

□

**Theorem 6.6.** *Suppose that $G$ has a non-real 2-block with defect group $D$, such that a Sylow 2-subgroup of $W := \mathbf{N}(D)/D$ has order 8. Then $G$ either has a real 2-block with defect group $D$ or it has a non-principal 2-block with defect group strictly containing $D$.*

*Proof.* Proposition 4.6 implies $G$ has a real 2-block with defect group $D$ unless possibly $W$ has a dihedral Sylow 2-subgroup. We may therefore suppose that $W$ has a dihedral Sylow 2-subgroup of order 8. Now Theorem 6.3 implies that $\mathbf{N}(D)$ has a non-trivial 2-regular element $g$ with defect group $E$ strictly containing $D$.

If $E$ is a Sylow 2-subgroup of $\mathbf{N}(D)$, then $\mathbf{N}(D)$ and hence $G$ has a non-principal 2-block with defect group strictly containing $D$. If $E$ has defect 1 less than maximal, then a standard argument shows that $\mathbf{N}(D)$ has a non-principal 2-block with defect group $E$, and hence $G$ has a non-principal 2-block with defect group strictly containing $D$.

If $[E : D] = 2$, then $\mathbf{N}_W(E/D)$ has a normal 2-complement, using Lemma 6.4. So $\mathbf{N}(D)$ has a real non-principal 2-block whose defect group contains $E$, by Proposition 6.5. It follows that $G$ has a real non-principal 2-block with a defect group which strictly contains $D$.

□

## References

[B76] R. Brauer, *Notes on representations of finite groups, I*, J. London Math. Soc. **90** (1976), 162-166.

[BF55] R. Brauer, K. A. Fowler, *On groups of even order*, Ann. Math. **62** (1955), 565–583.

[F62] P. Fong, *Solvable groups and modular representation theory*, Trans. Amer. Math. Soc. **103** (1962), 484–494.

[GL74] D. M. Goldschmidt, *2-Fusion in finite groups*, Ann. Math. **99** (1974), 70–117.

[G88] R. Gow, *Real 2-blocks of characters of finite groups*, Osaka J. Math. **25** (1988), 135–147.

[H68] K. Harada, *On groups all of whose 2-blocks have the highest defects*, Nagoya Math. J. 32 (1968), 283–286.

[I94] I. M. Isaacs, *Character Theory of Finite Groups*, Dover Publ., Inc., New York (1994).
[M99] J. Murray, *Blocks of defect zero and products of elements of order p*, J. Algebra 214 (1999), 385–399.
[NT89] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, New York, English Ed. (1989).
[O55] M. Osima, *Notes on blocks of group characters*, Math. J. Okayama Univ. **4** (1955), 175–188.
[O66] M. Osima, *On block idempotents of modular group rings*, Nagoya Math. J. **27** (1966), 429–433.
[T71] Y. Tsushima, *On the block of defect zero*, Nagoya Math. J. **44** (1971), 57–59.
[T74] Y. Tsushima, *On the existence of characters of defect zero*, Osaka J. Math. **11** (1974), 417–423.
[T77] Y. Tsushima, *On the weakly regular p-blocks with respect to* $O_{p'}(G)$, Osaka Math. J. 14 (1977), 465–470.
[W77] T. Wada, *On the existence of p-blocks with given defect groups* Hokkaido Math. J. 6 (1977), 243–248.

Roderick Gow,
Mathematics Department,
University College Dublin,
Belfield, Dublin 4.
email: rod.gow@ucd.ie

John Murray,
Mathematics Department,
University College Dublin,
Belfield, Dublin 4.
email: jcmurray@eircom.net