ON A CERTAIN IDEAL OF KÜLSHAMMER IN THE CENTRE OF A GROUP ALGEBRA

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Abstract. Let G be a finite group and let F be a splitting field of characteristic $p > 0$. We show that $I^2 = E_0$, where I is a certain ideal of the centre Z of FG, and E_0 is the span of the block idempotents of defect zero.

Let G be a finite group and let F be a field of characteristic p. We shall assume that F is a splitting field for G. Let λ denote the linear map $FG \to F$, given by

$$\lambda(\sum_{g\in G} a_g g) = a_1,$$

for $\sum_{g\in G} a_g g \in FG$. The map $B : FG \times FG \to F$,

$$B(a, b) = \lambda(ab), \quad \text{for } a, b \in FG,$$

is a nondegenerate symmetric bilinear form on FG. In fact B is *associative* in the sense that

$$B(ab, c) = B(a, bc), \quad \text{for } a, b, c \in FG.$$

If V is an F-subspace of FG, then $V^\perp$ will denote the dual space

$$V^\perp := \{a \in FG \mid B(a, b) = 0, \text{ for all } b \in V\}.$$

If A is a subalgebra of FG, and V is a right A-module, then $V^\perp$ is a left A-module. Let $K := Z^\perp$, where Z is the centre of FG. Then by the above remarks, K is a Z-submodule of FG. It is clear that

$$(1) \qquad K = \{\sum_{g\in G} a_g g \mid \sum_{g\in\mathcal{K}} a_g = 0, \text{ for all conjugacy classes } \mathcal{K} \text{ of } G\},$$

as the class sums $\mathcal{K}^+ := \sum_{g\in\mathcal{K}} g$ form an F-basis for Z. The following lemma is due to R. Brauer [B56]:

Lemma 2.

$$a \in K \implies a^p \in K,$$

$$(a+b)^p \equiv a^p + b^p \bmod K,$$

for all $a, b \in FG$.

Set

$$N := \begin{cases} 4, & \text{if } p = 2, \\ p, & \text{if } p \text{ is an odd prime,} \end{cases}$$

and define

$$T := \{x \in FG \mid x^N \in K\}.$$

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Then Lemma 2 implies that T is a subspace of FG which contains K. An easy argument shows that T is a Z-submodule of FG. Since $T \supseteq K$, it follows that

$$I := T^{\perp}$$

is an ideal of Z.

For each conjugacy class $\mathcal{K}$, set

$$\Omega(\mathcal{K}) := \{g \in G \mid g^N \in \mathcal{K}\}.$$

So $\Omega(\mathcal{K})$ is a union of conjugacy classes of G. We have the following (see (38) of [K91]):

Lemma 3.

$$T = \{\sum_{g \in G} a_g g \mid \sum_{g \in \Omega(\mathcal{K})} a_g = 0, \textit{ for all classes } \mathcal{K}\}.$$

Thus $\{\Omega(\mathcal{K})^+ \mid \mathcal{K}$ *a conjugacy class of* G, $\Omega(\mathcal{K}) \neq \phi\}$ *forms a basis for* I.

Proof. Say $a = \sum_{g \in G} a_g g \in FG$. Then

$$\begin{aligned} a \in T &\iff (\sum_{g \in G} a_g g)^N \in K, \text{ by definition of } T \\ &\iff \sum_{g \in G} a_g^N g^N \in K, \text{ using Lemma 2} \\ &\iff \sum_{g^N \in \mathcal{K}} a_g^N = 0, \text{ for all classes } \mathcal{K}, \text{ by (1)} \\ &\iff \sum_{g \in \Omega(\mathcal{K})} a_g = 0, \text{ for all classes } \mathcal{K}, \text{ as } F \text{ has characteristic } p. \end{aligned}$$

The last statement now follows from the first. □

Proposition 4. *Let* $z \in Z$ *and suppose that* $z^N = 0$. *Then* $Iz = zI = 0$.

Proof. Let $i \in I$ and $x \in FG$. It follows from the hypothesis that $zx \in T$. Thus $B(iz, x) = B(i, zx) = 0$. Since $x \in FG$ was arbitrary, the nondegeneracy of λ implies that $iz = 0$. □

Let Z_0 denote the F-subspace of Z spanned by the class sums of p-defect zero, and let E_0 denote the F-subspace of Z_0 spanned by the block idempotents of defect zero. Then Z_0 is an ideal of Z, using an argument due to R. Brauer. Moreover Iizuka and Watanabe [IW73] have shown that

$$(Z_0)^2 = E_0, \tag{5}$$

and

$$Z_0\, J(FG) = 0, \tag{6}$$

where $J(FG)$ is the Jacobson radical of FG. See (1.E) of [O80] also. A proof of the following result was indicated in [M99]:

Lemma 7. $I^2 \subseteq Z_0$.

Proof. Let $\mathcal{K}, \mathcal{L}$ and $\mathcal{M}$ be classes of G. The coefficient of $\mathcal{M}^+$ in $\Omega(\mathcal{K})^+\Omega(\mathcal{L})^+$ is given as the cardinality, modulo p, of the set

$$\Phi(\mathcal{M}) := \{(k,l) \in \Omega(\mathcal{K}) \times \Omega(\mathcal{L}) \mid kl = m\},$$

where m is a fixed element of $\mathcal{M}$. Let D be a defect group of m. Then D acts by conjugation on the pairs in $\Phi(\mathcal{M})$. So $|\Phi(\mathcal{M})| \equiv |\Phi_D(\mathcal{M})| \bmod p$, where

$$\Phi_D(\mathcal{M}) := \{(k,l) \in (C(D) \cap \Omega(\mathcal{K})) \times (C(D) \cap \Omega(\mathcal{L})) \mid kl = m\}.$$

Now let $\Omega(Z(D))$ be the subgroup of $Z(D)$ consisting of all $z \in Z(D)$ such that $z^N = 1$. Then $\Omega(Z(D))$ acts freely on $\Phi_D(\mathcal{M})$ via

$$(k,l) \to (kz, z^{-1}l), \quad \text{for } (k,l) \in \Phi_D(\mathcal{K}), \text{ and } z \in \Omega(Z(D)).$$

It follows that $|\Phi_D(\mathcal{M})| \equiv 0 \bmod p$ unless $\Omega(Z(D)) = \{1\} \iff D = \{1\}$ i.e. unless $\mathcal{M}$ has p-defect zero. The lemma follows. □

Corollary 8. *$I(I \cap J(FG)) = 0$ and hence $(I \cap J(FG))^2 = 0$.*

Proof. Suppose that $j \in I \cap J(FG)$. Then $j \in J(FG)$, as $I \subseteq Z$. Also $j^2 \in Z_0$, by Lemma 7. So $j^3 = j(j^2) = 0$, using (6). But then $j^N = j^{N-3}j^3 = 0$. Proposition 4 now implies that $I(I \cap J(FG)) = 0$. The equality $(I \cap J(FG))^2 = 0$ follows immediately. □

We can now prove our main result.

Theorem 9. $I^2 = E_0$.

Proof. Let E denote the F-subspace of Z spanned by the block idempotents. Then

$$Z = E \oplus J,$$

as F-algebras, where $J = Z \cap J(FG)$ is the Jacobson radical of Z. Now J is nil and the map $x \to x^p$ is an automorphism of F. It follows that there exists $m \geq 0$ such that $e^{p^m} = e$ and $j^{p^m} = 0$, for all $e \in E$ and $j \in J$.

If $i_1, i_2 \in I$, write

$$i_k = e_k + j_k, \quad (k = 1, 2),$$

where $e_k \in E$ and $j_k \in J$. Then $e_k = e_k^{p^m} + j_k^{p^m} = i_k^{p^m} \in I$. It follows that $e_k \in I$ and $j_k \in I \cap J(FG)$. So

$$i_1 i_2 = e_1 e_2 + e_1 j_2 + j_1 e_2 + j_1 j_2 = e_1 e_2,$$

using Corollary 8. Thus $I^2 \subseteq E \cap Z_0 = E_0$, using Lemma 7.

The oppositite inequality $I^2 \supseteq E_0$ follows from $I \supseteq Z_0$ and (5). □

We also have:

Proposition 10. *Let $\mathcal{K}$ be a p-singular class of G. Then $\Omega(\mathcal{K})^+ \in J(FG)$. In particular, $\Omega(\mathcal{K})^+\Omega(\mathcal{L})^+ = 0$, for each class $\mathcal{L}$ of G.*

Proof. Let B be a p-block of G, with associated central character ω. If B has positive defect, then $\omega((\Omega(\mathcal{K})^+)^2) = 0$, using Lemma 7, and so $\omega(\Omega(\mathcal{K})^+) = 0$. On the other hand, if B has defect zero, then $\omega(\Omega(\mathcal{K})^+) = 0$, as $\Omega(\mathcal{K})$ is a union of p-singular classes. We deduce that $\Omega(\mathcal{K})^+ \in J(FG)$. The last statement now follows from Corollary 8. □

If $g \in G$, we may write $g = g_p g_{p'} = g_{p'} g_p$, for a unique p-element g_p and a unique p-regular element $g_{p'}$. We call g_p the *p-part of g* and $g_{p'}$ the *p-regular part of g*. Let $\mathcal{K}$ be a p-regular class of G. The p-regular section $S(\mathcal{K})$ of G which contains $\mathcal{K}$ is defined as

$$S(\mathcal{K}) := \{g \in G \mid g_{p'} \in \mathcal{K}\}.$$

Setting $\mathcal{L}^N = \{g^N \mid g \in \mathcal{L}\}$, for each class $\mathcal{L}$ of G, we note that

$$S(\mathcal{K}) = \bigcup_{\mathcal{L} \subset S(\mathcal{K})} \Omega(\mathcal{L}^N).$$

The p-regular section sums $S(\mathcal{K})^+$ span an ideal R of Z, known as *Reynolds Ideal.* We have the following chain of ideals of Z:

$$E_0 \subseteq Z_0 \subseteq R \subseteq I.$$

Now $R = J(FG)^\perp \cap Z$, by (39) of [K91]. It follows easily that

$$R^2 = E_0.$$

So Theorem 9 is an improvement on this fact.

Corollary 11. *Suppose that $\mathcal{K}, \mathcal{L}$ are p-regular classes of G. Then*

$$S(\mathcal{K})^+ S(\mathcal{L})^+ = \Omega(\mathcal{K}^N)^+ \Omega(\mathcal{L}^N)^+.$$

Proof. This follows from Proposition 10, and the fact that $\mathcal{K}$ and $\mathcal{L}$ are the only p-regular classes in $S(\mathcal{K})$ and $S(\mathcal{L})$, respectively. □

The following extends results in [IW73] and [M99]:

Corollary 12. *G has a p-block of defect zero if and only if there exists p-regular classes $\mathcal{K}, \mathcal{L}$ of G such that $\Omega(\mathcal{K})^+ \Omega(\mathcal{L})^+ \neq 0$, i.e. there exists $g \in G$ (necessarily of p-defect zero) such that the cardinality of the set*

$$\{(x, y) \in G \times G \mid x^N \in \mathcal{K}, y^N \in \mathcal{L}, xy = g\}$$

is nonzero modulo p.

We now give some examples in the exceptional case where $p = 2$ and F has characteristic 2. Set

$$T_1 := \{x \in FG \mid x^2 \in K\}.$$

Then

$$I_1 := T_1^\perp$$

is an ideal of Z, and has as F-basis $\{\Omega_1(\mathcal{K})^+\}$, where $\mathcal{K}$ ranges over the conjugacy classes of G, and

$$\Omega_1(\mathcal{K}) := \{g \in G \mid g^2 \in \mathcal{K}\}.$$

Although $E_0 \subseteq I_1^2 \subseteq Z_0$, and our results can be extended to show that $E_0 = I_1^3$, it is not generally true that $E_0 = (I_1)^2$. For instance, if $G = \mathcal{S}_7$, the symmetric group on 7-symbols, then $0 = E_0 \subset I_1^2 = Z_0$, while if $G = M_{23}$, the Mathieu group of degree 23, then $0 \subset E_0 \subset I_1^2 \subset Z_0$. On the other hand, if $G = \mathcal{S}_3$, then $E_0 = I_1^2 = Z_0$.

Let $\mathcal{R}$ denote the set of elements of G which have 2-defect zero and which are conjugate to their inverses. We showed in [M99] that

$$(\Omega_1(1_G)^+)^2 = \mathcal{R}^+.$$

It follows that $\mathcal{R}^+ Z \subseteq I_1^2$. We have not found an example where $\mathcal{R}^+ Z \neq I_1^2$.

References

[B56] Richard Brauer, *Zur Darstellungstheorie der Gruppen endlicher Ordnung* Math. Zeit. 63 (1956), 406–444.

[IW73] K. Iizuka, A. Watanabe, *On the number of blocks of irreducible characters of a finite group with a given defect group*, Kumamoto J. Sci. (Math.) 9 (1973), 55–61.

[K91] Burkhard Külshammer, *Group-theoretical descriptions of ring-theoretical invariants of group algebras*, Progress in Math. 95 (1991), 425-442.

[M99] John Murray, *Blocks of defect zero and products of elements of order p*, J. Algebra 214 (1999), 385–399.

[O80] T. Okuyama, *Some studies on group algebras*, Hokkaido Mathematical J. 9 (1980), 217-221.

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