Letters in Mathematical Physics **36**: 403–413, 1996.

The Existence of Generalised Self-Dual Chern–Simons Vortices

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(Received: 31 March 1995)

Abstract. We establish the existence and uniqueness of radially symmetric self-dual topological vortices in the $p = 2$ members of the hierarchies of (generalised) Chern–Simons Higgs and Abelian Higgs models. We also obtain all possible symmetric nontopological vortices in the Chern–Simons model characterised by an additional parameter governing the decay rates of the fields.

Mathematics Subject Classifications (1991): 81T13, 35Q51, 34A26.

Key words: Chern–Simons vortices, Abelian Higgs models.

1. Introduction

Vortices of the Abelian Higgs model [1] play an important role in various areas of physics and their mathematical analysis has attracted considerable interest. These vortices are finite energy and topologically stable solutions of the static Hamiltonian in 2 Euclidean dimensions, pertaining to the Abelian Higgs model in $(2+1)$-dimensional Minkowski space. More recently, it was discovered [2] that a particular U(1) Higgs model in which the Maxwell term was replaced by the Chern–Simons (CS) term, also had vortex solutions. In addition to having a different kinetic term, the CS Higgs model [2] differs from the former [1] also in that it has a different potential term – the usual quartic symmetry-breaking Higgs self-interaction potential of the Abelian Higgs model is replaced by a sextic symmetry breaking potential. As a consequence, the CS Higgs model has, in addition to topologically stable vortex solutions, another category of solutions for which the magnitude of the Higgs field vanishes at infinity. The latter are the nontopological vortices [3].

One of the most important features of the Abelian and CS Higgs vortices is that they can be self-dual, namely that the energies are minimised absolutely by a set of first-order Bogomol'nyi equations which saturate the corresponding

* Supported in part by CEC under grant HCM–ERBCHRXCT930362.
** Supported in part by NSF under grant DMS–9400243.

topological inequalities. In both models [1, 2], this saturation is possible only when the dimensionless coupling strength λ of the respective Higgs self-interaction potential equals one. In these cases, the stress tensor vanishes identically and there can be no forces between such vortices. In turn, one of the most important features of self-dual vortices is that when λ is not equal to one, there is a force between the vortices of the Abelian Higgs model which is attractive or repulsive according to whether λ is smaller or greater than one, respectively [4]. It is expected that a similar situation holds qualitatively for the CS Higgs vortices also. Thus the self-dual vortices demarcate the attractive and repulsive vortices. This is a very important physical property.

The vortices of both U(1) models [1, 2] are exponentially localised to an absolute scale, namely the asymptotic (vacuum) value of the magnitude of the Higgs field at infinity. The detailed properties of these vortices depend on the specific dynamics of the model in question. While the choice of the symmetry-breaking Higgs self-interaction potential in each case [1, 2] is determined by the requirement of topological stability, it is possible to add other positive definite terms to the system without invalidating the corresponding topological inequalities. Candidates for such terms are additional potential terms satisfying the same vacuum conditions and Skyrme-like kinetic terms involving higher derivatives of both the U(1) and the Higgs fields. The latter must satisfy the condition that no higher power than the square of the 'velocity' is featured, which means that these Skyrme-like terms must be the squares of some totally anti-symmetric forms, which in two dimensions means that the only acceptable Skyrme term is the quartic kinetic term. Such a modification of the respective models [1, 2] will result in a certain quantitative difference in the detailed properties of the vortices, and is interesting for this reason. If performed in an arbitrary manner, this procedure will lead to systems for which it is not possible to saturate the topological inequalities and find noninteracting self-dual vortices. Vortices of such models are not expected to be endowed with attractive and repulsive phases and, hence, are of little physical interest. Besides this, an arbitrary modification of the CS Higgs model would sacrifice the necessary topological inequalities as a consequence of solving the Gauss law constraint in this case.

The problems of modifying the two U(1) Higgs models, with Maxwell [1] and CS [2] dynamics, respectively, with appropriate Skyrme terms so that the generalised models have topologically stable and self-dual vortex solutions, were solved in [6] and in [5], respectively. We refer to these as generalised Abelian Higgs and the generalised CS Higgs models, respectively. In [5] and [6], no analytic proof for the existence of the radially symmetric solutions discussed there was given. This is the aim of the present Letter, with which we proceed below.

2. Formulation of the Problem

Recall that the generalised self-dual Chern–Simons Higgs equations obtained in [5] are

$$\begin{aligned} &(1-|\phi|^2)F_{12} = \mathrm{i}(D_1\phi[D_2\phi]^* - [D_1\phi]^*D_2\phi) + \tfrac{1}{2}\lambda^2(1-|\phi|^2)^2|\phi|^2,\\ &D_1\phi = \mathrm{i}D_2\phi, \end{aligned} \tag{1}$$

where

$$D_j\phi = \partial_j\phi + \mathrm{i}A_j\phi \quad \text{and} \quad F_{jk} = \partial_j A_k - \partial_k A_j.$$

We are to look for an N-vortex solution of (1) so that, counting algebraic multiplicities, ϕ has N zeros, say $p_1, \ldots, p_N$.

Using the second equation in (1), we see that the first equation in (1) takes the form

$$(1-|\phi|^2)F_{12} + |D_1\phi|^2 + |D_2\phi|^2 = \tfrac{1}{2}\lambda^2(1-|\phi|^2)^2|\phi|^2. \tag{2}$$

On the other hand, with the notation

$$\partial = \tfrac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad A = A_1 + \mathrm{i}A_2,$$

we can rewrite the second equation in (1) as

$$2\mathrm{i}\partial\phi = A^*\phi \quad \text{or} \quad A^* = 2\mathrm{i}\partial \ln\phi. \tag{3}$$

Therefore

$$\begin{aligned} F_{12} &= -\mathrm{i}(\partial A - \partial^* A^*)\\ &= -2(\partial\partial^* \ln\phi^* + \partial^*\partial \ln\phi)\\ &= -2\partial\partial^* \ln|\phi|^2 = -\tfrac{1}{2}\Delta \ln|\phi|^2. \end{aligned}$$

Inserting this result into (2), we obtain

$$-(1-|\phi|^2)\Delta \ln|\phi|^2 + 2(|D_1\phi|^2 + |D_2\phi|^2) = \lambda^2(1-|\phi|^2)^2|\phi|^2. \tag{4}$$

Next, we observe that, if ϕ is represented locally as $\phi = \mathrm{e}^{\sigma+\mathrm{i}\omega}$, where σ and ω are real-valued functions, then, by (3), we have

$$\begin{aligned} D_1\phi &= (\partial + \partial^*)\phi + \mathrm{i}\left(\mathrm{i}\frac{\partial\phi}{\phi} - \mathrm{i}\frac{\partial^*\phi^*}{\phi^*}\right)\phi\\ &= 2\phi(\partial^*\sigma),\\ D_2\phi &= \mathrm{i}(\partial - \partial^*)\phi + \mathrm{i}\left(-\frac{\partial\phi}{\phi} - \frac{\partial^*\phi^*}{\phi^*}\right)\phi\\ &= -2\mathrm{i}\phi\partial^*\sigma. \end{aligned} \tag{5}$$

Let $|\phi|^2 = \mathrm{e}^u$. From (5), we have

$$\begin{aligned} |D_1\phi|^2 + |D_2\phi|^2 &= 4|\phi|^2(|\partial\sigma|^2 + |\partial^*\sigma|^2) \\ &= \tfrac{1}{2}\,\mathrm{e}^u|\nabla u|^2. \end{aligned} \tag{6}$$

Substituting (6) into (4), we find

$$(1-\mathrm{e}^u)\Delta u - \mathrm{e}^u|\nabla u|^2 = -\lambda^2(1-\mathrm{e}^u)^2\,\mathrm{e}^u + 4\pi\sum_{j=1}^{N}\delta_{p_j} \quad \text{in } \mathbb{R}^2. \tag{7}$$

It is more convenient to rewrite (7) as

$$\Delta(1+u-\mathrm{e}^u) = -\lambda^2(1-\mathrm{e}^u)^2\,\mathrm{e}^u + 4\pi\sum_{j=1}^{N}\delta_{p_j} \quad \text{in } \mathbb{R}^2. \tag{8}$$

It is easily checked that

$$w = q(u) = 1 + u - \mathrm{e}^u \tag{9}$$

is a strictly increasing function on $(-\infty, 0]$ and the range of $q(\cdot)$ is also $(-\infty, 0]$. Denote by $Q(w)$ the inverse function of q: $(-\infty, 0] \to (-\infty, 0]$. Then $Q(w)$ satisfies

$$w = 1 + Q(w) - \mathrm{e}^{Q(w)}, \qquad -\infty < w \leqslant 0. \tag{10}$$

With w being defined in (9) or (10), we rewrite (8) as

$$\Delta w = -g(w) + 4\pi\sum_{j=1}^{N}\delta_{p_j} \quad \text{in } \mathbb{R}^2, \tag{11}$$

where $g(w) = \lambda^2(1-\mathrm{e}^{Q(w)})^2\,\mathrm{e}^{Q(w)}$, $w \leqslant 0$.

From (10), we have

$$Q'(w) = (1-\mathrm{e}^{Q(w)})^{-1}, \quad w < 0.$$

Hence

$$g'(w) = \lambda^2(1 - 3\,\mathrm{e}^{Q(w)})\,\mathrm{e}^{Q(w)}, \quad w < 0. \tag{12}$$

Consider now the left derivative of $g(w)$ at $w = 0$. Since $Q(w) \to 0^-$ as $w \to 0^-$, Equation (10) gives us the asymptotic formula

$$Q(w) = \sqrt{2}C(w)\sqrt{-w} \qquad \text{for } w < 0 \text{ small}, \tag{13}$$

where $C(w) \to 1$ as $w \to 0^-$. Hence, by the definition of g and (12),

$$\lim_{w\to 0^-}\frac{g(w)}{w} = -2\lambda^2 = \lim_{w\to 0^-} g'(w). \tag{14}$$

This result allows us to extend the domain of definition of $g(\cdot)$ to a bounded C^1-function (by this we mean both g and g' are bounded) on entire $(-\infty,\infty)$ so that

$$g(w) = \lambda^2(1 - e^{Q(w)})^2 e^{Q(w)} \quad \text{for } w \leqslant 0 \quad \text{and} \quad g'(0) = -2\lambda^2.$$

In the sequel, we will always assume that g already has such an extension.

3. Proof of Existence for the Reduced Equation

From now on, we only consider radially symmetric solutions of (11) with $p_1, \dots, p_N$ = the origin of $\mathbb{R}^2$. Thus (11) becomes, by a simple application of the removable singularity theorem (see [8]) and the L'Hôpital rule, the following initial-value problem of an ordinary differential equation

$$\begin{aligned} w_{rr} + \frac{1}{r} w_r &= -g(w), \quad r > 0, \\ \lim_{r\to 0+} \frac{w(r)}{\ln r} &= \lim_{r \to 0+} r w_r(r) = 2N. \end{aligned} \tag{15}$$

We are in a position to quote a general result in [7].

Consider the second-order ordinary differential equation

$$\begin{aligned} &w'' + f(t) g(w) = 0, \qquad -\infty < t < \infty, \\ &w(t) = \alpha t + \mathrm{O}(1) \quad \text{as } t \to -\infty. \end{aligned} \tag{16}$$

The following are the basic hypotheses we are to make for (16) in order to obtain useful results in classical field theory.

(H1) The functions $f, g \in C^1(\mathbb{R})$ and

$$\int_{-\infty}^{0} |t f(t)| \, \mathrm{d}t < \infty,$$

$$\sup_{w\in\mathbb{R}} \{ |g(w)| + |g'(w)| \} < \infty.$$

(H2) There hold the properties

$$f(t) > 0, \quad t \in \mathbb{R}; \quad g(w) > 0, \quad w < 0;$$

$$\lim_{t\to\infty} f(t) = \infty; \quad g(0) = 0.$$

(H3) $f'(t) \geqslant 0$ for all $t \in \mathbb{R}$.

(H4) There exists $M > 0$ such that $g'(w) > 0$ when $w < -M$ and

$$\int_0^\infty f(t)g(-Mt)\,\mathrm{d}t < \infty.$$

(H5) If one defines

$$M_0 = \inf\left\{\sigma > 0 \,\middle|\, \int_0^\infty f(t)g(-\sigma t)\,\mathrm{d}t < \infty\right\},$$

then

$$\int_0^\infty f(t)g(-M_0 t)\,\mathrm{d}t = \infty.$$

In addition, for every $c > 0$,

$$\inf_{t>0} \frac{f(t-c)}{f(t)} > 0.$$

(H6) Let

$$G_0(w) = \int_{-\infty}^{w} g(v)\,\mathrm{d}v.$$

Note that the assumptions (H2)–(H4) would imply the finiteness of $G_0(w)$ for each $w \in \mathbb{R}$. Define

$$F_1(t) = \frac{f'(t)}{f(t)}, \quad G_1(w) = \frac{G_0(w)}{g(w)}.$$

Then both $f_1 = \lim_{t\to\infty} F_1(t)$ and $g_1 = \lim_{w\to-\infty} G_1(w)$ exist and are finite.

(H7) The functions F_1 and G_1 defined in the assumption (H6) satisfy $F_1(t) \geqslant f_1$ for all $t \in \mathbb{R}$ and $G_1(w) \geqslant g_1$ for all $w \in (-\infty, 0)$.

(H8) There is some $\delta > 0$ such that $g'(w) \leqslant 0$ in $[-\delta, 0]$.

The following result concerning the system (16) was established in [7].

THEOREM 1. *Suppose that* $\alpha \geqslant 0$ *and* f, g *satisfy* (H1)–(H3). *Then* (16) *has at least one solution* w *satisfying*

$$w \leqslant 0, \quad w' \geqslant 0, \quad w'' \leqslant 0 \quad \textit{in } \mathbb{R}$$

and

$$\lim_{t\to\infty} w(t) = 0. \tag{17}$$

If, in addition, (H8) *is fulfilled, the solution of* (16) *satisfying* (17) *is unique.*

If f, g *satisfy also* (H4)–(H6), *then for each* $\beta \in (\alpha + 2f_1 g_1, \infty)$, *Equation* (16) *has at least one solution* w *such that* $w < 0, w'' < 0$ *in* $\mathbb{R}$ *and*

$$\lim_{t\to\infty} w'(t) = -\beta. \tag{18}$$

If in addition (H7) *holds, then for any nonpositive solution of* (16) *satisfying* $\liminf_{t\to\infty} w(t) < 0$, *there exists some* $\beta \in (\alpha + 2f_1 g_1, \infty)$ *to achieve* (18).

We now show that the hypotheses (H1)–(H8) all hold for (15) under the new variable

$$t = \ln r, \tag{19}$$

which transforms (15) into the following equivalent system

$$\begin{aligned} &w'' + e^{2t} g(w) = 0, && -\infty < t < \infty, \\ &w(t) = 2Nt + O(1) && \text{as } t \to -\infty. \end{aligned} \tag{20}$$

Of course, we have $f(t) = e^{2t}, g = g(w), \alpha = 2N$, when we identify (20) with (16). It is straightforward to verify that f, g satisfy (H1)–(H3).

From (12), we see that when

$$w < Q^{-1}(-\ln 3) = g(-\ln 3) = 1 - \tfrac{1}{3} - \ln 3 = -M_1 < 0, \tag{21}$$

we have $g'(w) > 0$. Besides, (10) says that

$$w = Q(w) + (1 - e^{Q(w)}) \geqslant Q(w) \quad \text{for } w \leqslant 0.$$

Hence $e^{Q(w)} \leqslant e^w$ for $w \leqslant 0$. This result shows that

$$\int_0^\infty e^{2t} g(-Mt)\, dt < \infty$$

whenever $M > 2$. Of course, $2 > M_1$. Therefore (H4) holds.

To examine (H5), we note by using (10) that

$$\lim_{w\to-\infty} \frac{e^{Q(w)}}{e^w} = e^{-1}.$$

Consequently, there is a sufficiently large number $M > 0$ so that

$$\tfrac{1}{2} e^{-1} e^w < e^{Q(w)} < 2e^{-1} e^w \tag{22}$$

whenever $w < -M$. The inequality (22) implies immediately that

$$\inf\left\{\sigma > 0 \,\middle|\, \int_0^\infty e^{2t} g(-\sigma t)\, dt < \infty\right\} = 2$$

and

$$\int_0^\infty e^{2t} g(-2t)\,dt = \infty.$$

Therefore (H5) is also fulfilled.

In (H6), we have $F_1(t) \equiv 2$ and $f_1 = 2$. Besides, using the L'Hôpital rule,

$$\begin{aligned} g_1 &= \lim_{w\to-\infty} \frac{\int_{-\infty}^{w} g(v)\,dv}{g(w)} \\ &= \lim_{w\to\infty} \frac{g(w)}{g'(w)} = 1. \end{aligned}$$

In particular, (H6) holds as well.

For (H7), we want to know whether $G_1(w) \geqslant g_1 = 1$ for $w < 0$ or, equivalently,

$$\int_{-\infty}^{w} g(v)\,dv \geqslant g(w), \quad w < 0. \tag{23}$$

In fact, setting

$$F(w) = \int_{-\infty}^{w} g(v)\,dv - g(w), \quad w \leqslant 0,$$

we can check that

$$F(-\infty) = 0$$

and

$$F'(w) = g(w) - g'(w) = \lambda^2 e^{2Q(w)}(1 + e^{Q(w)}) > 0, \quad w < 0.$$

So $F(w) > 0$ for $-\infty < w < 0$. In other words, (23) is true and (H7) is fulfilled.

Finally (H8) follows from (12).

Using (20) and Theorem 1, we see that the following conclusions are obtained for (15).

THEOREM 2. *Consider nonnegative solutions of* (15).

(i) *There is a unique solution w such that $w \to 0$ as $r \to \infty$. This solution is strictly monotonically increasing. In particular, $w < 0$ everywhere.*

(ii) *For each $\beta \in (2N + 4, \infty)$, there is a solution w such that*

$$w(r) \to -\infty \quad \text{as } r \to \infty \quad \text{and} \lim_{r\to\infty} r w_r(r) = -\beta. \tag{24}$$

(iii) *If* w *is a solution of* (15) *such that* $w \to -\infty$ *as* $r \to \infty$, *then there is a suitable* $\beta \in (2N+4, \infty)$ *so that* w *satisfies the second property stated in* (24) *as well.*

Proof. We only need to show the monotonicity assertion in part (i). In fact, Theorem 1 shows already w is nondecreasing. If there were $r_1 < r_2$ so that $w(r_1) = w(r_2)$, then there would exist an $r_0 \in (r_1, r_2)$ so that $w_0 = w(r_0)$ were a local minimum. Hence, $w_{rr}(r_0) \geqslant 0, w_r(r_0) = 0$. Using this information in (15) and the property $g(w) > 0$ for $w < 0$, we would obtain $g(w_0) = 0$ or $w_0 = 0$. The uniqueness theorem of the initial-value problem of ordinary differential equations then would say that $w(r) = 0$ for all $r > 0$ because $w = 0$ is also a solution of the equation in (15). This contradiction proves that $w(r_1) < w(r_2)$ whenever $0 < r_1 < r_2$.

4. Existence of Vortices and Asymptotic Behaviour

Note that, in (7) or (8), the sector $u \to 0$ as $|x| \to \infty$ and the sector $u \to -\infty$ as $|x| \to \infty$ correspond to topological and nontopological vortex solutions, respectively. If there were a point at which $u > 0$, then u would have a positive maximum at a point $p \in \mathbb{R}^2$. Of course, $p \neq p_j$ for any $j = 1, \cdots, N$. However, inserting this information into (7), we would have $(\Delta u)(p) > 0$, which is false. Thus, $u \leqslant 0$ everywhere and the transformation (9) into the new variable w is general enough for us to obtain all possible finite-energy solutions, topological and nontopological. In this sense, Theorem 2 says that we have constructed all possible topological and nontopological radially symmetric N-vortex solutions.

To see that the solutions presented in Theorem 2 all carry finite energies and to get generalised magnetic and electric charges, we need to study their (implied) asymptotic behaviour as well. To this end, we consider the radial function $|\phi(r)|$ for the radially symmetric field configuration, corresponding to $|\phi| = e^{\frac{1}{2}u} = e^{\frac{1}{2}Q(w)}$. For the topological vortices, the behaviour of $|\phi|$ in the $|x| = r \gg 1$ region is already given in [5] which may be cited as follows:

$$|\phi(r)| = 1 - \gamma\sqrt{K_0(\sqrt{2}\,r)}, \tag{25}$$

where K_0 is the modified Bessel function K_ν with $\nu = 0$, which has the exponential decay resulting in the finiteness of the energy. The constant γ in (25) is determined by the behaviour of $|\phi(r)|$ in the $r \ll 1$ region. In the case of the nontopological vortices, the $r \gg 1$ behaviour of the function $|\phi|$ is again consistent with the finiteness of the energy but in this case it does not have an exponential but rather a power decay. It has been shown that this behaviour is the same for both the $p = 2$ and the $p = 1$ models, given by [3]

$$|\phi(r)| = \frac{C}{r^{\alpha_0}} - \frac{C^3}{8(\alpha_0 - 1)^2 r^{3\alpha_0 - 2}} + O(r^{-5\alpha_0 + 4}) \tag{26}$$

where the parameter α_0 is subject to $\alpha_0 \geqslant 1$. Interpreting our asymptotic results stated in Theorem 2, we find that

$$|\phi(r)| = e^{\frac{1}{2}u} = e^{\frac{1}{2}Q(w)} = O(r^{-\alpha_0}) \quad \text{for large } r = |x|, \tag{27}$$

where α_0, which is related to β in Theorem 2 through $\alpha_0 = \frac{1}{2}\beta$, may be arbitrarily chosen from the interval $(N+2, \infty)$ but other α_0's are prohibited. This restriction on the range of α_0 was arrived at in [3] using numerical methods. Given that our solutions are all self-dual, the energy equals the magnetic charge Φ which is not quantised since it is parametrised by the above arbitrary quantity $\alpha_0 > N + 2$ through

$$\Phi = 2\pi(N + \alpha_0). \tag{28}$$

We summarise the study in this section as follows.

THEOREM 3. *There exist two families of finite-energy radially symmetric vortices in the $p = 2$ generalised self-dual Chern–Simons model: topological and nontopological. The topological vortices are uniquely determined by their vortex number N and the interaction densities decay exponentially at infinity so that the associated electric and magnetic charges and the energy are all quantised. The nontopological vortices are not unique for a given vortex number N but are labelled by an additional parameter $\alpha_0 > N + 2$. These solutions are a continuous family which decay at infinity according to the sharp expression* (27) *and the corresponding magnetic charge may be determined by formula* (28).

5. Generalised Abelian Higgs Vortices

We then consider the generalised Abelian Higgs model proposed in [6]. For $p = 2$, the self-dual system is similar to (1) where only the Higgs self-interaction term is replaced by

$$3(1 - |\phi|^2)^2.$$

Thus the reduced elliptic vortex equation still takes the same form (11) but $g(w)$ is now defined by

$$g(w) = 6(1 - e^{Q(w)})^2, \quad w \leqslant 0.$$

With the above function in mind, we can verify that (H1)–(H3) and (H8) all hold. Hence, for the Abelian Higgs model with $p = 2$, there is a unique radially symmetric topological solution for each N. Its asymptotic behaviour and energy/flux quantisation are already addressed in [6].

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