# Assessment of controller output saturation in dynamic systems: a case of performance, efficiency and system stress trade-off


Matheus Mulherchen Golz Costa
*Dept. of Electronic Engineering*
*National University of Ireland Maynooth*
Maynooth, Ireland
matheus.costa.2024@mumail.ie

Josefredo Gadelha da Silva
*Dept. of Electronic Engineering*
*National University of Ireland Maynooth*
Maynooth, Ireland
josefredo.silva.2024@mumail.ie

Thalita Emanuelle de Nazaré
*Dept. of Electronic Engineering*
*National University of Ireland Maynooth*
Maynooth, Ireland
thalita.nazare.2023@mumail.ie

Erivelton Nepomuceno
*Dept. of Electronic Engineering*
*National University of Ireland Maynooth*
Maynooth, Ireland
erivelton.nepomuceno@mu.ie



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***Abstract*—This paper aims to assess the impact of incorporating limitations into the controller output. The study is conducted on a proportional-derivative controller (PD) applied in the stabilisation of an inverted pendulum system. The findings indicate the presence of a trade-off relationship between the limit imposed, the power of the control output, actuator stress and system performance. It's discussed how the power of the control actions changes with the limitation, presenting a threshold for stability. A relation between limit percentage and oscillation amplitude is also discussed. The findings may help to better understand the trade-off relation between actuator stresses, stability and power consumption, given it's direct relation to the controller output. It can also aid in to assess the implications of saturation on real world systems. As a secondary effect, the limitation proposed can also be presented as a simple method to change the system behaviour with only one parameter change, without changing the gains of the controller.**



***Index Terms*—control, saturation, efficiency, stress, performance**


## I. Introduction

The process of controller design typically entails the careful consideration of multiple factors, including but not limited to system stability, robustness, error analysis, and various performance metrics [1]. Control applications in the real world encompass the task of directing a system towards a desired state, while frequently encountering external disturbances, noise, and various sources of uncertainty. Furthermore, it is important to acknowledge that these applications are bound by certain limitations and constraints. One of the limitations that must be considered is power, which refers to the fact that real systems function within the constraints of a finite power utilisation and the damage it may arise in it's maximum values. Input saturation can have a substantial impact on the performance of a system and, in extreme cases, can lead to instability in the control system [2]. This underscores the significance of integrating saturation analysis into the design of controllers. Assessing the relationship between saturation and energy consumption of the controller is of paramount importance, especially in scenarios where prioritising energy efficiency is imperative. An efficient controller should effectively achieve the desired control objectives while minimising energy consumption, even in the presence of constraints.

Some work has been developed addressing the trade-off between energy expenditure and system performance [3]. Additionally, efforts have been made to improve control algorithms aiming to save energy [4], and evaluation of Iterative Learning Control (ILC) under input saturation has been also studied [5]–[8]. Similar techniques also have been presented, relying on the more efficient-range action of actuators [9]. Another approach involves evaluating the tuning of weighting matrices for the Linear Quadratic Regulator (LQR) and its relation to energy consumption [10]. Finally, some works address control systems powered by intermittent energy sources, such as renewable energy sources with is characterised by transient behaviour and saturation conditions [11] [12]. However, these approaches are rare and should be more widely adopted. In fact, there is a lack of papers addressing the correlation between saturation, energy consumption and stability or closed loop systems.

Among it's various types, controlling nonlinear systems is typically the most challenging as it requires more complex techniques. The main objective of this investigation is to examine the relation between the power of the control signal while implementing different limit values to it and analyse it's impact on the system response. The system chosen to be controlled is the inverted pendulum on a cart. This particular dynamic system is distinguished by its nonlinear properties, inherent instability, and susceptibility to external disturbances. Several control techniques have been employed to stabilise an inverted pendulum in a cart, including Model Predictive Control (MPC) [13] [14], Robust Control [15], Proportional-Integral-Derivative (PID) [16], among others. Bakaráč et. al. compares various control techniques to stabilise the inverted pendulum [17]. Once again, to the best knowledge of the authors, few papers specifically address the control of dynamic systems under controller output saturation conditions. This highlights the inverted pendulum in a cart as an ideal benchmark system for investigating the topic of saturation in controller design.

In this paper, it's presented an approach that utilises Proportional-Derivative Control (PD) to stabilise the inverted pendulum system. The effect of adding saturation to the control output is evaluated. By imposing limitations on the power of the control signal it is observed a difference in the system response and the averaged power of the control signal. The average controller power output power changes in a complex way, presenting a region of more or less power usage depending of the limit value imposed. Instability is reached when the limitations are too aggressive.

The findings of this study demonstrate the existence of a trade-off relationship between the power, efficiency and performance. It is possible to fulfil the control objective and reduce energy use by imposing limits on the controller, at the cost of system performance. It can also be understood as a method to reduce the stress and wear of actuators, given that the occurrence of stress (mechanical, thermal...) on such devices. Additionally, the findings may help in understand the impact of controllers that can operate effectively under saturation conditions.

## II. Technical Background

### A. Proportional-Derivative (PD) Control

The proportional-derivative (PD) control is a widely used control technique in control engineering to improve the dynamic response and stability of dynamic systems [18]. It is an extension of classical proportional control that incorporates an additional derivative action to enhance system performance. The sum of the proportional control signal and the derivative control signal yields the final control signal of the PD controller.

The final signal $u(t)$ of PD control is given by (1):

$$u(t) = P_c(t) + D(t). \tag{1}$$

$P_c(t)$ and $D(t)$ are defined in the next sections [19]. PD control provides a faster and damped response compared to simple proportional control. The proportional action responds immediately to the current error, while the derivative action anticipates future changes in the error, allowing for more precise and effective correction [19].

*1) Proportional Control ($P_c$):* Proportional control is a basic control method that generates a control signal proportional to the error between the desired set point and the measured value of the system. This action is achieved by multiplying the error by the proportional gain ($K_p$). The proportional control signal provides a correction that is directly proportional to the magnitude of the error. The proportional action is defined by the following equation:

$$P_c(t) = K_p \times e(t), \tag{2}$$

where $P_c(t)$ is the proportional control signal at time $t$, $K_p$ is the proportional gain, and $e(t)$ is the error at time $t$.

In proportional control, a higher value of $K_p$ increases the proportional correction provided by the controller, resulting in a faster response. However, a very high proportional gain can lead to oscillations and system instability.

*2) Derivative Action (D):* The derivative action, added to proportional control, is designed to take into account the rate of change of the error over time. This allows the PD controller to predict the future trend of the error and take early corrective actions. The derivative action is proportional to the derivative of the error with respect to time $\frac{de(t)}{dt}$. This derivative of the error is multiplied by the derivative gain ($K_d$) to produce the derivative control signal. The derivative action is defined by the following equation:

$$D(t) = K_d \times \frac{de(t)}{dt}, \tag{3}$$

where $D(t)$ is the derivative control signal at time $t$, $K_d$ is the derivative gain, and $\frac{de(t)}{dt}$ is the derivative of the error with respect to time at time $t$.

The derivative action adjusts the controller's response based on the error trend. It contributes to system stability and damping by reducing undesired oscillations. However, an excessively high derivative gain can lead to overly sensitive responses to noise and disturbances in the system.

### B. Power Calculation

In the field of signals and systems, it is customary to establish the energy of a signal. The energy of the signal $x(t)$ can be determined through the utilisation of (4):

$$E = \int_{-\infty}^{\infty} x^2(t)dt. \tag{4}$$

In instances where the signal $x(t)$ does not exhibit convergence to zero as $t$ tends to infinity, the energy of the signal will be deemed infinite due to the absence of convergence in the integral. In this particular scenario, the assessment of the mean energy yields a more suitable measure, referred to as signal strength, for obtaining accurate information. The power of a signal $x(t)$ is defined as (5):

$$P = \lim_{x\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt. \tag{5}$$

Considering the controller's immediate action at $t = 0$, it becomes imperative to adjust the calculation to adhere to (6):

$$P = \lim_{x\to\infty} \frac{1}{T} \int_{0}^{T} x^2(t)dt, \tag{6}$$

where $T$ represents the simulation time for the closed-loop system that will be further discussed. Therefore, a metric is established to assess various controllers based on their control output power metric.

### C. Inverted Pendulum

The inverted pendulum on a cart system is a well-known problem in the field of control theory, serving as a classical example of a non-linear and intrinsically unstable system. The utilisation of this system is prevalent in both academic settings and industrial applications owing to its resemblance to diverse control problems encountered in real-world scenarios, such as the stabilisation of a bipedal robot, the control of a rocket's stability during launch, or the management of an unmanned bicycle. The problem's simplicity and practicality render it an enticing platform for evaluating diverse control strategies.

The system consists of a pendulum that is connected to a cart that is able to move freely, as show in the Fig. 1. The pendulum is initially oriented in an upward position, and the objective is to sustain this precarious state of equilibrium by horizontally displacing the cart.

For a simple model of an inverted pendulum consisting of a mass $m$ attached to a rod of length $l$ with no mass. The pendulum is free to rotate around it's contact point with the cart, with the angle $\theta$ measured from the vertical position. The force of gravity acts vertically downward on the mass $m$. Its magnitude is given by (7):

$$F_{\text{gravity}} = m \times g, \tag{7}$$

where $g$ is the acceleration due to gravity. The tension in the rod applies a force on the mass $m$ tangential to the circular path, however, since the rod has no mass, this force does not contribute to the torque equation. When the pendulum is in motion, there is a centripetal force acting on the mass $m$ due to its circular path. This force is directed towards the pivot point and has a magnitude given by (8):

$$F_{\text{centrifugal}} = m \times l \times \dot{\theta}^2, \tag{8}$$

where $\dot{\theta}$ is the angular velocity of the pendulum. The torque acting on the pendulum is the net torque due to the forces mentioned above. Since the rod has no mass, the torque equation simplifies as shown in (9):

$$\tau = -m \times l \times g \times \sin(\theta), \tag{9}$$

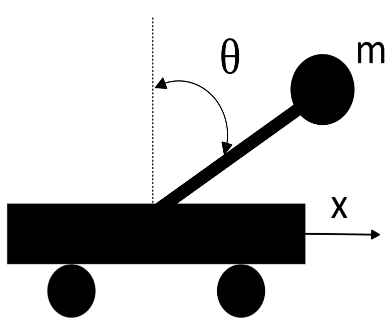

Fig. 1. Inverted pendulum on a cart system.

where $\tau$ is the torque. Applying Newton's second law for rotational motion, $\tau = I \times \alpha$, where $I$ is the moment of inertia of the pendulum and $\alpha$ is the angular acceleration. For a simple pendulum, the moment of inertia about the pivot point is $I = m \times l^2$.

Combining the torque equation and the moment of inertia results in (10):

$$-m \times l \times g \times \sin(\theta) = m \times l^2 \times \ddot{\theta}. \tag{10}$$

Rearranging the equation results in the equation of motion for the inverted pendulum as in (11):

$$\ddot{\theta} = -\frac{g}{l} \times \sin(\theta). \tag{11}$$

This is a second-order nonlinear differential equation that describes the dynamics of the inverted pendulum. It's important to note that the simplified model assumes certain assumptions and neglects factors such as friction, air resistance, and the mass of the rod. More complex models may consider these factors and lead to additional terms in the equations of motion, but such discussion does not belong in the scope of this work.

## III. Methodology

In order to examine the effects of constraints in the closed loop control system, several steps were undertaken. Initially, a widely recognised system from the existing body of control systems literature was selected. The selection of the inverted pendulum on a cart as an example was based on its widespread recognition and the intriguing potential for interpreting the outcomes in relation to stability and actuator stress. The system must be maintained in an upright position, which is inherently unstable, and thus necessitates reliance on a control system. In order to incorporate the concept of an output limit within the controller, the overall control block diagram was revised, as depicted in Fig. 2, where a limit block is positioned after the controller.

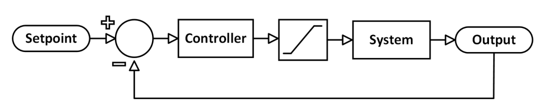

Fig. 2. Block diagram of the control system used.

Once a comprehensive understanding of the entire system was obtained, it became necessary to develop an algorithm for the purpose of implementing the system's dynamics, control system, and limitation logic. The system dynamics were acquired through the utilisation of differential equations, which were derived by the application of Newton's laws of motion and the analysis of the forces and torques exerted on the system. To control the pendulum, a proportional-derivative (PD) controller was chosen to stabilise the system and achieve the desired behaviour.

*A. Simulation*

The Matlab software was utilised for the purpose of implementing the simulation of the closed loop system and generating visual representations of the obtained results. All the functionalities were integrated into a single code, encompassing system dynamics, controller model, control limitation, data storage, and graph figure generation. Fig. 3 shows a flowchart that illustrates the general functionalities and structure of the program.

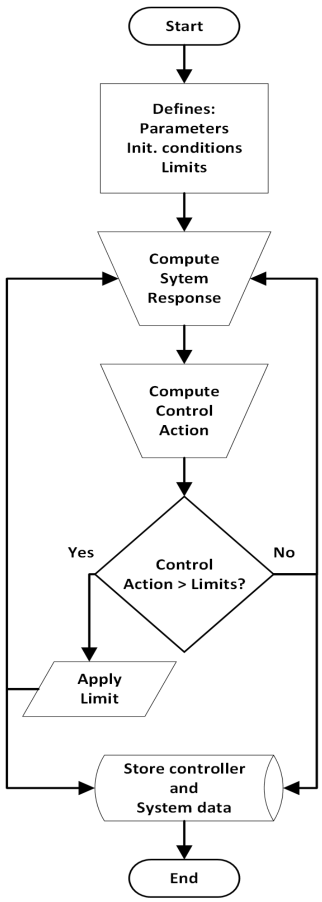


Fig. 3. Algorithm flowchart.

*B. Data analysis*

From the data obtained, a few analysis were conducted to understand the impacts of the limitation on the closed loop system. Initially, in order to assess the influence of constraints on the system's response, two graphs were generated. The first graph depicted the system's behaviour with minimal limitations imposed on the controller, while the second graph illustrated the system's response when the limits were significantly increased, aiming to investigate the consequences of excessively high constraints. In light of the anticipated variation in system response, a plot illustrating the temporal evolution of the power's instantaneous value in the control output was generated to ascertain the presence of any notable disparities among the values.

After obtaining the time response curves, an additional analysis was performed to ascertain the correlation between the average power of the control action and the imposed limit value on the systems. This evaluation aimed to examine the relationship between these parameters.

The data and code utilised to substantiate the findings of this study can be accessed via the DOI: osf.io/T8JA7.

## IV. Results

When the limits imposed were not too high, the methodology outlined resulted in the system responses depicted in Fig. 4.

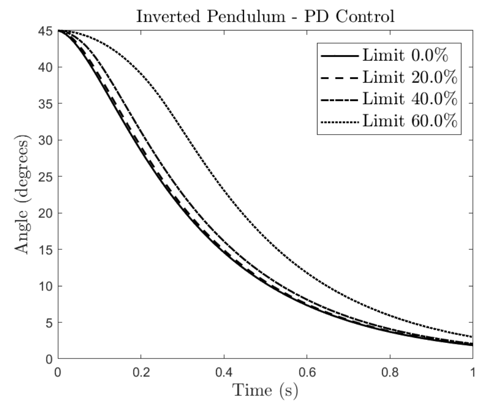


Fig. 4. System response with small limits.

It is evident that all systems achieve the designated set point, differing solely in the rate at which this task is accomplished. In this particular scenario, the instantaneous power over time of said systems is depicted in Fig. 5.

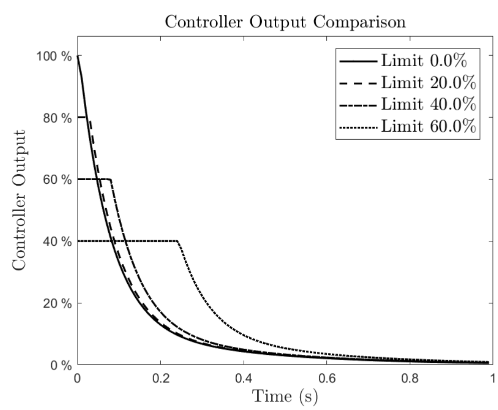


Fig. 5. Power of the control signal over time with small limits.

Initially, the controller employs the utmost level of response it is capable of generating. In systems with limited control outputs, it can be observed that the control system takes longer in stabilising the system. Furthermore, when the control action begins to decrease, the limited systems sustain an higher control output till the end.

As an expected behaviour, if the the limits are excessively high, closed loop control fails and the system presents an

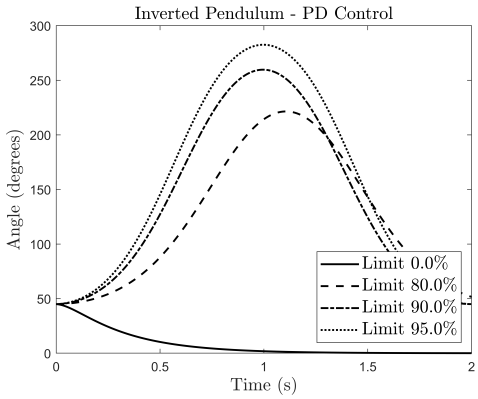


Fig. 6. System response with big limits.

oscillatory response towards a natural equilibrium point. This is illustrated in Fig. 6.

It is apparent that beyond a certain point, the controller experiences a loss in its capacity to achieve the desired set point, resulting in a system response characterised by oscillation. The amplitude of these oscillations is directly correlated to the imposed limit. In this scenario, the power of the control signal exhibits a consistent and constant level, with all constrained systems demonstrating a saturated response, assuming the maximum value, throughout the entire duration of the simulation.

## V. Discussion

Considering the particular attributes of the signal power curves, it is reasonable to inquire about the influence of the constraint value on the mean power of the control signal, as well as their interrelationship. To address this question, a graph illustrating the relationship between power averages and the percentage limit imposed is presented in Fig. 7.

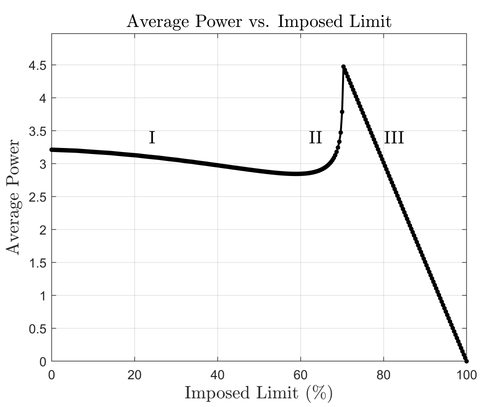


Fig. 7. Average power change with limit percentage.

Initially, in region I, the relation between the power of the control signal and the magnitude of the imposed limit is inversely proportional, as the limitation is increased the power decreases. In II the behaviour changes, it is observed a non-linear increase of the power with the increase of the limit.

TABLE I
Average Power relation to limit values.

| Limit imposed (%) | Average Power | Stable? |
|---|---|---|
| no limit | 3.21 | Yes |
| 10 | 3.18 | Yes |
| 20 | 3.13 | Yes |
| 30 | 3.06 | Yes |
| 40 | 2.98 | Yes |
| 50 | 2.89 | Yes |
| 60 | 2.85 | Yes |
| 70 | 3.79 | Yes |
| 80 | 3.02 | No |
| 90 | 1.51 | No |
| 100 | 0 | No |

Once a specific threshold is reached, which is dependant upon the control strategy, controlled system and control gains, the behaviour undergoes a sudden change. This change leads to III, where the control power diminishes as imposed by the limitations, leading to an oscillatory system that is already unable to sustain the desired set point. The numerical representation of these results is displayed through the selected data points outlined in Table I.

The analysis of this data indicates an complex relationship between the system's performance, power usage and stability. The imposed limit in this study reached 70% before ruining the control action. This result begs for more studies towards understanding how to fine tune the controller gains, power consumption and performance of a given closed loop system.

The adoption of this methodology presents certain difficulties, the findings indicate that the determination of stable control limits is not a simple procedure, but rather a complex and intricate one. The comprehensive understanding of a system's response necessitates validation specific to that system, highlighting the intricate nature of implementing the approach on a global scale.

## VI. Conclusions

This study presents a simple method of limiting power usage without the need for controller gain re-calibration. There is an interplay between the limitation of the controller, it's power level and system stability. If limits are increased, the system response becomes slower, but less wear and strain on the actuators may be obtained, given the lower maximum values of the actuator input (controller output). The limitation helps to lower the average control signal overall, which may relate to a lower average power consumption. In this aspect, the application of limits may increase the efficiency of the controller, in the sense that a lower average power signal is able to maintain the set point. The system may become unstable if a threshold to the limit value is surpassed, leaving the system in oscillation whose amplitude is proportional to the limit imposed.

In general, it's presented significant and unseen contributions to the understanding of the impact of limitations applied to the control output on dynamic systems. Subsequent investigations are necessary to help develop control systems that are both more efficient, resilient and capable of meeting the diverse needs of various applications.

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