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RESEARCH ARTICLE

Input-To-State Stable Tracking Control Design for Fully Actuated Mechanical Systems Using Position Measurements Only

Joel Ferguson[1,2] | Naoki Sakata[3] | Kenji Fujimoto[3]

[1]Maynooth International Engineering College, Maynooth University, Maynooth, Ireland | [2]School of Engineering, The University of Newcastle, Newcastle, New South Wales, Australia | [3]Graduate School of Engineering, Kyoto University, Kyoto, Japan

Correspondence: Joel Ferguson (joel.ferguson@mu.ie)

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ABSTRACT

In this article, we consider tracking control design for fully actuated mechanical systems using position measurements only. A recently developed hybrid momentum observer is used, which has the property that the momentum estimation error is a passive output from the estimation error dynamics. To complement this, a tracking error system is constructed with error coordinates defined between the momentum estimate and the desired momentum. The tracking error dynamics are similarly passive with the momentum estimation error as the passive input to the tracking error system. Exploiting the passivity of both the observer and tracking controller subsystems, a passive interconnection is constructed which results in a storage function for the joint observer and controller systems. It is shown that the joint system is Input-to-State Stable (ISS) with respect to external disturbances and the effect of the disturbance can be attenuated via tuning gains. The results are numerically demonstrated on a 2-link manipulator system.

1 | Introduction

Tracking control for mechanical systems has broad practical applicability, from robotic manipulators to underwater vehicles and satellite systems. Methods for tracking control have been well studied over the past decades from multiple approaches. These approaches have included PID [1], back-stepping [2], geometric [3], and energy-based approaches [4]. One drawback of many such nonlinear approaches is the underlying assumption that the full state vector, including position and velocity information, is available for control purposes. In contrast to this assumption, many practical systems have position measurements available but do not have direct measurements of the system's velocities.

In response to this limitation, many researcher have investigated methods for tracking control using only position information. One of the earliest works on the topic [5] considered point-to-point and tracking control for Euler–Lagrange systems. To overcome the lack of velocity information, a velocity observer was developed and a corresponding regulation control law proposed that ensured asymptotic stability with a bounded region of attraction, provided that the controller gains were suitably chosen. The design of velocity observers for nonlinear mechanical systems is known to be technically difficult which has motivated several authors to investigate tracking control methods without the use of observers. In the works [6–8] implemented adaptive schemes using a filtered variable in place of a velocity observer to achieve uniformly ultimately bounded stability. Semiglobal asymptotic tracking was later achieved using a similar approach in [9]. Similar stability properties were later achieved with the use of a discontinuous velocity observer in the work [10].

The first truly global solution to the tracking problem using filtered quantities in place of velocity observers was introduced in [11] where a so-called *dirty derivative* term was used. Some mild conditions on the controller gains were introduced as part of the technical proof to ensure global asymptotic stability of the tracking system. This idea was further considered in the subsequent work [12], which similarly achieved global asymptotic stability. The geometry of tracking systems with revolute joints was considered in [13], resulting in semiglobal asymptotic stability of the tracking system. Researchers have continued to explore tracking control methods using filtered quantities in [13, 14], which yielded similar stability properties with variations to controller structure and complexity. An alternate approach using passivity was proposed in [15] using passivity theory, which made use of a filtered quantity in place of an observer and also achieved global asymptotic stability of the tracking system. Tangential to these works, several authors have investigated extensions to the tracking problem, such as [16] which extended these ideas to flexible links while also considering actuator dynamics.

Despite these significant advances in the development of tracking control strategies, stability properties remained asymptotic at best. A significant step forward was made in the work [4], which used the Immersion and Invariance technique to develop a control scheme that rendered the target trajectory uniformly globally exponentially stable. The primary drawback of the approach was that the control system had a dimension $3n + 1$, where n is the dimension of the configuration space. A simpler method ensuring global exponential stability of the tracking system using a filtered quantity in place of a velocity observer was introduced in [17] by exploiting some structural properties of Euler-Lagrange systems. More recently, the work [18] proposed a control scheme that achieves global finite time convergence and local exponential stability while satisfying actuator constraints.

These works have resulted in significant progress toward tracking methods using only position information that result in strong stability guarantees. While many of these works have considered the role of model uncertainty, however, only a few authors have directly considered the impact of unmodeled perturbations on the control system. A constant unknown disturbance was considered in [8] where the disturbance was compensated for by an adaptive scheme. The role of an arbitrary bounded external perturbation was explicitly considered in [19] where a sliding mode controller was used to guarantee global asymptotic stability, rejecting the effects of certain types of model uncertainty and bounded external perturbations. The trade-off for this guarantee was chattering in the control signal. This limitation was relaxed in the later work [20] where robust and adaptive methods were combined to achieved asymptotic tracking with better experimental performance when compared to prior solutions.

A general summary of the surveyed works related to tracking control systems using only position measurements is included below. There is significant variation between the models used and the type of stability considered, so we have highlighted the control method, type of stability considered and if the models considered external disturbances.

With the exception of [7, 8, 19], these prior works have excluded analysis on the impact of external disturbances on the tracking control schemes. Additionally, these works are all subject to conditions on the controller gains to ensure the claimed stability results hold. In this work, we significantly extend the stability properties of tracking control systems by proposing a tracking scheme that is Input-to-State stable with respect to external perturbations. The control scheme utilizes the hybrid momentum observer proposed in [21] to estimate the system's momentum online. An interesting property of this observer is that the estimation error dynamics are passive, with the estimation error being a passive output. This property is exploited in this work to interconnect the observer with the proposed tracking error dynamics that are similarly designed to be a passive system. The result of this interconnection approach is a joint Lyapunov function for the observer and tracking controller which is valid for all positive choices of controller gains, avoiding the need for small gain theorems [22] or separation principles [23, 24]. It is additionally shown that the impact of bounded disturbances can be attenuated arbitrarily via a suitable selection of controller gains.

The remainder of the paper is structured as follows: In Section 2, some requisite background materials on mechanical and hybrid systems is introduced before formally stating the problem being addressed. The aforementioned momentum observer, which is fundamental to the main result, is revised in Section 3. A tracking control law that results in passive tracking error dynamics is proposed and interconnected with the observer in Section 4. The results are then demonstrated numerically on a 2 degree-of-freedom manipulator system in Section 5 before conclusions are drawn in Section 6.

Notation. Function arguments are declared upon definition and are omitted for subsequent use. $0_{n\times m}$ denotes a $n \times m$ matrix where each entry is equal to zero and I_n is a $n \times n$ identity matrix. For $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$, $P > 0$ we define $\|x\|_P^2 = x^\top Px$. For $S \in \mathbb{R}^{n\times q}$ we use the symbol $\|S\|$ to denote its induced norm. For mappings $H : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ we denote the transposed gradient as $\nabla H := \left(\frac{\partial H}{\partial x}\right)^\top$, the transposed Jacobian matrix as $\nabla_x G(x, y) := \left(\frac{\partial G}{\partial x}\right)^\top$. For a real matrix $A \in \mathbb{R}^{n\times n}$, we denote the symmetric component as $\mathrm{symm}(A) = \frac{1}{2}(A + A^\top)$. For a discrete event occurring at time T and a time-varying parameter $\phi(t)$, $\phi^- = \lim_{t\to T^-} \phi(t)$ whereas $\phi^+ = \phi(T)$.

2 | Background and Problem Formulation

In this section, some key background materials are summarized, and the problem being addressed within the article is formally stated.

2.1 | System Model

This work considers the class of fully actuated mechanical systems described within the port-Hamiltonian formalism, described by

$$
\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p}_0 \end{bmatrix} &= \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & -D_0(q) \end{bmatrix} \begin{bmatrix} \nabla_q H_0 \\ \nabla_{p_0} H_0 \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ G_0(q) \end{bmatrix} u - \begin{bmatrix} 0_{n\times 1} \\ \delta_{p_0}(t) \end{bmatrix} \\
y &= G_0^\top(q) \nabla_{p_0} H_0 \\
H_0(q, p_0) &= \frac{1}{2} p_0^\top M^{-1}(q) p_0 + V(q)
\end{aligned} \tag{1}
$$

where $q \in \mathbb{R}^n$ is the configuration, $p_0 \in \mathbb{R}^n$ is the canonical momentum, $V(q) \in \mathbb{R}^+$ is the potential energy, $M(q) = M^\top(q) \in \mathbb{R}^{n\times n}$ is the uniformly positive definite mass matrix satisfying

$$
\underline{m} I_n \leq M(q) = M^\top(q) \leq \overline{m} I_n \tag{2}
$$

for some $\overline{m} > \underline{m} > 0$, $H_0(q, p_0) \in \mathbb{R}^+$ is the Hamiltonian, $G_0(q) \in \mathbb{R}^{n\times n}$ is the full rank input mapping matrix, $D_0(q) = D_0^\top(q) \in \mathbb{R}^{n\times n}$ is the open-loop damping matrix which is positive semidefinite and $u, y \in \mathbb{R}^n$ are the input and corresponding passive output, respectively. The mass matrix is assumed to be differentiable for all q with continuous derivatives. The system is subject to an external disturbance $\delta_{p_0}(t)$, which satisfies the bound

$$
\gamma = \sup_t \left\| \delta_{p_0}(t) \right\| \tag{3}
$$

for some $\gamma \geq 0$. This disturbance could represent, for example, external forces, input signal imperfections, or modelling errors. While the bound γ is assumed to exist, it does not need to be known for control implementation.

2.2 | Passive Hybrid Systems

A hybrid system with state $x \in \mathbb{R}^n$ and input (disturbance) term $w \in \mathbb{R}^m$ is described by

$$
\begin{aligned}
\dot{x} &= f(x, w) \quad \text{for } (x, w) \in C \\
x^+ &= g(x, w) \quad \text{for } (x, w) \in D
\end{aligned} \tag{4}
$$

where $C, D \subset \mathbb{R}^n \times \mathbb{R}^m$ that describe the domains of continuous and discrete dynamics, respectively. Solutions of hybrid systems, $x(t, j)$, are defined on hybrid time domains $E \subset \mathbb{R}_+ \times \mathbb{Z}_+$ where $E = \cup_{j=0}^{J-1} \left([t_j, t_{j+1}], j \right)$ for some, possibly infinite, sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$ [25, Definition 2.3].

The solutions of the hybrid system contain aspects of both continuous-time and discrete-time systems. In this work, the momentum observer is described by hybrid dynamics, and care must be given for such systems to the domain on which a solution exists. For a given hybrid time domain E, we define

$$
\sup_t E = \sup \left\{ t \in \mathbb{R}_+ : \exists j \in \mathbb{N} \text{ such that } (t, j) \in E \right\} \tag{5}
$$

Other properties of solutions such as Zeno and eventually discrete behaviors can be found in [25, Chapter 2].

There are several notions of passivity for hybrid systems that are similar to the continuous-time and discrete-time definitions. Considering the hybrid system (4), we introduce a storage function $S : \mathbb{R}^n \to \mathbb{R}$ and output $y(x) \in \mathbb{R}^m$. In this work, we are interested in flow strictly passive systems, which are defined as follows:

Definition 1 ([26, Definition 9.4]). The hybrid system (4) with storage function S and output y is flow passive if

$$
\begin{aligned}
\dot{S}(x) &\leq -\rho(x) + y^\top w \text{ for } (x, w) \in C \\
S^+(x) &= S^-(x) \qquad \text{for } (x, w) \in D
\end{aligned} \tag{6}
$$

where $\rho(x)$ is a positive semidefinite function of x. It is flow strictly passive if $\rho(x)$ is positive definite.

2.3 | Problem Approach and Contributions

In this note, we propose a trajectory tracking control law for the fully actuated mechanical system (1) that uses position measurements only. Given a twice differentiable trajectory,

$$
q_d(t) \in \mathbb{R}^n \tag{7}
$$

we propose a dynamic control law that requires only measurements of $q(t)$ such that the tracking error $q_e(t) \triangleq q(t) - q_d(t)$ satisfies

- In the absence of a disturbance, $\delta_{p_0}(t) = 0_{n\times 1}$, the tracking error $q_e(t)$ converges at an exponential rate that can be tuned via controller gains. In contrast to the exponentially stable controllers proposed in [4, 17], the stability properties in this work hold for all positive choices of controller gains, simplifying tuning and implementation.
- The tracking error $q_e(t)$ is ISS with respect to the unknown disturbance $\delta_{p_0}(t)$. Moreover, the influence of the disturbances can be attenuated via a choice of controller gains. The ISS property is a stronger robustness property than has been considered by the prior works summarized in Table 1. Moreover, the ISS property similarly holds for all positive choices of controller gains.

To the best of the authorŠs knowledge, this is the first tracking control scheme using only position measurements that is ISS with respect to external disturbances.

The proposed control scheme utilizes the momentum observer reported in [21], which is ISS with respect to external disturbances and has passive error dynamics with the momentum estimation error being a passive output. The approach taken here is to construct a tracking controller using the momentum estimate of the aforementioned observer such that the tracking error dynamics are also a passive system with the estimation error being a passive input. This enables the observer and tracking controller to be interconnected via a power-preserving interconnection, resulting in a storage function for the joint observer and controller system. The result of this interconnection is that the joint system is stable for all positive choices of controller gains both within the observer and tracking controller.

TABLE 1 | A summary of related works considering tracking control of mechanical systems using position information only.

Paper	Control structure	Stability type	External disturbances
[5]	State feedback with velocity observer	Asymptotic stability	No
[7]	Adaptive state feedback with filtered quantity	Uniform ultimate bound	Yes
[6]	Adaptive state feedback with filtered quantity	Uniform ultimate bound	No
[8]	Adaptive state feedback with filtered quantity	Uniform ultimate bound	Yes
[10]	Sliding control	Semiglobal asymptotic stability	No
[27]	Passivity-based state feedback with filtered quantity	Semiglobal asymptotic stability	No
[4]	Passivity-based state feedback with I&I velocity observer	Global exponential stability	No
[11]	State feedback with filtered quantity	Global asymptotic stability	No
[12]	State feedback with filtered quantity	Global asymptotic stability	No
[13]	State feedback with filtered quantity	Semiglobal asymptotic stability	No
[14]	State feedback with filtered quantity	Semiglobal asymptotic stability	No
[17]	State feedback with filtered quantity	Global exponential stability	No
[19]	Sliding control	Global asymptotic stability	Yes
[20]	Adaptive sliding control	Global asymptotic stability	No
[15]	Passivity-based state feedback with filtered quantity	Global asymptotic stability	No
[18]	State feedback with filtered quantity	Global finite-time stability	No

Note: The controller proposed in this work is exponentially stable and ISS with respect to external disturbances.

3 | Momentum Observer

In this section, we summarize the hybrid momentum observer for mechanical systems studied in [21]. The observer exhibits a passivity property that is exploited in the sequel to achieve tracking control.

3.1 | Momentum Transformation

Similar to the prior works [4, 21, 28], the observer and tracking controller are defined in a transformed set of coordinates with the special property that the kinetic energy is independent of the configuration vector. The mass matrix has a unique, symmetric, and uniformly positive-definite square root that satisfies

$$M^{-1}(q) = T^2(q) = T(q)T(q) \tag{8}$$

where

$$\overline{m}^{-\frac{1}{2}} I_n \leq T(q) = T^\top(q) \leq \underline{m}^{-\frac{1}{2}} I_n \tag{9}$$

Using this square root, a momentum transformation is introduced as

$$p := T(q)p_0 \tag{10}$$

As $M(q)$ is differentiable with continuous derivatives, $T(q)$ and $T^{-1}(q)$ similarly are differentiable with continuous derivatives (see [21] for details).

Under the momentum transformation (10), the dynamics (1) are described by

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_{n\times n} & T(q) \\ -T(q) & S(q,p) - D(q) \end{bmatrix} \begin{bmatrix} \nabla_q V \\ p \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ G(q) \end{bmatrix} u \\ &\quad - \begin{bmatrix} 0_{n\times n} \\ T(q) \end{bmatrix} \delta_{p_0} \\ H(q,p) &= \frac{1}{2} p^\top p + V(q) \end{aligned} \tag{11}$$

where

$$\begin{aligned} D(q) &= T(q) D_0(q) T(q) \\ S(q,p) &= T(q) \left[\frac{\partial^\top}{\partial q} (T^{-1}(q)p) - \frac{\partial}{\partial q} (T^{-1}(q)p) \right] T(q) \\ G(q) &= T(q) G_0(q) \end{aligned} \tag{12}$$

In the transformed coordinates, $D(q)$ is the open-loop damping matrix, which satisfies $D(q) = D(q)^\top \geq 0$ and $G(q)$ is the input mapping matrix. As both $G_0(q)$ and $T(q)$ are full-rank, $G(q)$ is full-rank also.

The matrix $S(q,p)$ plays a significant role in the definition of the momentum observer, so its properties should be highlighted. The matrix is skew-symmetric and is linear an its second argument p. Due to this linearity, we can implicitly define a matrix $\overline{S}(q,p) \in \mathbb{R}^{n\times n}$ satisfying

$$\overline{S}(q,a)b = S(q,b)a \tag{13}$$

for any two vectors $a, b \in \mathbb{R}^n$. While $S(q,p)$ is skew-symmetric, $\overline{S}(q,p)$ is not necessarily skew-symmetric. The matrix $\overline{S}(q,p)$ is, however, linear in its second argument and all elements are continuous with respect to q, p.

Remark 1. The proposed tracking control scheme requires the matrices $T(q)$, $S(q,p)$, and $\overline{S}(q,p)$ for implementation. While these terms are often difficult to compute in closed form, they can be computed numerically point-wise for implementation. First, the matrix $T(q)$ can be evaluated as the unique positive-definite matrix square-root of $M^{-1}(q)$. Evaluation of $S(q,p)$ requires the partial derivatives $\frac{\partial}{\partial q_i}\left(T^{-1}(q)\right)$ which can be evaluated point-wise as the solution to the Lyapunov equation

$$\frac{\partial}{\partial q_i}(M(q)) = \frac{\partial}{\partial q_i}\left(T^{-1}(q)\right)T^{-1}(q) + T^{-1}(q)\frac{\partial}{\partial q_i}\left(T^{-1}(q)\right) \tag{14}$$

Using these partial derivatives, we construct the matrix

$$A(q,p) = \left[\frac{\partial}{\partial q_1}\left(T^{-1}(q)\right)p \ \cdots \ \frac{\partial}{\partial q_n}\left(T^{-1}(q)\right)p\right] \tag{15}$$

which allows evaluation of $S(q,p)$ as

$$S(q,p) = T(q)\left[A^\top(q,p) - A(q,p)\right]T(q) \tag{16}$$

Due to the linearity of $S(q,p)$, it can be represented as

$$S(q,p) = \sum_{i=1}^{n} S(q,e_i)p_i \tag{17}$$

which allows for evaluation of $\overline{S}(q,p)$ as

$$\overline{S}(q,p) = \left[S(q,e_1)p \ \ S(q,e_2)p \ \cdots \ S(q,e_n)p\right] \tag{18}$$

3.2 | Momentum Observer

The proposed tracking control scheme assumes the use of the hybrid momentum observer previously proposed in [21, 29]. The observer requires direct measurement of the configuration vector q and generates an estimate for the transformed momentum vector p, given by $\hat{p} \in \mathbb{R}^n$. The particular properties of this observer that are required for subsequent developments are that the momentum estimation error is ISS with respect to external disturbances, and the error dynamics are passive with the estimation error being the passive output. These properties are exploited in the sequel to extend similar properties to the proposed tracking system.

The hybrid observer dynamics has state dimension $n+1$, where n is the number of degrees-of-freedom of the underlying mechanical system. The state vector is partitioned into $x_p \in \mathbb{R}^n$ which is a piece-wise continuous state and $\phi \in \mathbb{R}_+$ which is a piece-wise constant state. These two terms are used together to form an estimate for the momentum, $\hat{p}$. The behavior of the observer are defined by the hybrid dynamics

$$\begin{cases} [\dot{x}_p, \dot{\phi}] = [f_{x_p}(q, \hat{p}, \phi), 0], & (x_p, \phi, q) \in C \\ [x_p^+, \phi^+] = [x_p - \kappa q, \phi + \kappa], & (x_p, \phi, q) \in C^c \\ \hat{p}(x_p, \phi, q) = x_p + \phi q \end{cases} \tag{19}$$

where

$$\begin{aligned} C &:= \left\{(x_p, \phi, q) \mid \phi T(q) - \operatorname{symm}(\overline{S}(q,\hat{p})) \geq \kappa I_n\right\} \\ f_{x_p}(q, \hat{p}, \phi, u, u_o) &:= [S(q,\hat{p}) - D(q) - \phi T(q)]\hat{p} \\ &\quad - T(q)\nabla_q V(q) + G(q)\left[u + u_o\right] \end{aligned} \tag{20}$$

The system model terms used in the observer definition are defined in (11), (12). The input u is the same input used for the plant (11), whereas $u_o \in \mathbb{R}^n$ is an additional observer input that is utilized as part of the tracking control stage to complete a passive interconnection between the observer and tracking controller. The solution to the observer (19) is defined on a hybrid time domain denoted by $\mathcal{E}_o$.

The behavior of the observer is tuned by a single user-defined constant $\kappa \in \mathbb{R}_+$ which specifies a minimum amount of damping that should be present in the observer. Note that the observer stability properties hold for all positive values of κ. The domains of continuous and discrete dynamics are defined by the set C which is a matrix inequality as a function of κ. The term on the left of the inequality $\phi T(q) - \operatorname{symm}(\overline{S}(q,\hat{p}))$ defines the damping of the observer error dynamics and must be positive to ensure convergence of the solution. Recalling that $T(q) > \underline{m}^{-\frac{1}{2}} I_n$, the inequality can always be satisfied for ϕ large enough. The role of the discrete dynamics is to ensure that the observer state ϕ is always sufficiently large so that the damping term is always greater than the user-specified minimum damping κI_n. The discrete dynamics for x_p ensure continuity of the momentum estimate $\hat{p}$ at any discrete event of the momentum observer.

By ensuring a minimum level of damping for the observer, it is possible to verify ISS of the estimation error with respect to external disturbances and exponential stability when there is no disturbance. The key properties of the observer are summarized in the following proposition. For further details of the derivations and extended discussion, please refer to the prior work [21].

Proposition 1 ([21, Proposition 1]). *Consider the hybrid momentum observer (19) for estimating the momentum of the mechanical system (11) and assume that (11) has a solution $q(t), p(t)$ on some time domain $\mathcal{D}_L = \left[0, T_L\right)$ with $T_L \leq \infty$. The resulting estimation error system has the following properties:*

a. *The momentum estimation error $\tilde{p} := \hat{p} - p$ has the dynamics*

$$\begin{aligned} \dot{\tilde{p}} &= [S(q,p) + \overline{S}(q,\hat{p}) - D(q) - \phi T(q)] \\ &\quad \times \nabla_{\tilde{p}} H_o + G(q)u_o + T(q)\delta_{p_0}, (x_p, \phi, q) \in C \\ \tilde{p}^+ &= \tilde{p}^-, \ (x_p, \phi, q) \in C^c \\ y_o &= G^\top(q)\tilde{p} \\ H_o(\tilde{p}) &= \frac{1}{2}\|\tilde{p}\|^2 \end{aligned} \tag{21}$$

and both $\tilde{p}(t)$, $H_o(t)$ are continuous on $\mathcal{E}_o$.

b. *The observer error dynamics (21) are flow strictly passive with input-output pairs (u_o, y_o), (δ_{p_0}, y_δ) and storage function*

$H_o(\tilde{p})$, *satisfying*

$$\dot{H}_o \leq -2\kappa H_o + \underbrace{\tilde{p}^\top G(q)}_{:=y_o^\top} u_o + \underbrace{\tilde{p}^\top T(q)}_{:=y_\delta^\top} \delta_{p_0}, (x_p, \phi, q) \in C$$
$$H_o^+ = H_o^-, \ (x_p, \phi, q) \in C^c \tag{22}$$

c. *If $u_o = 0_{n\times1}$, the momentum estimate $\hat{p}(t)$ exists for all $t \in \mathcal{D}_L$ and the observer estimation error $\tilde{p}(t)$ is ISS with respect to an unknown input δ_{p_0}, satisfying the bound*

$$\|\tilde{p}(t)\| \leq \sqrt{2H_o(0)}e^{-\frac{1}{2}\kappa t} + \underline{m}^{-\frac{1}{2}}\kappa^{-1}\gamma \tag{23}$$

for all $t \in \mathcal{D}_L$. If $\delta_{p_0} = 0_{n\times1}$, $\tilde{p} = 0_{n\times1}$ is a globally exponentially stable equilibrium.

In the sequel, a tracking controller will be defined that utilizes the momentum estimate $\hat{p}$ as part of the tracking error definition. For that development, it will be useful to describe the dynamics of the momentum estimate directly. Considering the definition of $\hat{p}$ in (19) and the dynamics of x_p, ϕ and q, it can be verified that $\hat{p}$ evolves according to

$$\begin{aligned}\dot{\hat{p}} &= [S(q,\hat{p}) - D(q)]\hat{p} - T(q)\nabla_q V(q) + G(q)\left[u + u_o\right] \\ &\quad - \phi T(q)\tilde{p}, \ (x_p, \phi, q) \in C \\ \hat{p}^+ &= \hat{p}^-, \ (x_p, \phi, q) \in C^c\end{aligned} \tag{24}$$

Note that the momentum estimate is continuous as a function of time, but has a discontinuous derivative at jump events. When the dynamics of $\hat{p}$ are compared to the dynamics of p in (11), it can be seen that the momentum estimate dynamics are a copy of the underlying plant dynamics with an additional correction term $-\phi T(q)\tilde{p}$ and an additional input term u_o.

4 | Tracking Control Design

In this section, a tracking control law is proposed that controls the mechanical system (1) to the target trajectory (7). The tracking control law assumes that the momentum observer (19) is used to estimate the systemŠs momentum from position measurements. The novelty of this approach is the construction of a tracking error system that is defined by the difference between the target momentum and the estimated momentum. This is in contrast with most tracking approaches that construct the error coordinate between the target momentum and the true momentum (or velocities). The advantage in constructing the error dynamics in this way is twofold. First, the unknown disturbance term $\delta_{p_0}(t)$ does not appear in the tracking error dynamics, and second, the resulting error dynamics allow for joint stability analysis of the observer and controller systems.

4.1 | Feed-Forward Tracking Control

In order to stabilize the target trajectory, a suitable error coordinate is introduced, and a feed-forward controller is proposed. To this end, we define the tracking error as

$$q_e(q,t) := q - q_d(t) \tag{25}$$

where q is the plant's configuration (11) and q_d is the target trajectory (7). The error coordinate q_e evolves according to

$$\begin{aligned}\dot{q}_e = \dot{q} - \dot{q}_d &= T\left[p - T^{-1}\dot{q}_d\right] \\ &= -T\tilde{p} + T\left[\hat{p} - T^{-1}\dot{q}_d\right]\end{aligned} \tag{26}$$

where $\hat{p}$ is the estimate of the momentum from the observer (19) and $\tilde{p}$ is the observer's estimation error.

We similarly require a tracking error for the momentum coordinate to be used for the tracking controller definition. For a given trajectory $q_d(t)$, the target momentum is defined to be

$$p_d(q,t) := T^{-1}(q)\dot{q}_d(t) \tag{27}$$

It is tempting to then define the momentum error as the difference between the system's momentum $p(t)$ and the target momentum. Contrary to this, however, we define the momentum tracking error as the difference between the estimated momentum $\hat{p}$, defined in (19), and the desired momentum

$$p_e(q, x_p, \phi, t) := \hat{p}(q, x_p, \phi) - p_d(q,t) \tag{28}$$

By defining the momentum error in this way, the unknown disturbance term $\delta_{p_0}(t)$ does not appear in the resulting tracking error dynamics. Additionally, the momentum estimation error $\tilde{p} = \hat{p} - p$ appears linearly in the resulting dynamics and can be treated as an input for the purpose of interconnecting the observer and controller systems. The dynamics of the tracking error dynamics and a feed-forward controller are defined in the following proposition.

Proposition 2. *Consider the mechanical system (11) together with the hybrid momentum observer (19). Defining the gravity compensation and tracking feed-forward control law*

$$\begin{aligned}u = -u_o + G^{-1}(q)\Big\{ &- [S(q,\hat{p}) - D(q)]p_d + T(q)\nabla_q V \\ &+ \frac{\partial p_d}{\partial q}T(q)\hat{p} + T^{-1}(q)\ddot{q}_d + v\Big\}\end{aligned} \tag{29}$$

results in the tracking error dynamics

$$\begin{aligned}\begin{bmatrix}\dot{q}_e \\ \dot{p}_e\end{bmatrix} &= \begin{bmatrix}0_{n\times n} & T(q) \\ -T(q) & S(q,\hat{p}) - D(q)\end{bmatrix}\begin{bmatrix}\nabla_{q_e} H_e \\ \nabla_{p_e} H_e\end{bmatrix} + \begin{bmatrix}0_{n\times1} \\ v\end{bmatrix} \\ &\quad - \begin{bmatrix}T(q) \\ \phi T(q) - \frac{\partial p_d}{\partial q}(q,t)T(q)\end{bmatrix}\tilde{p}, \ (x_p, \phi, q) \in C \\ \begin{bmatrix}q_e^+ \\ p_e^+\end{bmatrix} &= \begin{bmatrix}q_e^- \\ p_e^-\end{bmatrix}, \ (x_p, \phi, q) \in C^c \\ H_e &= \frac{1}{2}\|p_e\|^2\end{aligned} \tag{30}$$

where $v \in \mathbb{R}^n$ is an additional input for subsequent control design.

Proof. Verification of the q_e dynamics on C follows from (26), together with the definition of p_d in Equation (27) and p_e in Equation (28). Verification of the p_e dynamics follows from taking the time derivative of Equation (28) and substituting the expressions (27), (24), yielding

$$
\begin{aligned}
\dot{p}_e &= \dot{\hat{p}} - \dot{p}_d \\
&= [S(q,\hat{p}) - D(q)]\hat{p} - T(q)\nabla_q V(q) + G(q)\big[u + u_o\big] \\
&\quad - \phi T(q)\tilde{p} - \frac{\partial p_d}{\partial q} T(q) p - T^{-1}(q)\ddot{q}_d
\end{aligned}
\tag{31}
$$

Substituting the control law (29) into this expression recovers the dynamics in Equation (30). The signal q_e is continuous on the set C^c as both q, q_d are continuous. Continuity of p_e is inherited from the continuity of $\hat{p}$. □

Considering the tracking error system (30) the advantage of taking the momentum error as the difference between the momentum estimate and desired momentum is evident. The states q_e, p_e, storage function H_e, and the interconnection and damping structures are exactly known for control purposes. The only uncertain term is the momentum estimation error $\tilde{p}$ which enters the tracking error dynamics linearly. Note that the unknown disturbance term $\delta_{p_0}(t)$ does not appear in these dynamics.

4.2 | Kinetic-Potential Energy Shaping

We now consider secondary control design to render the origin of the tracking error system (30) attractive. As the state vector, energy function and interconnection and damping structures of the tracking system (30) are exactly known for control purposes, the control objective can be achieved by direct application of a variety of energy-shaping methods. Here, we utilize the method of kinetic-potential energy shaping [28] to inject damping into the configuration tracking error coordinates, resulting in a strictly flow passive property that will be required for joint analysis of the observer and controller systems.

Proposition 3. *Consider the tracking error system (30) in closed loop with the KPES control law*[1]

$$
\begin{aligned}
v &= \alpha\big[S(q,\hat{p}) - D(q) - K_d\big]K_p\big[q_e + \alpha p_e\big] \\
&\quad - T(q)K_p\big[q_e + \alpha p_e\big] - K_d p_e
\end{aligned}
\tag{32}
$$

where $K_p \in \mathbb{R}^{n\times n}$, $K_d \in \mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}_+$ are positive definite tuning parameters. The following properties hold

a. *The closed-loop dynamics have the form*

$$
\begin{aligned}
\begin{bmatrix} \dot{q}_e \\ \dot{p}_e \end{bmatrix} &= \underbrace{\begin{bmatrix} -\alpha T(q) & T(q) \\ -T(q) & S(q,\hat{p}) - D(q) - K_d \end{bmatrix}}_{:=F_e(q,\hat{p})} \begin{bmatrix} \nabla_{q_e}\tilde{H}_e \\ \nabla_{p_e}\tilde{H}_e \end{bmatrix} \\
&\quad - \begin{bmatrix} T(q) \\ \phi T(q) - \frac{\partial p_d}{\partial q}(q,t)T(q) \end{bmatrix}\tilde{p}, \ (x_p, \phi, q) \in C \\
\begin{bmatrix} q_e^+ \\ p_e^+ \end{bmatrix} &= \begin{bmatrix} q_e^- \\ p_e^- \end{bmatrix}, \ (x_p, \phi, q) \in C^c
\end{aligned}
$$

$$
\begin{aligned}
\tilde{y}_e &= -T(q)\Big[I_n, \ \phi I_n - \tfrac{\partial^\top p_d}{\partial q}(q,t)\Big]\nabla\tilde{H}_e \\
\tilde{H}_e(q_e,p_e) &= \frac{1}{2}\|p_e\|^2 + \underbrace{\frac{1}{2}\|q_e + \alpha p_e\|^2_{K_p}}_{:=V_d(q_e,p_e)}
\end{aligned}
\tag{33}
$$

b. *The closed-loop dynamics (33) are flow strictly passive with storage function $\tilde{H}_e$, input $\tilde{p}$ and output $\tilde{y}_e$, satisfying the inequality*

$$
\begin{aligned}
\dot{\tilde{H}}_e &\le -2\frac{\lambda_{min}(QR_{cl}Q)}{\lambda_{max}(Q)}\tilde{H}_e + \tilde{y}_e^\top\tilde{p}, \ (x_p,\phi,q) \in C \\
\tilde{H}_e^+ &= \tilde{H}_e^-, \ (x_p,\phi,q) \in C^c
\end{aligned}
\tag{34}
$$

where

$$
Q = \begin{bmatrix} K_p & \alpha K_p \\ \alpha K_p & I_n + \alpha^2 K_p \end{bmatrix}, \quad R_{cl} = \begin{bmatrix} \alpha T(q) & 0_{n\times n} \\ 0_{n\times n} & D(q) + K_d \end{bmatrix}
\tag{35}
$$

Proof. To verify Claim a), note that the gradients of the kinetic-potential function V_d satisfy

$$
\begin{aligned}
\nabla_{q_e}V_d &= K_p(q_e + \alpha p_e) \\
\nabla_{p_e}V_d &= \alpha K_p(q_e + \alpha p_e)
\end{aligned}
\tag{36}
$$

As $-\alpha T(q)\nabla_{q_e}V_d + T(q)\nabla_{p_e}V_d = 0_{n\times 1}$, the dynamics of q_e can be seen to match (30). Verification of the p_e dynamics follows from direct substitution of the control law (32) into (30). Continuity of q_e, p_e at jump events is inherited from Equation (30).

Considering Claim b), first note that the function $\tilde{H}_e$ satisfies

$$
\tilde{H}_e = \frac{1}{2}\begin{bmatrix} q_e \\ p_e \end{bmatrix}^\top \underbrace{\begin{bmatrix} K_p & \alpha K_p \\ \alpha K_p & I_n + \alpha^2 K_p \end{bmatrix}}_{Q} \begin{bmatrix} q_e \\ p_e \end{bmatrix}, \quad \nabla\tilde{H}_e = Q\begin{bmatrix} q_e \\ p_e \end{bmatrix}
\tag{37}
$$

The time evolution of $\tilde{H}_e$ along the flow dynamics of Equation (33) satisfies

$$
\begin{aligned}
\dot{\tilde{H}}_e &= -\nabla^\top\tilde{H}_e \underbrace{\begin{bmatrix} \alpha T(q) & 0_{n\times n} \\ 0_{n\times n} & D(q) + K_d \end{bmatrix}}_{R_{cl}} \nabla\tilde{H}_e \\
&\quad - \nabla^\top\tilde{H}_e \begin{bmatrix} T(q) \\ \phi T(q) + \frac{\partial p_d}{\partial q}(q,t)T(q) \end{bmatrix}\tilde{p} \\
&= -\begin{bmatrix} q_e \\ p_e \end{bmatrix}^\top QR_{cl}Q \begin{bmatrix} q_e \\ p_e \end{bmatrix} + \tilde{y}_e^\top\tilde{p}
\end{aligned}
\tag{38}
$$

Combining the expressions (37), (38) results in the passivity inequality (34). Continuity of $\tilde{H}_e$ at a jump event is inherited from the continuity of q_e, p_e at jump events. □

From Equation (34) it is clear that the tracking error dynamics are flow strictly passive where the rate of energy dissipation is

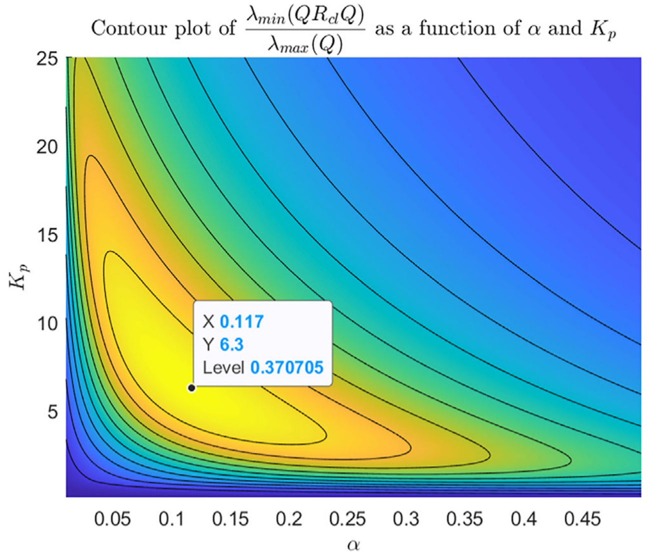

FIGURE 1 | Minimum dissipation rate of the tracking error dynamics as a function of α and K_p. The gain K_d has been set to 10 for this plot.

lower-bounded by the quantity $\frac{\lambda_{min}(QR_{cl}Q)}{\lambda_{max}(Q)}$. Considering the definitions of Q, R_{cl} in Equation (35), it can be seen that the selection of K_p and α influences both the numerator and denominator of this expression, making a simple tuning guideline difficult to establish. Rather, a contour plot of the ratio $\frac{\lambda_{min}(QR_{cl}Q)}{\lambda_{max}(Q)}$ was generated for the scalar case ($n = 1$) in Figure 1 for the special case $T(q) = 1$, $D(q) = 1$, $K_d = 10$. This plot suggests that for a given choice of K_d, the tuning gains α, K_p can be chosen within a region relatively small to maximize the dissipation rate. A similar profile is recovered for different choices of K_d.

4.3 | Passive Interconnection of Observer and Controller

We now consider the joint behavior of the momentum observer (19) with the tracking control law (32) by exploiting the passivity properties of each system. From Proposition 1, the momentum observer is flow strictly passive, satisfying the passivity inequality (22). In particular, note that the passive output y_o is a linear mapping of the momentum estimation error $\tilde{p}$. Conversely, Proposition 3 states that the tracking error dynamics are flow strictly passive, satisfying the passivity inequality (34). This passivity inequality has the momentum estimation error as the passive input. The common appearance of the momentum estimation error within the passivity inequalities of the observer and tracking control systems allows the two systems to be passively interconnected.

The passive input to the tracking error dynamics is the momentum estimation error, which is the passive output of the momentum observer. This implies that one direction of the passivity interconnection is already established through the definition of the observer and tracking control dynamics. To complete the passive interconnection of the two systems, the passive output of the tracking error dynamics, $\tilde{y}_e$, needs to be interconnected

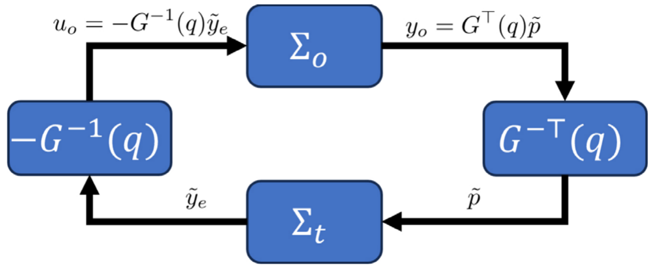

FIGURE 2 | Passive interconnection of the momentum observer system with the tracking control system. Σ_o represents the observer system in Equation (21) and Σ_t represents the tracking system in Equation (33).

with the passive input to the observer dynamics u_o. The total power-preserving interconnection between the two systems is given by

$$\begin{bmatrix} \tilde{p} \\ u_o \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & G^{-\top}(q) \\ -G^{-1}(q) & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \tilde{y}_e \\ y_o \end{bmatrix} \tag{39}$$

The resulting feedback interconnection is shown in Figure 2.

The feedback interconnection of two passive systems results in a passive system with storage function that is the sum of the individual storage function. The resulting storage function for the interconnected observer and controller systems is defined as

$$W(\tilde{p}, q_e, p_e) = H_o(\tilde{p}) + \tilde{H}_e(q_e, p_e) \tag{40}$$

The resulting passivity inequality is obtained by taking the time derivative of this expression and substituting in the inequalities (22), (34), together with the interconnection inputs (39). Making these substitutions on the flow dynamics results in

$$\begin{aligned} \dot{W} &= \dot{H}_o + \dot{\tilde{H}}_e \\ &\leq -2\kappa H_o + \underbrace{\tilde{p}^\top G(q)}_{y_o^\top} u_o + y_\delta^\top \delta_{p_0} \\ &\quad - 2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}\tilde{H}_e + \tilde{y}_e^\top \tilde{p}, \ (x_p, \phi, q) \in C \\ &\leq -2\kappa H_o - 2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}\tilde{H}_e + y_\delta^\top \delta_{p_0}, \ (x_p, \phi, q) \in C \end{aligned} \tag{41}$$

As both $H_o, \tilde{H}_e$ are continuous at any jump event of the observer, the interconnected observer and controller system is flow strictly passive with storage function W, input δ_{p_0} and output y_δ. The following proposition formalizes the stability properties of the joint system.

Proposition 4. *Consider the mechanical system (11) with the hybrid momentum observer (19) and tracking control law (29), (32). Setting the observer input as $u_o = -G^{-1}(q)\tilde{y}_e$ results in the following*:

a. *The joint observer and controller system is flow strictly passive with storage function W, input δ_{p_0} and output y_δ.*

b. *The solution to the interconnected system exists for all time and is ISS with respect to the external disturbance* $\delta_{p_0}(t)$, *satisfying the bound*

$$W(t) \le W(0)e^{-\varphi t} + \frac{1}{2\underline{m}\kappa\varphi}\gamma^2 \tag{42}$$

where

$$\varphi = \min\left\{\kappa, 2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}\right\} \tag{43}$$

c. *In the absence of an external disturbance,* $\delta_{p_0}(t) = 0_{n\times1}$, *the equilibrium point* $(\tilde{p}, q_e, p_e) = (0_{n\times1}, 0_{n\times1}, 0_{n\times1})$ *is globally exponentially stable. Moreover, the function* $W(t)$ *is a strict Lyapunov function satisfying the bound*

$$W(t) \le W(0)e^{-\varphi t} \tag{44}$$

Proof. Claim a) follows from (41), together with the fact that $H_o, \tilde{H}_e$ are continuous at any jump event.

Now, we consider Claim b) by first verifying that the forward solution of the joint observer and controller system exists for all time. Note that this is nontrivial as hybrid system can exhibit Zeno or eventually discrete behaviors that should be excluded. To do this, we first consider the evolution of the joint storage function $W(t)$. From (41), the time derivative of $W(t)$ along the flow dynamics satisfies

$$\dot{W} \le -2\kappa H_o - 2\frac{\lambda_{min}(QR_{cl}Q)}{\lambda_{max}(Q)}\tilde{H}_e + \frac{c}{2}\underbrace{\tilde{p}^\top T(q)T(q)\tilde{p}}_{y_\delta^\top y_\delta} + \frac{1}{2c}\left\|\delta_{p_0}\right\|^2, \ (x_p, \phi, q) \in C \tag{45}$$

where c is an arbitrary constant from application of Young's inequality. Recalling the Definitions (8) and (9), we take $c = \underline{m}\kappa$ which results in

$$\dot{W} \le \underbrace{-\min\left\{\kappa, 2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}\right\}}_{\varphi}W + \frac{1}{2\underline{m}\kappa}\left\|\delta_{p_0}\right\|^2 \quad (x_p, \phi, q) \in C \tag{46}$$

We now construct a bound for the storage function along the forward solution of the joint observer controller system via an induction argument. Suppose that jump events occur at times $t_1, t_2, \dots, t_N$ and assume that the storage function $W(t)$ satisfies

$$W(t) \le W(0)e^{-\varphi t} + e^{-\varphi t}\int_0^t e^{\varphi\tau}\frac{1}{2\underline{m}\kappa}\left\|\delta_{p_0}(\tau)\right\|^2 d\tau \tag{47}$$

for all $t \in [0, t_j]$. By the comparison [30, Lemma 3.4] and the solution to a LTI system [30, Chapter 4.9], the solution of the expression (46) on the interval $[t_j, t_{j+1}]$ satisfies the bound

$$W(t) \le W(t_j)e^{-\varphi(t-t_j)} + e^{-\varphi t}\int_{t_j}^t e^{\varphi\tau}\frac{1}{2\underline{m}\kappa}\left\|\delta_{p_0}(\tau)\right\|^2 d\tau \tag{48}$$

for $t \in [t_j, t_{j+1}]$. Evaluating the bound (47) at $t = t_j$ and substituting into (48) recovers the bound (47) on the interval $[0, t_{j+1}]$. As the bound (47) holds on the interval $[0, t_1]$, it follows by induction that this bound holds on the hybrid time domain. Recalling that the external force disturbance $\delta_{p_0}(t)$ satisfies the bound (3), the inequality (47) can be resolved as

$$W(t) \le W(0)e^{-\varphi t} + \frac{1}{2\underline{m}\kappa}\gamma^2 e^{-\varphi t}\int_0^t e^{\varphi\tau}\, d\tau \tag{49}$$

which can be evaluated to find the expression (42).

Now, we verify that the forward solution of the joint observer and controller system exists for all time via a contradiction argument. For the sake of contradiction, assume that there exists some T at which the observer exhibits Zeno or eventually discrete behavior. Along the forward solution on the interval $[0, T)$ the storage function $W(t)$ satisfies the bound (42), implying that $\tilde{p}(t) = p(t) - \hat{p}(t), p_e(t) = \hat{p}(t) - p_d(t)$ and $q_e(t) = q(t) - q_d(t)$ are all bounded. From the definition of the flow domain in (20), the system is in the flow domain if ϕ satisfies

$$\phi T(q) - \mathrm{symm}(\overline{S}(q, \hat{p})) \ge \kappa I_n \tag{50}$$

At any given point t, $\hat{p}(t)$ exists within a closed and bounded neighborhood of $p_d(t)$ and $q(t)$ is within a closed and bounded neighborhood of $q_d(t)$. As the elements of $\overline{S}(q, \hat{p})$ are continuous functions of $q, \hat{p}$, each element of $\mathrm{symm}[\overline{S}(q, \hat{p})]$ is bounded for $t \in [0, T)$. This implies that there exists some $\phi_{\max}^T$ such that the inequality (50) holds for all $\phi \ge \phi_{\max}^T$. Consequently, only finitely many jump events can occur on the interval $[0, T)$, which is a contradiction. We conclude that the forward solution of the joint observer and controller systems exists for all time.

Claim c) follows as a special case of Claim b). Setting $\gamma = 0$ in Equation (42) verifies an exponential rate of convergence for the joint storage function $W = H_o + \tilde{H}_e$. As both H_o and $\tilde{H}_e$ are quadratic in the state variables, exponential convergence of the states $\tilde{p}, q_e, p_e$ follows. □

It has now been established that the proposed tracking control scheme is ISS with respect to external disturbances and converges at an exponential rate in the absence of external perturbations. As the ISS Lyapunov function $W(t)$, defined in Equation (40), is the sum of the observer and tracking controller storage functions, this implies similar properties hold for both the tracking and observer systems. More specifically, in the absence of external disturbances, the momentum observer estimate converges exponentially to the true momentum value, and the tracking error converges to zero at an exponential rate. In the case that there is an unknown disturbance acting on the system, the error in the momentum estimate and the tracking error remain bounded with the bound dependent on the magnitude of the disturbance.

An overview of the total control structure is provided in Figure 3. In summary of the system roles, the momentum observer (19) utilizes measurements of the systems configuration and the known input to generate an estimate of the systems momentum that is ISS with respect to unknown disturbances. The feed-forward controller (29) generates the tracking error system (30) about reference trajectory. This system is then controlled to be strictly passive with respect to the input-output pair $(\tilde{p}, \tilde{y}_e)$ via the tracking controller (32). Finally, a passive interconnection

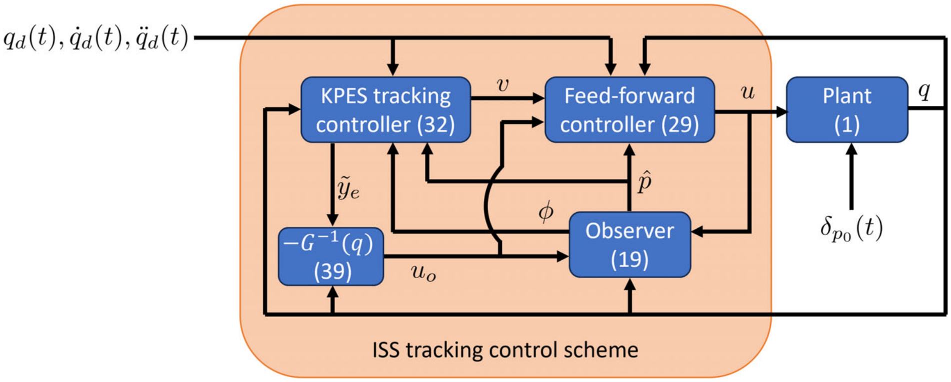

FIGURE 3 | Block diagram overview of the total ISS tracking control scheme.

between the controller and observer is completed by setting the observer input as per (39). The closed-loop tracking system resulting from this interconnection possesses the proved stability properties due to the fact that the observer and tracking control systems are individually strictly passive.

5 | Numerical Example

In this section, we numerically demonstrate the key results of this paper on a 2-degree-of-freedom manipulator system (See Figure 4). It is verified that in the absence of a disturbance acting on the system, the joint storage function (40) is a strict Lyapunov function for the joint observer and controller system. It is additionally demonstrated that when a disturbance is acting, the observer gains κ can be used to attenuate the asymptotic effect of the disturbance. Evaluation of all terms related to the momentum transformation in Equation (11) was done numerically point-wise using the methods detailed in Remark 1. All of the source code used to generate these results is available via https://github.com/JoelFerguson/ISS_Tracking_for_Mechanical_Systems.

5.1 | Dynamic Equations and Reference Trajectory

The dynamic equations of the robotic manipulator can be expressed in the form (1) where $q = (\theta_1, \theta_2)$ are the angular orientations of the first and second manipulator links with respect to the horizontal plane, respectively. The canonical momentum is defined by $p_0 = M_0(q)\dot{q}$ with the mass matrix given by

$$M_0(q) = \begin{bmatrix} J_1 + \frac{1}{4}m_1 l_1^2 + m_2 l_1^2 & \frac{1}{2}l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2}l_1 l_2 m_2 \cos(\theta_1 - \theta_2) & J_2 + \frac{1}{4}m_2 l_2^2 \end{bmatrix} \tag{51}$$

where $l_1, l_2, m_1, m_2, J_1, J_2$ are the lengths, masses, and moments of inertia of the rigid links. The potential energy has the form

$$V(q) = m_2 g\left(l_1 \sin\theta_1 + \frac{1}{2}l_2 \sin\theta_2\right) + \frac{1}{2}m_1 g l_1 \sin\theta_1 \tag{52}$$

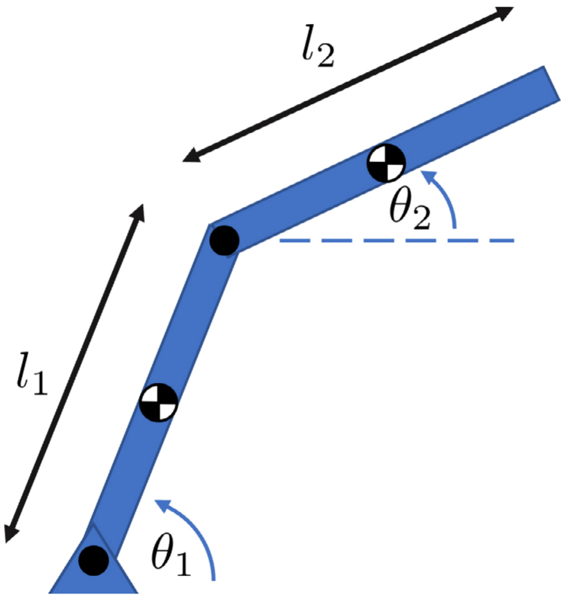

FIGURE 4 | Vertical 2 degree-of-freedom manipulator.

where g is the acceleration due to gravity. The input mapping matrix and the damping matrix are given by

$$G_0 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad D_0(q) = \begin{bmatrix} d_{j_1}(q) + d_{j_2}(q) & -d_{j_2}(q) \\ -d_{j_2}(q) & d_{j_2}(q) \end{bmatrix} \tag{53}$$

where $d_{j_1}(q)$ and $d_{j_2}(q)$ are the state-dependent coefficients of the friction of the first and second joints, respectively. The detailed physical parameters are listed in Table 2. Finally, for all simulations, the system was initialized from the configuration $q(0) = [0, 0]^\top$ with the initial canonical momentum $p_0(0) = [-1, 2]^\top$. Both the observer and tracking controller definitions require the system to be described in the noncanonical coordinates (11). All of the required terms were evaluated numerically point-wise using the methods described in Remark 1.

For all of the subsequent examples, the reference trajectory and its derivatives are given by

$$q_d(t) = \begin{bmatrix} \sin(t) \\ \cos(2t) \end{bmatrix}, \quad \dot{q}_d(t) = \begin{bmatrix} \cos(t) \\ -2\sin(2t) \end{bmatrix}, \quad \ddot{q}_d(t) = \begin{bmatrix} -\sin(t) \\ -4\cos(2t) \end{bmatrix} \tag{54}$$

5.2 | Trajectory Tracking With No Disturbances

First, we consider the tracking performance of the joint observer and controller system in the special case that no disturbance is acting on the system. That is, $\delta_{p_0}(t) = 0_{2\times 1}$. From Proposition 4.c it is expected that the tracking error converges to zero at an exponential rate, defined by the parameter φ. It is additionally expected that the joint storage function $W(t)$ is a strict Lyapunov function for the joint system. The 2-degree-of-freedom manipulator system was simulated using the control parameters $\alpha = 0.1, K_d = 10I_2, K_p = 6I_2$ and for several values of observer tuning gains κ. As the parameter κ appears directly in the expression for φ, defined in Equation (43), it is expected that this parameter would impact the rate of convergence.

The resulting trajectories for each of the controller gains, the momentum estimation error, and the solution of the strict Lyapunov function $W(t)$ are provided in Figure 5. In each case, it is clear that $W(t)$ decreases strictly as expected. Considering the definition of φ in Equation (43), it would be reasonable to presume that larger values of κ would result in faster rates of decay of the function $W(t)$. While it can be seen in Figure 5 that during the initial transient larger values of κ do translate into faster decay rates, it can be seen that this trend quickly vanishes. As the value of $2\frac{\lambda_{min}(QR_{cl}Q)}{\lambda_{max}(Q)}$ is equal to 0.7294 for this example, this term becomes the limiting factor is decay rates for values of κ greater than this value. The simulation results suggest that taking κ close the value of $2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}$ results in faster long-term decay rates. In each case, however, the tracking error quickly converges toward zero. It is also notable that the momentum estimation error converges to zero at a similar rate as the trajectory converges to the desired trajectory. This is shown to emphasize that there is no timescale separation between the observer and tracking controller dynamics in the proposed approach.

TABLE 2 | Physical parameters.

	Link 1	Link 2
Mass	$m_1 = 3$	$m_2 = 3$
Length	$l_1 = 1$	$l_2 = 1$
Moment of inertia	$J_1 = 3/12$	$J_2 = 3/12$
Coefficient of friction	$d_{j_1} = 1$	$d_{j_2} = 1$
Gravitational acceleration	$g = 9.8$	

5.3 | Trajectory Tracking With Disturbances

The utility of the observer tuning gain κ is more apparent when considering the response of the joint observer and tracking controller in the presence of disturbances. In this section, the 2-degree-of-freedom manipulator was simulated with the constant bounded disturbance term $\delta_{p_0}(t) = [1, 1]^\top$ for a variety of values for κ. From Proposition 4.b, the solution of the joint observer and tracking controller system is ISS with respect to external disturbances, satisfying the bound (42). Notably, the observer tuning gain κ appears in the denominator of the second term, which implies that increasing the value of κ should have the effect of attenuating the asymptotic effects of bounded disturbances.

To test this result, the system was simulated using the tuning gains $\alpha = 0.1, K_d = 10I_2, K_p = 6I_2$ and for several values for κ given by $\kappa = 1, 5, 10$. The resulting trajectories for each set of tuning gains, the momentum estimation error and the value of the ISS Lyapunov function $W(t)$ are plotted Figure 6. In each case, it is immediately clear that increasing the value of the observer gain κ has the intended effect of attenuating the asymptotic effect of the disturbance. The function $W(t)$ initially decays and asymptotically falls below a steady-state bound where the bound is smaller for larger choices of κ. It can additionally be seen that in the first 0.1 s larger values of κ result in faster initial convergence toward the reference trajectory. Somewhat counter-intuitively, however, is the observation that for the period 0.1–1.5 s the convergence rate is faster for smaller values of κ. As the value of $2\frac{\lambda_{\min}(QR_{cl}Q)}{\lambda_{\max}(Q)}$ is 0.7294 for this example, the authors conjecture that increased values of κ encourage energy to be routed to the tracking subsystem for dissipation, rather than being dissipated by the observer subsystem resulting in the observed trajectories of the ISS Lyapunov function. This observation align with the observation in [31, Remark 2] in that injecting additional damping into passive

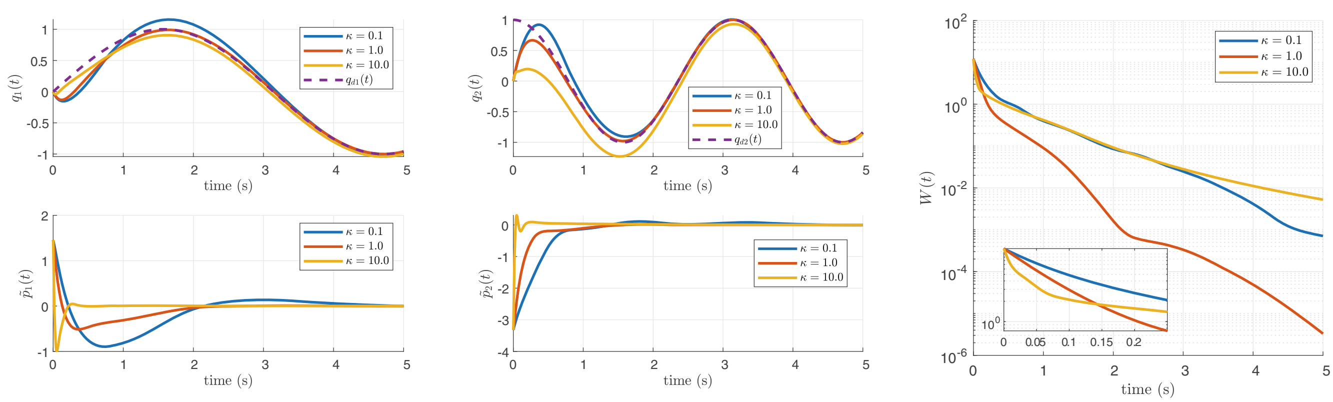

FIGURE 5 | Performance of tracking controller for several values of observer tuning gains κ with no disturbance applied to the system. The top two plots show the tracking results and the bottom two plots show the momentum estimation error, defined in (21). The right plot shows the value of the strict Lyapunov function $W(t)$, defined in (40). The exploded view of the first 0.25 s shows that increasing the tuning gain κ increases the initial rate of convergence to the desired trajectory.

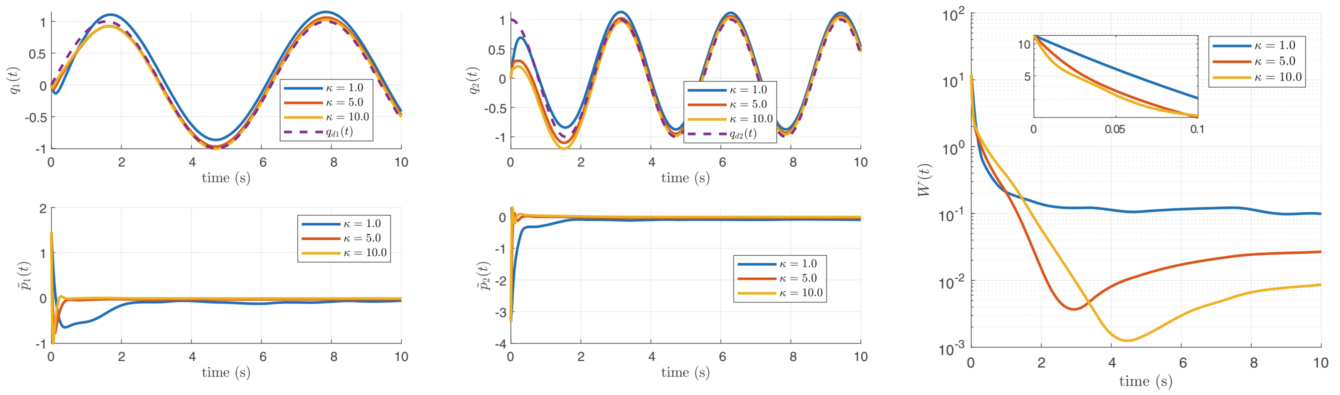

FIGURE 6 | Performance of tracking controller for several values of observer tuning gains κ with an unknown disturbance acting on the system. The top two plots show the tracking results, and the bottom two show the momentum observer estimation error, defined in Equation (21). The right plot shows the value of the ISS Lyapunov function $W(t)$, defined in Equation (40). It is shown that increasing the tuning gain κ attenuates the effect of an unknown disturbance on the tracking performance.

systems does not always result in a faster transient convergence. It can be seen from the plots of configuration $q(t)$ and momentum estimation error $p_e(t)$ that the bounded disturbance has caused a bounded perturbation in both of these quantities. The magnitude of the perturbation, however, is modulated by the tuning parameter κ with larger values of κ leading to smaller steady-state errors.

5.4 | Comparative Study

The proposed control scheme was directly compared numerically against the controllers proposed in [17–19]. These works were chosen as they are recent iterations of tracking control using filtered quantity in place of velocity observer, robust sliding mode methods and finite time methods, respectively. As the alternate tracking controllers do not have a similar ISS property, comparison was only performed for the scenario that there is no external disturbance acting. That is, $\delta_{p_0}(t) = 0_{2\times 1}$ for the following simulations. We note that the work [19] utilizes a sliding mode technique and can reject unknown disturbances up to a predefined bound. This form of robustness is different from the ISS property established in this work which holds for disturbances of arbitrary (but bounded) size. The problem setup used throughout this comparative study is the same as was defined in Section 5.1.

As all of the comparison works utilize an Euler–Lagrange system representation, we note that mechanical systems of the form (1) can be equivalently written in for form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D_0(q)\dot{q} + \nabla_q V(q_d(t)) = G_0(q)u - \delta_{p_0}(t) \quad (55)$$

where $C(q,\dot{q})$ is the centripetal-Coriolis matrix and all other terms are as defined in Section 2.1. Details on this system representation can be found in [32]. For the robotic manipulator in Section 5.1, the centripetal-Coriolis matrix has the form

$$C(q,\dot{q}) = \begin{bmatrix} 0 & \frac{1}{2} l_1 l_2 m_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2 \\ \frac{1}{2} l_1 l_2 m_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix} \quad (56)$$

For the sake of comparison, both the control input and tracking performance was considered. Along the nominal trajectory there is a nominal control input to maintain tracking, given by

$$\begin{aligned} u_d(t) = G_0^{-1}(q_d(t))\big[& M(q_d(t))\ddot{q}_d(t) + C(q_d(t),\dot{q}_d(t))\dot{q}_d(t) \\ & + D_0(q_d(t))\dot{q}_d(t) + \nabla_q V(q_d(t))\big] \end{aligned} \quad (57)$$

For each of the control laws, the deviation of the computed control from this nominal control input was computed and compared as a measure of how control-intensive the scheme is during the transient to the desired trajectory. Alongside this, the tracking error was considered as a measure of transient performance.

Two different tunings of the tracking control proposed in this paper were used. The first set of control gains were chosen as $\kappa = 1, \alpha = 0.1, K_d = 10I_2, K_p = 6I_2$ whereas the second set were chosen to be $\kappa = 0.3, \alpha = 0.18, K_d = 0.3I_2, K_p = 1.4I_2$. The reason for including two sets of tuning parameters is to emphasize that the comparative outcomes is largely dependent on the chosen tuning gains. It should be noted that this is equally true that the comparative tracking schemes of [17–19] could be tuned differently to obtain a different response. The authors have attempted to detune the comparison methods so as not to artificially generate large control inputs. Despite this limitation in direct comparison, the comparative simulation plots are useful for understanding the differences in behaviors of the considered techniques.

The resulting output plots from the comparison study is shown in Figure 7. The first plot shows the size of deviation that the control action has from the nominal input (57). An exploded view of the first 0.5 s is included for each plot to highlight the initial transient performance. All control schemes produced in the initial control action of similar magnitude, with the largest initial control response being from the proposed method with tuning gains 1, and the smallest control input being from the proposed method with tuning 2. This difference in performance highlights the role of tuning gains in system performance, emphasizing that only qualitative performance results should be taken from the figures. A comparatively small control input to the proposed scheme with tuning gains 2 was generated by the scheme of [18] which explicitly considers actuator limits within the control design. The scheme of [19] is in green and it can be seen that it maintains a constant deviation from the nominal trajectory

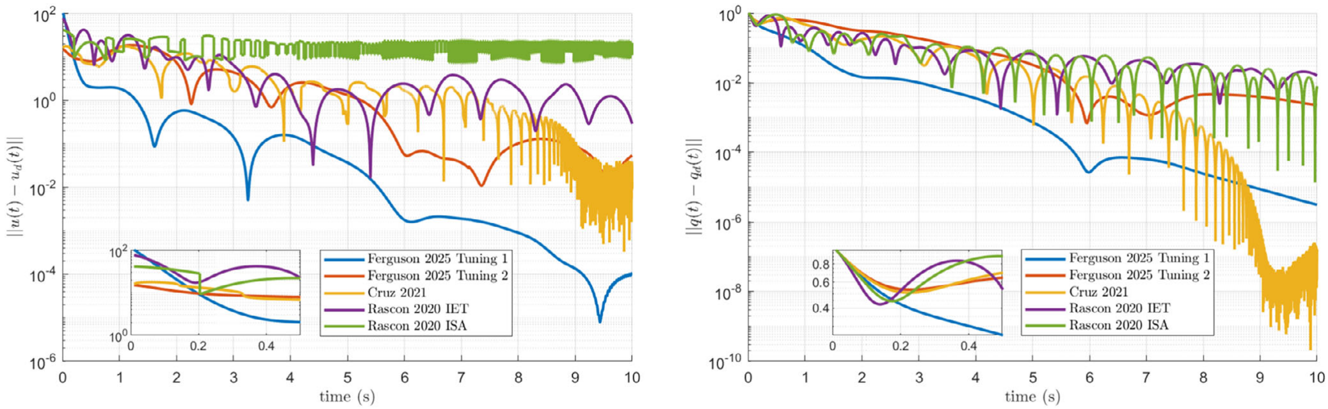

FIGURE 7 | Simulation results comparing the tracking controller proposed in this paper (blue, red), the finite-time controller of [18], the tracking controller using a filtered quantity in place of a velocity observer [17] and the robust sliding mode tracking controller of [19]. The first figure shows the deviation of each control signal from the nominal control trajectory (57). The second plot shows the norm of the tracking error.

for the full simulation time. As the scheme [19] uses a sliding method, it generates chattering of the control input which is the cause of this constant deviation from the nominal control trajectory.

The second plot of Figure 7 shows the norm of the tracking error in response to each control scheme. The proposed scheme with tuning 1 provides the superior tracking performance, whereas all other methods remain compatible for the majority of the simulation. It is interesting to note that the scheme [18] is a finite-time method, and it converges to the exact trajectory around 9 s into the simulation. The residual error after this point is due to limitations in numerical precision. The proposed method guarantees exponential stability for all choices of controller gains, and it can be verified from the resulting tracking error that the system is indeed converging at an exponential rate. The scheme of [17] in purple is guaranteed to have exponential convergence, but the control gains must be suitably chosen for the result to apply. It can be seen that the proposed scheme outperforms this method with both choices of controller gains.

6 | Conclusion

In this note, we have proposed a method for trajectory tracking of the fully actuated mechanical system using only position measurements. The approach combined a hybrid momentum observer with the property that the estimation error is a passive output of the estimation error dynamics. To exploit this property, a tracking error system was devised with the property that the momentum estimation error is a passive input to the error dynamics. Utilizing these properties, a power-preserving interconnection was formed between the observer and controller systems, resulting in a storage function for the joint systems. It was shown that the resulting interconnected system is ISS with respect to external disturbances.

Motivated by similar works on the topic, several interesting questions remain for future research. First, many tracking control works consider the plant model to be unknown and utilize an adaptive scheme to estimate plant parameters online while maintaining tracking. It is interesting to investigate if the proposed methods can be modified to include an adaptation law to estimate plant parameters online while also guaranteeing the ISS property. Second, the scheme of [18] ensures finite time convergence whilst satisfying actuator constraints. The authors believe that it may be possible to modify the shape of the storage functions used by the observer and tracking systems to achieve similar properties whilst maintaining the ISS property. Further investigation of these extensions will occur in future research.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are openly available in GitHub at https://github.com/JoelFerguson/ISS_Tracking_for_Mechanical_Systems.

Endnotes

[1] This is the same control law from [28, Proposition 2], applied to the tracking error system.

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