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# Dirac–Yang Monopoles in all Dimensions and Their Regular Counterparts*

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**Abstract**—The Dirac–Yang monopoles are singular Yang–Mills field configurations in all Euclidean dimensions. The regular counterpart of the Dirac monopole in $D = 3$ is the 't Hooft–Polyakov monopole, the former being simply a gauge transform of the asymptotic fields of the latter. Here, regular counterparts of Dirac–Yang monopoles in all dimensions are described. In the first part of this work, the hierarchy of Dirac–Yang monopoles will be defined; in the second part, the motivation to study these in a topical context will be briefly presented; and in the last part, two classes of regular counterparts to the Dirac–Yang hierarchy will be presented.

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## 1. THE DIRAC–YANG HIERARCHY IN $D \geq 3$

The Dirac [1] monopole can be constructed by gauge-transforming the asymptotic 't Hooft–Polyakov monopole [2] in $D = 3$, which can be taken to be spherically symmetric,[1)] such that the $SO(3)$ isovector Higgs field is gauged to a (trivial) constant, and the $SU(2) \sim SO(3)$ gauge group of the Yang–Mills (YM) connection breaks down to $U(1) \sim SO(2)$, the resulting Abelian connection developing a line singularity on the positive or negative $(x_3 = z)$ axis.

In exactly the same way, the Yang [3] monopole can be constructed by gauge transforming the asymptotic $(D = 5)$ dimensional "monopole" [4–6][2)] such that the $SO(5)$ isovector Higgs field is gauged to a (trivial) constant, and the $SO(5)$ gauge group of the YM connection breaks down to $SO(4)$, the resulting non-Abelian connection developing a line singularity on the positive or negative $x_5$ axis. In fact, the residual non-Abelian connection can take its values in one or other chiral representations of $SU(2)$, as formulated by Yang [3], but this is a low-dimensional accident which does not apply to the higher dimensional analog to be defined below, all of which are $SO(D-1)$ connections. Just like the 't Hooft–Polyakov monopole, and the monopole is the regular counterpart of the Dirac monopole, so is the $(D = 5)$ dimensional "monopole" [4–6] the regular counterpart of the Yang monopole.

The above two definitions of the Dirac and of the Yang monopoles will be the template for our definition of what we will refer to as the hierarchy of Dirac–Yang (DY) monopoles in all dimensions. The two examples just given are both in odd ($D = 3$ and $D = 5$) dimensions, but the DY hierarchy is in fact defined in all, including even, dimensions.

Just as the Dirac monopole can be defined as a gauge transform of the asymptotic spherically symmetric 't Hooft–Polyakov monopole, our definition for the DY fields in arbitrary $D$ dimensions starts from the (non-Abelian) $SO(D)$ YM field $A_i$ and the $D$-tuplet Higgs field $\Phi$:

$$A_i^{(\pm)} = \frac{1}{r}\Sigma_{ij}^{(\pm)}\hat{x}_j, \tag{1}$$

$$\Phi = \hat{x}_i\Sigma_{i,D+1}^{(\pm)} \text{ for odd } D; \tag{2}$$

$$A_i^{(\pm)} = \frac{1}{r}\Gamma_{ij}\hat{x}_j,$$

$$\Phi = \hat{x}_i\Gamma_{i,D+1} \text{ for even } D;$$

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1) It is not in fact necessary to restrict ourselves to spherically symmetric fields only. By choosing to start with the asymptotic axially symmetric fields characterized with vorticity $n$, the gauge-transformed connection is just $n$ times the usual Dirac monopole field.

2) Such models were first introduced in [4] and most recently constructed numerically in [5] and in [6].

$\hat{x}_i = x_i/r$, $i = 1, 2, \ldots, D$, is the unit radius vector; $\Gamma_i$ are the Dirac gamma matrices in $D$ dimensions with the chiral matrix $\Gamma_{D+1}$ for even $D$, so that

$$\Gamma_{ij} = -\frac{1}{4}\left[\Gamma_i, \Gamma_j\right]$$

are the Dirac representations of $SO(D)$. The matrices $\Sigma_{ij}$, employed only in the odd-$D$ case, are

$$\Sigma_{ij}^{(\pm)} = -\frac{1}{4}\left(\frac{1\!\!1 \pm \Gamma_{D+2}}{2}\right)\left[\Gamma_i, \Gamma_j\right],$$

$\Gamma_{D+1}$ being the chiral matrix in $D+1$ dimensions and $\Sigma_{ij}^{(\pm)}$ being one or the other of the two possible chiral representations of the $SO(D)$ subgroup of $SO(D+1)$.

That (1) and (2) are the asymptotic fields of regular monopoles in $D$ dimensions is the subject of Section 3, while in the next section we will argue why such regular finite-energy monopoles are relevant. Here we define the DY field configurations as gauge transforms of (1) and (2).

The DY monopoles result from the action of the following $SO(D)$ gauge group element:

$$g_\pm = \frac{(1 \pm \cos\theta_1)1\!\!1 \pm \Gamma_D \Gamma_\alpha \hat{x}_\alpha \sin\theta_1}{\sqrt{2(1 \pm \cos\theta_1)}}, \tag{3}$$

having parametrized the $\mathbb{R}^D$ coordinate $x_i = (x_\alpha, x_D)$ in terms of the radial variable $r$ and the polar angles

$$(\theta_1, \theta_2, \ldots, \theta_{D-2}, \varphi), \tag{4}$$

with the index $\alpha$ running over $\alpha = 1, 2, \ldots, D-1$. The meaning of the "$\pm$" sign in (3) is as follows: In choosing these signs, the Dirac line singularity will be along the negative or positive $x_D$ axis, respectively. (In the case of odd $D$, if we choose the opposite sign on $\Sigma$ in (1), the situation will be reversed.) In other words, the DY field will be the $SO(D-1)$ connection on the upper or lower half $D-1$ sphere, $S^{D-1}$, respectively, the transition gauge transformation being given by $g_+ g_-^{-1}$. Notice that the dimensionality of the matrices $g$, given by (3), and those of both (1) and (2) match in each case.

In $D > 3$ dimensions, the gauge group element (3) was first employed in [7, 8] in $D = 4$ and was subsequestly extended to all $D$ in [9, 10].

The result of the action of (3) on (1) or (2),

$$A_i \to g A_i g^{-1} + g \partial_i g^{-1},$$
$$\Phi \to g \Phi g^{-1},$$

yields the required DY fields $\hat{A}_i^{(\pm)} = (\hat{A}_\alpha^{(\pm)}, \hat{A}_D^{(\pm)})$:

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1 \pm \cos\theta_1)}\Sigma_{\alpha\beta}\hat{x}_\beta, \tag{5}$$
$$\hat{A}_D^{(\pm)} = 0 \ \text{ for odd } \ D;$$

$$\hat{A}_\alpha^{(\pm)} = \frac{1}{r(1 \pm \cos\theta_1)}\Gamma_{\alpha\beta}\hat{x}_\beta, \tag{6}$$
$$\hat{A}_D^{(\pm)} = 0 \ \text{ for even } \ D;$$

and the Higgs field is gauged to a constant; i.e., it is trivialized.

The components of the DY curvature $\hat{F}_{ij}^{(\pm)} = (\hat{F}_{\alpha\beta}^{(\pm)}, \hat{F}_{\alpha D}^{(\pm)})$ follow from (5) and (6) straightforwardly. To save space, we give only the curvature corresponding to (5):

$$\hat{F}_{\alpha\beta}^{(\pm)} = -\frac{1}{r^2}\left[\Gamma_{\alpha\beta} + \frac{1}{(1 \pm \cos\theta_1)}\hat{x}_{[\alpha}\Gamma_{\beta]\gamma}\hat{x}_\gamma\right], \tag{7}$$

$$\hat{F}_{\alpha D}^{(\pm)} = \pm\frac{1}{r^2}\Gamma_{\alpha\gamma}\hat{x}_\gamma, \tag{8}$$

where the notation $[\alpha\beta]$ implies the antisymmetrization of the indices, and the components of the curvature for even $D$ corresponding to (6) follow by replacing $\Gamma$ in (7) and (8) with $\Sigma^{(\pm)}$. The parametrization (5), (6) and (7), (8) for the DY field appeared in [9, 10].

That the DY field (5) and (6) in $D$ dimensions, constructed by gauge transforming the asymptotic fields (1) and (2) of a $SO(D)$ EYM system, is a $SO(D-1)$ YM field is obvious. For $D = 3$ and $D = 5$, these are the Dirac [1] and Yang [3] monopoles, respectively.

In retrospect, we point out that, to construct DY monopoles, it is not even necessary to start from a YMHiggs system, but ignoring the Higgs field and simply applying the gauge transformation (3) to the YM members of (1) and (2) results in the DY monopoles (5) and (6). In other words, the only function of the Higgs fields in (1) and (2) is the definition of the gauge group element (3) designed to trivialize it.

We will henceforth restrict our detailed considerations concerning the regular counterparts of the DY monopoles to the first two lowest dimensions, namely, $D = 3$ and $D = 4$. This excludes even the Yang monopole itself, but it is more instructive since we then deal with both an odd and even $D$. Before that, however, we will motivate briefly the role of the regular monopoles in the next section.

## 2. MOTIVATION

Field theory solitons in higher dimensions find application [11] as the $D$ branes of string theory and, also, for open heterotic strings [12] in the absence of gravity. As solitons of string theory, $D$ branes must be finite-energy/mass solutions of the appropriate gravitating field theories.

When non-Abelian matter gravitates, there occur both regular and black-hole solutions with finite mass/energy, in contrast with Abelian matter, where only black-hole solutions exist. In $3+1$ spacetime dimensions, the gravitating YM field, both in the absence [13] and in the presence [14, 15] of the isovector Higgs, field, has been intensively studied. The Dirac monopole field features in these solutions a limiting field configuration in the form of Reissner–Nordström (RN) solutions of the Einstein–Maxwell system.

In $D+1$ spacetime dimensions, with $D \geq 4$, the gravitating YM field again has both regular [16] and black-hole [17] solutions with finite mass/energy. The situation is the same also in the presence of a negative [18] and a positive [19] cosmological constant. Again, higher dimensional RN solutions appear as limiting solutions [20], but the latter feature non-Abelian gauge fields now, unlike in the $D=3$ case, where the gauge sector of the RN field is the usual, Abelian, Maxwell field. These are the DY monopoles introduced above.

The fields DY (5), (6) and (7), (8) appeared recently in [21], where it was shown that they satisfy the gravitating YM equation (for the usual $p=1$ YM system), and it is satisfied by them in all dimensions $D$, with or without a cosmological constant. This is not surprising, since *in the presence of gravity* the second-order field equations to YM systems consisting of the superposition of all possible members of $p$ hierarchy [defined below by (11)] are satisfied by DY fields.

In [16, 17], we have constructed finite mass solutions to the $(p=1)+(p=2)$ YM model in $D=4,5,6,7$ or spacetimes $d=5,6,7,8$ for the spherically symmetric $SO(D)$ YM connection

$$A_i = \frac{1-w(r)}{r}\Sigma_{ij}\hat{x}_j. \tag{9}$$

Setting the function $w(r)=0$ by hand reduces (9) to the singular Wu–Yang (WY) part of the field (1) and (2), which we know are gauge equivalent to the DY fields (5) and (6), and hence equations satified by the WY fields are also satisfied by the DY fields. This result carries through to the full superposed YM hierarchy in any given dimension, subject to satisfying finite-energy scaling requirements. Of course, when gravitating YM solutions are constructed, $w(r)=0$ is not set by hand. These are the DY fields which arise as the RN configurations as limiting solutions [20].

It remains to be seen what the interesting properties of the gravitating WY [with $w=0$ in (9)] fields are. Clearly, these have to be black-hole solutions since the WY fields are singular at the origin. In [18], we have given the mass function $m(r)$ [first member of Eq. (24) therein] for field (9) in arbitrary dimensions for the gravitating YM system consisting of the full superposition of $p$ YM terms. In the WY limit, i.e., with $w=0$, this is

$$m' = \sum_{p=1}^{P} \frac{\tau_p}{2(2p-1)!} \times \frac{(d-3)!}{(d-2[p+1])!} r^{-(4p-d+2)}, \tag{10}$$

where $d = D+1$ is the dimension of the spacetime. Obviously, the mass, namely, the integral of (10), will diverge for certain combinations of $p$ and $d$. Most importantly, for $d \geq 5$ (i.e., for "higher dimensions"), the usual $p=1$ YM term will result in infinite mass, and for the mass to be finite, the least nonlinear YM term must be the $p=2$ one. Thus, restricting ourselves to the usual YM term as in [21] leads to infinite mass!

In [21], it is commented that the advantage of employing singular DY (or alternatively WY as seen above) solutions is that they are evaluated in closed form, unlike the regular gravitating matter solutions as, e.g., [13] in $D=3$ and [16] in $D \geq 4$. To retain this feature—of closed-form black-hole gravitating non-Abelian matter solutions—and to have finite mass, the appropriate $p$ YM rather than (usual) $(p=1)$ YM terms must be employed.

Strictly speaking, for the purposes of picking out the correct $p$ YM terms in (10), there is no need to start from the full theory that supports regular finite-energy topologically stable counterparts of the DY monopoles. One could simply consider the (singular) black-hole solution featuring the DY fields (5) and (6) or even more directly the corresponding WY fields [9, 10, 22].

## 3. THE REGULAR COUNTERPARTS OF DY FIELDS

Regular solutions to gravitating non-Abelian (YM) matter fall into two main classes. The first of these is simply the solutions to the models described by the Lagrangians consisting of the superposition of (possibly) all members of the YM hierarchy [3)] [25]:

$$\mathcal{L}_P = \sqrt{-\det g} \sum_{p=1}^{P} \frac{\tau_p}{2(2p)!} \operatorname{Tr} F(2p)^2, \tag{11}$$

[3)] The YM hierarchy of $SO(4p)$ gauge fields in the chiral (Dirac matrix) representations consisting only of the $p$ YM term in (11) was first introduced in [23] to construct self-dual instantons in $4p$ dimensions. These are spherically symmetric solutions generalizing the usual four-dimensional BPST instantons [24]. (A more general overview covering all even dimensions is reviewed in [25].) The self-duality equation for the $p=2$ case was solved independently in [26], whose authors subsequently stated in their erratum that this solution was the instanton of the $p=2$ member of the hierarchy introduced earlier in [23].

$F(2p)$ denoting the $p$-fold totally antisymmetrized product

$$F(2p) \equiv F_{\mu_1\mu_2\ldots\mu_{2p}} = F \wedge F \wedge \ldots \wedge F, \quad p \text{ times},$$

of the YM curvature, $F(2) = F_{\mu\nu}$, in this notation. Clearly, the highest value $P$ of $p$ in (11) is finite and depends on the dimensionality $d = D + 1$ of the spacetime. To complete the definition of models (11), the gauge group $G$ must be specified. With our aim in the present paper of constructing static spherically symmetric solutions in $d = D + 1$ spacetime dimensions, the smallest such gauge group is $G = SO(d-1) = SO(D)$. When restricting the sum in (11) to one term $p$, in $4p$ dimensions, the resulting self-duality equations are overdetermined [27] except for $p = 1$, and only when subjected to spherical or axial symmetry are these not overdetermined for $p \geq 2$. The instantons of the generic system (11), while stable, are not self-dual and cannot be evaluated in closed form and are constructed numerically [28].

(There have been other attempts at constructing higher dimensional YM systems, mostly by generalizing the self-duality equations. We do not refer to these here since the self-duality equations they employ are merely first-order equations solving the second-order field equations and do not lead to finite action/energy, which is our overarching criterion here. An exception is another hierarchy of YM models, which coincide with the hierarchy of [23] in $4p$ dimensions, but are also defined in $4p + 2$ dimensions [29, 30], supporting finite-action (instanton) solutions. While it is straightforward to construct spherically symmetric solutions with gauge group $SO(4p + 2)$ in the chiral Dirac representations, these self-duality equations are even more overdetermined than those of the $4p$-dimensional hierarchy. The action densities of the systems [29, 30] are not positive definite, so that, while the self-duality equations do solve the second-order field equations, they do not saturate a Bogomol'nyi bound and hence are not necessarily stable.)

To (11) is added some gravitational Lagrangian, e.g., Einstein–Hilbert or Gauss–Bonnet, or a superposition of these, or possibly even a dilaton term. Many such studies [16–18, 20] were carried out and the regular solutions were constructed. In [20], in particular, it was pointed out that the $SO(2)$ RN fixed point occurring in $d = 3 + 1$ has its $SO(D - 1)$ analog for all $D$. These are indeed the DY monopole fields discussed in Section 1, although in [20] we did not use that nomenclature, referring to these simply as RN fields. Before proceeding to the second class of models, we end our discussion of the present class by pointing out that the finite mass/energy solutions they support do not *always* survive the decoupling of gravity, e.g., in the $d = 4$ case [13].

The second class of models consists of YM fields, viz. (11), interacting with scalar matter. By far, the most prominent of these are the gauged Higgs (YMH) models [4)] whose solitons are stabilized by monopole charges. In $D = 3$, these are the celebrated 't Hooft–Polyakov monopoles, and in $D$ dimensions, those defined in [6], which will be illustrated below, in the nongravitating case. All these models feature a $D$-component isovector Higgs field which is instrumental (but not essential) in our definition of DY fields in Section 1. The main difference of the solutions of gauged Higgs systems from those of (11) without Higgs fields is that they *always* survive the decoupling of gravity.

While the Dirac monopole [1] and the Yang monopole [3] are defined in $D = 3$ and $D = 5$, here we will choose the dimensions $D = 3$ and $D = 4$ for our illustrations, with the purpose of displaying both an odd-$D$ and an even-$D$ example. Even in this restictive catchment, there are two ways of constructing YMH models. The first of these is via the dimensional reduction of $p$ YM systems on a product space $\mathbb{R}^D \times S^{4p-D}$, while the second one is more ad hoc and it relies on the fact that the topology of a YMH system is encoded in the Higgs field exclusively [10, 32]. The relation between these two procedures was explored in some detail in [33], so we give just a summary here. In both procedures, the all-important quantities are the *topological charges*, for whose definitions we refer to [33], which enable the statement of Bogomol'nyi inequalities leading to the $D$-dimensional models. In the first case, these are the magnetic monopole charges descending from the $(2p)$th Chern–Pontryagin (CP) charge defined on $\mathbb{R}^D \times S^{4p-D}$, while in the second case, the topological charges are the winding numbers of the Higgs field, suitably reexpressed so that the winding numbers are the integrals of *gauge-invariant* densities.

We will first consider the descended CP-topological-charge case and then the covariantized winding-number case for $D = 3$ and $D = 4$. In each case, we will define the charge density, followed by the resulting models whose solutions support regular monopoles.

In any given dimension $D$, the descended CP density can be constructed from any $p$ YM system on any $\mathbb{R}^D \times S^{4p-D}$. Naturally, the examples we give here are the simplest possibilities, pertaining to the smallest possible choice for this $p$. Descending from the second CP density on $\mathbb{R}^D \times S^{4-D}$ and the fourth

[4)] There are other gravitating YM-scalar matter models, e.g., the gauged Grassmannian model in $d = 5$ [31].

CP density on $\mathbb{R}^D \times S^{8-D}$ for $D=3$ and $D=4$, respectively, the two reduced CP (or magnetic charge) densities [7, 8] are

$$\varrho_{\rm CP}^{(3)} = \frac{1}{16\pi}\varepsilon_{ijk}{\rm Tr}F_{ij}D_k\Phi \tag{12}$$

and

$$\varrho_{\rm CP}^{(3)} = \frac{1}{16\pi}\varepsilon_{ijk}\partial_k{\rm Tr}F_{ij}\Phi, \quad i=1,2,3, \tag{13}$$

$$\varrho_{\rm CP}^{(4)} = \frac{1}{64\pi^2}\varepsilon_{ijkl} \tag{14}$$

$$\times\, {\rm Tr}\Gamma_5\Big[S^2F_{ijkl} + 4\{S, D_i\Phi\}\{F_{[jk}, D_{l]}\Phi\}$$
$$+\, 3\left(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi]\right)$$
$$\times \left(\{S, F_{kl}\} + [D_k\Phi, D_l\Phi]\right)\Big]$$

and

$$\varrho_{\rm CP}^{(4)} = \frac{1}{64\pi^2}\varepsilon_{ijkl} \tag{15}$$

$$\times\, \partial_i{\rm Tr}\Gamma_5\left[\eta^4 A_j\left(F_{kl} - \frac{2}{3}A_kA_l\right)\right.$$
$$+\, \frac{1}{6}\eta^2\Phi\{F_{[jk}, D_{l]}\Phi\}$$
$$\left. +\, \frac{1}{6}\Phi\left(\{S, F_{kl}\} + [D_k\Phi, D_l\Phi]\right)D_j\Phi\right],$$
$$i = 1,2,3,4,$$

where $F_{ijkl} = \{F_{i[j}, F_{kl]}\}$ is the curvature 4-form, and we have used the notation $S = \left(\eta^2 - \Phi^2\right)$.

In passing, (13) and (15) demonstrate the fact that the topological current of a reduced CP charge density is *gauge invariant* for odd $D$ and is *gauge variant* for even $D$.

In this case, the resulting action/energy density that supports regular finite-action/energy topologically stable solutions follows *uniquely* from the same dimensional descent that yielded the charge densities (12)–(14), now applied to the action density of the $p$ YM system on $\mathbb{R}^D \times S^{2p-D}$ (with $p=1$ for $D=3$ and $p=2$ for $D=4$). The descended Bogomol'nyi inequalities can be saturated only in the $p=1$ case, so that the solutions in question are only those to the second-order field equations for $p \geq 2$.

The energy/action densities bounded from below by (12)–(14), with this bound actually saturated in the $D=3$ case, are

$$\mathcal{S}^{(3)} = \frac{1}{4}{\rm Tr}\left(F_{ij}^2 + 2D_i\Phi^2\right) \tag{16}$$

and

$$\mathcal{S}^{(3)} = \frac{1}{48}{\rm Tr}\Big(F_{ijkl}^2 + 4\{F_{[jk,D_l]}\Phi\}^2 \tag{17}$$
$$+\, 18\left(\{S, F_{ij}\} + [D_i\Phi, D_j\Phi]\right)^2$$
$$+\, 5\{S, F_{kl}\}^2 + 54S^4\Big).$$

The DY gauge here has a particularly enlightening application. In this gauge, all Higgs-dependent terms in (16) and (17) vanish and all we are left with are the 2-form and 4-form YM terms. What is more is that this shows that the asymptotic behavior of any of these monopoles is such that the curvature 2-form decays as $r^{-2}$, unlike instantons.

It may be interesting here to remark that, in $D=4$, where we performed the descent over $\mathbb{R}^4 \times S^4$ yielding (14) and (17), we could have opted instead to descend over the six-dimensional space $\mathbb{R}^4 \times S^2$. In that case, the appropriate 6-dimensional YM system [5)] would have been

$${\rm Tr}\left(\frac{1}{4}F_{\mu\nu}^2 + \frac{\kappa}{48}F_{\mu\nu\rho\sigma}^2\right),$$

if the residual action is to be bounded from below by a topological charge, in this case the third CP charge. But then the residual model would have featured an $F_{ij}^2$ term whose volume integral diverges by virtue of the asymptotics explained in the previous paragraph.

Next, we give the suitably gauge covariantized [33] winding-number densities in terms of the usual winding-number density

$$\varrho_0^{(D)} = \varepsilon_{i_1i_2\ldots i_D}\varepsilon^{a_1a_2\ldots a_D} \tag{18}$$
$$\times\, \partial_{i_1}\phi^{a_1}\partial_{i_2}\phi^{a_2}\ldots\partial_{i_D}\phi^{a_D},$$

which is not gauge invariant, and the gauge-invariant density

$$\varrho_G^{(D)} = \varepsilon_{i_1i_2\ldots i_D}\varepsilon^{a_1a_2\ldots a_D} \tag{19}$$
$$\times\, D_{i_1}\phi^{a_1}D_{i_2}\phi^{a_2}\ldots D_{i_D}\phi^{a_D},$$

[5)] Departing from our brief for a moment and considering a monopole in $D=5$, on the other hand, it is indeed possible to descend from a purely $p=2$ YM term on $\mathbb{R}^5 \times S^1$, so the residual system in this case would feature only an $F_{ijkl}^2$ term with a valid topological lower bound [4, 5].

which is not a total divergence. For the purpose at hand, it is more convenient to use a component notation for the $SO(D)$ YM connection and the $D$-plet Higgs field

$$A_i = -\frac{1}{2}A_i^{aa'}\Sigma_{aa'}, \quad \Phi = -\frac{1}{2}\phi^a\Sigma_{aD+1}$$

for odd $D$, with $\Sigma$ replaced by $\Gamma$ for even $D$. These charge densities are

$$\varrho^{(3)}_{\text{wind}} = \varrho^{(3)}_0 + \frac{1}{4\pi}\frac{3}{2}\varepsilon_{ijk}\varepsilon^{baa'} \times \partial_i\left(A_j^{aa'}\phi^b\partial_k|\phi^c|^2\right)x \quad (20)$$

and

$$\varrho^{(3)}_{\text{wind}} = \varrho^{(3)}_G + \frac{1}{4\pi}\frac{3}{2}\varepsilon_{ijk}\varepsilon^{baa'}F_{ij}^{aa'}\phi^b\partial_k|\phi^c|^2 \quad (21)$$

for $D = 3$, and for $D = 4$

$$\varrho^{(4)}_{\text{wind}} = \varrho^{(4)}_0 - \partial_i\left(|\vec{\phi}|^2\partial_j\Omega_{ij}\right) - \frac{3}{8}\varepsilon_{ijkl}\varepsilon^{bb'cc'} \times \partial_i\left\{(\eta^4 - |\vec{\phi}|^4)A_j^{cc'}\left[\partial_k A_l^{bb'} + \frac{2}{3}\left(A_\rho A_l\right)^{bb'}\right]\right\} \quad (22)$$

and

$$\varrho^{(4)}_{\text{wind}} = \varrho^{(4)}_G + \frac{3}{2}\varepsilon_{ijkl}\varepsilon^{bb'cc'} \times \left\{\left(\partial_i|\vec{\phi}|^2\right)F_{kl}^{cc'}\phi^b D_j\phi^{b'} + \frac{1}{16}\left(\eta^4 - |\vec{\phi}|^4\right)F_{ij}^{bb'}F_{kl}^{cc'}\right\}, \quad (23)$$

where $\Omega_{ij}$ denotes the *gauge-covariant* tensor quantity:

$$\Omega_{ij} = \frac{3}{2}\varepsilon_{ijkl}\varepsilon^{bb'cc'}A_l^{cc'}\phi^b\left(\partial_k\phi^{b'} + D_k\phi^{b'}\right), \quad (24)$$

which vanishes when subjected to spherical symmetry irrespective of the detailed asymptotic decay of the fields. The surface integrals of the total divergence term in (13) and (15) vanish for suitable finite-energy/action boundary conditions, so that the topological charge here is simply the winding number. The Bogomol'nyi inequalities are constructed from the gauge-covariant charge densities (21) and (23). This is quite a straightforward procedure, but increasingly nonunique with increasing dimension. The only caveat is to exclude those possibilities not consistent with finite-energy/action requirements for a Higgs model. We will not list these here as they are not particularly instructive and are rather cumbersome, the $D = 3$ case being given in [33]. Perhaps, the main distinctive feature of energy/action densities bounded by (12)–(14) versus those bounded by (21)–(23) instead is that the energy/action of the models constructed via dimensional descent always has smaller energy/action than those arrived at directly via winding-number considerations.

## REFERENCES


1. P. A. M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931).
2. G. 't Hooft, Nucl. Phys. B **79**, 276 (1974); A. M. Polyakov, JETP Lett. **20**, 194 (1974).
3. C. N. Yang, J. Math. Phys. **19**, 320 (1978).
4. D. H. Tchrakian, J. Math. Phys. **21**, 166 (1980).
5. H. Kihara, Y. Hosotani, and M. Nitta, Phys. Rev. D **71**, 041701 (2005), hep-th/0408068.
6. E. Radu and D. H. Tchrakian, Phys. Rev. D **71**, 125013 (2005); hep-th/0502025.
7. G. M. O'Brien and D. H. Tchrakian, Mod. Phys. Lett. A **4**, 1389 (1989).
8. K. Arthur, G. M. O'Brien, and D. H. Tchrakian, J. Math. Phys. **38**, 4403 (1997).
9. Zhong-Qi Ma and D. H. Tchrakian, Lett. Math. Phys. **26**, 179 (1992).
10. D. H. Tchrakian and F. Zimmerschied, Phys. Rev. D **62**, 045002 (2000); hep-th/9912056.
11. N. Sakai and D. Tong, J. High Energy Phys. **0503**, 019 (2005); K. Hashimoto and D. Tong, hep-th/0506022.
12. J. Polchinski, hep-th/0510033.
13. R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988).
14. K. Lee, V. P. Nair, and E. J. Weinberg, Phys. Rev. **45**, 2751 (1992).
15. P. Breitenlohner, P. Forgacs, and D. Maison, Nucl. Phys. B **383**, 357, (1992); **442**, 126 (1995).
16. Y. Brihaye, A. Chakrabarti, and D. H. Tchrakian, Class. Quantum Grav. **20**, 2765 (2003), hep-th/0202141.
17. Y. Brihaye, A. Chakrabarti, B. Hartmann, and D. H. Tchrakian, Phys. Lett. B **561**, 161 (2003), hep-th/0212288.
18. E. Radu and D. H. Tchrakian, Phys. Rev. D **73**, 024006 (2006), gr-qc/0508033.
19. Y. Brihaye, E. Radu, and D. H. Tchrakian, gr-qc/0610087.

20. P. Breitenlohner, D. Maison, and D. H. Tchrakian, Class. Quantum Grav. **22**, 5201 (2005), gr-qc/0508027.
21. G. W. Gibbons and P. K. Townsend, Class. Quantum Grav. **23**, 4873 (2006), hep-th/0604024.
22. A. C. T. Wu and T. T. Wu, J. Math. Phys. **15**, 53 (1974).
23. D. H. Tchrakian, Phys. Lett. B **150**, 360 (1985).
24. A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. B **59**, 85 (1975).
25. D. H. Tchrakian, *Yang–Mills Hierarchy*, in *Proceedings of the 21st International Conference on Differential Geometric Methods in Theoretical Physics*, Ed. by C. N. Yang, M. L. Ge, and X. W. Zhou; Int. J. Mod. Phys. A (Proc. Suppl.) **3**, 584 (1993).
26. B. Grossman, T. W. Kephart, and J. D. Stasheff, Commun. Math. Phys. **96**, 431 (1984); **100**, 311 (E) (1985).
27. D. H. Tchrakian and A. Chakrabarti, J. Math. Phys. **32**, 2532 (1991).
28. J. Burzlaff and D. H. Tchrakian, J. Phys. A **26**, L1053 (1993).
29. C. Saclioglu, Nucl. Phys. B **277**, 487 (1986).
30. K. Fujii, Lett. Math. Phys. **12**, 363, 371 (1986).
31. Y. Brihaye, E. Radu, and D. H. Tchrakian, Int. J. Mod. Phys. A **19**, 5085 (2004), hep-th/0405255.
32. J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. **16**, 433 (1975).
33. T. Tchrakian, *Winding Number Versus Chern–Pontryagin Charge*, in a volume in Honor of Sergei Matinyan, Ed. by V. G. Gurzadyan and A. G. Sedrakian (World Sci., Singapore, 2002), hep-th/0204040.