Isotopy and Concordance in Intermediate Ricci Curvatures

By

Jacqueline Birkett

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Submitted to:
Maynooth University
Department of Mathematics and Statistics

Submitted:
9th September 2025

Head of Department:
Professor Stephen Buckley

Supervised by
Dr. Mark Walsh

This research was supported by funding through the Maynooth University Hume Doctoral Awards.

When I consider your heavens,
    the work of your fingers,
the moon and the stars,
    which you have set in place,
what is mankind that you are mindful of them,
    human beings that you care for them?

Lord, our Lord,
    how majestic is your name in all the earth!

Psalm 8: 3-4, 9

Declaration

I hereby affirm that this thesis, submitted for evaluation as part of the Doctor of Philosophy program, is solely my own work. I have taken appropriate care to ensure its originality and, to the best of my knowledge, it does not violate any copyright laws. Any use of others' work has been properly cited and acknowledged within the text. The thesis work was conducted from October 2020 to September 2025 under the supervision of Dr. Mark Walsh, Department of Mathematics and Statistics, Maynooth University, and was funded by the Maynooth University John and Pat Hume Doctoral Scholarship.

Student ID: 11250519

Date: September 2025

Acknowledgements

I can't express my gratitude to my supervisor, Dr. Mark Walsh, enough. He took me on as an (extremely) mature student and he has been nothing but supportive, encouraging and wise. I am in awe of his knowledge and creativity. He has given me so much of his time and energy. He has also been a shoulder to cry on, which is more than he's paid for. How he has had the patience over all these years I don't know. And on the way he has also taught me loads!

I must also express my gratitude to my first supervisor, Dr. Anthony Small, who initially encouraged me to become a Ph.D. student. He tried very hard to instill in me the first principles of algebraic geometry and I thoroughly enjoyed my time with him. He sadly had to retire through ill health and I can only hope that I was not a major contributor to his problems.

I would not have been studying for this degree if it had not been for the very generous support for my research through the John and Pat Hume Doctoral Scholarship. Moreover, Maynooth University has been more than generous in waiving fees to help me through all the disasters that have occurred throughout this time. I must also acknowledge the support and encouragement I have received from the Department of Mathematics and Statistics at Maynooth University. They not only supported my application but have also shown great patience with me over the years.

Acknowledgement and thanks are due to my wonderful family who have encouraged me while making it very clear that they think I'm bonkers. I promise to tidy up the study eventually and also not to explain this research to them at length.

And lastly thanks to God, who has carried me through this time.

Abstract

In this thesis we extend a result of M. Walsh that showed that, under reasonable conditions, positive scalar curvature metrics which are Gromov-Lawson concordant are in fact isotopic. This thesis generalises this result by proving that Gromov–Lawson concordance implies isotopy in the space of Riemannian metrics on simply connected, smooth, closed manifolds with positive Ricci-(k, n) curvature for certain k at least 3 when $n \geq 5$. To do this, we use a strengthening of the Gromov-Lawson surgery technique for extending a positive scalar curvature metric over the trace of a codimension ≥ 3 surgery to a positive scalar curvature metric which is a product near the boundary. We extend this to positive Ricci-(k, n) curvature metrics making use of theorems of Wolfson and Kordass. We also compute the Ricci-(k, n) curvature on a variety of standard metrics on the sphere and disc including so-called mixed torpedo metrics. In addition we give the conditions under which these standard metrics are isotopic in the space of positive Ricci-(k, n) curvature metrics.

Moreover we extend a theorem of Carr to show that the space of positive Ricci-$(k, 4n-1)$ curvature metrics on a $(4n-1)$-dimensional, smooth, closed, spin manifold, $n \geq 2$, has infinitely many path components.

Contents

List of Figures

Page

Chapter 1

Introduction

1.1 Types of curvature

The topology of a manifold places restrictions on the Riemannian metrics admissible on that manifold and, through the Levi-Civita connection, the curvature.

Classically there are three types of curvature: sectional, scalar and Ricci curvatures. In two dimensions, the sectional curvature, K, is simply the Gaussian curvature. Where the dimension, n, of the manifold M, is at least 3, the sectional curvature, K, at a point $x \in M$ is a map, which assigns to each 2-dimensional subspace, V, of the tangent space T_xM, the Gaussian curvature at x of the 2-dimensional submanifold specified locally by V. To say that a manifold has positive sectional curvature is to say that for every $x \in M$ and every 2-dimensional subspace V in T_xM, the corresponding sectional curvature $K(x, V) > 0$. This is a very strong form of curvature. The *Ricci curvature*, *Ric*, is a partial averaging of the sectional curvature and the *scalar curvature*, *scal*, is a complete average of the sectional curvature; see section 3.1. This research focuses on a form of curvature intermediate between Ricci and scalar curvature, defined by Wolfson [53], that of Ricci-(k, n) curvature, $k \in \{1, \cdots, n\}$, and in particular, positive Ricci-(k, n) curvature. Ricci-$(1, n)$ curvature is the same as Ricci curvature and Ricci-(n, n) curvature is the same as scalar curvature.

For 2-dimensional manifolds the three classical types of curvature give the same amount of information. The Gauss-Bonnet theorem [25] shows the limitations imposed on compact, oriented, Riemannian surfaces. It relates the curvature and topology of a 2-dimensional manifold, via its Euler characteristic, $\chi = 2 - 2g$, here g being the genus of the manifold:

$$\int_M K dA = 2\pi\chi(M).$$

Therefore the torus, where $g = 1$, does not admit a metric of positive sectional curvature.

For general dimensions, n, conditions such as positivity of the curvature place limits on the topology of the underlying manifold. This is most severe in the case of positive sectional curvature. For example, Bonnet's Theorem [25] states that a complete, connected Riemannian manifold with curvature bounded below by $\frac{1}{R^2}$, $R > 0$, is compact, has a finite fundamental group and has a diameter less than or equal to πR.

The analogue for Ricci curvature to Bonnet's Theorem for sectional curvature is Myers' theorem [25]. If (M, g) is a complete, connected Riemannian n-dimensional manifold, and the Ricci curvature, $Ric(v)$, is bounded below by $\frac{n-1}{R^2}|v|^2$ for any tangent vector v, then the manifold is compact, has finite fundamental group and has diameter $\leq \pi R$.

However conditions on scalar curvature, being an average, does not limit the topology of a manifold in the same way. A manifold has a metric with *positive scalar curvature*, *psc*, if the scalar curvature at each point of the manifold is positive. Many techniques for generating manifolds with positive scalar curvature are known and some of these are discussed in section 1.3. However many of these techniques, for example Gromov-Lawson surgery, are, in general, not suitable to generate manifolds with positive sectional curvature or even positive Ricci curvature. This thesis extends the use of these techniques from positive scalar curvature to positive Ricci-(k, n) curvature.

In recent years there has been research into notions of curvature other than the classical ones: those intermediate between sectional and Ricci and those intermediate between Ricci and scalar curvature. We describe below some of those which have been the focus of research.

(i) Wu [57], Shen [38] and Wilhelm [52], among others, have researched k^{th}-*intermediate Ricci curvature*, $k \in \{1, \cdots, n-1\}$, a type of curvature intermediate between sectional and Ricci curvature. A manifold has positive k^{th}-intermediate Ricci curvature provided that for any choice $\{v, w_1, w_2, \cdots, w_k\}$ of an orthonormal $(k+1)$-frame the sum of sectional curvatures $\Sigma_{i=1}^{k} K(v, w_i) > 0$ [52].

(ii) Labbi [23] has researched *p-curvature*. In this work we use the term (l, n)-intermediate scalar curvature and denote it as $s_{l,n}(x, P)$. This is a type of curvature intermediate between sectional and scalar curvature. It is defined as follows [7]. Let (M, g) be an n-dimensional Riemannian manifold, with possibly non-empty boundary, and let $l \in \{0, 1, \cdots, n-2\}$. The $(l, n)-$ intermediate scalar curvature of M is the function $s_{l,n} : Gr_l(M) \to \mathbb{R}$ defined for $x \in M$, P an l-plane in T_xM and $\{e_1, \cdots, e_{n-l}\}$, an orthonormal basis of the orthogonal complement $P^\perp$ of P in T_xM, by

$$s_{l,n}(x,P) = \sum_{i,j} K_x(e_i, e_j)$$

where $K_x(e_i, e_j)$ is the sectional curvature at x of the subspace of T_xM spanned by the vectors e_i and e_j. A metric g has positive (l,n)-intermediate scalar curvature if for all l-planes in TM, $s_{l,n}(x,P) > 0$.

(iii) A type of curvature intermediate between Ricci and scalar curvature, *m-intermediate curvature*, $\mathcal{C}_m(e_1, \cdots, e_m)$, was introduced by Brendle, Hirsch and Johne [4] and they defined it as follows. Let $N^n = M^{n-m} \times \mathbb{T}^m$ be a Riemannian manifold. Then N has positive m-intermediate curvature at $x \in M$, if the inequality

$$\mathcal{C}_m(e_1, \cdots, e_m) := \sum_{i=1}^{m} \sum_{j=i+1}^{n} Rm(e_i, e_j, e_i, e_j) > 0$$

holds for every orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM. N has positive m-intermediate curvature, if it has positive m-intermediate curvature for all $x \in M$.

(iv) *Bi-Ricci curvature*, $BRic(v,w)$, may be calculated as follows:

$$BRic(v,w) = Ric(v) + Ric(w) - K(v,w),$$

where v and w are orthonormal vectors. This has been the subject of much research by, for example, Shen and Ye [37] and Assimos, Savas-Halilaj and Smoczyk [1].

(v) Wolfson [53] researched *Ricci-(k,n) curvature*, $Ric_{(k,n)}$, which is defined as the sum of the k smallest eigenvalues of the Ricci tensor. We say that a metric has *positive Ricci-(k,n) curvature* if the Ricci-(k,n) curvature of the metric is positive for each $x \in M$.

We wish to give a sense of how the sectional curvatures feature in the curvatures discussed above. To this end we schematically depict these curvatures in figure 1.1, for an ordered orthonormal basis, $\{e_1, \cdots, e_7\}$, at an arbitrary point x of a Riemannian manifold. Each 7×7 grid represents the sectional curvature, $K_{i,j}$, of the plane spanned by the vectors e_i and e_j, by a small square. The squares shaded in grey represent the sectional curvatures that feature in the relevant curvature. We will be interested in metrics of positive Ricci-(k,n) curvature. We note that positive Ricci-(n,n) curvature is equivalent to positive scalar curvature and positive

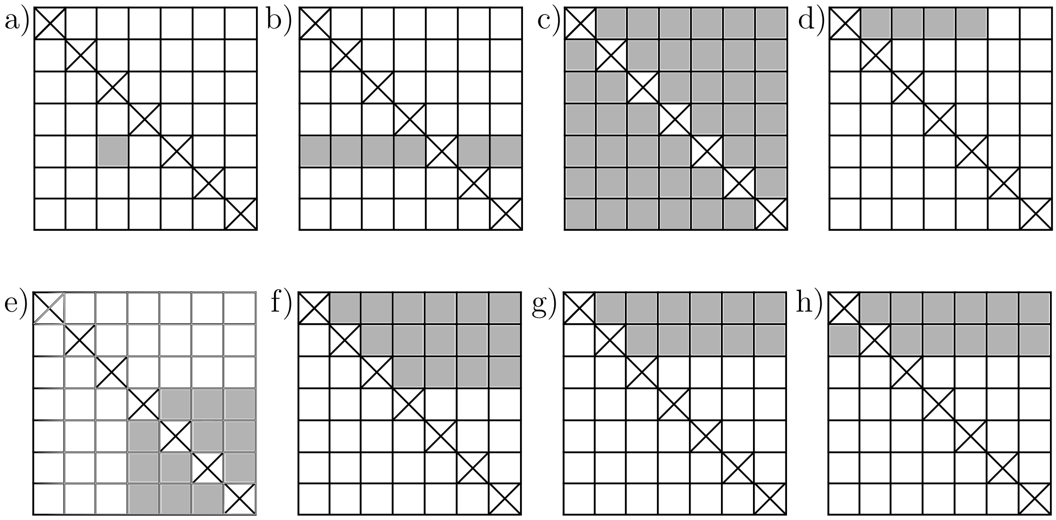

Figure 1.1: a) Sectional curvature, $K_{5,3}$; b) Ricci curvature, $Ric(e_5)$; c) Scalar curvature, $scal$; d) 4^{th}-intermediate Ricci curvature; e) $(3,7)$-intermediate scalar curvature, $s_{3,7}$; f) 3-intermediate curvature, $\mathcal{C}_3(e_1, \cdots, e_7)$ for $M^4 \times T^3$; g) Bi-Ricci curvature; and h) Ricci-(2,7) curvature.

Ricci-$(1,n)$ curvature is equivalent to positive Ricci curvature. Moreover bi-Ricci curvature and Ricci-$(2,n)$ curvature are closely related:

$$Ric_{(2,n)} = BRic(e_1, e_2) + K_{21}.$$

The figure indicates that there are other relationships between these curvatures which could be investigated.

1.2 Topological obstructions

Given a topological manifold, we want to know if it is possible to find a metric on the manifold which satisfies a given curvature condition. As positive scalar curvature is the least restrictive curvature condition, most success has been found in classifying which manifolds admit positive scalar curvature metrics and which do not. This is something we will return to shortly.

1.2.1 Dimension and other obstructions

Many theorems concerning curvature conditions on a manifold have some form of dimensional constraint. As is well known, one-dimensional manifolds have no intrinsic curvature, positive scalar curvature conditions on two dimensional manifolds have been classified, and higher dimensional manifolds may or may not admit metrics with specified curvature conditions.

However, there are no obstructions for an n-dimensional manifold, $n \geq 3$, to admit a negative scalar curvature metric [2] or a complete metric with negative Ricci curvature [27]. The negative sectional curvature condition imposes severe topological restrictions. The Cartan-Hadamard Theorem [25] states that a complete, connected manifold with non-positive sectional curvature is always aspherical, that is it has a contractible universal cover. Positive curvature conditions offer more of a challenge in terms of existence and construction. We thus present some results for positive curvatures of different types.

For positive scalar curvature, some dimension constraints are

(i) When $n = 2$, the manifold admits a complete Riemannian metric of psc if and only if the manifold is diffeomorphic to $\mathbb{R}^2$, S^2 or $\mathbb{R}P^2$ [25].

(ii) Some positive scalar curvature results on product manifolds are given. For a compact, enlargeable manifold M [17] then a complete metric on:

   (a) $M \times \mathbb{R}$ cannot admit a psc metric;

   (b) $M \times \mathbb{R}^2$ cannot admit a uniformly psc metric; and

   (c) $M \times \mathbb{R}^3$ can admit a uniformly psc metric.

Naturally if a manifold does not admit a positive scalar curvature metric, it will not admit a metric with positive sectional or intermediate curvatures. Hence the dimensional constraints listed above in relation to psc also apply to all the other types of curvature.

1.2.2 Obstructions to positive scalar curvature on spin manifolds

A simply-connected manifold, $n \geq 5$, is a spin manifold if every embedded 2-dimensional sphere has trivial normal bundle, Theorem 2.10 in [24]. Every compact, simply-connected n-manifold, $n \geq 5$, which is not spin, admits a metric of psc [16].

However when a manifold has spin it may or may not admit a psc-metric. There are certain index theoretic obstructions. The most well known is the $\hat{A}$-genus, which depends only on the spin cobordism type of the manifold. In order for a closed, spin manifold, M, of dimension $4n$ to admit a psc-metric, it is necessary that $\hat{A}(M) = 0$ [26]. Note that if a manifold does not admit a psc-metric, then it will not admit a metric with positive Ricci-(k, n) curvature.

Let $\{W; X_0, X_1\}$ be an $(n+1)$-dimensional cobordism; see figure 1.2 and section 2 for the definition. Then X_0 and X_1 are said to be *cobordant*, and $\{W : X_0, X_1\}$ a *cobordism*. Two closed manifolds are said to be *spin cobordant* if they are the disjoint boundary of a spin manifold with consistent spin structures. Gromov and Lawson [16] showed that any compact simply-connected spin manifold of dimension

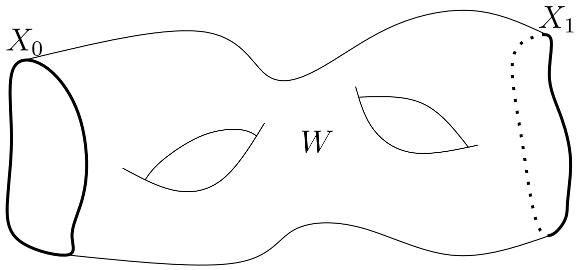

Figure 1.2: Cobordism $\{W; X_0, X_1\}$.

$n \geq 5$ which is spin cobordant to a manifold which admits a metric of psc, also admits a metric of psc.

There is an equivalence relation between the spin cobordant, closed n-dimensional manifolds forming the disjoint boundary, with the equivalence classes forming a group. Milnor defined a homomorphism, α, from the group to the real K-homology of a point [31]. Hitchin showed that if a closed, spin manifold admits a psc metric, then $\alpha([M]) = 0$ [19]. Incidentally, when $n \equiv 0 \mod 4$, then $\alpha([M]) = \hat{A}(M)$. Hence if $\hat{A}(M) \neq 0$, then there is an obstruction to the closed, spin manifold M^{4n} having psc; and consequently there is an obstruction to positive Ricci curvature and to positive sectional curvature.

Hitchin had shown that in order for a closed spin manifold M to have any chance of admitting a psc-metric, the spin cobordism class $[M]$ must lie in the kernel of α. Following the work of Gromov and Lawson in [16], Stolz in [39] completed the following classification.

Theorem 1.2.1 (Gromov-Lawson, Stolz)**.** *Let M be a closed smooth simply connected manifold with dimension $n \geq 5$. Then M admits a psc-metric if and only if M is either not spin, or, M is spin with $\alpha([M]) = 0$.*

Crowley and Wraith [10] proved that all 2-connected manifolds, M^7, admit a metric with positive Ricci curvature. (This is also true for 4-connected manifolds, M^{11}.)

Of particular interest to us, is the case of positive Ricci-(k, n) curvature metrics. Wolfson proved that every compact simply-connected non-spin n-manifold, $n \geq 5$, carries a metric with positive Ricci-$(n-1, n)$ curvature [53]. This is analogous to the Gromov-Lawson result [16] for psc curvature, i.e. positive Ricci-(n, n) curvature. Similarly Wolfson showed that every compact simply-connected spin n-manifold, M, with $\alpha(M) = 0$ carries a metric with positive Ricci-$(n-1, n)$ curvature. This is analogous to part of the result proved by Stolz [39] for psc metrics.

A similar result for positive (l, n)-intermediate scalar curvature has been proved by Labbi [23]. A compact, 2-connected manifold, M, (and therefore spin), where $n \geq 7$, admits a metric with positive (l, n)-intermediate scalar curvature if and only if $\alpha([M]) = 0$. This is very similar to positive Ricci-$(n-1, n)$ curvature; see figure 1.3 which shows a grid representing the sectional curvatures for a 7-dimensional orthonormal basis. The squares coloured grey are those that are included in a) (l, n)-intermediate scalar curvature for $l = 1$, $n = 7$ and b) Ricci-$(n-1, n)$ curvature for $n = 7$.

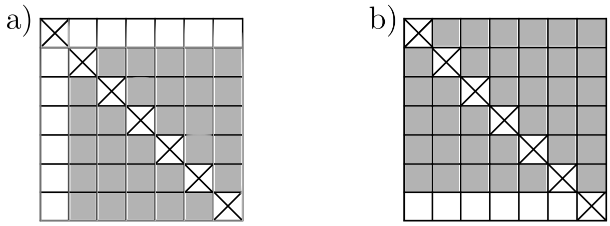

Figure 1.3: a) (1,7)-curvature; and b) Ricci-(6,7) curvature.

1.3 Construction of new metrics

In section 1.2 we have considered some of the topological obstructions to the existence of a metric with a certain curvature condition. In this section, we consider how to construct a manifold with a metric satisfying a particular curvature condition, where no obstructions exist. For strong curvature notions, like the sectional (and to a lesser extent the Ricci), constructive techniques for positive curvature are few and far between. In the case of positive scalar curvature, however, the topological technique of surgery leads to a powerful method for constructing new psc-metrics. This method plays a central role in the classification results above.

1.3.1 Surgery

We describe a p-surgery on an n-dimensional manifold, where $p+q+1 = n$, as follows. A p-surgery removes a disc-sphere product $S^p \times D^{q+1}$ embedded in the manifold, M, and replaces it with a sphere-disc product $D^{p+1} \times S^q$ to give the manifold, M'. The manifolds, M and M', may be topologically very different. For example figure, 1.4 shows that 0-surgery on the sphere, S^2, gives the torus T^2. A 0-surgery can also be used to connect disjoint n- manifolds, a technique called *connected sum.*

1.3.1.1 Gromov-Lawson surgery

Separately Gromov-Lawson [16] and Schoen-Yau [35] showed that compact n-manifolds, with a psc metric may be constructed from existing manifolds with psc metrics using codimension ≥ 3 surgery. We describe Gromov-Lawson surgery briefly. A sphere-disc product, $S^p \times D^{q+1}$, where $p+q+1 = n$, $q+1 \geq 3$, is embedded

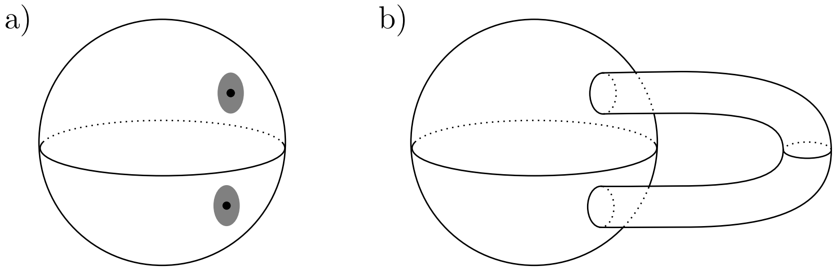

Figure 1.4: 0-surgery on the sphere, S^2. a) Embedded sphere-disc product $S^0 \times D^2$; and b) $D^1 \times S^1$ attached to S^2.

in a compact psc manifold M by a map ϕ. The metric on $\phi(S^p \times D^{q+1})$ is gradually adjusted while retaining the original metric on $M \setminus (\phi(S^p \times D^{q+1}))$. Figure 1.5 shows the result of deforming the original metric on the embedded sphere-disc product to give a smooth, surgery-ready manifold which has a psc metric throughout. The metric near $S^p \times \{0\}$ becomes a standard psc metric using the method described below. Let r denote the radial distance from $S^p \times \{0\}$ in $\phi(S^p \times D^{q+1})$. Around $S^p \times \{0\}$, as r tends to zero, the metric on the geodesic spheres $S^q(r)$ changes through an isotopy of psc metrics to one approaching the standard round metric. The geodesic spheres are "pushed out" to obtain a hypersurface of $(\phi(S^p \times D^{q+1})) \times [0, \infty)$, where t is the coordinate orthogonal to the original embedding $\phi(S^p \times D^{q+1})$. The pushing out of the geodesic spheres is done in such a way as to maintain a psc metric. The curve in the $r - t$ plane made by such a process is called a *Gromov-Lawson curve.* A necessary condition to maintain psc in this process is that $q \geq 2$. As the radius of the q-spheres reduces the contribution to psc increases and thus compensates for the negative contribution of the adjustments required. There is a final isotopy of the metric near $S^p \times \{0\}$ to a standard "surgery-ready" metric.

Figure 1.5 depicts this process schematically. Figure 1.5(a) shows the original metric on $M \setminus (\phi(S^p \times D^{q+1}))$ which is, by assumption, a psc metric. As the geodesic spheres approach $S^p \times \{0\}$ the metric on them becomes more standardised, see figure 1.5(b). Adjustments occurs using a Gromov-Lawson curve which ensures that a psc metric is maintained throughout; see figure 1.5(c) and (d). The metric is adjusted to a standard metric near $S^p \times \{0\}$, see figure 1.5(d), which is a rotationally symmetric product metric of the standard sphere metric on S^p with a torpedo metric on D^{q+1}. The manifold is now surgery-ready and we complete this "geometric surgery" by removing the metrically standard sphere disc product, $S^p \times D^{q+1}$, and attaching a new standard piece, $D^{p+1} \times S^q$.

Hence we can construct a manifold with a metric having psc from an existing psc manifold by a p-surgery provided that the surgery has codimension $q + 1 \geq 3$.

This surgery is described in more detail by Walsh [44] and in [11]. This surgery may be reversed by performing a complementary q-surgery to give a manifold with

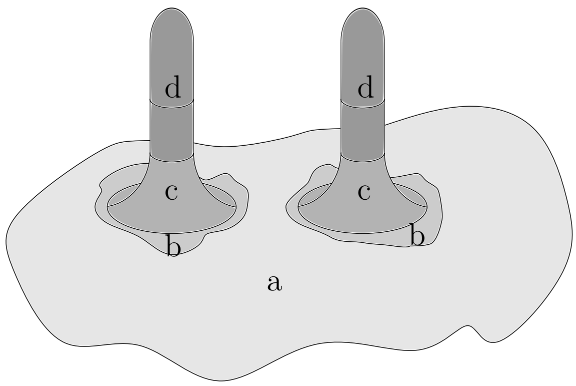

Figure 1.5: Regions of surgery-ready manifold with a) Original psc metric; b) Boundary with radial spheres of S^q; c) Shrinking S^q along part of a Gromov-Lawson curve; and d) Regions with standard torpedo metric.

the same topology as M. Similarly, the cancellation theorem 6.4 of [30] gives conditions whereby a $(p+1)$-surgery on M' gives M'', which is topologically the same as M. Despite this, the M'' will have a different psc-metric to the original manifold M, following the Gromov-Lawson construction.

1.3.1.2 Preserving curvature conditions after surgery

As stated above, Gromov-Lawson [16] and Schoen-Yau [35] showed that compact manifolds with a psc metric may be constructed from manifolds with psc metrics using codimension ≥ 3 surgery. We say that a metric with a certain curvature condition which is preserved after codimension c-surgery is called a *c-surgery stable* condition. Hence, psc is a c-surgery stable condition, $c \geq 3$, on a compact manifold.

Other curvature conditions are surgery stable. Labbi [23] showed that a manifold obtained by surgery in codimension $l+3$, from a manifold admitting a metric of positive (l,n)-intermediate scalar curvature, also admits a metric of positive (l,n)-intermediate scalar curvature.

Of great relevance to us, Wolfson [53] showed that positive Ricci-(k,n) curvature, $2 \leq k \leq n$, is a surgery stable condition on a compact n-dimensional manifold, M, provided that the codimension of the surgery, $q+1 \geq \max\{n+2-k, 3\}$.

Hence psc, positive (l,n)-intermediate scalar curvature and positive Ricci-(k,n) curvature are called *surgery stable curvature conditions*. Hoelzel [20] showed that surgery stable curvature conditions are open, convex $O(n)$-invariant cones in the space of algebraic curvature operators stable under surgeries of codimension $\geq c$ provided that the cone contains the operator corresponding to $S^{c-1} \times \mathbb{R}^{n-c+1}$, $c \geq 3$. Therefore, curvature conditions that are not surgery stable include positive sectional and positive Ricci curvatures.

1.3.1.3 The trace of a surgery

Gajer [15] and Walsh [44] extended the theorem of Gromov and Lawson [16] and Schoen and Yau [35] to the $(n+1)$-dimensional trace, W, of a surgery $\{W; M, M'\}$. The disjoint n-dimensional boundaries of W are M, the manifold to be operated on and M', the result of p-surgery on M. Gajer [15] and Walsh [44] showed the trace of a p-surgery, on a closed manifold M with a metric of psc, where $q \geq 2$, admits a metric which has psc and is a product near the boundary extending the original metric on M.

Unfortunately Gromov-Lawson surgery may not be used on manifolds with positive sectional curvature or positive Ricci curvature to give traces with positive sectional curvature or positive Ricci curvature, respectively, as these curvature conditions are too restrictive. However Burkemper, Searle and Walsh [7] showed that an $(l, n) > 0$ metric on M could be extended over the trace of a p surgery on M to an $(l, n+1) > 0$ metric which restricts to a product of $(l, n) > 0$ metrics near the boundary, provided $l \in 0, 1, ..., q+1-3$ where $p+q+1 = n$.

We will extend these results to the trace of a p-surgery on a compact manifold with another intermediate curvature, Ricci-(k, n) curvature; see Theorem A.

1.3.1.4 Connected sum

The connected sum joins two manifolds together and is p-surgery where $p = 0$. Let M_1^n and M_2^n be two n-dimensional manifolds and $\phi_i : \mathbb{R}^n \to M_i^n$, $i \in \{1, 2\}$ be two embeddings with ϕ_1 and ϕ_2 orientation preserving or reversing, respectively. Let α be an orientation reversing map. Then, the connected sum $M_1 \# M_2 := M_1 \# M_2(\phi_1, \phi_2, \alpha)$ is the manifold obtained by the disjoint union of $M_1 - \phi_1(0)$ and $M_2 - \phi_2(0)$ by identifying $h_1(v)$ with $h_2(\alpha(v))$ [22].

Gromov and Lawson [16] showed that the connected sum of two compact, n-dimensional manifolds with psc, where $n \geq 3$, has a metric with psc.

In general the connected sums between Ricci positive manifolds will not in general support a metric of positive Ricci curvature [55].

However the following connected sums of manifolds with positive Ricci curvature results in a manifold with positive Ricci curvature:

(i) The connected sum $\#_{i=1}^r S^n \times S^m$ of r-copies of $S^n \times S^m$ for $n, m \geq 2$, where S^p is the standard p-dimensional sphere [36].

(ii) This is generalised in [55] to factor spheres with different dimensions in Theorem A of [55]. The manifold $(S^{n_1} \times S^{m_1}) \# (S^{n_2} \times S^{m_2}) \# \cdots \# (S^{n_r} \times S^{m_r})$ admits a metric of positive Ricci curvature for any $n_i, m_i \geq 3$ such that $n_i + m_i = n_j + m_j$ for all $1 \leq i, j \leq r$.

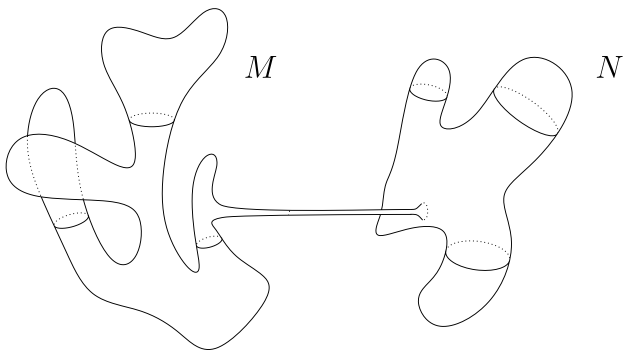

Figure 1.6: Connected sum $M\#N$.

This has been extended in [5] by Burdick. He defines a product of spheres as $\mathbf{S}^\alpha = \prod_{1\geq i\geq|\alpha|} S^{\alpha_i}$ where α is a multi-index $\alpha = (\alpha_1, \cdots, \alpha_{|\alpha|})$. Then for any sequence

$$M_i^n \in \{S^k \times \mathbf{S}^\alpha, \mathbb{C}P^{\frac{k}{2}} \times \mathbf{S}^\alpha, \mathbb{H}P^{\frac{k}{4}} \times \mathbf{S}^\alpha, \mathbb{O}P^{\frac{(k=16)}{8}} \times \mathbf{S}^\alpha : \\ k + |\alpha| = n, k \geq 2, \text{ and } \alpha_i \geq 3\},$$

the connected sum, $\#M_i^n$, admits a metric of positive Ricci curvature.

(iii) Crowley and Wraith proved that where a manifold, M^{4j-1}, $j \geq 2$, is highly connected and, where $j \equiv 1 \bmod 4$ is also $(2j-1)$-parallelisable, then there is a homotopy sphere $\sum^{4j-1}$ such that $M\#\sum$ admits a Ricci positive metric [10]. They proved this by showing that these manifolds may be described as the boundaries of explicit plumbings.

However the intermediate curvature, Ricci-(k, n), is less restrictive. Wolfson [53] has shown that where M_1 and M_2 are compact n-manifolds with metrics of positive Ricci-(k, n) curvature, $2 \geq k \geq n$, then their connected sum $M_1\#M_2$ also carries a metric with positive Ricci-(k, n) curvature.

An example of the difficulty of extending manifolds with positive sectional or Ricci curvature metrics to connected sums is given in the introduction to [43]. Real projective space, $\mathbb{R}P^n$, $n \geq 2$, being locally isometric to the round metric on S^n, has a metric with positive Ricci curvature and positive sectional curvature. By Van Kampen, the connected sum, $\mathbb{R}P^n\#\mathbb{R}P^n$, $n \geq 3$, has fundamental group the free product, $\mathbb{Z}_2 \star \mathbb{Z}_2$, an infinite fundamental group. By Bonnet's Theorem, Theorem 11.7 of [25], positive sectional curvature on a complete, connected manifold forces the fundamental group to be finite. Similarly by Myers's Theorem, Theorem 11.8 of [25], positive Ricci curvature on a complete, connected manifold forces the fundamental group to be finite. Hence, $\mathbb{R}P^n\#\mathbb{R}P^n$ does not admit a metric with positive sectional or positive Ricci curvature. However by the surgery theorem, it does admit a metric with psc.

1.4 The space of Riemannian metrics

We now consider the space of metrics on a smooth, closed manifold, X^n, $n \geq 2$, with a particular curvature condition. The space of Riemannian metrics on X with C^∞ topology, $\mathcal{R}\mathrm{iem}(X)$, is an infinite-dimensional Fréchet space which is convex, see chapter 1 of [42]. However once a curvature condition, C, is specified far less is known about the subspace of Riemannian metrics which have that curvature condition, $\mathcal{R}\mathrm{iem}^C(X) \subset \mathcal{R}\mathrm{iem}(X)$. In particular we are interested in the subspace of Riemannian metrics which have positive Ricci-(k,n) curvature, $\mathcal{R}\mathrm{iem}^{Ric^+_{k,n}}(X) \subset \mathcal{R}\mathrm{iem}(X)$. Assuming that the space is not empty, we would like to know more about the different topological properties such as homotopy type or even just path connectedness of the space of such metrics.

1.4.1 Homotopy

The homotopy class of the space of psc metrics may not change after certain types of surgery. Chernysh in [9] and Walsh in [45] prove that where a manifold N is obtained from a smooth, compact manifold M^n, $n \geq 5$, by p-surgery, $2 \leq p \leq n-3$, then the spaces $Riem^+(M)$ and $Riem^+(N)$ are homotopy equivalent. Indeed where M is a simply connected spin manifold, $n \geq 5$, then the homotopy type of the space of psc-metrics is a spin-cobordism invariant [45].

Kordass in [21] generalised this result to surgery on a sphere, S^p, with trivial normal bundle, embedded in a manifold with a "$q+1$-surgery-stable curvature condition", C [20]. Let N be obtained from the closed n-dimensional manifold M from such a p-surgery, where $p \geq \frac{n-1}{2}$. Then, the spaces of Riemannian metrics with that curvature condition, C, $\mathcal{R}\mathrm{iem}_C(M)$ and $\mathcal{R}\mathrm{iem}_C(N)$, are homotopy equivalent. Kordass gives examples of surgery stable curvature conditions:

(i) Gromov and Lawson in [16] and Schoen and Yau in [35] showed that psc is a surgery stable curvature condition provided $q \geq 2$.

(ii) Wolfson in [53] showed that positive Ricci-(k,n) curvature is a surgery stable curvature condition where $k \geq 2$, $q \geq \max\{n+1-k, 2\}$, i.e. $p \leq \min\{k-2, n-3\}$.

(iii) Labbi in [23] showed that positive (l,n)-intermediate scalar curvature is a surgery stable curvature condition provided $q \geq l+2$.

Kordass's Theorem above largely follows from an initial Theorem he proved in [21]. For the manifolds above with a compact p-dimensional submanifold, N^p, then the inclusion of metrics standard near N^p, $\mathcal{R}\mathrm{iem}^{std}_C(M) \subset \mathcal{R}\mathrm{iem}_C(M)$ is a weak homotopy equivalence. A similar result was proved in [9] in the case where $p \leq n-3$.

Incidentally, a corollary of this is Chernysh's Surgery Theorem. We note that sectional curvature is not a stable curvature condition, nor is Ricci curvature which is equivalent to Ricci-$(1, n)$ curvature.

Naturally, to give more information, the manifold must be specified. Walsh in [46], proved that $\mathcal{R}\text{iem}^+(S^n)$, $n \geq 3$ is homotopy equivalent to an H-space, with a homotopy commutative and homotopy associative product operation; and provided $n \neq 4$ it is weakly homotopy equivalent to an n-fold loop space. Ebert and Randal-Williams [12] prove that many spaces of psc metrics have the homotopy type of infinite loop spaces, in particular the path component of the round metric in $\mathcal{R}\text{iem}^+(S^n)$, $n \geq 6$.

Similarly, Walsh and Wraith in [49] proved that the space of positive Ricci-(k, n) curvature metrics on the n-sphere, $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, $n \geq 3$, $k \geq 2$, has the structure of an H-space with a homotopy commutative, homotopy associative product operation. Moreover the path component of this space containing the round metric is weakly homotopy equivalent to an n-fold loop space.

1.4.2 Path-Connectedness

Two metrics on an n-manifold, X, that lie in the same path component of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ are said to be *isotopic*. Two positive Ricci-(k, n) curvature metrics, g_0 and g_1, are said to be *concordant* if there is an $(n+1)$-dimensional cylinder, $X \times I$, equipped with a positive Ricci-$(k+1, n+1)$ curvature metric, $\overline{g}$ with a product metric at the boundary and $g_0 = \overline{g}|_{X\times\{0\}}$ and $g_1 = \overline{g}|_{X\times\{1\}}$. We show that isotopic metrics are concordant in section 3.3. However whether concordance implies isotopy is a difficult open question. Walsh [44] showed that in the case of positive scalar curvature metrics on closed simply connected manifolds of dimension $n \geq 5$ concordance implies isotopy in the case of certain types of concordances. We extend this result to the case of positive Ricci-(k, n) curvature metrics.

In [41], Tuschmann gave a survey of path-connectedness of the spaces and moduli spaces of Riemannian metrics with certain curvature conditions. Manifolds may admit metrics with a certain type of curvature which are not isotopic in the space of metrics with that curvature. For positive scalar curvature metrics on certain manifolds, classified by dimension, the following is known:

(i) It has long been known that the space, $\mathcal{R}\text{iem}^+(S^2)$, is path connected [51]; indeed Rosenberg and Stolz proved that it is contractible in Theorem 3.4 of [33]. In this theory they also proved that $\mathcal{R}\text{iem}^+(\mathbb{R}P^2)$ is path-connected and contractible.

(ii) Codá Marques in [28] proved that the space, $\mathcal{R}\text{iem}^+(S^3)$, is path-connected.

(iii) For dimensions of $8n$ and $8n+1$, Hitchin in [19] showed that $\mathcal{R}\text{iem}^+(M)$ for a closed spin manifold, M, admitting a psc metric has more than one path component and has nontrivial fundamental group.

(iv) As noted earlier, the $\hat{A}$ genus is an obstruction for $\mathcal{R}\text{iem}^+(M^{4n}) \neq \emptyset$, where M is a closed spin manifold. Carr [8] used the $\hat{A}$ genus of certain manifolds diffeomorphic to S^{4n-1} to show that $\mathcal{R}\text{iem}^+(S^{4n-1})$, $n \geq 2$, has infinitely many path components. Ebert and Wiemeler in [13] showed that where $\mathcal{R}\text{iem}^+(M^n)$ is not empty for M^n a simply-connected, closed spin manifold, where $n \geq 5$ then $\mathcal{R}\text{iem}^+(M^n)$ is homotopic to $\mathcal{R}\text{iem}^+(S^n)$, where S^n is the n-dimensional sphere with standard smooth structure. Hence $\mathcal{R}\text{iem}^+(M^{4n-1})$, $n \geq 2$, has infinitely many path components. Further, Tuschmann and Wraith deduce in Theorem 4.2.2.2 that $\mathcal{R}iem^+(M^{4n-1})$, $n \geq 2$ has infinitely many path components, where M is any closed spin manifold admitting a single psc metric [42].

We note that when a manifold, M, admits a psc metric, then $\mathcal{R}\text{iem}^+_{sec}(M) \subset \mathcal{R}\text{iem}^+_{Ric}(M) \subset \mathcal{R}\text{iem}^+(M)$ and, hence, if $\mathcal{R}\text{iem}^+_{Ric}(M) = \varnothing$, then $\mathcal{R}\text{iem}^+_{sec}(M) = \varnothing$.

There are analogous results for other curvature conditions. Wraith in [54] showed that homotopy spheres bounding parallelizable manifolds admit a metric of positive Ricci curvature. The space of positive Ricci curvature metrics for such homotopy spheres in dimension $4n-1$, $\mathcal{R}\text{iem}^+_{Ric}(S^{4n-1})$, $n \geq 2$, has infinitely many path components [56]. Burdick in [6] showed that this result could be extended from $\mathcal{R}\text{iem}^+_{Ric}(S^{4n-1})$ to $\mathcal{R}\text{iem}^+_{Ric}(M^{4n-1})$, $n \geq 2$, where M is a spin manifold admitting a particular type of positive Ricci curvature metric.

Burkemper, Searle and Walsh in [7] proved an analogous result for positive $(l, 4n-1)$ intermediate scalar curvature metrics, $n \geq 2$ on smooth, closed, spin manifolds to show that the space $\mathcal{R}iem^{s_{l,4n-1}>0}(M)$ has infinitely many path components.

1.5 Main Results

Gajer [15], and Walsh [43] used a construction to build psc metrics on a compact cobordism $\{W; X_0, X_1\}$ which is the trace of a surgery in codimension of at least 3, giving a product metric near the boundary. This construction was used by Burkemper, Searle and Walsh [7] to build positive (l, n)-intermediate scalar curvature metrics on the trace. In chapter 6, we use this construction to build positive Ricci-(k, n) curvature metrics on the trace of the surgery and prove the following Theorem.

Theorem A. *Let X be a smooth n-dimensional closed manifold with $n \geq 3$, $\varphi : S^p \times D^{q+1} \hookrightarrow X$ an embedding where $p + q + 1 = n$ and let $W_\varphi := \{W_\varphi; X, X_\varphi\}$*

denote the trace of a p-surgery on φ. Suppose g is a Riemannian metric on X which has positive Ricci-(k,n) curvature when $2 \leq k \leq n$ and $p \neq 1$, or $3 \leq k \leq n$ when $p = 1$. Then provided $q \geq max\{n+1-k, 2\}$, there is a metric $\overline{g}_{\varphi}$ on W_{φ} so that

a) $\overline{g}_{\varphi}|_X = g$*;*

b) $\overline{g}_{\varphi}$ *is a product near the boundary* ∂W_{φ}*; and*

c) $\overline{g}_{\varphi}$ *has positive Ricci-*$(k+1, n+1)$ *curvature.*

In order to prove this theorem we first prove Lemma 3.3.1 that a smooth path of metrics in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ gives rise to a positive Ricci-$(k+1, n+1)$ metric on the cylinder $X \times I$. Thus isotopic positive Ricci-(k,n) metrics are concordant.

In chapter 6 we use Theorem 6.0.1 proved by Wolfson [53] to give the conditions for the co-dimension of the surgery on a manifold admitting a positive Ricci-(k,n) metric so that the post-surgery manifold also admits a positive Ricci-(k,n) metric. We also use Theorem 6.0.2 proved by Kordass [21] which shows that under certain conditions the inclusion of the space of metrics with positive Ricci-(k,n) curvature which are standard near compact submanifolds in the space of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ is a weak homotopy equivalence.

In chapter 4 we calculate the curvature of certain standard metrics on the disc, D^n, and sphere, S^n, and determine for which k, the metrics have positive Ricci-(k,n) curvature. Hence, given an initial manifold admitting a positive Ricci-(k,n) metric we can determine for which k' the trace of the surgery on this manifold admits a metric with positive Ricci-$(k', n+1)$ curvature and the post-surgery manifold admits a metric with positive Ricci-(k,n) curvature.

Corollary 3.3.1.1 is used to study the path-connectedness of the space of positive Ricci-(k,n) curvature metrics on some manifolds. In chapter 7 we use this Theorem to prove one application extending a theorem of Carr [8] in relation to psc metrics on S^{4n-1}, $n \geq 2$. We use the method of connected sums used by Burkemper, Searle and Walsh [7] for $\mathcal{R}\text{iem}^{s_{l,n}>0}(M)$.

Theorem B. *Let M be a $(4n-1)$-dimensional, smooth, closed, spin manifold, $n \geq 2$, which admits a positive Ricci-(k,4n-1) curvature metric, for $k \geq 2n+1$. Then the space of Riemannian metrics of M, with positive Ricci-(k,4n-1) curvature, $\mathcal{R}iem^{Ric^+_{(k,4n-1)}}(M)$ has infinitely many path components.*

Remark. *This theorem also follows from the results of Frenck and Kordass in [14]. However, the construction we use is quite different and far more geometrically explicit.*

In chapter 8 we consider the space of metrics with positive Ricci-(k,n) curvature on X, $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$, and we show that under reasonable conditions, the metrics on the boundary of a Gromov-Lawson concordance are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$. To do so we use the well-known Cancellation Theorems from Morse-Smale theory in sections 5 and 6 of [30] which gives conditions where a $(p+1)$-surgery cancels a p-surgery. In this we equip a union of the traces of the two surgeries with a Morse triple, $f = (f, \mathfrak{m}, V)$ where f is the Morse function, $\mathfrak{m}$ is the background metric and V the gradient-like vector field; see section 2.2 for more details.

We prove the following theorem in chapter 8. An admissible Morse function with respect to Ricci-(k, n) curvature is one in which all the critical points of the function have index $\lambda \leq \min\{k-1, n-2\}$. This ensures the surgeries associated to the critical points satisfy the codimension conditions of Wolfson.

Theorem C. *Let X be simply connected, smooth, closed manifold of dimension $n \geq 5$. Let $f = (f, \mathfrak{m}, V)$ be a k-admissible Morse triple, $k \geq 3$, on $X \times I$, satisfying:*

1. *The conditions of Morse-Smale's Cancellation Theorems 2.3.1;*
2. *$n - \lambda_i \geq max\{n+1-k, 2\}$ where λ_i, $i \in \{1, 2\}$ are the indices of the two Morse critical points; and*
3. *If $\lambda_i = 2$, $k \geq 4$. In all other cases $k \geq 3$.*

Then for any positive Ricci-(k, n) curvature metric, g_0 on X and corresponding Gromov-Lawson concordance, $\overline{g} = \overline{g}(g_0, f)$, the metrics g_0 and $\overline{g}|_{X\times\{1\}}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$.

We note in chapter 6 that the surgery construction of Gromov and Lawson [16], Schoen and Yau [35], Gajer [15], and Walsh [43] uses standard metrics on the neighbourhood of the surgery sphere embedded in the manifold. We show that each of the spaces of these standard metrics with Ricci-(k, n) positive curvature are path-connected subspaces of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$. These standard metrics include so-called mixed torpedo metrics, defined in section 4.6. In section 5.3 using a particular isotopy we give the conditions that the standard mixed torpedo metrics are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$.

We then consider the union of the traces of two cancelling surgeries, a p-surgery and a $(p+1)$-surgery, $\{W; X, X_1\}$, where the p-surgery takes place on the embedded sphere-disc product in (X, g_0).

A Morse function $f : W \to I$ is used to decompose the cobordism in to a union of traces of surgeries. The metric $\overline{g} = \overline{g}(g_0, f)$ on W with respect to g_0 and f, extends g_0 and has a product structure near the boundary. The resulting metric, $\overline{g} = \overline{g}(g_0, f)$, is called a *Gromov-Lawson cobordism.* In the case where the cobordism is the cylinder, $X \times I$ then $\overline{g}$ is called a *Gromov-Lawson concordance* between g_0 and $g_1 = \overline{g}|_{X\times\{1\}}$.

Walsh in [44] proved that in the case of psc metrics and X having dimension ≥ 5, then for a Gromov-Lawson concordance the metrics g_0 and $\overline{g}|_{X_0 \times \{1\}}$ are isotopic. He proved this in the case of two cancelling critical points. We prove the analogous theorem C in the case of metrics of positive Ricci-(k, n) curvature metrics, where $k \geq 3$ when $p \neq 1$, and $k \geq 4$ when $p = 1$.

To get a sense of this cancellation, consider the example where X is diffeomorphic to S^n and equip the trace W with a k-admissible Morse function with two critical points with Morse indices $p+1$ and $p+2$. The index $p+1$ critical point gives rise to a p-surgery resulting in a manifold diffeomorphic to $S^{p+1} \times S^q$. The subsequent index $p+2$ critical point "cancels" out the previous surgery and results in a manifold diffeomorphic to S^n. In figure 1.8 we show two methods to achieve the metric

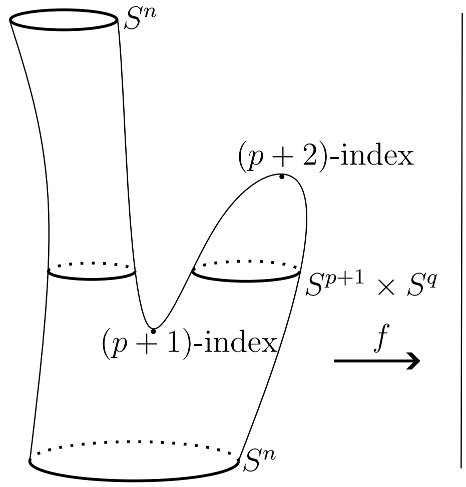

Figure 1.7: Morse function, f, on trace with critical points of index $p+1$ and $p+2$.

schematically depicted in figure 1.8(d)(iii). Path (d) uses p-surgery on an embedded p-sphere followed by a $(p+1)$-surgery on the embedded $(p+1)$-sphere; and Path (c) is an isotopy which uses positive Ricci-(k, n) standard metrics. This involves a detailed analysis of the metrics used during the isotopy and the proof of Theorem C is given in Chapter 8.

This may be extended to the case where the Gromov-Lawson concordance, $\overline{g}(g_0, f)$, is the result of more than one pair of cancelling surgeries:

Theorem D. *Let X be a simply connected, smooth, closed manifold of dimension $n \geq 5$, equipped with a metric of positive Ricci-(k, n) curvature, g_0. Let f be a k-admissible Morse function $f : X \times I \to I$, giving a Gromov-Lawson concordance, $\overline{g}(g_0, f)$ on the cylinder $X \times I$. Then the metrics g_0 and $g_1 = \overline{g}|_{X \times \{1\}}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$ if $4 \leq k \leq n$ or for k satisfying $3 \leq k \leq n$ if f has no critical points of Morse index 2.*

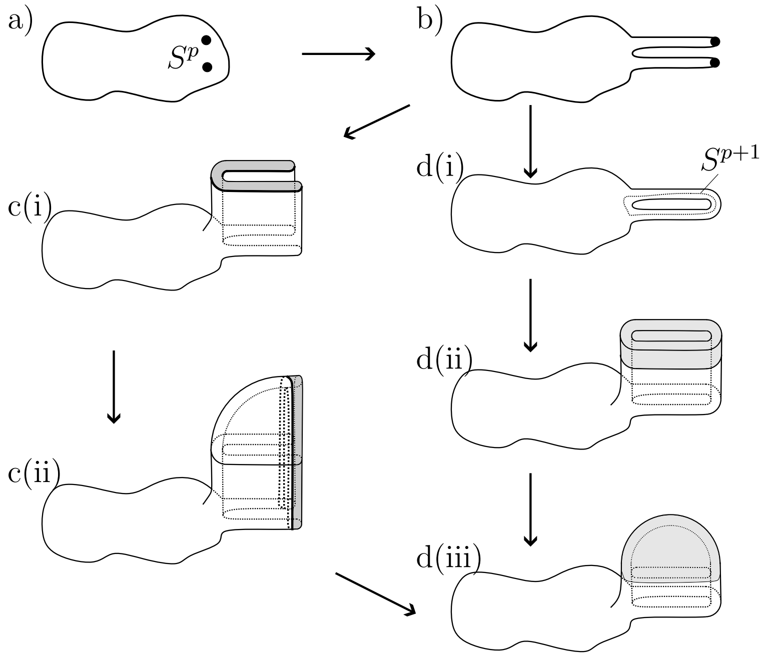

Figure 1.8: a) Original manifold; b) p-surgery ready manifold; c(i) and c(ii) Isotopy of metric; d(i) Manifold after p- surgery; d(ii) $(p+1)$-surgery ready manifold; and d(iii) Manifold after $(p+1)$-surgery.

Chapter 2

Smooth Topological Preliminaries

2.1 Surgery and Cobordism

Let X be an n-dimensional, smooth, closed manifold. Let φ be an embedding:

$$\varphi : S^p \times D^{q+1} \hookrightarrow X,$$

where $n = p + q + 1$. As usual the *sphere* $S^p(\rho) := \{x \in \mathbb{R}^{p+1}; |x| = \rho\}$, $S^p := S^p(1)$ and the *disc* $D^{q+1}(\rho) := \{x \in \mathbb{R}^{q+1}; |x| \leq \rho\}$, $D^{q+1} := D^{q+1}(1)$; figure 2.1(a).

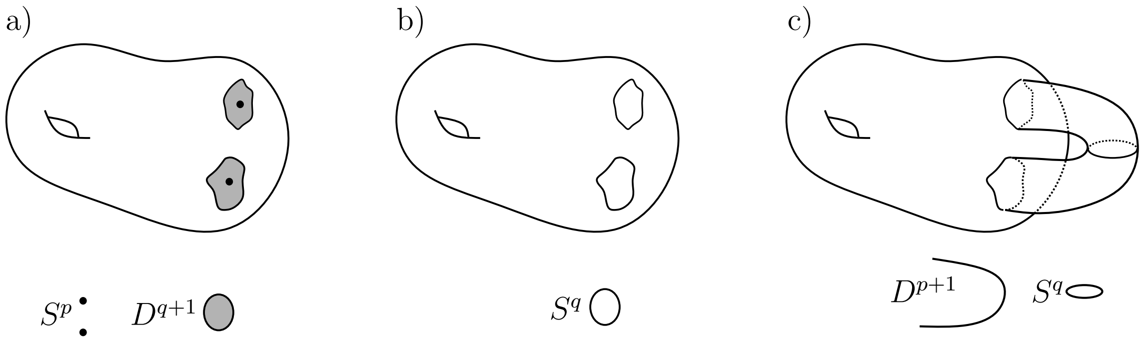

Figure 2.1: a) Embedding, $\varphi : S^p \times D^{q+1} \hookrightarrow X$; b) $\overline{X \setminus \varphi(S^p \times D^{q+1})}$; and c) X_φ.

We denote as X_φ the result of surgery on the embedding φ.

$$X_\varphi := (\overline{X \setminus \varphi(S^p \times D^{q+1})}) \cup_\varphi (D^{p+1} \times S^q).$$

Thus X_φ is obtained by first removing the embedded sphere-disc product, $\varphi(S^p \times D^{q+1})$, leaving the closed manifold $\overline{X \setminus \varphi(S^p \times D^{q+1})}$ with boundary diffeomorphic to $S^p \times S^q$; figure 2.1(b). We then use the embedding, φ, to glue in a second sphere-disc product $D^{p+1} \times S^q$, which has boundary $S^p \times S^q$ and use the embedding map φ restricted to the boundary, to give the appropriate gluing instructions as an attaching map so that the boundaries of $\overline{X \setminus \varphi(S^p \times D^{q+1})}$ and $D^{p+1} \times S^q$ are identified; figure 2.1(c). This is a smooth manifold after standard smoothing of corners at the attachment. This process is an example of a *p-surgery* (or a *co-dimension (q+1)-surgery)* on the manifold X.

Given X_φ, the result of surgery on φ, we can construct a smooth $(n+1)$-dimensional manifold, W_φ, with boundary, ∂W_φ, equal to the disjoint union of X and X_φ, known as the *trace* of the surgery on φ. It is formed by taking $X \times I$, where I is the interval $[0,1]$ and gluing the disc product $D^{p+1} \times D^{q+1}$ to $X \times \{1\}$ using the embedding map $\varphi : S^p \times D^{q+1} \hookrightarrow X$; figure 2.2(a). We schematically depict X_φ in figure 2.2(b) as the right hand boundary of W_φ.

Two closed smooth n-dimensional manifolds X_0 and X_1 are said to be *cobordant* if there exists a compact smooth $(n+1)$-dimensional manifold W with disjoint boundary $\partial W = X_0 \sqcup X_1$. The triple $\{W; X_0, X_1\}$ is called a *cobordism*. It is well known that cobordism provides an equivalence relation on closed smooth n-manifolds. It is clear from the above construction that the trace, W_φ, of surgery on the embedding φ forms a cobordism $\{W_\varphi; X, X_\varphi\}$. Such a cobordism, arising as the trace of a single surgery, is known as an *elementary cobordism*. By taking appropriate unions of cobordisms, and smoothing where required, new cobordisms may be formed. For example, $\{W; X_0, X_1\} \cup \{Z; X_1, X_2\} = \{W \cup Z : X_0, X_2\}$. Moreover, any compact cobordism $\{W; X_0, X_1\}$ may be decomposed as a union of finitely many elementary cobordisms. This fact can be proved using Morse Theory, a subject we now turn our attention to.

a) b)

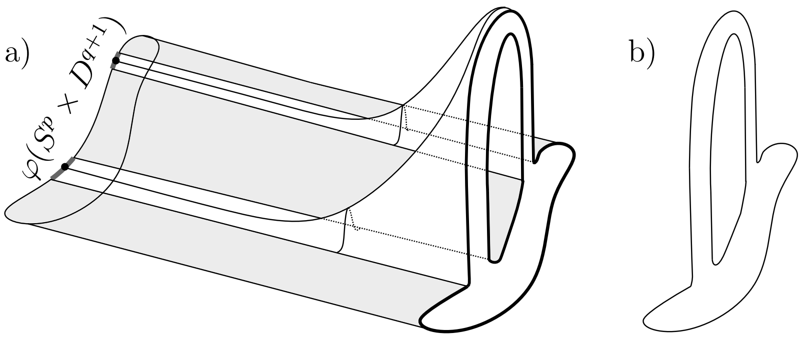

Figure 2.2: a) The trace of a surgery; and b) The right hand boundary, X_φ, of the trace of a surgery.

2.2 Morse Triples

Let ψ_0 and ψ_1 denote disjoint *collar* neighbourhood embeddings on a cobordism $\{W; X_0, X_1\}$:

$$\psi_0 : X_0 \times [0, \varepsilon) \hookrightarrow W \qquad \text{where } \psi_0(x, 0) = x \in X_0 \subset \partial W,$$
$$\psi_1 : X_1 \times (1-\varepsilon, 1] \hookrightarrow W \qquad \text{where } \psi_1(x, 1) = x \in X_1 \subset \partial W.$$

Let $\mathcal{F} = \mathcal{F}(W)$ denote the space of smooth functions $f : W \to I$ on the smooth, compact, $(n+1)$-dimensional cobordism $\{W; X_0, X_1\}$. We assume that $f \circ \psi_i(x,t) = t$, $i \in \{0,1\}$, for all $f \in \mathcal{F}(W)$. Thus there are no critical points near ∂W. The space $\mathcal{F}(W)$ is a subspace of the space of smooth functions on W with the usual C^∞ topology [18].

A critical point, w, of a smooth function $f \in \mathcal{F}(W)$ is called *degenerate* if the determinant of the Hessian, $det(D^2(f)(w)) = 0$ and *non-degenerate* if $det(D^2(f)(w)) \neq 0$. A non-degenerate critical point is also known as a *Morse critical point.* A *Morse function* is one in which all of the critical points are non-degenerate critical points. The Morse Lemma [29] states that a critical point, w, is a Morse critical point if and only if there exists a coordinate system $(x_1, x_2, \cdots, x_{n+1})$ about w such that

$$f(x_1, .., x_{n+1}) = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n+1} x_i^2$$

where $c = f(w)$. The term λ in this formula, known as the *Morse index of the Morse critical point* w, is the dimension of the negative eigenspace of the Hessian $D^2(f)(w)$ and is invariant of the choice of coordinates. Suppose W is equipped with a Morse function, $f : W \to I$, with only one critical point at $w \in W$, which has index λ. Then W is diffeomorphic to the trace of a surgery on an embedding $S^{\lambda-1} \times D^{n+1-\lambda} \hookrightarrow X_0$. In particular, for $\varepsilon > 0$, level sets $f^{-1}(c - \varepsilon)$ are diffeomorphic to X_0 while level sets $f^{-1}(c + \varepsilon)$ are diffeomorphic to X_1. For more details see Chapter 3 of [29]. As a consequence of the Morse Lemma, the critical points of a Morse function are isolated.

The space of Morse functions, $\mathcal{M}(W)$, is an open, dense subspace of $\mathcal{F}(W)$; see Theorem 2.7 of [30]. Figure 2.3 shows a Morse function f on W.

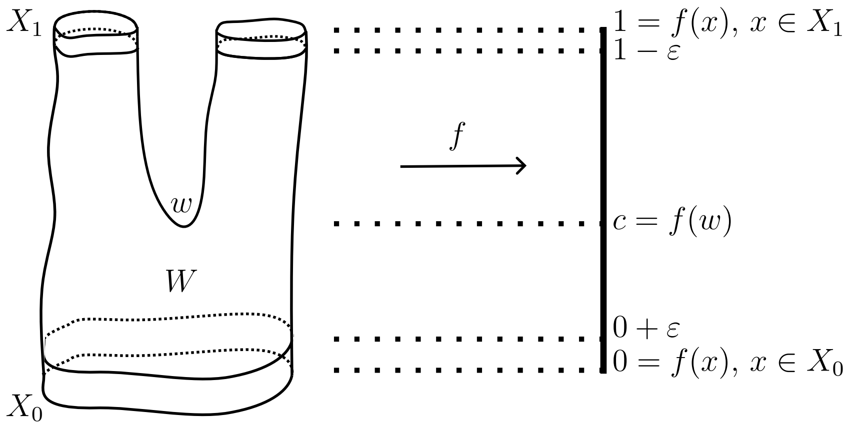

Figure 2.3: Morse function, f, on W showing the collar neighbourhoods.

We denote the set of Morse critical points of the Morse function, f, as $\sum f$, which is a finite set as W is compact. The *Morse number*, μ, of a general cobordism, W, is defined to be the minimum number of critical points over all Morse functions on

W.

We equip W with an arbitrary "background" Riemannian metric $\mathfrak{m}$ and define a *gradient-like vector field* V on W with respect to f and $\mathfrak{m}$ such that:

1. The directional derivative of f at x in the direction V_x, $V_x(f)$ is positive for all $x \in W \setminus \left(\sum f\right)$; and

2. Given any critical point w of f, there is a neighbourhood U of w such that for all $x \in U$, $V_x = grad_{\mathfrak{m}} f(x)$.

A *Morse triple*, $\{f, \mathfrak{m}, V\}$, on the $(n+1)$-dimensional manifold, W, is a triple such that f is a Morse function, $\mathfrak{m}$, the background metric, and V the related gradient-like vector field.

We abbreviate the Morse triple $\{f, \mathfrak{m}, V\}$ to f as the choice of metric and corresponding gradient-like vector field will make no difference to our work.

For simplicity we will consider the case where f is a Morse function on the trace of the embedding $\varphi : S^p \times D^{q+1} \hookrightarrow X$, $W_\varphi := \{W_\varphi; X, X_\varphi\}$, with only one critical point w of index $p+1$. Associated with this critical point is an incoming trajectory disc, $D_-^{p+1}(w)$, bounded by the incoming sphere, $S_-^p(w)$, where $S_-^p(w) = \varphi(S^p \times \{0\})$ is embedded in $X \subset \partial W_\varphi$. Exiting from the critical point there is an outgoing trajectory disc, $D_+^{q+1}(w)$, bounded by the outgoing sphere $S_+^q(w)$ which is embedded in $X_\varphi \subset \partial W_\varphi$ with trivial normal bundle. The disc $D_-^{p+1}(w)$ consists of integral curves of the gradient like vector field beginning at $S_-^p(w)$ and ending at w. Similarly, the disc $D_+^{q+1}(w)$ consists of integral curves of the gradient like vector field beginning at w and ending at $S_+^q(w)$; figure 2.4 (a). Figure 2.4(b) shows $N = \varphi(S^p \times D^{q+1}) \subset X \subset W_\varphi$, the embedding of S^p with trivial normal bundle in X. It also shows the neighbourhood U being the union of the trajectories emanating from N. Note that figure 2.4 shows the trace of a p-surgery and a Morse critical point of index $p+1$, w.

We now introduce some notation and definitions which we will use throughout the rest of this section. We denote

- $S_{t,-}^p(w)$ to be the incoming trajectory disc $D_-^{p+1}(w)$ restricted to the level set $f^{-1}(t)$, $S_{t,-}^p(w) = f^{-1}(t) \cap D_-^{p+1}(w)$, $t \in [0, f(w))$;
- $S_{t,+}^q(w)$ to be the outgoing trajectory disc $D_+^{q+1}(w)$ restricted to the level set $f^{-1}(t)$, $S_{t,+}^q(w) = f^{-1}(t) \cap D_+^{q+1}(w)$, $t \in (f(w), 1]$.

These definitions can be extended in the obvious way where there are more critical points.

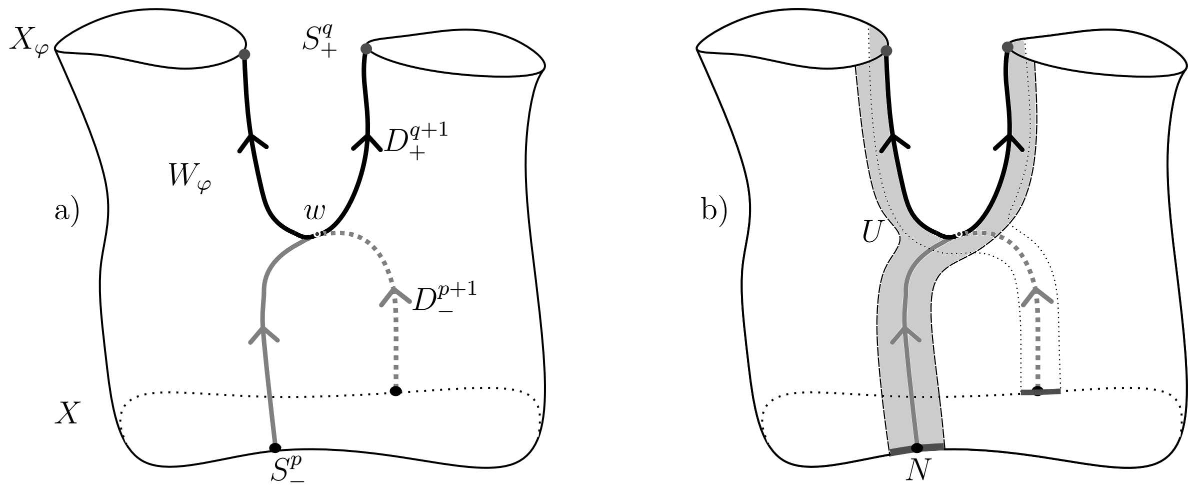

Figure 2.4: a) Trajectory discs and b) Neighbourhoods N and U.

2.3 Cancellation Theorems

An important special case for us concerns a cobordism formed by taking the union of a particular pair of elementary cobordisms. Let $W = W_{\varphi_1} \cup W_{\varphi_2}$ be the union of two elementary cobordisms, the trace of a p-surgery, $W_{\varphi_1} := \{W_{\varphi_1}; X, X_{\varphi_1}\}$, and the trace of a $(p+1)$-surgery, $W_{\varphi_2} := \{W_{\varphi_2}; X_{\varphi_1}, X_{\varphi_2}\}$, glued along the common boundary, X_{φ_1}. Here X_{φ_2} is the result of X after a p-surgery on $\varphi_1 : S^p \times D^{q+1} \hookrightarrow X$ followed by a $(p+1)$-surgery on $\varphi_2 : S^{p+1} \times D^q \hookrightarrow X_{\varphi_1}$. Let f be a Morse triple with $f : W \to I$, with $0 < f(w_1) < f(w_2) < 1$, where w_i are the critical points corresponding to the surgeries on φ_i, $i \in \{1, 2\}$. We assume transversal intersection of the outgoing sphere, $S^q_{t,+}(w_1)$, from the index $p+1$ critical point, w_1, and the incoming sphere, $S^{p+1}_{t,-}(w_2)$, going to the index $p+2$ critical point w_2, in each level set, $f^{-1}(t)$, $t \in (f(w_1), f(w_2))$. Note that $p + q + 1 = n$, the dimension of the level set, $f^{-1}(t)$. Thus for each t, the intersection, $S^q_{t,+}(w_1) \cap S^{p+1}_{t,-}(w_2)$, must consist of a finite collection of points, n_i. We assume that the normal bundle to $S^{p+1}_{t,-}(w_2)$ is oriented. We choose a positively oriented frame of q linearly independent vectors that span the tangent space of $S^q_{t,+}(w_1)$ at each n_i. In each case this also spans the normal space of $S^{p+1}_{t,-}(w_2)$ at n_i. The *intersection number* of $S^q_{t,+}(w_1)$ and $S^{p+1}_{t,-}(w_2)$ at n_i is $+1$ if the frame represents a positively oriented basis for the normal space of $S^{p+1}_{t,-}(w_2)$ at n_i and -1 otherwise. The intersection number $(S^q_{t,+}(w_1)).(S^{p+1}_{t,-}(w_2))$ is the sum of the intersection numbers over all the intersection points, n_i, in $f^{-1}(t)$. Under certain conditions, given below in Theorem 2.3.1, the gradient like vector field may be perturbed to give a transverse intersection of $S^q_{t,+}(w_1)$ and $S^{p+1}_{t,-}(w_2)$ so that there is one intersection point in each level set, $f^{-1}(t)$, $t \in (f(w_1), f(w_2))$.

Assuming this perturbation has been made, figure 2.5(a) shows the outward trajectory disc, $D^{q+1}_+(w_1)$, an arc of which emerges from critical point, w_1, and goes

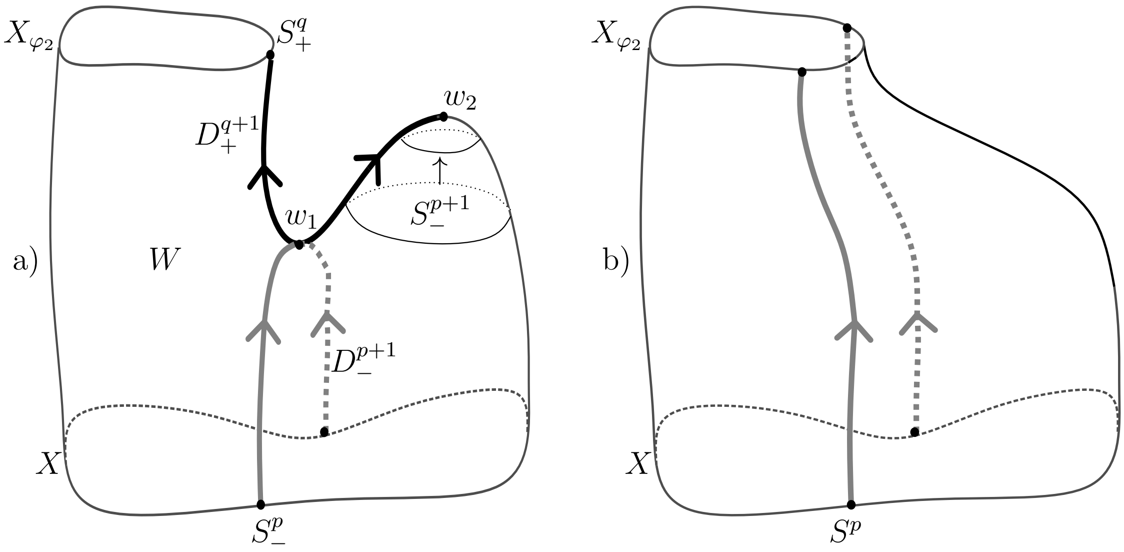

Figure 2.5: a) Trajectory discs on W; and b) Manifold where W is diffeomorphic to $X \times I$.

to critical point, w_2. There is also an incoming trajectory disc, $D_-^{p+2}(w_2)$ converging towards critical point w_2. The transverse intersection of $D_+^{q+1}(w_1)$ and $D_-^{p+2}(w_2)$, the union of the single intersection point in each level set, describes a one-dimensional curve, P, from w_1 to w_2. Note that the sum of the dimensions of the discs less the dimension of the trace is one, i.e. $(q+1)+(p+2)-(n+1)=1$. Figure 2.6(a) shows a vector field with two critical points, with a small neighbourhood, B, shown in grey, around the one-dimensional curve. In figure 2.6(b), the vector field in B around the two critical points is shown in more detail. Figure 2.6(c) shows the way that the vector field in B may be adjusted to give a nowhere zero vector field. The vector field on the rest of the trace, $W \setminus B$, remains unchanged.

It is important that the manifolds are simply connected so that a slight perturbation may reduce the number of points of intersection as indicated in figure 2.7. Figure 2.7(a) shows three intersection points but a small perturbation to the curve may be performed to leave only one intersection point as in figure 2.7(b). Figure 2.7(c) schematically shows the potential obstruction that not being simply connected can be to reducing the number of points of intersection.

In figures 2.5 and 2.6 we have shown the two critical points as corresponding to a p-surgery followed by a $(p+1)$-surgery. When the $(n+1)$-dimensional manifold resulting from the union of the traces of these surgeries is diffeomorphic to $X \times I$ we call the two surgeries *cancelling surgeries*; figure 2.5 (b). We note that it is possible for two surgeries to cancel in other ways. For example following a p-surgery by an $(n-p-1)$-surgery can have the effect of reversing the original surgery.

The following Theorem, proved in chapter 6 of [30], gives sufficient conditions in which the two surgeries are cancelling surgeries.

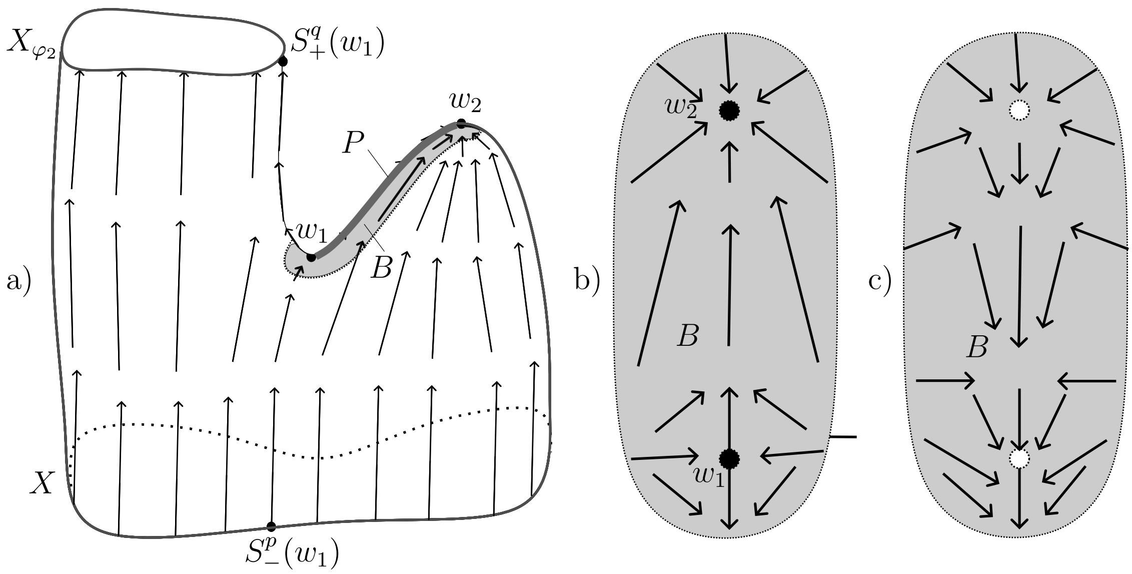

Figure 2.6: a) Manifold with two critical points; b) Vector field of B with two critical points; and c) Vector field of B without critical points and a nowhere zero vector field.

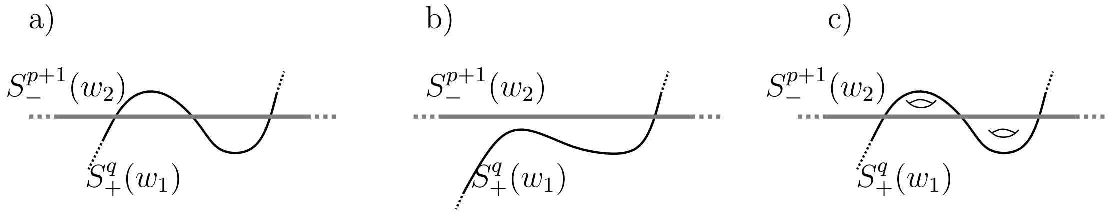

Figure 2.7: a) Three points of intersection; b) One point of intersection; and c) Obstruction to reducing number of points of intersection.

Theorem 2.3.1 ((Morse) Strong Cancellation Theorem)**.** *Let $\{W; X_0, X_1\}$ be a simply connected, smooth, compact cobordism. Let $f = (f, \mathfrak{m}, V)$ be a Morse triple, where $f : W \to I$, satisfies the following:*

(a) The function f has exactly two critical points w_1 and w_2 and $0 < f(w_1) < c < f(w_2) < 1$.

(b) The critical points w_1 and w_2 have Morse index $p + 1$ and $p + 2$, respectively, where $1 \leq p \leq n - 4$.

(c) For each $t \in (f(w_1), f(w_2))$, the trajectory sphere $S^q_{t,+}(w_1)$ emerging from the critical point w_1 and the trajectory sphere $S^{p+1}_{t,-}(w_2)$ converging towards the critical point w_2 have intersection number ± 1.

Then

(i) W is diffeomorphic to $X_0 \times I$.

(ii) The gradient-like vector field V may be altered near $f^{-1}(c)$ so that $S^q_{c,+}(w_1)$ and $S^{p+1}_{c,-}(w_2)$ intersect transversely at a single point in $f^{-1}(c)$. The union of these intersection points forms a trajectory arc from w_1 to w_2.

(iii) The gradient-like vector field V may be further perturbed in a small neighbourhood of the trajectory arc from critical point w_1 to w_2 giving a nowhere zero vector field, V'. All the trajectories now commence at X_0 and end at X_1.

(iv) The gradient-like vector field V' is now the gradient of a function f' without critical points which agrees with f outside a neighbourhood of the aforementioned trajectory arc between the critical points.

Chapter 3

Geometric Preliminaries

3.1 Curvature

Let (M, g) be an n-dimensional smooth Riemannian manifold with metric g and Levi-Civita connection ∇. We recall that the *Riemann curvature endomorphism,* R, is the $(3, 1)$ tensor field defined by the mapping

$$(X, Y, Z) \mapsto R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y and Z are vector fields. In turn the *Riemann Curvature Tensor*, Rm, is the $(4, 0)$-tensor field defined

$$Rm(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where W is also a vector field on M. The *sectional curvature* of (M, g) is defined as follows. Let $\sigma \subset T_pM$, $p \in M$, be a two-dimensional linear subspace of the tangent space, T_pM, at p. Then the sectional curvature of the two-dimensional linear subspace, σ, at p, $K(p, \sigma)$, is

$$K(p, \sigma) = K_{v,w}(p) = \frac{Rm(v, w, w, v)}{g(v, v)g(w, w) - g(v, w)g(w, v)},$$

where v and w span σ. We suppress, in our notation, the fact that v and w must be extended as vector fields near p. The sectional curvature is independent of the choice of v, w and any vector field extensions of v and w; see section 2.2.3 of [32]. For each $p \in M$, let $Gr_2(T_pM)$ be the Grassmann manifold of 2-dimensional subspaces of the tangent space T_pM and $Gr_2(M)$ be the corresponding Grassmann bundle over M. To say that a manifold has positive sectional curvature, means that the function, K:

$$\begin{aligned} K : Gr_2(M) &\to \mathbb{R} \\ (p, \sigma) &\mapsto K(p, \sigma) \end{aligned}$$

is positive for all points p in the manifold and 2-planes σ in T_pM.

For any orthonormal basis of T_pM, $\{e_i\}$, we define the *Ricci tensor*, *Ric*, by the formula:

$$Ric_p(v, w) = \sum_{i=1}^{n} g(R(e_i, v)w, e_i),$$

for $v, w \in T_pM$. This is a bilinear form and is the contraction of the Riemannian curvature tensor. Using the same notation we will often interpret it as a quadratic form given by

$$Ric_p(v) = \sum_{i=1}^{n} g(R(e_i, v)v, e_i).$$

The Ricci tensor is independent of the choice of orthonormal basis.

To say that a manifold has *positive Ricci curvature* means that $Ric_p(v) > 0$ for all $v \in T_pM$ at every point $p \in M$.

The *scalar curvature* at a point p in M, $scal(p)$, is the trace of the Ricci tensor and defined

$$scal(p) = \sum_{i,j=1,i\neq j}^{n} K_{e_i,e_j}(p) = \sum_{i=1}^{n} Ric_p(e_i),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis for T_pM. As in the case of Ricci curvature, the scalar curvature is independent of the choice of orthonormal basis. To say that a manifold has *positive scalar curvature*, psc, means that $scal(p) > 0$, for all $p \in M$. Where a manifold has positive sectional curvature, then the manifold also has positive Ricci curvature and consequently positive scalar curvature.

Hereafter we suppress the notation for the point p.

The *Ricci-(k,n) curvature*, $Ric_{(k,n)}$, is defined as follows. As the Ricci tensor is symmetric, its eigenvalues, λ_i, $i \in \{1, \cdots, n\}$ are real. The Ricci-(k, n) curvature is the sum of the k smallest eigenvalues. We say that a metric has *positive Ricci-(k, n) curvature* if the Ricci-(k, n) curvature of the metric is positive for each $p \in M$. Note that positive Ricci-(n, n) curvature is equivalent to positive scalar curvature and positive Ricci-$(1, n)$ curvature is equivalent to positive Ricci curvature.

Hence, for a manifold

$$K > 0 \Longrightarrow Ric > 0 \Longrightarrow Ric_{(k,n)} > 0, \forall\ k \Longrightarrow scal > 0.$$

In particular

$$Ric_{(k,n)} > 0 \Longrightarrow Ric_{(k+1,n)} > 0.$$

3.2 Spaces of Riemannian metrics, isotopy, cobordism and concordance

Let $\mathcal{R}\text{iem}(X)$ be the space of all Riemannian metrics on X, an n-dimensional, smooth, closed Riemannian manifold, with its standard C^∞ topology; see Chapter 1 of [42] for a detailed description. The space $\mathcal{R}\text{iem}(X)$ is convex. This follows from the fact that the set of $n \times n$ positive-definite symmetric matrices forms a convex cone in the space of all $n \times n$ matrices. Let

$$\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X) := \{g \in \mathcal{R}\text{iem}(X) : Ric^g_{(k,n)} > 0\},$$

be the open subset of $\mathcal{R}\text{iem}(X)$ consisting of metrics which have positive Ricci-(k,n) curvature on the n-dimensional manifold X. Here $Ric^g_{(k,n)} : X \to \mathbb{R}$ denotes the Ricci-(k,n) curvature of the metric g on X. Note that

$$\mathcal{R}\text{iem}^{Ric^+_{(1,n)}}(X) \subset \mathcal{R}\text{iem}^{Ric^+_{(2,n)}}(X) \subset \cdots \subset \mathcal{R}\text{iem}^{Ric^+_{(n,n)}}(X).$$

For a given k, the space $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ may or may not be empty. Assuming $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ to be non-empty, we may ask about its topology. The most basic question in this regard concerns path-connectivity. For example, Carr [8] has shown that the space of psc-metrics on the sphere, S^{4n-1}, $n \geq 2$ has infinitely many path-components. Similarly, Wraith [56] has shown that the space of positive Ricci-metrics on the sphere, S^{4n-1}, $n \geq 2$ has infinitely many path-components. This problem and more general topological questions necessitate the notions of isotopy and concordance. Other results concerning the topology of the spaces of Riemannian metrics have been referred to in section 1.4.

Versions of isotopy and concordance exist in many areas of mathematics and in many of the papers cited herein they refer to psc metrics. Here we define these notions with respect to metrics of positive Ricci-(k,n) curvature. A pair of metrics, g_0 and g_1, which lie in the same path component of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ are said to be $Ric_{(k,n)} > 0$ *isotopic*. That is, there exists a path, called a *positive Ricci-(k,n) isotopy*, which connects the metrics:

$$\begin{aligned} I &\to \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X) \\ t &\mapsto g_t, \end{aligned}$$

where I is the interval $[0,1]$.

Let W be a manifold with boundary ∂W, a smooth, closed manifold, equipped with a collar, $\psi : \partial W \times [0,\varepsilon) \hookrightarrow W$. We denote the space of Riemannian metrics on W such that the pull back of a metric on W restricted to the collar takes the form of a product,

$$\mathcal{R}\text{iem}(W, \partial W) := \{\bar{g} \in \mathcal{R}\text{iem}(W) : \psi^*\bar{g} = \bar{g}|_{\partial W} + dt^2\},$$

where $\overline{g}|_{\partial W}$ is the restriction of $\overline{g}$ to the boundary ∂W. We are interested in the case where W is a cobordism between two closed manifolds, X_0 and X_1, i.e. $W := \{W; X_0, X_1\}$. The boundary may be equipped with collars using the embeddings ψ_0 and ψ_1, $\psi_i : X_i \times [0, \varepsilon) \hookrightarrow W$, such that $\psi_i(x, 0) = x, x \in X_i$, $i \in \{0, 1\}$. In this case the space of Riemannian metrics

$$\mathcal{R}\mathrm{iem}(W, \partial W) = \mathcal{R}\mathrm{iem}(W, X_0 \sqcup X_1) := \{\overline{g} \in \mathcal{R}\mathrm{iem}(W) : \psi_i^* \overline{g} = \overline{g}|_{X_i} + dt^2, i \in \{0, 1\}\}.$$

Let us temporarily consider metrics which have positive scalar curvature. The subspace of $\mathcal{R}\mathrm{iem}(W, \partial W)$ consisting of metrics which have positive scalar curvature is denoted $\mathcal{R}\mathrm{iem}^+(W, \partial W)$. A special case of the cobordism $\{W; X_0, X_1\}$ occurs when $X_0 = X_1 = X$. Here $\partial W = \psi_0(X \times \{0\}) \sqcup \psi_1(X \times \{1\})$ and, for simplicity, we denote the manifold W as $\{W; X, X\}$. The pullback of the metrics on the boundary are $\psi_0^* g_{\psi_0(X \times \{0\})} = g_0$ and $\psi_1^* g_{\psi_1(X \times \{1\})} = g_1$, which are both metrics on X. The metrics g_0 and g_1 in $\mathcal{R}\mathrm{iem}^+(X)$, are *psc-cobordant* if there exists W a cobordism $\{W; \partial W\}$, where $\partial W = \psi_0(X \times \{0\}) \sqcup \psi_1(X \times \{1\})$, and $\overline{g} \in \mathcal{R}\mathrm{iem}^+(W, \partial W)$ so that $\overline{g}|_{X_0} = g_0$ and $\overline{g}|_{X_1} = g_1$ for some psc-metric $\overline{g} \in \mathcal{R}\mathrm{iem}^+(W, \partial W)$.

Let us further restrict to the case when W is the $(n+1)$-dimensional cylinder $X \times I$, where $I = [0, 1]$. Two metrics g_0 and g_1 in the space of psc-metrics on X, $\mathcal{R}\mathrm{iem}^+(X)$, are *psc-concordant* if there exists W diffeomorphic to $X \times I$ and $\overline{g} \in \mathcal{R}\mathrm{iem}^+(W, \partial W)$ so that $\overline{g}|_{X \times \{0\}} = g_0$ and $\overline{g}|_{X \times \{1\}} = g_1$ for some metric $\overline{g} \in \mathcal{R}\mathrm{iem}^+(W, \partial W)$. We say that $(W, \overline{g})$ is a *psc-concordance.*

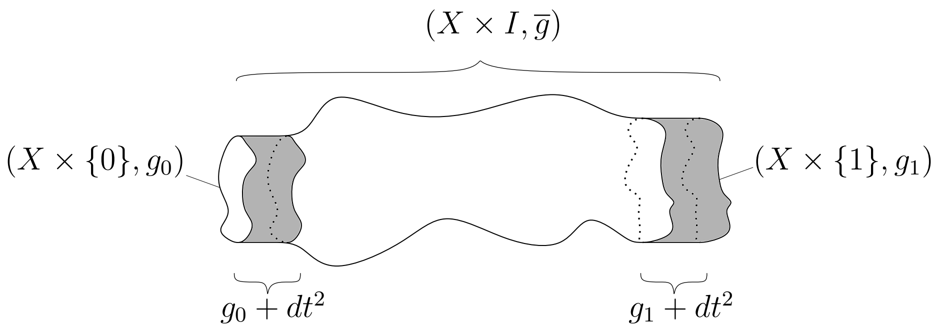

Figure 3.1: Concordant metrics g_0 and g_1

Clearly, if a pair of psc-metrics are psc-concordant they are also psc-cobordant. Moreover isotopic psc-metrics are psc-concordant [15], [16]. However psc-cobordant metrics may not be psc-concordant as Carr [8] showed in his work on psc-metrics on the spheres, S^{4n-1}, $n \geq 2$. He used this to show that there are infinitely many path-components in the space $\mathcal{R}\mathrm{iem}^+(S^{4n-1})$, as metrics which are not psc-concordant are not psc-isotopic. Moreover, psc-concordant metrics may not be psc-isotopic, as Rubermann [34] showed in the case of certain dimension 4 manifolds, by using the Seiberg-Witten invariant to detect the difference between psc-isotopy and psc-concordance. However, it is a difficult open question as to whether psc-concordance implies psc-isotopy [33] in the case of n-manifolds, $n \geq 5$. It is conjectured in the case of simply-connected, closed manifolds of dimension $n \geq 5$ that psc-concordance implies psc-isotopy but this remains a difficult open problem. The notion of Gromov-Lawson concordance, described below, was introduced to make the problem more tractable.

We wish to generalise the definition of concordance to positive Ricci-(k, n) curvature metrics. In the case when $k = n$, recall this is psc-concordance. We define two positive Ricci-(k, n) curvature metrics, g_0 and g_1, on X, to be $Ric_{(k,n)} > 0$ *concordant* if there is a positive Ricci-$(k+1, n+1)$ metric, $\overline{g}$, on the cylinder $X \times I$, so that $\overline{g} = g_0 + dt^2$ and $\overline{g} = g_1 + dt^2$ near $X \times \{0\}$ and $X \times \{1\}$, respectively. The metric, $\overline{g}$, is called a *positive Ricci-(k, n) concordance* and the metrics g_0 and g_1 are *positive Ricci-(k,n) concordant*. Assuming positive Ricci-(k, n) curvature is understood, then we will use the terms isotopic and concordant to mean positive Ricci-(k, n) *isotopic* and positive Ricci-(k, n) *concordant*, respectively. Where $k = n$, we refer to *psc-isotopic* and *psc-concordant metrics*.

Let $\varphi : S^p \times D^{q+1} \hookrightarrow X$ be an embedding into a closed n-dimensional manifold, X, where $p + q + 1 = n$. Let X_φ be the manifold resulting from surgery on φ; see chapter 2. In Theorem A of [16] and Corollary 6 of Theorem 4 [35], Gromov and Lawson, and Schoen and Yau describe a "geometric" surgery which allows under appropriate circumstances for the preservation of the psc condition under surgery. More precisely, if X admits a psc-metric and $q \geq 2$, then X_φ also admits a psc-metric. The theorem is constructive, in the sense that, given a psc-metric g on X, it produces a new psc-metric g_φ on X_φ. This is described in further detail in chapter 6.

Furthermore, and again assuming $q \geq 2$, the psc-metric g on X can be extended to a psc-metric $\overline{g}$ over the trace $\{W_\varphi; X, X_\varphi\}$ which takes a product structure near the boundary and restricts as g_φ at X_φ; see Walsh in Theorem 2.2 of [44] and Gajer in [15]. This trace equipped with the resulting metric $\overline{g}$ we call a *Gromov-Lawson trace*. Burkemper, Searle and Walsh generalized this construction to work for (l, n)-intermediate scalar curvature, where l refers to the dimension of a plane in the tangent space, for $0 \leq l \leq n-2$ for surgeries in codimension at least $l + 3$ [7]. This curvature interpolates between sectional curvature when $l = n - 2$ and scalar curvature when $l = 0$ and is due to Labbi [23]. In this work we extend this result to positive Ricci-(k, n) metrics.

As each Gromov-Lawson trace has a product metric on the boundary we may combine such traces of surgeries metrically to give a manifold which is a union of finitely many traces. It is possible that the union of traces may be topologically a cylinder, $X \times I$, admitting a psc metric with product psc metrics on the disjoint boundaries. This is an example of a psc concordance. Such a psc concordance is called a *Gromov-Lawson concordance*.

In the case of Gromov-Lawson concordances, Walsh [44] has shown that psc concordance implies psc isotopy for closed simply connected manifolds of dimension $n \geq 5$. It is an open question as to whether all concordances, where $n \geq 5$, adjusted via an isotopy, are a union of Gromov-Lawson concordances. We consider in this work concordances of positive Ricci-(k, n) metrics constructed using the Gromov-Lawson Surgery technique and, using a simplified proof, we will extend the Gromov-Lawson concordance implies isotopy result to positive Ricci-(k, n) metrics.

3.3 Isotopy implies concordance for positive Ricci-(k, n) metrics

It is well known that isotopic psc-metrics are psc-concordant [16], [44], [15]. This fact plays an essential role in the various surgery based metric constructions, for example in [44]. In this, Walsh shows the transition of the original psc metric on a manifold to an isotopic surgery ready psc metric. Moreover Carr in [8] proved that there were countably many path components of $\mathcal{R}\text{iem}^+(S^{4n-1})$, $n \geq 2$, by showing that certain metrics are not psc-concordant and therefore not psc-isotopic. We show that this result generalises to hold for isotopic positive Ricci-(k, n) metrics in the proof of Lemma 3.3.1. We adapt the proof of Walsh [43] for the case of psc metrics to the positive Ricci-(k, n) case.

Lemma 3.3.1. *Let $g_r, r \in I$, be a smooth path in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$, where X is compact. Then there exists a constant $0 < \Lambda \leq 1$ so that for every smooth function $f : \mathbb{R} \to [0, 1]$ with $|\dot{f}|, |\ddot{f}| \leq \Lambda$, the metric $G = g_{f(t)} + dt^2$ on $X \times \mathbb{R}$ has positive Ricci-(k+1, n+1) curvature.*

Proof. Choose a point $(x_0, t_0) \in X \times \mathbb{R}$. Denote by $(x_0^1, \cdots, x_0^n, x_0^{n+1} = t)$, coordinates around (x_0, t_0), where $x_0^1, \cdots, x_0^n$ are normal coordinates on X with respect to the metric $g_{f(t_0)}$. Without loss of generality, let $\partial_1, \cdots, \partial_n$, be eigenvectors of the Ricci tensor of $g_{f(t)}$, with eigenvalues, $\lambda_i \leq \cdots \leq \lambda_n$.

In these coordinates, the Ricci tensor at (x_0, t_0) takes the form of a diagonal $n \times n$ matrix. We denote this matrix $D_n(Ric)$. By assumption, the sum of the k smallest eigenvalues of the Ricci curvature of the metric $g_{f(t_0)}$ is positive, and by compactness we can assume that the sum of the k smallest eigenvalues of $g_{f(t)}$ are bounded below by a constant $c > 0$ for all $t \in I$.

Let $\overline{\nabla}$ denote the Levi-Civita connection with respect to the metric G on the $(n+1)$-dimensional cylinder $X \times \mathbb{R}$ and let ∇ denote the Levi-Civita connection of the metric $g_{f(t_0)}$ on the n-dimensional hypersurface $X \times \{t_0\}$. All of the calculations take place at the point (x_0, t_0).

We now calculate the Ricci curvature of G of the orthonormal vector fields $\partial_1, \cdots, \partial_n, \partial_{n+1}$, $\partial_{n+1} = \partial_t$, of $T_{(x_0,t_0)}(X \times \mathbb{R})$. The Christoffel symbols $\overline{\Gamma}^l_{i,j}$ of the connection $\overline{\nabla}$ are calculated using the formula

$$\bar{\Gamma}^l_{i,j} = \frac{1}{2}G^{lm}(\partial_j G_{im} + \partial_i G_{jm} - \partial_m G_{ij}),$$

to give

i	j	l	$\overline{\Gamma}^l_{i,j}$
$\leq n$	$\leq n$	$\leq n$	$\Gamma^l_{i,j}$
$\leq n$	$\leq n$	$n+1$	$-\frac{1}{2}\partial_r g_{r(ij)}(x_0, f(t_0)).\dot{f}(t_0)$
$\leq n$	$n+1$	$n+1$	0
$\leq n$	$n+1$	$\leq n$	$\frac{1}{2}G^{lm}\partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0)$
$n+1$	$n+1$	$\leq n$	0
$n+1$	$n+1$	$n+1$	0

Recollect that for a Levi-Civita connection $\overline{\Gamma}^l_{i,j} = \overline{\Gamma}^l_{j,i}$. From the table above it can be seen that $\overline{\Gamma}^{n+1}_{i,j}$ and $\overline{\Gamma}^l_{i,n+1}$ are both $O(|\dot{f}|)$.

Let $\overline{K}_{i,j}$ and $K_{i,j}$ denote the respective sectional curvatures for the metrics G and $g_{f(t_0)}$. The Gauss curvature equation gives $\overline{K}_{i,j}$ in terms of $K_{i,j}$ when $i, j \leq n$ as follows:

$$\overline{K}_{i,j} = K_{i,j} - G(\mathrm{II}(\partial_i, \partial_i), \mathrm{II}(\partial_j, \partial_j)) + G(\mathrm{II}(\partial_i, \partial_j), \mathrm{II}(\partial_i, \partial_j)),$$

where II denotes the second fundamental form on $X \times \{t_0\}$. As $i, j \leq n$, the second fundamental form may be expressed as

$$\begin{aligned}\mathrm{II}(\partial_i, \partial_j) &= G(\overline{\nabla}_{\partial_i}\partial_j, \partial_{n+1})\partial_{n+1} \\ &= \overline{\Gamma}^{n+1}_{i,j}\partial_{n+1},\end{aligned}$$

as $G_{ij} = \delta_{ij}$ at the point(x_0, t_0).

Therefore

$$\begin{aligned}\overline{K}_{i,j} &= K_{i,j} + \overline{\Gamma}^{n+1}_{ij}\overline{\Gamma}^{n+1}_{ij} - \overline{\Gamma}^{n+1}_{ii}\overline{\Gamma}^{n+1}_{jj} \\ &= K_{i,j} + \frac{1}{4}\Big((\partial_r g_{r(ij)}(x_0, f(t_0)))^2 - \partial_r g_{r(ii)}(x_0, f(t_0))\partial_r g_{r(jj)}(x_0, f(t_0))\Big)(\dot{f}(t_0))^2 \\ &= K_{i,j} + O(|\dot{f}|^2).\end{aligned}$$

Where $i \leq n$, $j = n+1$, the expression for $\overline{K}_{i,n+1}$ is as follows:

$$\overline{K}_{i,n+1} = \frac{G(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_{n+1}}\partial_{n+1} - \overline{\nabla}_{\partial_{n+1}}\overline{\nabla}_{\partial_i}\partial_{n+1}, \partial_i)}{G_{ii}G_{n+1,n+1} - G_{i,n+1}G_{i,n+1}}.$$

As $G_{ij} = \delta_{ij}$ at the point(x_0, t_0), the denominator is 1. Therefore

$$\begin{aligned}\overline{K}_{i,n+1} &= G(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_{n+1}}\partial_{n+1} - \overline{\nabla}_{\partial_{n+1}}\overline{\nabla}_{\partial_i}\partial_{n+1}, \partial_i) \\ &= G(\overline{\nabla}_{\partial_i}(\overline{\Gamma}^l_{n+1,n+1}\partial_l) - \overline{\nabla}_{\partial_{n+1}}(\overline{\Gamma}^l_{i,n+1}\partial_l), \partial_i) \\ &= G(\partial_i(\overline{\Gamma}^l_{n+1,n+1})\partial_l + (\overline{\Gamma}^l_{n+1,n+1}\overline{\Gamma}^m_{i,l})\partial_m - \partial_{n+1}(\overline{\Gamma}^l_{i,n+1})\partial_l - \overline{\Gamma}^l_{i,n+1}\overline{\Gamma}^m_{n+1,l}\partial_m, \partial_i) \\ &= \partial_i(\overline{\Gamma}^i_{n+1,n+1}) + \overline{\Gamma}^l_{n+1,n+1}\overline{\Gamma}^i_{i,l} - \partial_{n+1}(\overline{\Gamma}^i_{i,n+1}) - \overline{\Gamma}^l_{i,n+1}\overline{\Gamma}^i_{n+1,l}.\end{aligned}$$

As $\partial_{n+1} = \partial_t$

$$\begin{aligned}\overline{K}_{i,n+1} = 0 + 0 &- \frac{1}{2}\partial_t(G^{im}\partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0)) \\ &- \frac{1}{4}\big(G^{lm}\partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0)G^{ip}\partial_r g_{r(lp)}(x_0, f(t_0)).\dot{f}(t_0)\big).\end{aligned}$$

As

$$\begin{aligned}\frac{1}{2}\partial_t(G^{im}\partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0)) =&\frac{1}{2}\partial_t(G^{im}\partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0) \\ &+ \frac{1}{2}G^{im}\partial_r g_{r(im)}(x_0, f(t_0)).\ddot{f}(t_0),\end{aligned}$$

then

$$\overline{K}_{i,n+1} = O(|\dot{f}|^2) + O(|\ddot{f}|).$$

Let the Ricci curvature of the normal vector fields be denoted $\overline{Ric}(\partial_i)$ and $Ric(\partial_i)$ for the metrics G and $g_{f(t_0)}$. The Ricci curvature may be calculated from the sectional curvature as follows:

$$\overline{Ric}(\partial_i)(x_0, t_0) = \sum_{i \neq j} \overline{K}_{i,j}(x_0, t_0), \text{ where } i \text{ is fixed.}$$

Therefore, where $i \leq n$ the Ricci curvature is:

$$\begin{aligned}\overline{Ric}(\partial_i) &= \overline{K}_{i,n+1} + \sum_{i \neq j}^{n} \overline{K}_{i,j}, \text{ where } i \text{ is fixed,} \\ &= O(|\dot{f}|^2) + O(|\ddot{f}|) + \sum_{i \neq j}^{n} \big(K_{i,j} + O(|\dot{f}|^2)\big), \\ &= Ric(\partial_i) + O(|\dot{f}|^2) + O(|\ddot{f}|).\end{aligned}$$

Hence $\overline{Ric}(\partial_i)$ and $Ric(\partial_i)$ differ by terms of $O(|\ddot{f}|)$ and $O(|\dot{f}|^2)$.
Where $i = n + 1$ the Ricci curvature is:

$$\begin{aligned}\overline{Ric}(\partial_{n+1}) &= \sum_{i=1}^{n} \overline{K}_{i,n+1}, \\ &= \sum_{i=1}^{n} \big(O(|\dot{f}|^2) + O(|\ddot{f}|)\big), \\ &= O(|\dot{f}|^2) + O(|\ddot{f}|).\end{aligned}$$

Therefore $\overline{Ric}(\partial_{n+1})$ is an expression which has only terms involving $O(|\dot{f}|^2)$ and $O(|\ddot{f}|)$.

Having calculated the quadratic form of the Ricci curvature of the coordinate vector fields, $\overline{Ric}(\partial_i)$, we can now calculate the bilinear terms, $\overline{Ric}(\partial_i, \partial_j)$, of the Ricci tensor as follows:

$$\overline{Ric}(\partial_i, \partial_j) = \partial_l \overline{\Gamma}^l_{i,j} - \partial_j \overline{\Gamma}^l_{i,l} + \overline{\Gamma}^l_{l,m} \overline{\Gamma}^m_{i,j} - \overline{\Gamma}^l_{j,m} \overline{\Gamma}^m_{l,i}.$$

We calculate the bilinear terms where $i, j \leq n$

$$\begin{aligned}
\overline{Ric}(\partial_i, \partial_j) = {} & \partial_l \Big(\Gamma^l_{i,j} - \frac{1}{2} \partial_r g_{r(ij)}(x_0, f(t_0)).\dot{f}(t_0) \Big) - \partial_j \Gamma^l_{i,l} \\
& + \Big(\Gamma^l_{l,m} + \frac{1}{2} G^{ls} \partial_r g_{r(ls)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \Big(\Gamma^m_{i,j} - \frac{1}{2} \partial_r g_{r(ij)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \\
& - \Big(\Gamma^l_{j,m} - \frac{1}{2} \partial_r g_{r(jm)}(x_0, f(t_0)).\dot{f}(t_0) + \frac{1}{2} G^{ls} \partial_r g_{r(js)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \\
& \Big(\Gamma^m_{l,i} - \frac{1}{2} \partial_r g_{r(il)}(x_0, f(t_0)).\dot{f}(t_0) + \frac{1}{2} G^{ms} \partial_r g_{r(is)}(x_0, f(t_0)).\dot{f}(t_0) \Big).
\end{aligned}$$

The off diagonal elements of the Ricci tensor $D_n(Ric)$ are equal to zero and hence

$$Ric(\partial_i, \partial_j) = \partial_l \Gamma^l_{i,j} - \partial_j \Gamma^l_{i,l} + \Gamma^l_{l,m} \Gamma^m_{i,j} - \Gamma^l_{j,m} \Gamma^m_{l,i} = 0.$$

Therefore, by inspection it can be seen that

$$\overline{Ric}(\partial_i, \partial_j) = O(|\dot{f}|) + O(|\dot{f}|^2),$$

when $i, j \leq n$.

We calculate the bilinear terms when $i \leq n$ and $j = n + 1$.

$$\begin{aligned}
\overline{Ric}(\partial_i, \partial_{n+1}) = {} & \partial_l \Big(\frac{1}{2} G^{lm} \partial_r g_{r(im)}(x_0, f(t_0)).\dot{f}(t_0) \Big) - \partial_t \Gamma^l_{i,l} \\
& + \Big(\Gamma^l_{l,m} + \frac{1}{2} G^{ls} \partial_r g_{r(is)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \Big(\frac{1}{2} G^{lm} \partial_r g_{r(il)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \\
& - \Big(\frac{1}{2} G^{ls} \partial_r g_{r(ms)}(x_0, f(t_0)).\dot{f}(t_0) \Big) \\
& \Big(\Gamma^m_{l,i} - \frac{1}{2} \partial_r g_{r(il)}(x_0, f(t_0)).\dot{f}(t_0) + \frac{1}{2} G^{ms} \partial_r g_{r(is)}(x_0, f(t_0)).\dot{f}(t_0) \Big).
\end{aligned}$$

By inspection it can be seen that

$$\overline{Ric}(\partial_i, \partial_{n+1}) = O(|\dot{f}|) + O(|\dot{f}|^2),$$

when $i \leq n$ and $j = n + 1$. Therefore the symmetric $(n + 1) \times (n + 1)$ matrix, consisting of the components of the Ricci tensor, $\overline{Ric}_{i,j}$, $Mat(\overline{Ric}(n+1))$ is the sum of two matrices:

$$Mat(\overline{Ric}(n + 1)) = D_{n+1}(Ric) + A$$

where $D_{n+1}(Ric)$ is a diagonal matrix with diagonal elements, which are the eigenvalues of the Ricci tensor, $D_n(Ric)$, and all other entries are zero and A is an $(n+1) \times (n+1)$ matrix whose non-zero entries consist solely of terms of $O(|\dot{f}|)$, $O(|\dot{f}|^2)$ and $O(|\ddot{f}|)$ evaluated at (x_0, t_0).

The matrix $Mat(\overline{Ric}(n+1))$ is a real, symmetric matrix and it may therefore be diagonalised to give real eigenvalues. This diagonal matrix we denote as $D_{n+1}(\overline{Ric})$ with eigenvalues, $\overline{\lambda}_i$, which are the values of the principal Ricci curvatures of the orthonormal eigenvectors given by the columns of $\overline{P}$.

$$\begin{aligned} D_{n+1}(\overline{Ric}) &= \overline{P}^{-1}\ Mat(\overline{Ric}(n+1))\ \overline{P} \\ &= \overline{P}^{-1}\ D_{n+1}(Ric)\ \overline{P} + \overline{P}^{-1}\ A\ \overline{P}, \end{aligned}$$

We order the eigenvalues, $\overline{\lambda}_i$, of $D_{n+1}(\overline{Ric})$ in increasing size. We order the diagonal entries, λ_i and zero, of $D_{n+1}(Ric)$, in increasing size and denote these values λ_i'. Note that it is now possible that

$$\sum_{i=1}^{k} \lambda_i' \leq 0.$$

However

$$\sum_{i=1}^{k+1} \lambda_i' \geq 0.$$

The eigenvalues $\overline{\lambda}_i$ and λ_i' are related by the following inequality [40]:

$$|\overline{\lambda}_i - \lambda_i'| \leq \|A\|_{op}$$

where $\|A\|_{op}$ is the operator norm of A, being the square root of the maximum value of the eigenvalues of A^2. We use the relation [50]:

$$\|A\|_{op} \leq \|A\|_1 \|A\|_\infty$$

where

$$\|A\|_1 = \max_j \Big\{ \sum_{i=1}^{n+1} |A_{ij}| \Big\}; \qquad \|A\|_\infty = \max_i \Big\{ \sum_{j=1}^{n+1} |A_{ij}| \Big\}.$$

As A is symmetric $\|A\|_1 = \|A\|_\infty$. Let $\|A\|_1 = \|A\|_\infty = b$ where $b > 0$. Then

$$|\overline{\lambda}_i - \lambda_i'| \leq b^2 \text{ and } (n+1)|\overline{\lambda}_i - \lambda_i'| \leq (n+1)b^2.$$

Hence if each $A_{ij} < \frac{1}{n+1}\sqrt{\frac{c}{n+1}}$, the sum of the first $(k+1)$ eigenvalues of $Mat(\overline{Ric}(n+1))$ will be positive. This can be ensured by adjusting Λ as required.

Hence the metric $G = g_{f(t)} + dt^2$ on $X \times \mathbb{R}$ has positive Ricci-$(k+1, n+1)$ curvature. $\square$

The case where $k = n$ has previously been proved [43].

The corollary of this lemma for psc metrics is that isotopic metrics are concordant as proved in [43]. The corollary for Ricci-(k, n) metrics is given below.

Corollary 3.3.1.1. *Ricci-(k, n) metrics which are isotopic for some arbitrary compact n-dimensional manifold, X, are concordant.*

We follow the proof of [43], sometimes verbatim.

Proof. Let g_0 and g_1 be two Ricci-(k, n) metrics in the same path component of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ connected by the path $g_r \subset \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$. This may be approximated by a smooth path in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ on the interval $[0, 1]$. Let f be a smoothly increasing function such that

$$f(s) = \begin{cases} 1 & \text{when } s \geq l_2 \\ 0 & \text{when } s \leq l_1. \end{cases}$$

The restriction of the derivatives $|\dot{f}|$ and $|\ddot{f}|$ to less than Λ as in Lemma 3.3.1 above can be obtained by increasing the difference between l_1 and l_2 as required. Choose A_1 and A_2 so that $0 < A_1 < l_1 < l_2 < A_2 < 1$ so that from Lemma 3.3.1, the metric $g_{f(s)} + ds^2$ on $X \times [A_I, A_2]$ has positive Ricci-$(k + 1, n + 1)$ curvature. This metric can be pulled back to give the desired concordance on $X \times I$, thus showing that isotopic positive Ricci-(k, n) metrics are concordant.

□

Chapter 4

Standard metrics on the sphere and the disc

In this chapter we review some standard metric constructions on the sphere and disc. It is important for our later work to understand for which k these metrics satisfy the $Ric_{(k,n)} > 0$ curvature condition and when this positive curvature can be made arbitrarily large.

We will need to describe metrics on various parts of the standard sphere and disc and hence define the regions as follows. The sphere, $S^n = \{\bar{x} \in \mathbb{R}^{n+1} : |\bar{x}| = 1\}$, where $\bar{x} = (x_1, \cdots, x_{n+1})$, maybe decomposed into two regions, the upper half sphere, $S^n_+ = \{\bar{x} \in \mathbb{R}^{n+1} : |\bar{x}| = 1, x_{n+1} \geq 0\}$ and the lower half sphere, $S^n_- = \{\bar{x} \in \mathbb{R}^{n+1} : |\bar{x}| = 1, x_{n+1} \leq 0\}$. The equator, $S^n_{eq} = S^{n-1} \hookrightarrow S^n$, is $S^n_{eq} = \{\bar{x} \in S^n : x_{n+1} = 0\}$. Note that $S^n_{eq} = S^n_+ \cap S^n_-$.

The disc, $D^n = \{\bar{x} \in \mathbb{R}^n : |\bar{x}| \leq 1\}$ may be decomposed into two regions, the upper half disc, $D^n_+ = \{\bar{x} \in \mathbb{R}^n : |\bar{x}| \leq 1, x_n \geq 0\}$ and the lower half disc, $D^n_- = \{\bar{x} \in \mathbb{R}^n : |\bar{x}| \leq 1, x_n \leq 0\}$. The equator, $D^n_{eq} = D^{n-1} \hookrightarrow D^n$, is $D^n_{eq} = \{\bar{x} \in D^n : x_n = 0\}$.

4.1 Round metric on the sphere

The *standard round metric on the sphere*, $S^n \subset \mathbb{R}^n$, is denoted ds_n^2 and takes the form of a warped producct

$$ds_n^2 = dt^2 + \sin^2(t) ds_{n-1}^2,$$

as the pullback of the Euclidean metric via the embedding

$$\begin{aligned} \varphi : (0,\pi) \times S^{n-1} &\to \mathbb{R}^{n+1} \\ (t,\theta) &\mapsto (\cos(t), \sin(t).\theta). \end{aligned}$$

In these coordinates, the corresponding round metric of radius r is $dt^2 + r^2 \sin^2(\frac{t}{r})ds_{n-1}^2$, the curvatures of which are as follows:

$$\begin{aligned} K &= \frac{1}{r^2}, \\ Ric &= \frac{n-1}{r^2}, \\ Ric_{(k,n)} &= \frac{k(n-1)}{r^2}, \quad 1 \leq k \leq n, \\ scal &= \frac{n(n-1)}{r^2}. \end{aligned}$$

4.2 Torpedo metrics on the disc

We specify a family of functions $\eta = \eta_{\delta,\lambda} : [0, \delta\frac{\pi}{2} + \lambda] \to [0, 1]$, $\delta > 0, \lambda > 0$. To aid the reader we suppress the δ, λ notation. The function η has the following properties:

(i) $\eta(r) = \delta \sin \frac{r}{\delta}$ when r is near 0.

(ii) $\eta(r) = \delta$ when $r \geq \delta\frac{\pi}{2}$.

(iii) $\ddot{\eta}(r) \leq 0$,

(iv) The m^{th} derivative at $\delta\frac{\pi}{2}$, $\eta(r)^{(m)} = 0$ for all $m \geq 1$,

Such functions are easy to construct and are called *torpedo functions* (or more precisely $\delta - \lambda$ *torpedo functions)*. Such a function is illustrated in figure 4.1 (a).

Using a torpedo function, η, an important standard metric on the disc, $D^n(\delta\frac{\pi}{2} + \lambda)$, arises from the embedding:

$$\begin{aligned} torp : (0, \delta\frac{\pi}{2} + \lambda) \times S^{n-1} &\to \mathbb{R} \times \mathbb{R}^n \\ (r, \theta) &\mapsto (\alpha(r), \eta(r).\theta), \end{aligned}$$

where $\alpha = \alpha_{\delta,\lambda}$ and has the following properties:

1. $\alpha(r) = \alpha_0 - \int_0^r \sqrt{1 - \dot{\eta}(u)^2}du$, where $\alpha_0 = \int_0^{\frac{\pi}{2}} \sqrt{1 - \dot{\eta}(u)^2}du$ and
2. $\dot{\alpha}^2 + \dot{\eta}^2 = 1$.

The metric arising from this embedding we call the $\delta - \lambda$ *torpedo metric*, $g_{torp}^n(\delta)_\lambda$, on the disc $D^n(\delta\frac{\pi}{2} + \lambda)$:

$$g_{torp}^n(\delta)_\lambda = dr^2 + \eta_{\delta,\lambda}(r)^2 ds_{n-1}^2.$$

The torpedo metric consists of a round hemispherical cap of radius δ joined smoothly to a cylindrical tube of length λ; figure 4.1 (b).

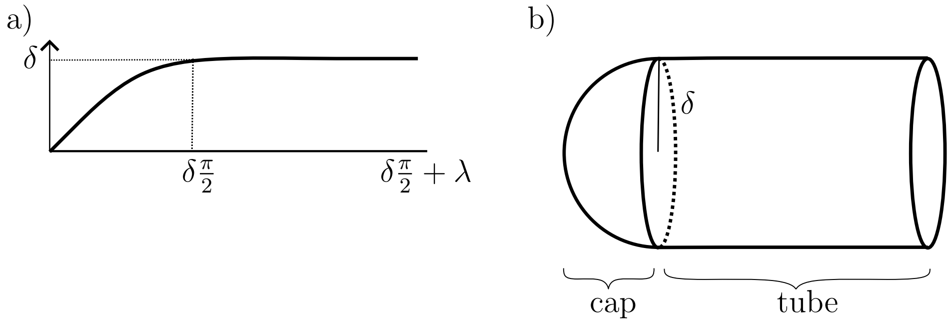

Figure 4.1: a) A $\delta - \lambda$ torpedo function, $\eta_{\delta,\lambda}$; and b) Resulting torpedo metric, $g^n_{tor}(\delta)_\lambda$ on the disc

The torpedo metric is an example of a *warped product metric* on the disc with the torpedo function being an example of a *warping function.* It is easy to show that the torpedo metric has positive scalar curvature when $n \geq 3$; see section 3 of chapter 1 in [44]. We show for which k the torpedo metric has positive Ricci-(k, n) curvature.

Lemma 4.2.1. *Let D^n be equipped with a $\delta - \lambda$ torpedo metric, g^n_{torp}, where $n \geq 3$. Then for $k \geq 2$, and any $\lambda \geq 0$, the Ricci-(k, n) curvature can be made arbitrarily large by choosing $\delta > 0$ sufficiently small.*

Proof. Let $(r, \theta_1, \cdots, \theta_{n-1})$ be standard polar coordinates on D^n so that $(\partial_r, \partial\theta_1, \cdots, \partial\theta_{n-1})$ denote the corresponding coordinate vector fields. Let $g := g^n_{torp} = dr^2 + \eta^2 d\theta^2$, where $d\theta^2$ is the standard round metric on S^{n-1} and $\eta = \eta_{\delta,\lambda}(r)$. Some properties of η are summarised in the table below:

function \ r	0	$(0, \delta\frac{\pi}{2})$	$[\delta\frac{\pi}{2}, \delta\frac{\pi}{2} + \lambda]$
η	0	> 0	δ
$\dot{\eta}$	> 0	> 0	0
$\ddot{\eta}$	0	< 0	0
$\dddot{\eta}$	< 0	< 0	0

From this the Christoffel symbols associated to the Levi-Civita connection arising from this metric connection are:

$$\Gamma^r_{\theta_{n-1},\theta_{n-1}} = -\dot{\eta}\eta \qquad \Gamma^r_{\theta_i,\theta_i} = -\dot{\eta}\eta \sin^2\theta_{i+1} \cdots \sin^2\theta_{n-1}$$

$$\Gamma^{\theta_i}_{r,\theta_i} = \Gamma^{\theta_i}_{\theta_i,r} = \frac{\dot{\eta}}{\eta} \qquad \Gamma^{\theta_i}_{\theta_j,\theta_i} = \Gamma^{\theta_i}_{\theta_i,\theta_j} = \frac{\cos\theta_j}{\sin\theta_j}, \quad i < j.$$

$$\Gamma^{\theta_i+1}_{\theta_i,\theta_i} = -\sin\theta_{i+1}\cos\theta_{i+1} \qquad \Gamma^{\theta_i+j}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1}) \cdots \sin^2(\theta_{i+j-1}).$$

All other $\Gamma^l_{i,j}$ are zero.

Thus sectional curvature, $K_{i,j}$, is

$$K_{r,\theta_i} = -\frac{\ddot{\eta}}{\eta}; \quad K_{\theta_i,\theta_j} = \frac{1-\dot{\eta}^2}{\eta^2}.$$

From this we obtain the Ricci curvature of the torpedo metric, $Ric(v)$, of the coordinate vector fields:

$$Ric(v) = \begin{cases} -(n-1)\frac{\ddot{\eta}}{\eta} & \text{when } v = \partial_r \\ (n-2)\frac{1-\dot{\eta}^2}{\eta^2} - \frac{\ddot{\eta}}{\eta} & \text{when } v = \partial\theta_i, 1 \leq i \leq n-1. \end{cases}$$

The functions involved in the Ricci curvatures, may be summarised:

function \ r	0	$(0, \delta\frac{\pi}{2})$	$[\delta\frac{\pi}{2}, \lambda + \delta\frac{\pi}{2}]$
$\frac{1-\dot{\eta}^2}{\eta^2}$	> 0	$> 0^\star$	> 0
$-\frac{\ddot{\eta}}{\eta}$	> 0	≥ 0	0

* Here the assumption is made that $0 < \dot{\eta} < 1$, for $r \in (0, \delta\frac{\pi}{2})$.

Hence when $r \in [\delta\frac{\pi}{2}, \lambda + \delta\frac{\pi}{2}]$, $Ric(\partial_r) = 0$.

Provided $n \geq 3$ all other $Ric(e_i) > 0$.

Hence the torpedo metric has positive Ricci-$(2, n)$ curvature, provided $n \geq 3$. □

4.3 Double torpedo metric on the sphere

A *double torpedo function*, $\bar{\eta}$, is a smooth function on $[0, b]$ such that:

$$\bar{\eta}(r) = \begin{cases} \eta(r) & \text{when } r \in [0, \frac{b}{2}] \\ \eta(b-r) & \text{when } r \in [\frac{b}{2}, b], \end{cases}$$

where η is a torpedo function with the properties given in section 4.2 and $\frac{b}{2} > \frac{\delta}{2}$; figure 4.2 (a). We can obtain a *double torpedo metric*, $g^n_{Dtorp}(\delta)$, on S^n, defined $g^n_{Dtorp}(\delta) = dr^2 + \bar{\eta}(r)^2 ds^2_{n-1}$, by using a double torpedo function on the domain $[0, b]$, as shown in figure 4.2 (b).

Corollary 4.3.0.1. *Let S^n be equipped with a $\delta - \lambda$ double torpedo metric, g^n_{Dtorp}, where $p + q + 1 = n$ and $n \geq 3$. Then for $k \geq 2$, and any $\lambda \geq 0$, the Ricci-(k, n) curvature can be made arbitrarily large by choosing $\delta > 0$ sufficiently small.*

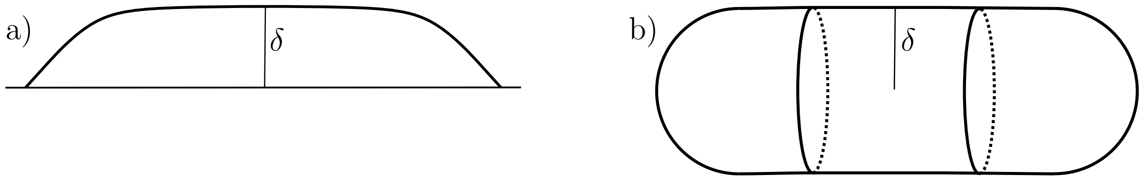

Figure 4.2: a) Double torpedo function, $\bar{\eta}$; and b) Corresponding double torpedo metric, $g^n_{Dtorp}(\delta)$, on S^n.

This is an example of a warped product metric inducing a metric on the sphere. More generally, consider $dt^2 + f(t)^2 ds^2_{n-1}$ as the warped product metric on $(0, b) \times S^{n-1}$ with warping function $f : (0, b) \to \mathbb{R}$. This extends uniquely to a smooth metric on S^n, provided that the properties of the function, f, are those determined by Petersen [32] and are as follows:

Lemma 4.3.1. *(Chapter 1, Section 1.4.1, [32]). If $f : (0, b) \to (0, \infty)$ is smooth and $f(0) = f(b) = 0$, then the warped product metric $dt^2 + f(t)^2 ds^2_{n-1}$ extends uniquely to a smooth metric on S^n if and only if $f^{(even)}(0) = f^{(even)}(b) = 0$, $\dot{f}(0) = 1$ and $\dot{f}(b) = -1$.*

4.4 Boot metric

Extensive use is made of the torpedo metric in defining the so-called *boot metric*, g_{boot}, [48]. This metric is defined on the manifold $(D^n \times I) \cup (S^{n-1} \times \mathcal{Q}) \cup (D^{n+1}_+)$, where $\mathcal{Q}$ is a piece of 2-dimensional Euclidean space. This is essentially a disc with corners. To aid the reader we give a schematic diagram of the boot metric in figure 4.3, where $R_1, \cdots, R_5$ are regions of the manifold satisfying $(D^n \times I) \cong R_1 \cup R_2 \cup R_3$, $S^{n-1} \times \mathcal{Q} \cong R_4$ and $D^{n+1}_+ = R^5$. These regions will be specified more precisely in due course. For now we will give a summary description before establishing that this metric satisfies relevant $Ric_{k,n} > 0$ curvature conditions. Further details may be found in [48]. Consider the embedding, *cyl*:

$$\begin{aligned} cyl : (0, b) \times S^{n-1} \times [0, L + 2] &\to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ (r, \theta, t) &\mapsto (\alpha(r), \beta(r).\theta, t). \end{aligned}$$

This induces a cylinder metric $g^n_{Dtorp} + dt^2 = dr^2 + \beta(r)^2 ds^2_{n-1} + dt^2$ on $S^n \times [0, L+2]$, where $\alpha : [0, b] \to [0, \infty)$ and $\beta : [0, b] \to [0, \infty)$ are smooth functions that satisfy the conditions given as follows:

(i) $\beta(r) > 0$, for all $r \in (0, b)$;
(ii) $\beta(0) = 0, \quad \dot{\beta}(0) = 1, \quad \beta^{(even)}(0) = 0$;

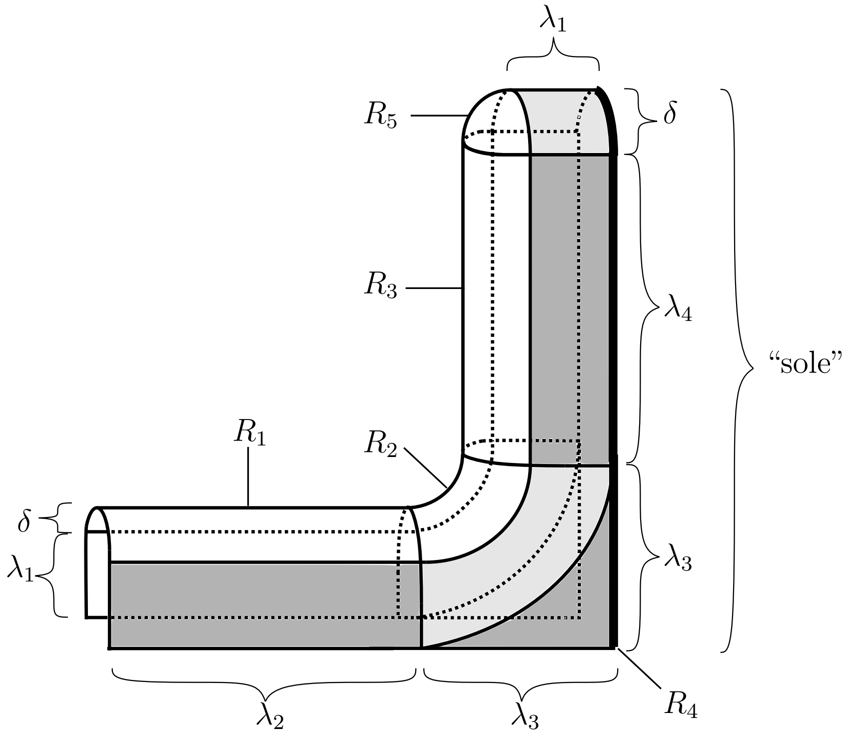

Figure 4.3: The boot metric.

(iii) $\beta(b) = 0.\quad \dot{\beta}(b) = -1, \quad \beta^{(even)}(b) = 0;$

(iv) $\ddot{\beta} \leq 0$ and $\dddot{\beta}(0) < 0;$

(v) When r is near but not at 0, $\ddot{\beta}(r) < 0;$

(vi) $\dddot{\beta}(b) > 0,$ while $\ddot{\beta}(r) < 0$ when r is near but not at b.

Moreover

$$\alpha(r) = \alpha_0 - \int_0^r \sqrt{1 - \dot{\beta}(u)^2} du, \text{ where } \alpha_0 = \int_0^{\frac{b}{2}} \sqrt{1 - \dot{\beta}(u)^2} du.$$

The functions α and β behave like cosine and sine functions respectively near $r = 0$. Moreover, α is determined completely by β so as to satisfy $\dot{\alpha}^2 + \dot{\beta}^2 = 1$.

In the case where β is the torpedo function, $\overline{\eta}_{\delta,\lambda}$, this embedding, cyl, by restriction to $(0, \frac{b}{2}) \times S^{n-1} \times [0, L+2]$, induces a cylinder metric, $g^n_{torp} + dt^2$, on $D^n \times [0, L+2]$. Effectively the image of this restricted embedding is the cylinder depicted in figure 4.4.

We specify collars, $D^n \times [0, 1]$ and $D^n \times [L+1, L+2]$. Near the boundary of the collars the metric is a product metric, $g^n_{torp} + dt^2$, and no adjustment is made to the metric on the collars. We wish to adjust the metric on the region $D^n \times [1, L+1]$ by bending it around an angle of $\frac{\pi}{2}$ in the $(x_0 - x_{n+1})$-plane while maintaining a positive Ricci-$(k, n+1)$ curvature for some k. In order to do so we will need to bend round a wide circular arc. We consider a family of embeddings parameterised by γ, $\gamma \in [0, \frac{\pi}{2}]$:

$$\begin{aligned} \mathrm{cyl}_{\beta,\gamma,\Lambda} : (0, b) \times S^{n-1} \times [0, \gamma] &\to \mathbb{R} \times \mathbb{R}^n \times [0, \infty] \\ (r, \theta, t) &\mapsto (\Lambda + \alpha(r), \beta(r).\theta, t), \end{aligned}$$

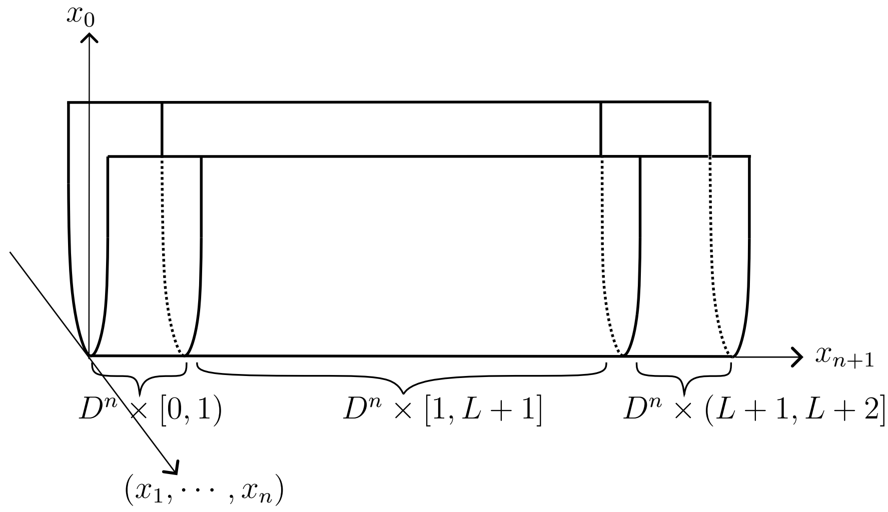

Figure 4.4: Cylinder of torpedo metrics on $D^n \times [0, L+2]$

where $\Lambda > |\alpha(b)|$ is a (potentially large) constant. The map, $cyl_{\beta,\gamma,\Lambda}$, ensures that the image of the embedding lies inside $(0,\infty) \times \mathbb{R}^n \times [0,\infty)$.

In order to bend it we compose this embedding with a map:

$$\begin{aligned} \text{bend} : (0,\infty) \times \mathbb{R}^n \times [0,\infty) &\to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ (x_0, (x_1, \cdots, x_n), x_{n+1}) &\mapsto (x_0 \cos(x_{n+1}), (x_1, \cdots, x_n), x_0 \sin(x_{n+1})), \end{aligned}$$

This bends the image of $\text{cyl}_{\beta,\gamma,\Lambda}$ in the $(x_0 - x_{n+1})$-plane by an angle of γ; see figure 4.5 with $\gamma = \frac{\pi}{2}$. Provided Λ is large enough, we show in the proof of Lemma 4.4.1 that this maintains a positive Ricci-$(k, n+1)$ curvature for some k.

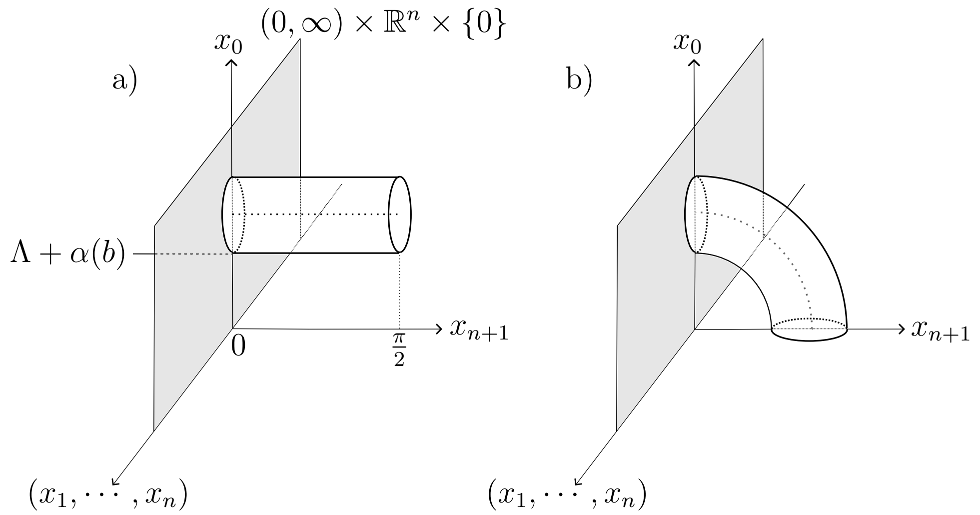

Figure 4.5: a) The image of the map $\text{cyl}_{\beta,\gamma,\Lambda}$; and b) The image of the composition map bend $\circ$ $\text{cyl}_{\beta,\gamma,\Lambda}$, $\gamma = \frac{\pi}{2}$.

Figure 4.6 shows the image of the composition map, bend $\circ$ $\text{cyl}_{\beta,\gamma,\Lambda}$, where $\gamma = \frac{\pi}{2}$, where the domain is restricted to $(0,b) \times D^{n-1} \times [0, \frac{\pi}{2}]$.

Let $g^{n+1}_{\beta,\gamma,\Lambda} := (\text{bend} \circ \text{cyl}_{\beta,\gamma,\Lambda})^\star g^{n+2}_{\text{Euc}}$ be the pull back of the Euclidean metric on

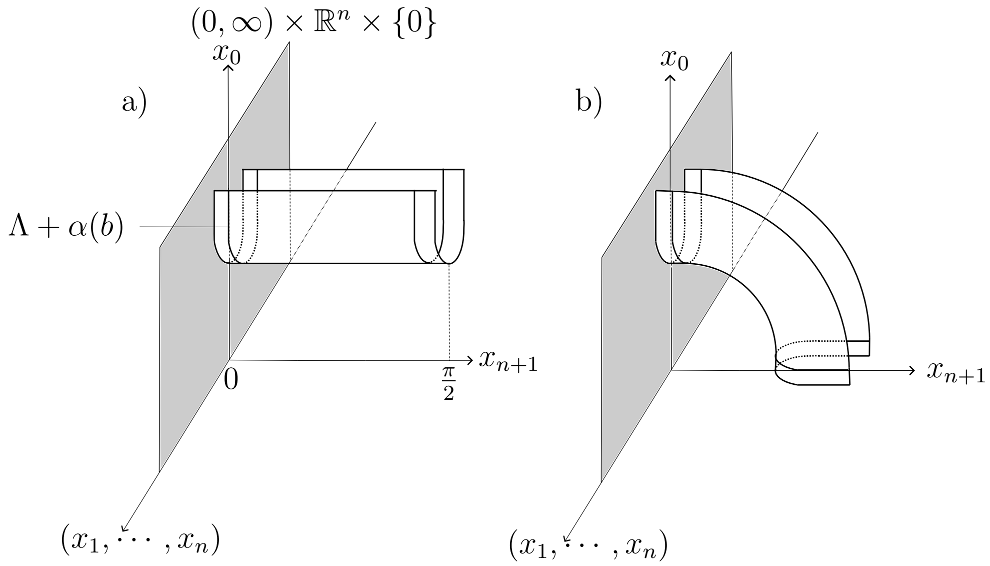

Figure 4.6: a) The image of the map $\mathrm{cyl}_{\beta,\gamma,\Lambda}$ on $D^n\times I$; and b) The image of the composition map bend $\circ$ $\mathrm{cyl}_{\beta,\gamma,\Lambda}$ on $D^n\times I$.

$\mathbb{R}^{n+2}$. A straightforward calculation (in [48]) yields

$$g^{n+1}_{\beta,\gamma,\Lambda} = dr^2 + \beta(r)^2 ds^2_{n-1} + (\Lambda+\alpha(r))^2 dt^2.$$

In Lemma 5.3 of [48] the scalar curvature of this metric, $s_{\eta,\gamma,\Lambda}$, is calculated to be

$$s_{\beta,\gamma,\Lambda} = (n-1)(n-2)\left[\frac{1-\dot{\beta}^2}{\beta^2}\right] - 2(n-1)\frac{\ddot{\beta}}{\beta} - \frac{2(n+1)}{\beta}\left[\frac{\dot{\alpha}\dot{\beta}}{\Lambda+\alpha}\right] - 2\frac{\ddot{\alpha}}{\Lambda+\alpha}.$$

The first and second terms of $s_{\beta,\gamma,\Lambda}$ are positive while the third and fourth terms have as denominator $\Lambda+\alpha(r)$. Provided Λ is large enough, the contributions of the third and fourth terms can be made arbitrarily small and hence this metric can be shown to have positive scalar curvature. Moreover in [48] it was shown that the map $\gamma \to g^{n+1}_{\beta,\gamma,\Lambda}$, where $\gamma\in[0,\frac{\pi}{2}]$, is an isotopy through metrics of positive scalar curvature. We will shortly calculate the Ricci curvature of the metric $g^{n+1}_{\beta,\gamma,\Lambda}$ in order to find the conditions for which the metric has positive Ricci-(k,n) curvature; see section 4.4.2.

Using this metric and other standard metrics, Walsh [48] constructed the boot metric as a union of the following components; figure 4.7:

(i) $(R_1 = D^n\times[0,\lambda_2], g^n_{torp}(\delta)_{\lambda_1}+dt^2)$: The product of an n-dimensional disc with an interval of length λ_2 together with a metric $g^n_{torp}(\delta)_{\lambda_1}+dt^2$, where $t\in[0,\lambda_2]$. As before δ refers to the radius of the torpedo with $\delta+\lambda_1$ being the length of the torpedo.

(ii) $(R_2 = D^n\times[0,\frac{\pi}{2}\lambda_3], g^{n+1}_{\beta,\gamma,\Lambda})$: The product of an n-dimensional disc with an interval with the metric $g^{n+1}_{\beta,\gamma,\Lambda}$ obtained by "bending" the cylinder metric $g^n_{torp}(\delta)_{\lambda_1}+dt^2$ around $\frac{\pi}{2}$.

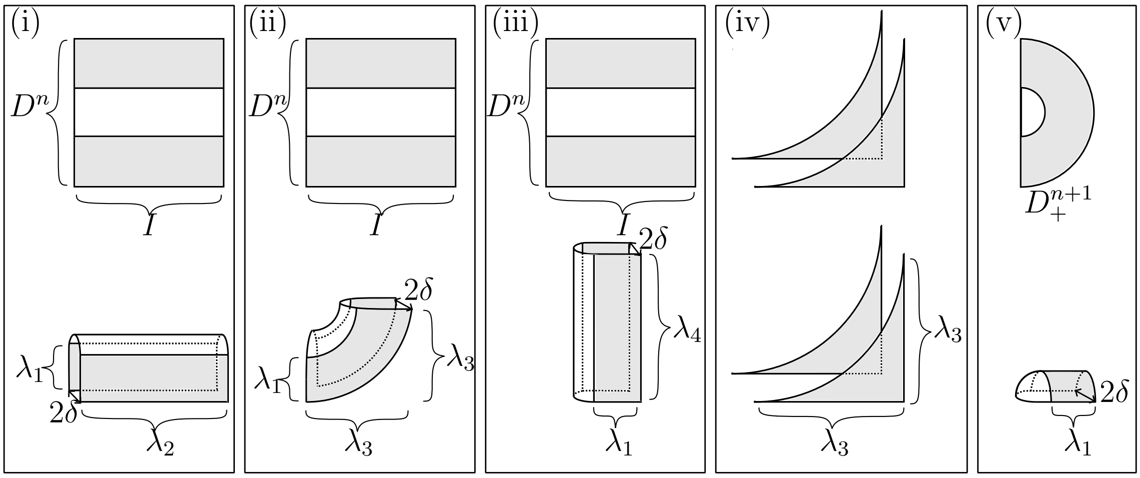

Figure 4.7: Regions of the boot metric: (i) $(R_1, g^n_{torp}(\delta)_{\lambda_1} + dt^2)$; (ii) $(R_2, g^{n+1}_{\beta,\gamma,\Lambda})$; (iii) $(R_3, g^n_{torp}(\delta)_{\lambda_1} + dr^2)$; (iv) $(R_4, \delta^2 ds^2_{n-1} + dr^2 + dt^2)$; and (v) $(R_5, g^{n+1}_{torp}(\delta)_{\lambda_1})$.

(iii) $(R_3 = D^n \times [0, \lambda_4], g^n_{torp}(\delta)_{\lambda_1} + dr^2)$: The product of an n-dimensional disc with an interval with the metric $g^n_{torp}(\delta)_{\lambda_1} + dr^2$, $r \in [0, \lambda_4]$. This is glued to the $(n+1)$-dimensional hemidisc described in (v) below.

(iv) $(R_4 = S^{n-1} \times Q(\lambda_3), \delta^2 ds^2_{n-1} + dr^2 + dt^2)$: This is the product of S^{n-1} with a piece of 2 dimensional Euclidean space, $Q(\lambda_3)$, which gives the "heel" of the boot metric. More precisely, the 2-dimensional piece of Euclidean space, $Q(\lambda_3)$, is a square of side λ_3, without a quarter 2-dimensional disc with radius λ_3.

(v) $(R_5 = D^{n+1}_+, g^{n+1}_{torp}(\delta)_{\lambda_1})$: D^{n+1}_+ which is defined as an $(n+1)$-dimensional hemidisc together with a metric $g^{n+1}_{torp}(\delta)_{\lambda_1}$ restricted to D^{n+1}_+.

The metrics given in (i) to (v) may be glued together smoothly and the resulting boot metric is shown in figure 4.3.

We denote the choice of parameters for a boot metric as the vector $\overline{\lambda} = (\lambda_1, \cdots, \lambda_4)$ and denote the boot metric on D^{n+1} as $g^{n+1}_{boot}(\delta)_{\Lambda,\overline{\lambda}}$. Note that the boot metric provides a collar neighbourhood with a product metric of width λ_1 on the "sole" of the boot, shown on the right hand side of figure 4.3.

Lemma 4.4.1. *Let $(D^n \times I) \cup (S^{n-1} \times \mathcal{Q}) \cup (D^{n+1}_+)$ be equipped with a boot metric, $g^{n+1}_{boot}(\delta)_{\Lambda,\overline{\lambda}}$, where $p+q+2 = n+1$, $q \geq 2$ and Λ is large. Then for $p \neq 1$, the metric has positive Ricci-$(k, n+1)$ curvature for $k \geq 3$. For $p = 1$, the metric has positive Ricci-$(k, n+1)$ curvature when $k \geq 4$.*

Proof. We calculate in the next five sections for which k, the metrics on the five regions of the boot have positive Ricci-$(k, n+1)$ curvature.

4.4.1 Curvature of the metric $g^n_{torp}(\delta)_{\lambda_1} + dt^2$ on Region 1 of the boot.

We here calculate for which k, the metric on R_1 of the boot, has positive Ricci-$(k+1, n+1)$ curvature. The metric on R_1 is

$$g^n_{torp}(\delta)_{\lambda_1} + dt^2 = \eta_{\delta,\lambda_1}(r)^2 ds^2_{n-1} + dr^2 + dt^2.$$

The relevant properties of the torpedo function, η_{δ,λ_1}, may be summarised as follows:

function \ r	0	$(0, \delta\frac{\pi}{2})$	$[\delta\frac{\pi}{2}, \lambda + \delta\frac{\pi}{2}]$
$\eta_{\delta,\lambda_1}(r)$	0	> 0	δ
$\dot{\eta}_{\delta,\lambda_1}(r)$	1	> 0	0
$\ddot{\eta}_{\delta,\lambda_1}(r)$	0	≤ 0	≤ 0

It should be noted that the function, η_{δ,λ_1}, behaves like a sine function near 0. In this subsection let $g := g^n_{tor}(\delta)_{\lambda_1} + dt^2$ and $\eta := \eta_{\delta,\lambda_1}(r)$. Let the region, $R_1 = D^n \times [0, \lambda_2]$ have coordinates $(r, \theta_1, \cdots, \theta_{n-1}, t)$, where $(r, \theta_1, \cdots, \theta_{n-1})$ are polar coordinates on D^n. Let $(\partial_r, \partial\theta_1 \cdots, \partial\theta_{n-1}, \partial_t)$ denote the corresponding coordinate vector fields. Then the metric on R_1 is

$$g = dt^2 + dr^2 + \eta^2 d\theta^2.$$

Where $i, j \in \{1, \cdots, n-1\}$, $i \neq j$, Christoffel symbols are calculated as:

$$\overline{\Gamma}^r_{\theta_{n-1},\theta_{n-1}} = -\dot{\eta}\eta \qquad \overline{\Gamma}^r_{\theta_i,\theta_i} = -\dot{\eta}\eta \sin^2(\theta_{i+1}) \cdots \sin^2(\theta_{n-1})$$

$$\overline{\Gamma}^{\theta_i}_{r,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,r} = \frac{\dot{\eta}}{\eta} \qquad \overline{\Gamma}^{\theta_i}_{\theta_i,\theta_j} = \overline{\Gamma}^{\theta_i}_{\theta_j,\theta_i} = \frac{\cos\theta_j}{\sin\theta_j}, i < j$$

$$\overline{\Gamma}^{\theta_{i+1}}_{\theta_i,\theta_i} = -\sin\theta_{i+1}\cos\theta_{i+1} \qquad \overline{\Gamma}^{\theta_{i+j}}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1}) \cdots sin^2(\theta_{i+j-1}).$$

All other $\overline{\Gamma}^l_{i,j}$ are zero.

Relevant sectional curvatures, $\overline{K}_{i,j}$, are calculated as:

$$\overline{K}_{i,j} = \frac{g(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\partial_j - \overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_i}\partial_j, \partial_i)}{g_{i,i}g_{j,j} - g^2_{i,j}},$$

giving

$$\overline{K}_{r,\theta_i} = -\frac{\ddot{\eta}}{\eta}; \quad \overline{K}_{\theta_i,\theta_j} = \frac{1-\dot{\eta}^2}{\eta^2}; \quad \overline{K}_{t,.} = \overline{K}_{.,t} = 0.$$

Hence the Ricci curvature, $\overline{Ric}(v)$, of the coordinate vector fields are as follows:

$$\overline{Ric}(v) = \begin{cases} 0 & \text{when } v = \partial_t \\ -(n-1)\frac{\ddot{\eta}}{\eta} & \text{when } v = \partial_r \\ (n-2)\frac{1-\dot{\eta}^2}{\eta^2} - \frac{\ddot{\eta}}{\eta} & \text{when } v = \partial\theta_i, 1 \leq i \leq n-1. \end{cases}$$

The functions involved in the Ricci curvatures, may be summarised:

function \ t	0	$(0, \delta\frac{\pi}{2})$	$[\delta\frac{\pi}{2}, \lambda_1 + \delta\frac{\pi}{2}]$
$\frac{1-\dot{\eta}^2}{\eta^2}$	>0	$> 0^\star$	> 0
$-\frac{\ddot{\eta}}{\eta}$	> 0	≥ 0	≥ 0

* Here the assumption is made that $0 < \dot{\eta}^2 < 1$.

Both $\overline{Ric}(\partial_r)$ and $\overline{Ric}(\partial_t)$ are always non-negative (though not necessarily positive) through the interval $(0, \lambda_1 + \delta\frac{\pi}{2}]$. All $\overline{Ric}(\partial\theta_i)$ are positive for all $t \in [0, \lambda_1 + \delta\frac{\pi}{2}]$, provided $n \geq 3$. Given these restrictions, Ricci-$(3, n+1)$ curvature is computed as

$$\begin{aligned} Ric_{(3,n+1)} &= \overline{Ric}(\partial_r) + \overline{Ric}(\partial_t) + \overline{Ric}(\partial\theta_i) \\ &= -(n-1)\frac{\ddot{\eta}}{\eta} + 0 + (n-2)\frac{1-\dot{\eta}^2}{\eta^2} - \frac{\ddot{\eta}}{\eta} \\ &= (n-2)\frac{1-\dot{\eta}^2}{\eta^2} - n\frac{\ddot{\eta}}{\eta}. \end{aligned}$$

Assuming $n > 2$ and since $\dot{\eta}^2 < 1$, the first summand is positive. The right hand summand is non-negative since $\ddot{\eta}$ is non-positive. Hence on $(0, \lambda_2]$, the metric, $g^n_{torp}(\delta)_{\lambda_1} + dt^2$ has positive Ricci-$(3, n+1)$ curvature, provided $n \geq 3$.

4.4.2 Curvature of the metric, $g^{n+1}_{\beta,\gamma,\lambda}$, on Region 2 of the boot.

We calculate below the Ricci-$(k, n+1)$ curvature of the metric on R_2,

$$g^{n+1}_{\beta,\gamma,\Lambda} = (\Lambda + \alpha(r))^2 dt^2 + dr^2 + \beta(r)^2 ds^2_{n-1}.$$

Let $\Lambda + \alpha(r) = \delta(r)$. Let $R_2 = D^n \times [0, \frac{\pi}{2}\lambda_3]$, have coordinates $(r, \theta_1, \cdots, \theta_{n-1}, t)$, where $(r, \theta_1, \cdots, \theta_{n-1})$ are polar coordinates on D^n. Let $(\partial_r, \partial\theta_1 \cdots, \partial\theta_{n-1}, \partial_t)$ denote the corresponding coordinate vector fields. Then the metric on R_2 is

$$g^{n+1}_{\beta,\gamma,\Lambda} = (\delta(r))^2 dt^2 + dr^2 + \beta(r)^2 d\theta^2.$$

In the case $l = m = r$, the Christoffel symbols are calculated as:

$$\overline{\Gamma}^r_{\theta_{n-1},\theta_{n-1}} = -\dot{\beta}\beta \qquad \overline{\Gamma}^r_{\theta_i,\theta_i} = -\dot{\beta}\beta\sin^2\theta_{i+1}\cdots\sin^2\theta_q \qquad \overline{\Gamma}^r_{t,t} = -\dot{\delta}\delta.$$

All other $\overline{\Gamma}^r_{i,j}$ equal zero.

Let $l = m = \theta_i$.

$$\overline{\Gamma}^{\theta_i}_{r,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,r} = \frac{\dot{\beta}}{\beta}; \qquad \overline{\Gamma}^{\theta_i}_{\theta_j,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,\theta_j} = \frac{\cos\theta_j}{\sin\theta_j},$$

$$\Gamma^{\theta_{i+1}}_{\theta_i,\theta_i} = -\sin\theta_{i+1}\cos\theta_{i+1} \qquad \Gamma^{\theta_{i+j}}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1})\cdots sin^2(\theta_{i+j-1}).$$

Let $l = m = t$.

$$\overline{\Gamma}^t_{r,t} = \overline{\Gamma}^t_{t,r} = \frac{\dot{\delta}}{\delta}.$$

All other $\overline{\Gamma}^l_{i,j}$ equal zero.

Relevant sectional curvatures are calculated as:

$$\overline{K}_{r,\theta_i} = -\frac{\ddot{\beta}}{\beta}; \qquad \overline{K}_{r,t} = -\frac{\ddot{\delta}}{\delta};$$

$$\overline{K}_{\theta_i,\theta_j} = \frac{1-\dot{\beta}^2}{\beta^2}; \qquad \overline{K}_{\theta_i,t} = -\frac{\dot{\beta}}{\beta}\frac{\dot{\delta}}{\delta}.$$

Recalling $\delta(r) = \Lambda + \alpha(r)$, the Ricci curvature, $\overline{Ric}(v)$, of the coordinate vector fields are as follows:

$$\overline{Ric}(v) = \begin{cases} -(n-1)\frac{\ddot{\beta}}{\beta} - \frac{\ddot{\alpha}}{\Lambda+\alpha} & \text{when } v = \partial_r \\ (n-2)\frac{1-\dot{\beta}^2}{\beta^2} - \frac{\ddot{\beta}}{\beta} - \frac{\dot{\alpha}\dot{\beta}}{\beta(\Lambda+\alpha)} & \text{when } v = \partial\theta_i,\ 1 \le i \le n-1 \\ -(n-1)\frac{\dot{\alpha}\dot{\beta}}{\beta(\Lambda+\alpha)} - \frac{\ddot{\alpha}}{\Lambda+\alpha} & \text{when } v = \partial_t \end{cases}$$

Provided Λ is positive and large enough, this may be summarised as follows:

function \ r	0	$(0,\varepsilon]$	$(\varepsilon, b-\varepsilon')$	$[b-\varepsilon', b)$	b
β	0	>0	>0	>0	0
$\dot{\beta}$	1	>0	$\lvert\dot{\beta}\rvert \geq 0$	<0	-1
$\ddot{\beta}$	0	<0	<0	<0	0
$\Lambda+\alpha$	>0	>0	>0	>0	>0
$\dot{\alpha}$	0	<0	<0	<0	0
$\ddot{\alpha}$	-1	<0	$\lvert\ddot{\alpha}\rvert \geq 0$	>0	1

Hence the functions involved in the Ricci curvatures, may be summarised:

function \ r	0	$(0,\varepsilon]$	$(\varepsilon, b-\varepsilon')$	$[b-\varepsilon', b)$	b
$\frac{1-\dot{\beta}^2}{\beta^2}$	>0	>0	>0	>0	>0
$-\frac{\ddot{\beta}}{\beta}$	>0	>0	>0	>0	>0
$-\frac{\ddot{\alpha}}{\Lambda+\alpha}$	>0	>0	$\lvert -\frac{\ddot{\alpha}}{\Lambda+\alpha}\rvert \geq 0$	<0	<0
$-\frac{\dot{\alpha}\dot{\beta}}{\beta(\Lambda+\alpha)}$	>0	>0	$\lvert -\frac{\dot{\alpha}\dot{\beta}}{\beta(\Lambda+\alpha)}\rvert \geq 0$	<0	<0

Therefore the Ricci curvatures, apart from $\overline{Ric}(\psi_i)$, are as follows:

Ric(e_i) \ r	0	$(0,\varepsilon]$	$(\varepsilon, b-\varepsilon')$	$[b-\varepsilon', b)$	b
$\overline{Ric}(\partial_r)$	>0	>0	*	*	*
$\overline{Ric}(\partial\theta_i)$	>0	>0	*	*	*
$\overline{Ric}(\partial_t)$	>0	>0	$\lvert\overline{Ric}(\partial_t)\rvert \geq 0$	<0	<0

*Can be made positive provided Λ is large enough.

We do not have positive Ricci-$(1, n+1)$ curvature as $\overline{Ric}(\partial_t)$ is not positive on all of the interval $r \in [0, b]$. Provided $p \geq 1$ then $\overline{Ric}(\partial\theta_i)$ is positive and provided Λ is big enough, $\overline{Ric}(\partial_r)$ is positive. Given these restrictions, Ricci-$(2, n+1)$ is computed as the lower of $\overline{Ric}(\partial_t) + \overline{Ric}(\partial_r)$ and $\overline{Ric}(\partial_t) + \overline{Ric}(\partial\theta_i)$, both of which are positive. Hence the metric $g^{n+1}_{\beta,\gamma,\Lambda}$ has positive Ricci-$(2, n+1)$ curvature when $n+1 > 3$. When $n+1 = 3$, the metric has positive $Ricci - (3,3)$ curvature, i.e. positive scalar curvature.

4.4.3 Curvature of the metric, $g^n_{torp}(\delta)_{\lambda_1} + dr^2$, on Region 3 of the boot.

We refer back to subsection 4.4.1 where we showed that the metric $g^n_{torp}(\delta)_{\lambda_1} + dt^2$ on Region 1 has Ricci-$(3, n+1)$ positive curvature, provided $n \geq 3$.

4.4.4 Curvature of the metric, $dt^2 + dr^2 + \delta^2 ds^2_{n-1}$, on Region 4 of the boot.

Let R_4, which recall is homeomorphic to $Q(\lambda_3) \times S^{n-1}$, have coordinates $(t, r, \theta_1, \cdots, \theta_{n-1})$ such that $(\partial_t, \partial_r, \partial_{\theta_1} \cdots, \partial_{\theta_{n-1}})$ are the corresponding coordinate vector fields. The metric with these coordinates is

$$g := dt^2 + dr^2 + \delta^2 ds^2_{n-1},$$

where $\delta > 0$.

From this the Christoffel symbols are:

$$\overline{\Gamma}^{\theta_{i+1}}_{\theta_i,\theta_i} = -\sin\theta_{i+1}\cos\theta_{i+1} \qquad \overline{\Gamma}^{\theta_{i+j}}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1})\cdots sin^2(\theta_{i+j-1})$$

$$\overline{\Gamma}^{\theta_i}_{\theta_j,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,\theta_j} = \frac{\cos(\theta_j)}{\sin(\theta_j)}, i < j.$$

All other $\overline{\Gamma}^{l}_{i,j}$ are zero.

Thus the sectional curvature, $\overline{K}_{\theta_i,\theta_j} = \frac{1}{\delta^2}$. All other $\overline{K}_{i,j}$ are zero. Hence the Ricci curvature, $\overline{Ric}(v)$ is as follows:

$$\overline{Ric}(v) = \begin{cases} \frac{n-2}{\delta^2} & \text{when } v = \partial\theta_i,\ 1 \leq i \leq n-1 \\ 0 & \text{when } v = \partial_r \text{ or when } v = \partial_t. \end{cases}$$

From this we can see that provided $n \geq 3$, then the metric $\delta^2 ds^2_{n-1} + dr^2 + dt^2$ is Ricci-$(3, n+1)$ positive.

4.4.5 Curvature of the metric, $g^{n+1}_{torp}(\delta)_{\lambda_1}$, on Region 5 of the boot.

For completeness we calculate the conditions for positive Ricci-$(k, n+1)$ curvature of the metric on R_5

$$g^{n+1}_{torp}(\delta)_{\lambda_1} = dt^2 + \eta_{\delta,\lambda_1}(t)^2 ds^2_n.$$

The properties of the torpedo function are given in section 4.2.

Let $R_5 = D^{n+1}_{+}$ have polar coordinates $(t, \theta_1, \cdots, \theta_n)$, such that $(\partial_t, \partial\theta_1, \cdots, \partial_{\theta n})$ denote the corresponding coordinate vector fields. The metric on R_5, may be stated as

$$g = dt^2 + \eta^2 d\theta^2.$$

Christoffel symbols are calculated as:

$$\overline{\Gamma}^{t}_{\theta_n,\theta_n} = -\dot{\eta}\eta \qquad \overline{\Gamma}^{t}_{\theta_i,\theta_i} = -\dot{\eta}\eta\sin^2\theta_{i+1}\cdots\sin^2\theta_n \qquad \overline{\Gamma}^{\theta_i}_{t,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,t} = \frac{\dot{\eta}}{\eta},$$

$$\overline{\Gamma}^{\theta_{i+1}}_{\theta_i,\theta_i} = -\sin\theta_{i+1}\cos\theta_{i+1} \qquad \overline{\Gamma}^{\theta_{i+j}}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1})\cdots sin^2(\theta_{i+j-1}),$$

$$\overline{\Gamma}^{\theta_i}_{\theta_j,\theta_i} = \overline{\Gamma}^{\theta_i}_{\theta_i,\theta_j} = \frac{\cos\theta_j}{\sin\theta_j}, i < j.$$

All other $\overline{\Gamma}^{l}_{i,j}$ are zero.

Sectional curvature, $\overline{K}_{i,j}$ is calculated to give:

$$\overline{K}_{t,\theta_i} = -\frac{\ddot{\eta}}{\eta}; \quad \overline{K}_{\theta_i,\theta_j} = \frac{1-\dot{\eta}^2}{\eta^2}.$$

Hence the Ricci curvature, $\overline{Ric}(v)$, of the coordinate vector fields are as follows:

$$\overline{Ric}(v) = \begin{cases} -n\frac{\ddot{\eta}}{\eta} & \text{when } v = \partial_t \\ (n-1)\frac{1-\dot{\eta}^2}{\eta^2} - \frac{\ddot{\eta}}{\eta} & \text{when } v = \partial\theta_i,\ 1 \leq i \leq n. \end{cases}$$

The functions involved in the Ricci curvatures were summarised in section 4.2.

$\overline{Ric}(\partial_t)$ is not positive for all $t \in [0, \lambda_1 + \delta\frac{\pi}{2}]$, and hence the torpedo metric is not Ricci-$(1, n+1)$ positive. However $\overline{Ric}(\partial\theta_i)$ has positive curvature, provided $n \geq 2$. Hence the metric, $g^{n+1}_{torp}(\delta)_{\lambda_1}$, has positive Ricci-$(2, n+1)$ curvature, provided $n \geq 2$.

4.4.6 Summary of the positive Ricci-$(k, n+1)$ curvature conditions of the boot metric.

We give in the table below the value of k required for the metric on the regions of the boot to have positive Ricci-$(k, n+1)$ curvature.

Region	Metric	k	n
Region 1	$g^{n}_{torp}(\delta)_{\lambda_1} + dt^2$	3	≥ 3
Region 2	$g^{n+1}_{\beta,\gamma,\lambda}$	3	≥ 3
Region 3	$g^{n}_{torp}(\delta)_{\lambda_1} + dr^2$	3	≥ 3
Region 4	$\delta^2 ds^2_{n-1} + dr^2 + dt^2$	3	≥ 3
Region 5	$g^{n+1}_{torp}(\delta)_{\lambda_1}$	2	≥ 2

Therefore the boot metric has positive Ricci-$(3, n+1)$ curvature when $n \geq 3$ and Λ is large enough. □

4.5 p, q-decomposition of S^n

Consider the embedding

$$\sigma : \left(0, \frac{\pi}{2}\right) \times S^p \times S^q \to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$$
$$(t, \Psi, \Theta) \mapsto (\cos t.\Psi, \sin t.\Theta)$$

where $p+q+1 = n$. The image of σ is (almost all of) the standard sphere $S^n \subset \mathbb{R}^{n+1}$. By restricting t to different subintervals of $(0, \frac{\pi}{2})$ we can decompose S^n into various subsets.

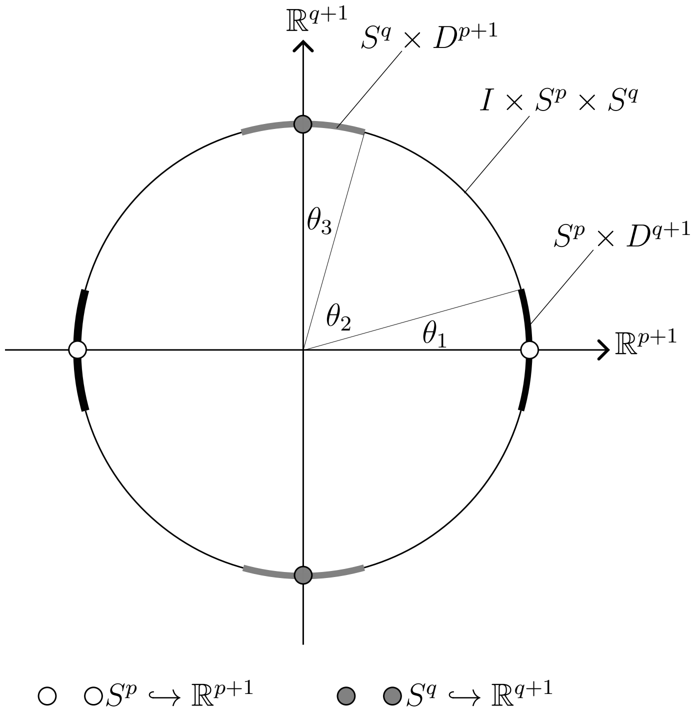

Figure 4.8: p, q decomposition of S^n.

The decomposition gives three regions:

$$R_1 = \text{Image } \sigma|_{t\in(0,\theta_1)} \cong S^p \times D^{q+1}$$
$$R_2 = \text{Image } \sigma|_{t\in[\theta_1,\theta_1+\theta_2]} \cong I \times S^p \times S^q$$
$$R_3 = \text{Image } \sigma|_{t\in(\theta_1+\theta_2,\frac{\pi}{2})} \cong D^{p+1} \times S^q$$

where $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2}$ and $\theta_1, \theta_2, \theta_3 > 0$. Thus the embedding, σ, gives a decomposition of S^n as:

$$S^n \cong (S^p \times D^{q+1}) \cup (I \times S^p \times S^q) \cup (S^q \times D^{p+1}).$$

Letting $\theta_2 = 0$ the decomposition becomes:

$$S^n \cong (S^p \times D^{q+1}) \cup_{S^p \times S^q} (S^q \times D^{p+1}).$$

The embedding, σ, induces the standard round metric, $dt^2+\cos^2(t)d\Psi^2+\sin^2(t)d\Theta^2$.

Recall the regions of the sphere and disc defined at the start of chapter 4. We now define the upper half of the n-sphere, with p,q-decomposition, with respect to the q-sphere, which we denote as $S^n_{q^+}$.

$$S^n_{q^+} = \text{Image } \sigma\big((0,\frac{\pi}{2}) \times S^p \times S^q_+\big) \cong (S^p \times D^{q+1}_+) \cup (D^{p+1} \times S^q_+).$$

Similarly, the lower half of the sphere, with p,q-decomposition, with respect to the q-sphere, which we denote as $S^n_{q^-}$ decomposes as:

$$S^n_{q^-} \cong (S^p \times D^{q+1}_-) \cup (D^{p+1} \times S^q_-).$$

The equator of the sphere, with p,q-decomposition, with respect to the q-sphere, which we denote as S^n_{eq} decomposes as:

$$S^n_{q,eq} = S^n_{q^+} \cap S^n_{q^-} \cong (S^p \times D^{q+1}_{eq}) \cup (D^{p+1} \times S^q_{eq}).$$

Note that $S^n_{q^+}$, $S^n_{q^-}$ and $S^n_{q,eq}$ are diffeomorphic to S^n_+, S^n_- and S^n_{eq} respectively.

In the p,q-decomposition, the embedding σ used the sine and cosine functions. These functions may be generalised to two warping functions $u,v : (0,b) \to (0,\infty)$ to give the metric $dt^2+u(t)^2ds_p^2+v(t)^2ds_q^2$. We give more details in the next section.

4.6 Mixed torpedo metrics

The double torpedo metric is an example of a warped product metric on the sphere. In this section we consider using more than one type of warping function on the sphere using the p,q-decomposition as described in section 4.5. Let u and v be two warping functions: $u,v : (0,b) \to (0,\infty)$. Let σ be the embedding:

$$\begin{aligned}\sigma : (0,b) \times S^p \times S^q &\to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+1}\\ (t,\Psi,\Theta) &\mapsto (u(t).\Psi, v(t).\Theta).\end{aligned}$$

The metric arising from this embedding is a *doubly warped product metric* $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2$. This extends uniquely to a smooth metric on S^n provided that the properties of the functions, u and v, are those determined by Petersen [32] in Lemmas 4.1 and 4.2 of section 1.4.1 which are given below:

Theorem 4.6.1. *(Section 1.4, [32]) Let u and v be smooth functions $u,v : (0,b) \to (0,\infty)$ where $u(b) = 0$ and $v(0) = 0$. Then we get a smooth doubly warped product metric at $t = 0$ and at $t = b$ if and only if*

$$u(0) > 0, \qquad u^{(odd)}(0) = 0, \qquad u^{(even)}(b) = 0, \qquad \dot{u}(b) = -1. \tag{4.1}$$

$$v(b) > 0, \qquad v^{(odd)}(b) = 0, \qquad v^{(even)}(0) = 0 \qquad \dot{v}(0) = 1. \tag{4.2}$$

In addition to the conditions listed in (4.1) and (4.2) of theorem 4.6.1, we have additional conditions, as required by Walsh [44], so that the metric has positive scalar curvature:
For the function $u : (0, b) \to (0, \infty)$:

$$\ddot{u} \leq 0, \qquad \ddot{u}(t) < 0 \text{ when } t \in (b - \varepsilon, b), \qquad \dddot{u}(b) > 0. \tag{4.3}$$

For the function $v : (0, b) \to (0, \infty)$

$$\ddot{v} \leq 0, \qquad \ddot{v}(t) < 0 \text{ when } t \in (0, \varepsilon), \qquad \dddot{v}(0) < 0. \tag{4.4}$$

A particular case of a doubly warped product metric is one in which the warping functions u and v are torpedo functions. Let $v(t) = \eta(t)$ and $u(t) = \eta(b-t)$ where η is a torpedo function with the properties given in section 4.2. Using p, q-decomposition, we can form the *mixed torpedo metric*, $g^{p,q}_{Mtorp}$ on S^n where:

$$g^{p,q}_{Mtorp} = dt^2 + u(t)^2 ds_p^2 + v(t)^2 ds_q^2.$$

The mixed torpedo metric combines

$$\varepsilon^2 ds_p^2 + g^{q+1}_{torp} = \varepsilon^2 ds_p^2 + dt^2 + v(t)^2 ds_q^2 \text{ on } S^p \times D^{q+1}$$

and

$$\delta^2 ds_q^2 + g^{p+1}_{torp} = \delta^2 ds_q^2 + dt^2 + u(t)^2 ds_p^2 \text{ on } D^{p+1} \times S^q.$$

We give a schematic illustration of the metrics, $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ on S^n in figure 4.9.

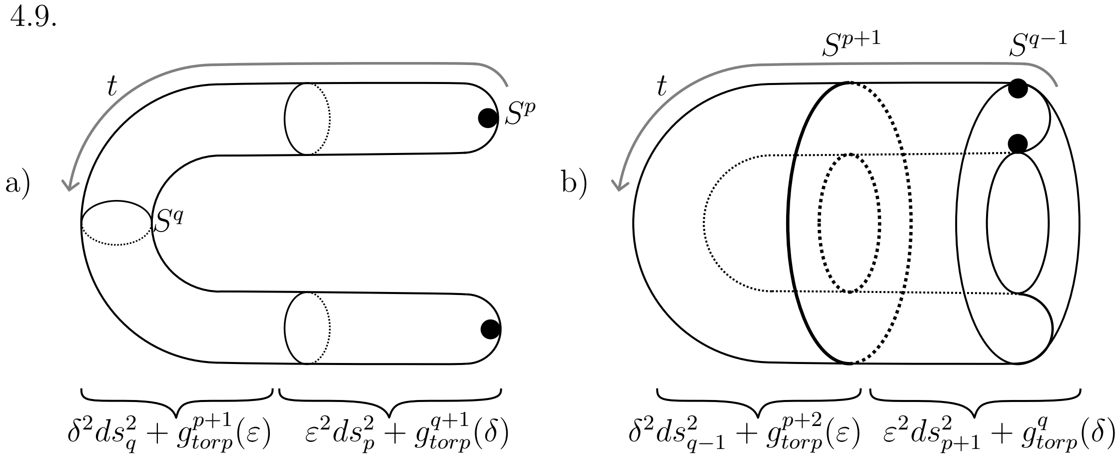

Figure 4.9: a) The mixed torpedo metric $g^{p,q}_{Mtorp}$; and b) The mixed torpedo metric $g^{p+1,q-1}_{Mtorp}$.

We calculate for which k, the mixed torpedo metric has positive Ricci-(k, n) curvature.

Lemma 4.6.2. *Let S^n be equipped with the mixed torpedo metric, $g^{p,q}_{Mtorp}$, where $p + q + 1 = n$. Where both p and $q \in \{0, 1\}$, the metric does not have positive* *scalar curvature. For $p \neq 1$ and $q \geq 2$ or $p \geq 2$ and $q \neq 1$, the metric has positive Ricci-(k,n) curvature for $k \geq 2$. When $p = 1$ and $q \geq 2$ or $p \geq 2$ and $q = 1$, the metric has positive Ricci-(k,n) curvature for $k \geq 3$.*

Proof. Petersen [32] gives the Ricci curvatures, $Ric(e_i)$, with respect to the coordinate vector fields $(\partial_t, e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q})$, tangent to $S^n = I \times S^p \times S^q$, where t is tangent to the interval I, $(e_1, \cdots, e_p)$ are tangent to S^p and $(e_{p+1}, \cdots, e_{p+q})$ are tangent to S^q:

$$Ric(e_i) = \begin{cases} -p\frac{\ddot{u}}{u} - q\frac{\ddot{v}}{v} & \text{when } i = t \\ -\frac{\ddot{u}}{u} + (p-1)\frac{1-\dot{u}^2}{u^2} - q\frac{\dot{u}}{u}\frac{\dot{v}}{v} & 1 \leq i \leq p \\ -\frac{\ddot{v}}{v} + (q-1)\frac{1-\dot{v}^2}{v^2} - p\frac{\dot{u}}{u}\frac{\dot{v}}{v} & p+1 \leq i \leq p+q. \end{cases}$$

For completeness we give the scalar curvature of the metric $g^{p,q}_{Mtorp}$, $s(g^{p,q}_{Mtorp})$:

$$\begin{aligned} s(g^{p,q}_{Mtor}) =& -p\frac{\ddot{u}}{u} - q\frac{\ddot{v}}{v} + p\big(-\frac{\ddot{u}}{u} + (p-1)\frac{1-\dot{u}^2}{u^2} - q\frac{\dot{u}}{u}\frac{\dot{v}}{v}\big) \\ &+ q\big(-\frac{\ddot{v}}{v} + (q-1)\frac{1-\dot{v}^2}{v^2} - p\frac{\dot{u}}{u}\frac{\dot{v}}{v}\big) \\ =& -2p\frac{\ddot{u}}{u} - 2q\frac{\ddot{v}}{v} + p(p-1)\frac{1-\dot{u}^2}{u^2} + q(q-1)\frac{1-\dot{v}^2}{v^2} - 2pq\frac{\dot{u}}{u}\frac{\dot{v}}{v}. \end{aligned}$$

The functions u and v vary with t as follows:

function \ t	0	$(0,\varepsilon)$	$[\varepsilon, b-\varepsilon]$	$(b-\varepsilon, b)$	b
u	>0	>0	>0	>0	0
$\dot{u}$	0	≤ 0	≤ 0	≤ 0	-1
$\ddot{u}$	≤ 0	≤ 0	≤ 0	<0	0
$\dddot{u}$					>0
v	0	>0	>0	>0	>0
$\dot{v}$	1	≥ 0	≥ 0	≥ 0	0
$\ddot{v}$	0	<0	≤ 0	≤ 0	≤ 0
$\dddot{v}$	<0				

Hence the functions involved in the Ricci curvatures, may be summarised as:

function \ t	0	$(0,\varepsilon)$	$[\varepsilon, b-\varepsilon]$	$(b-\varepsilon, b)$	b
$-\frac{\ddot{u}}{u}$	≥ 0	≥ 0	≥ 0	> 0	> 0
$-\frac{\ddot{v}}{v}$	> 0	> 0	≥ 0	≥ 0	≥ 0
$\frac{1-\dot{u}^2}{u^2}$	> 0	> 0	> 0	> 0	> 0
$\frac{1-\dot{v}^2}{v^2}$	> 0	> 0	> 0	> 0	> 0
$-\frac{\dot{u}}{u}\frac{\dot{v}}{v}$	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0

Therefore the Ricci curvatures are as follows:

$\mathrm{Ric}(e_i)$ \ t	0	$(0,\varepsilon)$	$[\varepsilon, b-\varepsilon]$	$(b-\varepsilon, b)$	b
$\mathrm{Ric}(e_t)$	> 0	> 0	≥ 0	> 0	> 0
$\mathrm{Ric}(e_i)$	> 0	> 0	> 0	> 0	> 0

Note that $\mathrm{Ric}(e_i)$, $i \in (1, \cdots, p+q)$ is positive provided $p \geq 2$ and $q \geq 2$.

We can see that all Ricci curvatures are non-negative but that $Ric(\partial_t)$ may be zero. Hence it can be seen that, provided $p \neq 1$ and $q \geq 2$, the mixed torpedo metric $g^{p,q}_{Mtorp}$ has positive Ricci-(k,n) curvature when $k \geq 2$ and $n \geq 5$. When $p = 1$, $Ric(e_1)$ may be zero and hence $g^{1,q}_{Mtorp}$ has positive Ricci-(k,n) curvature when $k \geq 3$ and $n \geq 4$. $\square$

Chapter 5

Isotopy of metrics

5.1 Isotopy of warped product metrics on the sphere

Recall from section 4.3 that a warped product metric on $(0, b) \times S^{n-1}$ is a metric of the form $dt^2 + f(t)^2 ds^2_{n-1}$ with warping function $f : (0, b) \to \mathbb{R}$. This extends uniquely to a smooth metric on S^n, provided that the function, f, satisfies the properties determined by Petersen [32] as given in Lemma 4.3.1.

Given a local orthonormal frame, $(\partial_t, e_1, \cdots, e_{n-1})$, on $(0, b) \times S^{n-1}$ where ∂_t is tangent to the interval $(0, b)$ and e_i is tangent to the sphere S^{n-1} then the Ricci curvatures of the metric, $dt^2 + f(t)^2 ds^2_{n-1}$, are given in [32] section 3.2.3 as:

$$Ric(\partial_t) = -(n-1)\frac{\ddot{f}}{f},$$
$$Ric(e_i) = (n-2)\frac{1-\dot{f}^2}{f^2} - \frac{\ddot{f}}{f}, \text{where } i \in \{1, \cdots, n-1\}.$$

We denote by $\mathcal{F}(0, b)$ the space of smooth functions, $(0, b) \to \mathbb{R}$, which have the properties listed in Lemma 4.3.1 together with the conditions $\ddot{f}(t) \leq 0, t \in [0, b]$, $\dddot{f}(0) < 0$ and $\dddot{f}(b) > 0$. Examples of elements of $\mathcal{F}(0, b)$ are shown in figure 5.1.

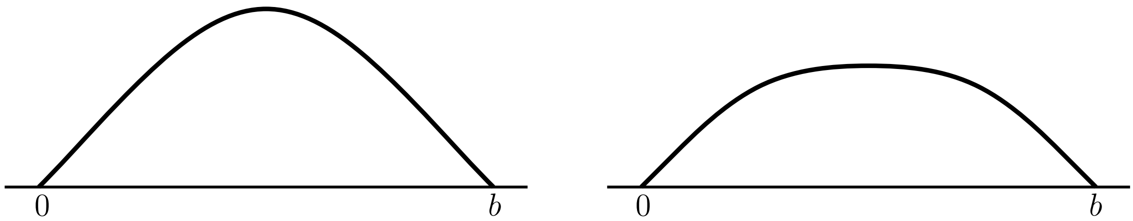

Figure 5.1: Examples of elements of $\mathcal{F}(0, b)$.

The associated smooth metrics, $dt^2 + f(t)^2 ds^2_{n-1}$, form a space, $\mathcal{W}(0, b)$, defined as $\mathcal{W}(0, b) := \{dt^2 + f(t)^2 ds^2_{n-1} : f \in \mathcal{F}(0, b)\}$. It has been shown that this space is a path-connected subspace of $\mathcal{R}\text{iem}^+(S^n)$; see Proposition 1.6 of [44]. We prove below the corresponding result for positive Ricci-(k,n) curvature:

Lemma 5.1.1. *The space $\mathcal{W}(0,b)$ is a path-connected subspace of $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$ where $k \geq 2$ and $n \geq 3$.*

Proof. By using the conditions given in Lemma 4.3.1 we ensure that each metric $g \in \mathcal{W}(0,b)$ uniquely determines a smooth metric on S^n.

In the table below we analyse the functions involved in the Ricci curvatures, noting that $\dot{f}(t) \in (-1,1)$ when $t \in (0,b)$.

function \ t	0	$(0,b)$	b
$\frac{1-\dot{f}^2}{f^2}$	>0	>0	>0
$-\frac{\ddot{f}}{f}$	>0	≥ 0	>0

Hence the Ricci curvatures satisfy

Ric \ t	0	$(0,b)$	b	Note
$Ric(\partial_t)$	>0	≥ 0	>0	1
$Ric(e_i)$	>0	>0	>0	2

Notes

1. Ricci curvature provided dimension, $n \geq 2$.

2. Ricci curvature provided dimension, $n \geq 3$.

Any metric $dt^2 + f(t)^2 ds^2_{n-1}$, where $f \in \mathcal{F}(0,b)$, has at least Ricci-$(2,n)$ positive curvature, provided $n \geq 3$. Hence the space $\mathcal{W}(0,b) \subset \mathcal{R}\text{iem}^{Ric^+_{(2,n)}}(S^n)$, when $n \geq 3$. Of course, for some f, the metric will also have Ricci-$(1,n)$ positive curvature, i.e. positive Ricci curvature, for example the standard round metric.

Path-connectivity of $\mathcal{W}(0,b)$ follows immediately from the fact that $\mathcal{F}(0,b)$ is convex. Proving this latter fact is an easy exercise. □

Note that as the round metric $dt^2 + (\frac{b}{\pi})^2 \sin^2(\frac{\pi t}{b}) ds^2_{n-1}$ of radius $(\frac{b}{\pi})$ is an element of $\mathcal{W}(0,b)$, then all the metrics in $\mathcal{W}(0,b)$ are isotopic to this round metric.

We now let b vary and consider the space $\mathcal{F} := \bigcup_{b\in(0,\infty)} \mathcal{F}(0,b)$ and the corresponding space of metrics $\mathcal{W} := \bigcup_{b\in(0,\infty)} \mathcal{W}(0,b)$, where $b > 0$. It has been shown that the space $\mathcal{W}$ is a path-connected subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$; see Proposition 1.7 [44]. Similarly,

Lemma 5.1.2. *The space $\mathcal{W}$ is a path-connected subspace of $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$, where $k \geq 2$ and $n \geq 3$.*

Proof. We consider elements $g_1 = dt^2 + f_1(t)^2 ds^2_{n-1}$ and $g_2 = dt^2 + f_2(t)^2 ds^2_{n-1}$ of $\mathcal{W}$, where $f_1 \in \mathcal{F}(0,b_1)$ and $f_2 \in \mathcal{F}(0,b_2)$, $b_1 > 0$, $b_2 > 0$, $b_1 \neq b_2$. The metrics, g_1 and g_2, are isotopic to the standard round metric and thus g_1 is isotopic to g_2. Therefore, the space $\mathcal{W}$ is a path-connected subspace of $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^n)$ where $k \geq 2$. and $n \geq 3$, the conditions required by theorem 5.1.1. □

We note that

- Any two torpedo metrics, $g^n_{torp}(\delta_1)_{\lambda_1}$ and $g^n_{torp}(\delta_2)_{\lambda_2}$, are isotopic by obvious rescaling.
- The double torpedo metric, $g^n_{Dtorp}(\delta) = dt^2 + \bar{\eta}^2_\delta(t) ds^2_{n-1}$, on S^n, has double torpedo function, $\bar{\eta} \in \mathcal{F}(0,b)$, for some b, and thus $g^n_{Dtorp}(\delta)$ is isotopic to the round metric ds^2_n. Therefore $g^n_{Dtorp}(\delta) \in \mathcal{W}$.

5.2 Isotopy of doubly warped products and mixed torpedo metrics

Let the mixed torpedo metric, $dt^2 + u(t)^2 ds^2_p + v(t)^2 ds^2_q$, be the metric on $(0,b) \times S^p \times S^q$ with warping functions u and v defined on an interval $(0,b)$. The warping functions, u and v, have the properties listed in (4.1) to (4.4) of section 4.6. The space of all such functions, u, is denoted $\mathcal{U}(0,b)$ and the space of all such functions, v is denoted $\mathcal{V}(0,b)$. The space of metrics, $g = dt^2 + u(t)^2 ds^2_p + v(t)^2 ds^2_q$, where $u \in \mathcal{U}(0,b)$ and $v \in \mathcal{V}(0,b)$ is denoted $\hat{\mathcal{W}}^{p,q}(0,b)$. It has been shown that $\hat{\mathcal{W}}^{p,q}(0,b)$ is a path-connected subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$, $n \geq 3$; see Lemma 1.9 of [44]. Below we generalise this theorem for positive Ricci-(k,n) curvature.

Lemma 5.2.1. *Let $n \geq 5$, $p \geq 1$ and $q \geq 2$ (or $p \geq 2$ and $q \geq 1$) with $p+q+1 = n$. Then the space $\hat{\mathcal{W}}^{p,q}(0,b)$ is a path-connected subspace of $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$, for $k \geq 2$.*

Proof. We note that an element $g \in \hat{\mathcal{W}}^{p,q}(0,b)$ uniquely determines a smooth metric on S^n as the functions u and v satisfy the conditions in Theorem 4.6.1. We have calculated that, provided $p \geq 2$ and $q \geq 2$, the generalised mixed torpedo metric has positive Ricci-(k,n) curvature when $k \geq 2$ and $n \geq 5$. Hence

$\hat{\mathcal{W}}^{p,q}(0,b) \subset \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, where $p \geq 2$, $q \geq 2$, $n \geq 5$ and $k \geq 2$. When $p = 1$, the generalised mixed torpedo metric has positive Ricci-(k, n) curvature when $k \geq 3$ and $n \geq 4$. Hence $\hat{\mathcal{W}}^{1,q}(0,b) \subset \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, where $q \geq 2$, $n \geq 4$ and $k \geq 3$.

It can easily be shown that the spaces $\mathcal{U}(0,b)$ and $\mathcal{V}(0,b)$ are convex and hence that the space $\mathcal{U}(0,b) \times \mathcal{V}(0,b)$ is convex. Hence $\hat{\mathcal{W}}^{p,q}(0,b)$ is path-connected. □

Note that

- When $p = 0$, an element of $\hat{\mathcal{W}}^{p,q}(0,b)$ is a warped product metric on S^n as discussed in section 5.1.
- When $n \geq 4$, $p = 1$ and $q \geq 2$ (or $p \geq 2$ and $q = 1$), the space $\hat{\mathcal{W}}^{p,q}(0,b)$ is a path-connected subspace of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, for $k \geq 3$.
- When $p = 1$, $q = 1$ and, therefore $n = 3$, the metric need not have positive scalar curvature and therefore may not have positive Ricci-(k, n) curvature for any $1 \leq k \leq 3$.

We now consider the spaces $\mathcal{U} \times \mathcal{V} := \bigcup_{b\in(0,\infty)} \mathcal{U}(0,b) \times \mathcal{V}(0,b)$, $\hat{\mathcal{W}}^{p,q} := \bigcup_{b\in(0,\infty)} \hat{\mathcal{W}}^{p,q}(0,b)$ and $\hat{\mathcal{W}} := \bigcup_{p+q+1=n} \hat{\mathcal{W}}^{p,q}$. Walsh proved that $\hat{\mathcal{W}}$ is a path-connected subspace of $\mathcal{R}iem^+(S^n)$ when $n \geq 3$; see Proposition 1.10 of [44].

Theorem 5.2.2. *The space $\hat{\mathcal{W}}$ is a path-connected subspace of $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$, for $n \geq 4$ and $k \geq 3$.*

Proof. We note that $\frac{b}{\pi}\cos\left(\frac{\pi t}{b}\right) \in \mathcal{U}(0,b)$ and $\frac{b}{\pi}\sin\left(\frac{\pi t}{b}\right) \in \mathcal{V}(0,b)$ and hence the round metric, $\left(\frac{b}{\pi}\right)^2 ds_n^2 = dt^2 + \left(\frac{b}{\pi}\right)^2\cos^2\left(\frac{\pi t}{b}\right)ds_p^2 + \left(\frac{b}{\pi}\right)^2\sin^2\left(\frac{\pi t}{b}\right)ds_q^2$ is an element of $\hat{\mathcal{W}}^{p,q}(0,b)$. By Theorem 5.2.1, any element $g \in \hat{\mathcal{W}}^{p,q}(0,b)$ is isotopic to $\left(\frac{b}{\pi}\right)^2 ds_n^2$. In turn, the metric $\left(\frac{b}{\pi}\right)^2 ds_n^2$ is isotopic to the round metric, $ds_n^2 = dt^2 + \cos^2 t\, ds_p^2 + \sin^2 t\, ds_q^2 \in \hat{\mathcal{W}}^{p,q}$, where $p + q + 1 = n$, by a rescaling. As the round metric ds_n^2 is an element of each space $\hat{\mathcal{W}}^{p,q}$, then the space $\hat{\mathcal{W}}$ is path-connected.

As elements of $\hat{\mathcal{W}}$ are elements of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, for $n \geq 4$ and $k \geq 3$, then the space $\hat{\mathcal{W}}$ is a path-connected subspace of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^n)$, for $n \geq 4$ and $k \geq 3$. □

Recalling the mixed torpedo metrics defined earlier in section 4.6, we obtain the following corollary.

Corollary 5.2.2.1. *Let the metrics $g^{p,q}_{Mtorp}$ and $g^{p',q'}_{Mtorp}$ have positive Ricci-(k, n) curvature where $p + q + 1 = p' + q' + 1 = n$, where $n \geq 4$ and $k \geq 3$. Then the metrics $g^{p,q}_{Mtorp}$ and $g^{p',q'}_{Mtorp}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}$.*

5.3 Relative isotopy of metrics $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$

Recall from section 4.5 that S^n may be decomposed as $(S^p \times D^{q+1}) \cup (D^{p+1} \times S^q)$. Moreover, the mixed torpedo metric on this decomposition is $dt^2+u(t)^2ds_p^2+v(t)^2ds_q^2$, denoted $g^{p,q}_{Mtorp}$, where u and v have the properties given in section 4.6. The mixed torpedo metric is schematically illustrated in the left hand side figure of 5.2 for $p=0$ and $q=1$. The equator of the sphere with respect to the q-sphere, $S^n_{q,eq}$, decomposes as $(S^p \times D^{q+1}_{eq}) \cup (D^{p+1} \times S^q_{eq})$ which is diffeomorphic to $(S^p \times D^q) \cup (D^{p+1} \times S^{q-1})$. The equator has a mixed torpedo metric, $g^{p,q}_{Mtorpeq}$, which is the restriction of $g^{p,q}_{Mtorp}$ to the equator and is $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_{q-1}^2$. This is shown emboldened in the left hand side figure of figure 5.2.

The analogously defined mixed torpedo metric, $g^{p+1,q-1}_{Mtorp}$, on $(S^{p+1} \times D^q) \cup (D^{p+2} \times S^{q-1})$, is schematically illustrated in the right hand side figure of figure 5.2 when $p = 1$ and $q = 0$. The equator of the sphere with respect to the p-sphere, $S^n_{p,eq}$, decomposes as $(S^{p+1}_{eq} \times D^q) \cup (D^{p+2}_{eq} \times S^{q-1})$ which is diffeomorphic to $(S^p \times D^q) \cup (D^{p+1} \times S^{q-1})$. The equator has a mixed torpedo metric, $g^{p+1,q-1}_{Mtorpeq}$, which is the restriction of $g^{p+1,q-1}_{Mtorp}$ to the equator and is $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_{q-1}^2$. This is shown emboldened in the right hand side figure of 5.2.

Thus both $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ have the same equator metric, $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_{q-1}^2$. Note also that $g^{p,q}_{Mtorpeq}$ and $g^{p+1,q-1}_{Mtorpeq}$ is isometric to $g^{p,q-1}_{Mtorp}$. While in Corollary 5.2.2.1 we show that $g^{p,q}_{Mtorp}$ and $g^{p',q'}_{Mtorp}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}$ where $n \geq 4$ and $k \geq 3$, we will be interested in a particular isotopy between $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$, a relative isotopy which **fixes the equator metric** $g^{p,q}_{Mtorpeq}$.

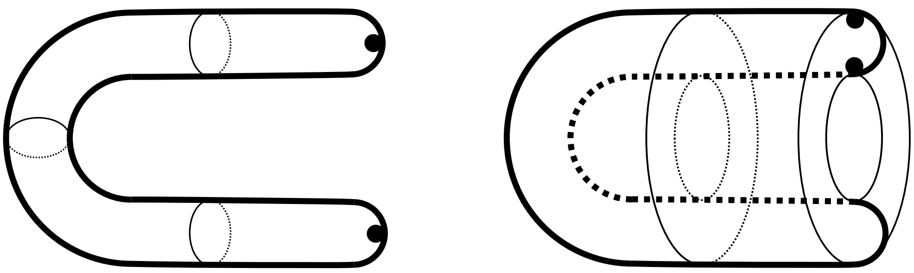

Figure 5.2: The equator $g^{p,q-1}_{Mtorp}$, denoted $g^{p,q}_{Mtorpeq}$, of a) $g^{(p,q)}_{Mtorp}$ and b) $g^{(p+1,q-1)}_{Mtorp}$.

Note that as $u \in \mathcal{U}(0,b)$ and $v \in \mathcal{V}(0,b)$, as defined in Theorem 4.6.1, by Theorem 5.2.2 the metric $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2$ is isotopic to the standard round metric. Henceforward we denote $\eta_\varepsilon(b-t)$ as $u(t)$ and $\eta_\delta(t)$ as $v(t)$.

We intend to prove in Theorem 5.3.1 that there is an isotopy between $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$, which keeps the equator metric $g^{p,q}_{Mtorpeq}$ fixed.

Theorem 5.3.1. *For any non-negative integers, p and q, with $p+q+1=n$ and $n \geq 5$, let $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ be mixed torpedo metrics on the sphere S^n both of which have the same equator metric, $g^{p,q}_{Mtorpeq}$. Then there exists an isotopy in $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$ between $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ which keeps the common equator metric, $g^{p,q}_{Mtorpeq}$, fixed for $k \geq 3$ when $p \neq 1$ or $q \neq 2$. When $p=1$ or $q=2$, $k \geq 4$.*

Proof. In Lemma 4.6.2 it was shown that where $p=0$ or $p \geq 2$ and $q \geq 2$, the metric $g^{p,q}_{Mtorp}$ has positive Ricci-(k,n) curvature metric for $k \geq 2$. When $p=1$ and $q \geq 2$, or when $p \geq 2$ and $q=1$, $g^{p,q}_{Mtorp}$ has positive Ricci-(k,n) curvature metric for $k \geq 3$. In Appendix A we give a table with the values of k for which both the metrics $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ have positive Ricci-(k,n) curvature.

We will construct an isotopy through $\mathcal{R}iem^{Ric^+_{(k,n)}}(S^n)$ from the metric $g^{p,q}_{Mtorp}$ to the metric $g^{p+1,q-1}_{Mtorp}$, which keeps the equator metric $g^{p,q}_{Mtorpeq}$ fixed.

We start with the mixed torpedo metric $g^{p,q}_{Mtorp}$ defined on S^n, with p,q-decomposition as described in section 4.5, arising from the embedding, denoted $\sigma_{p,q}$ to emphasise the decomposition, as follows:

$$\begin{aligned} S^n &\cong (S^p \times D^{q+1}) \cup (D^{p+1} \times S^q) \\ &= \Big((S^p \times D^{q+1}_+) \cup (D^{p+1} \times S^q_+)\Big) \cup \Big((S^p \times D^{q+1}_-) \cup (D^{p+1} \times S^q_-)\Big) \end{aligned}$$

where S^q_+ and S^q_- denote the upper and lower halves of the sphere, S^q, respectively. Recall that the embedding $\sigma_{p,q}$ is defined

$$\begin{aligned} \sigma_{p,q} : (0,b) \times S^p \times S^q &\to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+1} \\ (t, \Psi, \Theta) &\mapsto (u(t).\Psi, v(t).\Theta). \end{aligned}$$

The mixed torpedo metric on the upper half of the sphere, $S^n_+ \cong (S^p \times D^{q+1}_+) \cup (D^{p+1} \times S^q_+)$, is denoted $g^{(p,q)+}_{Mtorp}$ and on the lower half of the sphere, $S^n_- \cong (S^p \times D^{q+1}_-) \cup (D^{p+1} \times S^q_-)$, is denoted as $g^{(p,q)-}_{Mtorp}$. The equator of the sphere, S^n, is S^{n-1}, where

$$\begin{aligned} S^{n-1} &\cong (S^p \times D^q) \cup (D^{p+1} \times S^{q-1}), \text{ where} \\ S^q_+ \cap S^q_- &= S^{q-1} \subset \mathbb{R}^q \subset \mathbb{R}^{q+1}. \end{aligned}$$

The mixed torpedo metric on the equator is denoted $g^{p,q-1}_{Mtorp} = g^{p,q}_{Mtorpeq}$; see figure 5.6(a).

The metric $g^{p,q}_{Mtorp}$ on S^n is

$$\begin{aligned} g^{p,q}_{Mtorp} &= dt^2 + u(t)^2 ds_p^2 + v(t)^2 ds_q^2, \\ &= dt^2 + u(t)^2 ds_p^2 + v(t)^2 (dr^2 + \sin^2(r) ds_{q-1}^2). \end{aligned}$$

In order to smoothly transition to the metric, $g^{p+1,q-1}_{Mtorp}$, we will make use of an alternative description of the torpedo metric as described in the following section.

5.3.1 Alternative description of the torpedo

In order to describe the deformation required we will use an alternate description of the coordinates for a torpedo metric, g^n_{torp}, on D^n, described in [47] as follows. The coordinates are shown in figure 5.3 which also shows the upper half disc, D^n_+, the lower half disc, D^n_-, and the equator, D^{n-1}.

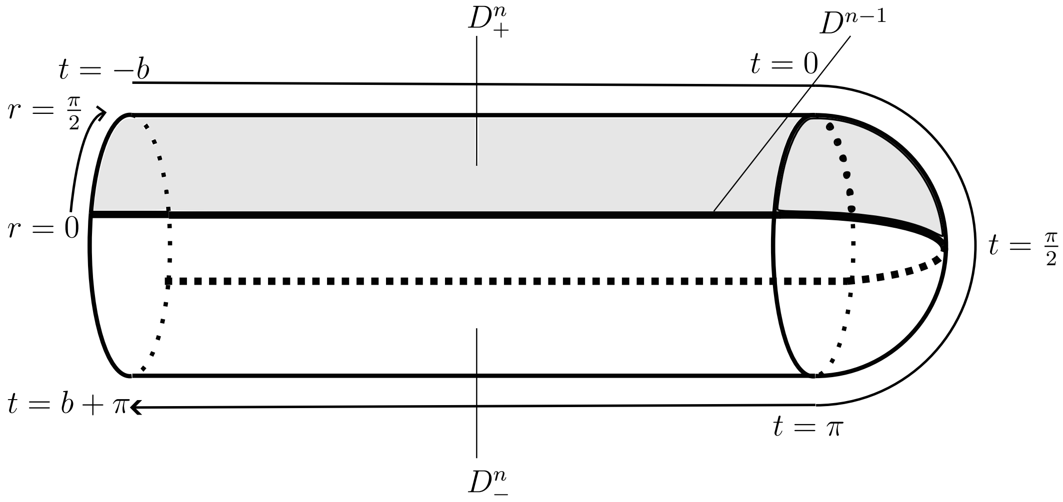

Figure 5.3: Alternative description of the torpedo metric.

An $(n-1)$-dimensional hemisphere is traced up one side of a cylinder, from $t = -b$ to $t = 0$, see figure 5.4(a), then bent round an angle of π to form an n-dimensional hemisphere, see figure 5.4(b), and then traced down the other side of the cylinder, see figure 5.4(c).

Figure 5.4: Tracing out the hemisphere a) $t \in [-b, 0]$; b) $t \in [-b, \pi]$; and c) $t \in [-b, b+\pi]$.

Let $r \in (0, \frac{\pi}{2})$, $t \in (-b, b+\pi)$ and $\alpha : (0, \frac{\pi}{2}) \times (-b, b+\pi) \to [0, 1]$. We use a bump function $\mu : (-b, \pi + b) \to [0, 1]$ that is zero outside $t \in [-\tau, \pi + \tau]$, τ arbitrarily small and $\tau > 0$, to give a smooth function α:

$$\alpha(r, t) = 1 - \mu(t) + \mu(t) \sin(r),$$

where $\alpha(r, t)$ is defined as

(i) $\alpha(r, t) = 1$ when $t \in (-b, -\tau)$,

(ii) $\alpha(r, t) = \sin r$ when $t \in (\tau, \pi - \tau)$,

(iii) $\alpha(r,t) = 1$ when $t \in (\pi + \tau, b + \pi)$.

The function is depicted in figure 5.5.

Recall from section 4.6 that the mixed torpedo metric $dr^2 + v(r)^2 ds_p^2 + u(r)^2 ds_q^2$ arises from the embedding:

$$\sigma : (0,b) \times S^p \times S^q \to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$$
$$(r, \Psi, \Theta) \mapsto (u(r).\Psi, v(r).\Theta)$$

where u and v are warped product metrics.

Analogously, let the embedding ρ be

$$\rho : (0,b) \times S^1 \times S^{n-2} \to \mathbb{R}^2 \times \mathbb{R}^{n-1}$$
$$(r, t, \Theta) \mapsto (\alpha(r,t).t, \cos(r).\Theta),$$

where α is the warping function described above. This embedding gives the torpedo metric

$$g^n_{torp} = dr^2 + \alpha(r,t)^2 dt^2 + \cos^2(r) ds^2_{n-2}.$$

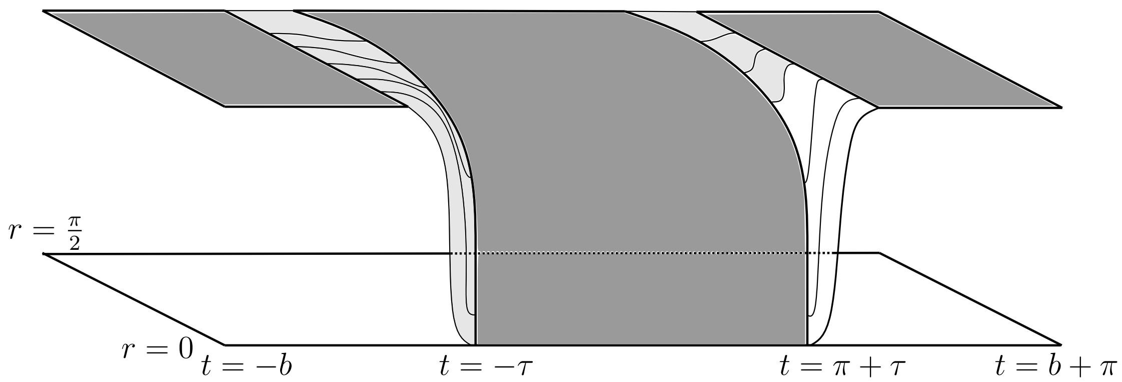

Figure 5.5: The function $\alpha : (0, \frac{\pi}{2}) \times (-b, b+\pi) \to [0,1]$

Below we describe how to decompose the mixed torpedo metric $g^{p,q}_{Mtorp}$ on S^n into three regions, two of which have sphere-disc products. We use these alternative coordinates on the relevant discs and show the deformation required in order to smoothly transition from $g^{p,q}_{Mtorp}$ to $g^{p+1,q-1}_{Mtorp}$, while keeping the equator metric, $g^{p,q-1}_{Mtorp}$, fixed.

5.3.2 Initial adjustment to the mixed torpedo metric, $g^{p,q}_{Mtorp}$

Assuming S^n is equipped with the metric $g^{p,q}_{Mtorp}$, let us consider a small tubular neighbourhood, $N_\tau(S^{n-1})$ of the equator, S^{n-1}, where the metric $g^{p,q}_{Mtorp}$ restricts on S^{n-1} as $g^{p,q-1}_{Mtorp}$, on this neighbourhood, and assume that the metric is the product metric $g^{p,q-1}_{Mtorp} + dr^2$ on $N_\tau(S^{n-1})$. We describe in sections 5.3.2.1 to 5.3.2.3 how we stretch the τ neighbourhood in the direction orthogonal to the equator (with coordinate r) to give the metric $g^{p,q-1}_{Mtorp} + dr^2$, $r \in [-c, c]$ as in figure 5.6. This requires changing the metric in the r direction from a standard round metric to a double torpedo metric. The resulting metric on S^n we denote $g^{p,q}_{Mtorp,stretch}$.

In section 5.3.2 we construct an isotopy from the metric, $g^{p,q}_{Mtorp}$, to the product metric, $g^{p,q}_{Mtorp,stretch}$, while keeping the equator metric, $g^{p,q-1}_{Mtorp}$ fixed.

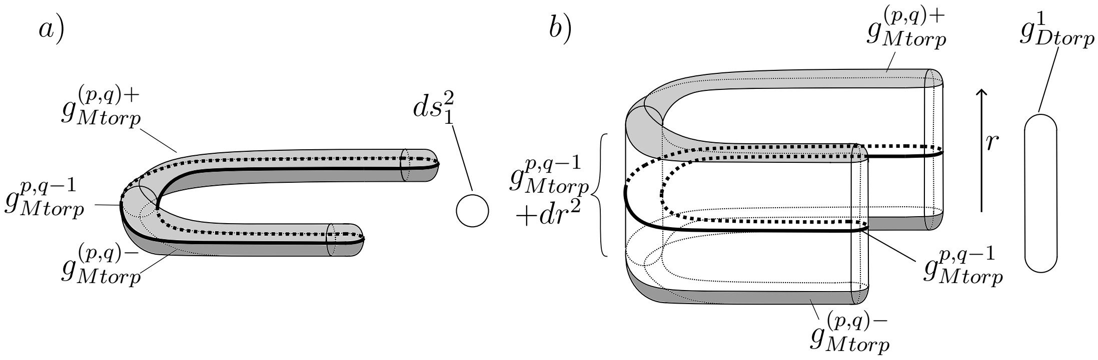

Figure 5.6: a) $g^{p,q}_{Mtorp}$ showing round metric on tube of torpedo; and b) $g^{p,q}_{Mtorp,stretch}$ showing double torpedo metric on tube of torpedo.

Recall from section 4.5 that using (p, q)-decomposition and the embedding $\sigma_{p,q}$, S^n can be decomposed into three regions where $R_1 \cong D^{q+1} \times S^p$, $R^2 \cong S^p \times S^q \times I$ and $R^3 \cong D^{p+1} \times S^q$. Hence the mixed torpedo metric on S^n can also be considered as the metric on the union of three regions:

$$(D^{q+1} \times S^p, dt^2 + \eta_\delta(t)^2 ds_q^2 + ds_p^2) \cup (S^p \times S^q \times I, dt^2 + ds_p^2 + ds_q^2)$$
$$\cup\, (D^{p+1} \times S^q, dt^2 + \eta_\varepsilon(t)^2 ds_p^2 + ds_q^2),$$

see figure 5.7. We consider below the adjustments required.

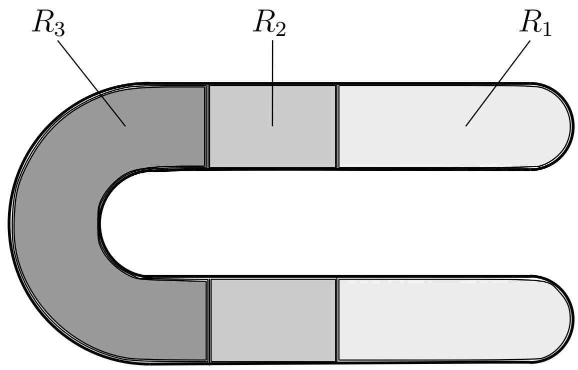

Figure 5.7: Analysis of the mixed torpedo metric $R_1 \cong (D^{q+1} \times S^p, dt^2 + ds_p^2 + \eta_\delta(t)^2 ds_q^2)$; $R_2 \cong S^p \times S^q \times I, dt^2 + ds_p^2 + ds_q^2)$; and $R_3 \cong (D^{p+1} \times S^q, dt^2 + \eta_\varepsilon(t)^2 ds_p^2 + ds_q^2)$.

5.3.2.1 Effect of stretching the mixed torpedo metric, $g^{p,q}_{Mtorp}$, on R_1.

Recall that the mixed torpedo metric, $g^{p,q}_{Mtorp}$, restricted to R_1 in figure 5.7 may be described as $(D^{q+1} \times S^p, dt^2 + v(t)^2 ds_q^2 + ds_p^2)$, where v is a torpedo metric. The equator restricted to R_1 is isometric to $(D^q \times S^p, dt^2 + v(t)^2 ds_{q-1}^2 + ds_p^2)$ and therefore the tubular neighbourhood of the equator restricted to R_1, $N_\tau(S^{n-1})|_{R_1} \subset S^n$ is isometric to $(I \times D^q \times S^p, dr^2 + dt^2 + v^2 ds_{q-1}^2 + ds_p^2)$. We change the coordinates of D^{q+1} factor of R_1 as described in section 5.3.1 for D^n where the metric now restricts as $dr^2 + \alpha(r,t)^2 dt^2 + \cos^2(r) ds_{q-1}^2 + ds_p^2$.

We consider the deformation required of the metric on R_1, which keeps the equator metric fixed, in the neighbourhood, $N_\tau(S^{n-1}) \cap R_1$, where

$$N_\tau(S^{n-1})|_{R_1} \cong I \times S^p \times S^{q-1} \times I.$$

We essentially wish to "flatten" the cos(r) function near $r = 0$ by replacing it with a homotopy of functions defined

$$f_s(r) = (1-s)\cos(r) + s\omega(r)$$

where $s \in [0,1]$ and $\omega : (-\frac{\pi}{2}, \frac{\pi}{2}) \to [0,1]$ is a smooth function such that

$$\omega(r) = \begin{cases} \cos(r) & \text{when } r \in (-\frac{\pi}{2}, -\xi], \\ 1 & \text{when } r \in (-\tau, \tau), \\ \cos(r) & \text{when } r \in [\xi, \frac{\pi}{2}) \end{cases}$$

where $0 < \tau < \xi$ and $\ddot{\omega}(r) \leq 0$.

Moreover we will wish to "flatten" the function α by using a transition function

$$\beta_s(r,t) = (1-s)\alpha(r,t) + s\gamma(r,t)$$

where $s \in [0,1]$ and the smooth function $\gamma(r,t)$ on $(0, \frac{\pi}{2}) \times (-b, b+\pi)$ satisfies:

$$\gamma(r,t) = \begin{cases} 1 & \text{when } t \in (-b, 0) \\ \sin(r) & \text{when } t \in (\tau, \frac{\pi}{2} - \tau) \\ 1 & \text{when } t \in (\frac{\pi}{2} - \frac{\tau}{4}, \frac{\pi}{2} + \frac{\tau}{4}) \\ \sin(r) & \text{when } t \in (\frac{\pi}{2} + \tau, \pi - \tau) \\ 1 & \text{when } t \in (\pi, b+\pi). \end{cases}$$

Smoothness of γ in t is obtained by way of a suitable cut off function, $\nu(t) : (-b, \pi + b) \to [0,1]$ where γ is defined by:

$$\gamma(r,t) = 1 - \nu(t) + \nu(t)\sin(r).$$

Henceforward we suppress the subscript s. The metric, $g^{p,q}_{Mtorp}$, restricted to $N_\tau(S^{n-1})|_{R_1}$ is adjusted to

$$g = dr^2 + \beta^2(r,t)dt^2 + f^2(r)ds^2_{q-1} + ds^2_p,$$

The metric, restricted to the D^{q+1} factor of R_1, $dr^2 + \beta^2(r,t)dt^2 + f^2(r)ds^2_{q-1}$, is shown in figure 5.8. Note that the equator metric is fixed.

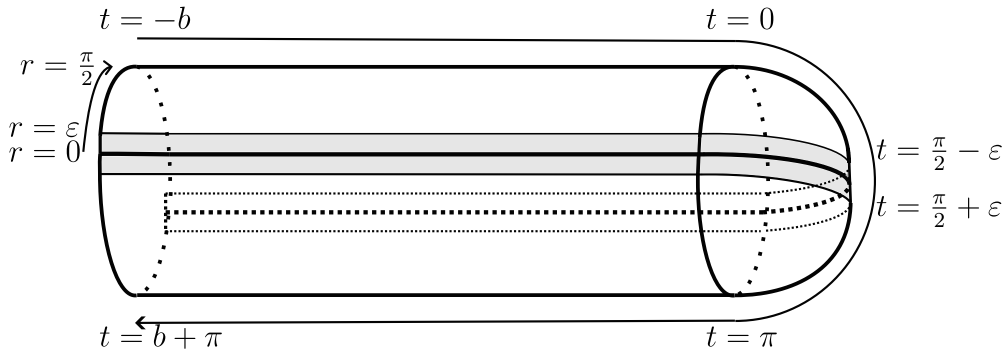

Figure 5.8: The metric, $dr^2 + \beta^2(r,t)dt^2 + f^2(r)ds^2_{q-1}$.

Christoffel symbols are calculated for the metric, $dr^2 + \beta^2(r,t)dt^2 + f^2(r)ds^2_{q-1} + ds^2_p$, as:

$$\Gamma^r_{t,t} = -\beta_r\beta \qquad \Gamma^r_{\theta_i,\theta_i} = -\dot{f}f\sin^2\theta_{i+1}\cdots\sin^2\theta_{q-1}$$

$$\Gamma^t_{r,t} = \frac{\beta_r}{\beta} \qquad \Gamma^t_{t,t} = \frac{\beta_t}{\beta} \qquad \Gamma^{\theta_i}_{r,\theta_i} = \frac{f_r}{f}$$

$$\Gamma^{\theta_{i+j}}_{\theta_i,\theta_i} = -\sin\theta_{i+j}\cos\theta_{i+j}\sin^2(\theta_{i+1})\cdots\sin^2(\theta_{i+j-1}) \qquad \Gamma^{\theta_i}_{\theta_j,\theta_i} = \frac{\cos\theta_j}{\sin\theta_j}, \quad i<j$$

$$\Gamma^{\psi_{i+j}}_{\psi_i,\psi_i} = -\sin\psi_{i+j}\cos\psi_{i+j}\sin^2(\psi_{i+1})\cdots\sin^2(\psi_{i+j-1}) \qquad \Gamma^{\psi_i}_{\psi_j,\psi_i} = \frac{\cos\psi_j}{\sin\psi_j}, \quad i<j.$$

All other $\Gamma^l_{i,j}$ are zero.

From these the sectional curvatures, $K_{i,j}$, are:

$$K_{r,t} = -\frac{\beta_{rr}}{\beta}; \quad K_{r,\theta_i} = -\frac{f_{rr}}{f}; \quad K_{r,\psi_i} = 0; \quad K_{t,\theta_i} = -\frac{f_r\beta_r}{f\beta};$$

$$K_{t,\psi_i} = 0; \quad K_{\theta_i,\theta_j} = \frac{1-f_r^2}{f^2}; K_{\theta_i,\psi_i} = 0; \quad K_{\psi_i,\psi_j} = 1$$

We evaluate the functions involved in the sectional curvatures. Note that all the derivatives are with respect to r and not with respect to t or s.

(i) We evaluate the function

$$-\frac{\beta_{rr}}{\beta} = -\frac{(1-s)\alpha_{rr} + s\gamma_{rr}}{(1-s)\alpha + s\gamma}$$
$$= \frac{(1-s)\mu(t)\sin(r) + s\nu\sin(r)}{(1-s)(1-\mu(t)+\mu(t)\sin(r)) + s(1-\nu(t)+\nu(t)\sin(r))}.$$

The table below shows the values of the function $-\frac{\beta_{rr}}{\beta}$ with respect to s, μ and ν.

	$\mu=0$	$\mu=1$	$\nu=0$	$\nu=1$
$s=0$	0	1	N/A	N/A
$s=1$	N/A	N/A	0	1

Hence $-\frac{\beta_{rr}}{\beta} \geq 0$.

(ii) We evaluate the function

$$-\frac{f_{rr}}{f} = \frac{(1-s)\cos(r) - s\omega_{rr}}{(1-s)\cos(r) + s\omega}.$$

The table below shows the values of the function $-\frac{f_{rr}}{f}$ with respect to s and ω.

	$r \in (-\frac{\pi}{2}, -\tau]$	$r \in (-\tau, \tau)$	$r \in [\tau, \frac{\pi}{2})$
$s=0$	1	1	1
$s=1$	1	0	1

Hence $-\frac{f_{rr}}{f} \geq 0$.

(iii) We evaluate the function

$$\begin{aligned} -\frac{f_r\beta_r}{f\beta} &= -\frac{\big((s-1)\sin(r) + s\omega_r\big)\big((1-s)\alpha_r + s\gamma_r)\big)}{\big((1-s)\cos(r) + s\omega\big)\big((1-s)\alpha + s\gamma)\big)} \\ &= \frac{\big((1-s)\sin(r) - s\omega_r\big)\big((1-s)(\mu\cos(r) + s\nu\cos(r))\big)}{\big((1-s)\cos(r) + s\omega\big)\big((1-s)(1-\mu+\mu\sin(r) + s(1-\nu+\nu\sin(r))\big)}. \end{aligned}$$

The table below shows the values of the function $-\frac{f_r\beta_r}{f\beta}$ with respect to s, μ and ν.

	$\mu=0$	$\mu=1$	$\nu=0$	$\nu=1$
$s=0$	0	1	N/A	N/A
$s=1$	N/A	N/A	0	1

Hence $-\frac{f_r\beta_r}{f\beta} \geq 0$.

(iv) The function

$$\frac{1-f_r^2}{f^2} = \frac{1 - \big((s-1)\sin(r) + s\omega_r\big)^2}{\big((1-s)\cos(r) + s\omega\big)^2} > 0,$$

for all values of $s \in [0,1]$ and $r \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Thus all of the terms for evaluating the sectional curvatures are non-negative.

The Ricci curvature, $Ric(v)$, of the coordinate vector fields are as follows:

$$Ric(v) = \begin{cases} -\frac{\beta_{rr}}{\beta} - (q-1)\frac{f_{rr}}{f} & \text{when } v = \partial_r \\ -\frac{\beta_{rr}}{\beta} - (q-1)\frac{f_r\beta_r}{f\beta} & \text{when } v = \partial_t \\ -\frac{f_{rr}}{f} - \frac{f_r\beta_r}{f\beta} + (q-2)\frac{1-f_r^2}{f^2} & \text{when } v = \partial\theta_i, 1 \leq i \leq q-1 \\ p-1 & \text{when } v = \partial\psi_i, 1 \leq i \leq p. \end{cases}$$

As both of the terms for $Ric(\partial_r)$ are not strictly positive, the best we can say is that $Ric(\partial_r) \geq 0$. Similarly both of the terms for $Ric(\partial_t)$ are not strictly positive, $Ric(\partial_t) \geq 0$. The first two terms for $Ric(\partial\theta_i)$ are not strictly positive. Provided $q > 2$, the third term is positive, hence $Ric(\partial\theta_i) > 0$. As $Ric(\partial\psi_i) = p - 1$, $Ric(\partial\psi_i) > 0$, provided $p \neq 1$. Where $p = 1$, $Ric(\partial\psi_i) = 0$.

Hence the metric $g = dr^2 + \beta^2(r,t)^2 dt^2 + f(r)^2 ds^2_{q-1} + ds^2_p$, has positive Ricci-(k,n) curvature, provided $q > 2$, when $k \geq 3$ and $p \neq 1$, and $k \geq 4$ when $p = 1$.

5.3.2.2 Effect of stretching the mixed torpedo metric, $g^{p,q}_{Mtorp}$, on R_2.

Recall that the mixed torpedo metric, $g^{p,q}_{Mtorp}$ restricted to R_2 in figure 5.7 may be described as $(S^p \times S^q \times I, dt^2 + ds^2_p + ds^2_q)$. We change the metric on R_2 from $ds^2_p + ds^2_q + dt^2 = ds^2_p + dr^2 + cos^2(r) ds^2_{q-1} + dt^2$ to $ds^2_p + dr^2 + f_s(r)^2 ds^2_{q-1} + dt^2$, which is a continuation of the metric on the tube of R_1 torpedo metric, $t \in ([-b, 0] \cup [\pi, b + \pi])$, see figure 5.8. It thus has positive Ricci-(k,n) curvature, provided $q > 2$, when $k \geq 3$ and $p \neq 1$ and $k \geq 4$ when $p = 1$.

5.3.2.3 Effect of stretching the mixed torpedo metric, $g^{p,q}_{Mtorp}$, on R_3.

The initial adjustment required to the metric on R_3 of the mixed torpedo metric is analogous to that required for R_1. Recall R_3 is diffeomorphic to $D^{p+1} \times S^q$, which is shown schematically in figure 5.7. Let $(\partial_t, \partial\psi_1, \cdots, \partial\psi_p, \partial_r, \partial\theta_1 \cdots, \partial\theta_{q-1})$ denote the corresponding coordinate vector fields on the tangent space of $D^{p+1} \times S^q$. The original metric on R_3 is $g^{p+1}_{torp} + ds^2_q = dt^2 + \eta_\varepsilon(t)^2 ds^2_p + ds^2_q$, where η_ε is a torpedo function as described in 4.2. We consider the neighbourhood of the equator, $N_\tau(S^{n-1})|_{R_3}$, where we wish to "flatten" the metric using a method similar to that used in R_1. The metric is adjusted to obtain $dt^2 + \eta_\varepsilon(t)^2 ds^2_p + dr^2 + h^2_s(r) ds^2_{q-1}$. Here $h_s(r) = (1-s)\sin(r) + s\overline{\eta}(r)$, where $\overline{\eta}$ is a double torpedo function as described in 4.3 with $\ddot{\overline{\eta}} \leq 0$, $\dot{\overline{\eta}} \leq 1$ and $s \in [0,1]$. When $s = 1$, the metric on R_3 is $g^{p+1}_{torp} + g^q_{Dtorp}$. Note that the metric on the disc factor D^{p+1} is not changed.

We note that $h_s \in \mathcal{F}$ (see subsection 5.1) and therefore $dr^2 + h^2_s(r) ds^2_{q-1} \in \mathcal{W}$ for all $s \in [0,1]$, where here $\mathcal{W}$ is a path-connected subspace of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(S^q)$ where $k \geq 2$ and $q \geq 3$.

Using the calculations in sections 4.2 and 4.3, the Ricci curvatures of the metric $dt^2 + \eta_\varepsilon(t)^2 ds^2_p + dr^2 + h_s(r)^2 ds^2_{q-1} = g^{p+1}_{torp} + g^q_{Dtorp}$ are given below. We denote $\eta_\varepsilon(t)$ as η and $h_s(r)$ as h.

$$Ric(v) = \begin{cases} -p\frac{\ddot{\eta}}{\eta} & \text{when } v = \partial_t \\ -\frac{\ddot{\eta}}{\eta} + (p-1)\frac{1-\dot{\eta}^2}{\eta^2} & \text{when } v = \partial\psi_i, 1 \leq i \leq p \\ -(q-1)\frac{\ddot{h}}{h} & \text{when } v = \partial_r \\ -\frac{\ddot{h}}{h} - (q-2)\frac{1-\dot{h}^2}{h^2} & \text{when } v = \partial\theta_i, 1 \leq i \leq q-1. \end{cases}$$

The Ricci curvatures $Ric(\partial_t)$ and $Ric(\partial_r)$ are both non-negative and $Ric(\partial\psi_i)$ and $Ric(\partial\theta_i)$ are both strictly positive provided $p > 2$ and $q > 3$ respectively. Hence it can be seen that the metric $g^{p+1}_{torp} + g^{q}_{Dtorp}$ has positive Ricci-(k, n) curvature for values of k as follows:

$$k \geq \begin{cases} 3 & \text{when } p = 0 \text{ and } q \geq 3 \\ 3 & \text{when } p \geq 2 \text{ and } q \geq 3 \\ 4 & \text{when } p \geq 2 \text{ and } q = 2 \\ 4 & \text{when } p = 1 \text{ and } q \geq 3. \end{cases}$$

We then stretch the τ-neighbourhood region orthogonally to the equator, while keeping the equator fixed, to $r = c$ and to $r = -c$ respectively, where $r = 0$ is at the equator. The resulting metric is denoted $g^{p,q}_{Mtorp,stretch}$; see figure 5.6(b).

5.3.3 Isotopy from $g^{p,q}_{Mtorp,stretch}$ to $g^{p+1,q-1}_{Mtorp,stretch}$

At this stage we have constructed isotopies from $g^{p,q}_{Mtorp}$ to $g^{p,q}_{Mtorp,stretch}$ and from $g^{p+1,q-1}_{Mtorp}$ to $g^{p+1,q-1}_{Mtorp,stretch}$. In the following sections, we complete our proof of Theorem 5.3.1 by constructing an isotopy between $g^{p,q}_{Mtorp,stretch}$ and $g^{p+1,q-1}_{Mtorp,stretch}$ which fixes the equator metric $g^{p,q-1}_{Mtorp}$.

Recall from Corollary 5.2.2.1 to Theorem 5.2.2 that the mixed torpedo metrics $g^{p,q}_{Mtorp}$ and $g^{p',q'}_{Mtorp}$, where $p + q = p' + q' = n - 1$, $n \geq 4$, are isotopic in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^n)$ for $k \geq 3$. This is because both of the mixed torpedo metrics are isotopic in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^n)$ to the standard round metric on S^n. Hence there are paths in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^n)$ from $g^{p,q}_{Mtorp}$ to ds^2_n. Recall also that the mixed torpedo metrics $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ have the same equator metric $g^{p,q}_{Mtorpeq}$; see figure 5.2. Moreover $g^{p,q}_{Mtorpeq}$ is itself a mixed torpedo metric $g^{p,q-1}_{Mtorp}$ which is isotopic to the standard round metric on S^{n-1} and hence there are paths in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^{n-1})$ from $g^{p,q-1}_{Mtorp}$ to ds^2_{n-1}. In the proof of Theorem 5.3.1 we will use a specific isotopy of metrics in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(S^n)$ between $g^{p,q}_{Mtorp}$ and ds^2_n, which restricts to the equator as an isotopy of metrics in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n-1)}}(S^{n-1})$ between $g^{p,q-1}_{Mtorp}$ and ds^2_{n-1}.

In section 4.5 we described the p, q-decomposition of $S^n \cong (S^p \times D^{q+1}) \cup_{S^p \times S^q} (D^{p+1} \times S^q)$ and $S^n_+ \cong (S^p \times D^{q+1}_+) \cup (D^{p+1} \times S^q_+)$. Recall that $g^{p,q}_{Mtorp} = dt^2 + u(t)^2 ds^2_p + v(t)^2 ds^2_q$ where u and v are torpedo functions with the properties given in Theorem 4.6.1. Hence near $t = 0$, u behaves like cosine and v like sine. As described in the proof of Theorem 5.2.2, the straight line homotopies between the functions u and cos in $\mathcal{U}(0,b)$ and v and sin in $\mathcal{V}(0,b)$ induce an isotopy between the mixed torpedo metric $g^{p,q}_{Mtorp}$ and the round metric ds_n^2 in the space of positive Ricci-(k,n) metrics when

- $p \geq 2$, $q \geq 2$, $n \geq 5$, then $k \geq 2$;
- $p = 1$, $q \geq 2$ or $q = 1$, $p \geq 2$ then $n \geq 4$ and $k \geq 3$; and
- $p = q = 1$, then the metric $g^{p,q}_{Mtorp}$ does not have positive scalar curvature.

In particular, under the usual coordinates on $(0,b) \times S^p \times S^q$, this isotopy is given by the formula:

$$g_l^n = dt^2 + \big[(1-l)u(t) + l\cos(t)\big]^2 ds_p^2 + \big[(1-l)v(t) + l\sin(t)\big]^2 ds_q^2.$$

Importantly, this isotopy has nice restrictions to the upper hemisphere S^n_+ and equator S^{n-1} by replacing ds_q^2 with ds_{q+}^2 and ds_{q-1}^2 in the above equation. More precisely we obtain, as restrictions, an isotopy

$$g_{l+}^n = dt^2 + \big[(1-l)u(t) + l\cos(t)\big]^2 ds_p^2 + \big[(1-l)v(t) + l\sin(t)\big]^2 ds_{q+}^2 \qquad (5.1)$$

between $g_{0+} = g^{p,q}_{Mtorp+}$ and $g_{1+} = ds^2_{n+}$, the round hemisphere, and an isotopy of the boundary metric on S^{n-1},

$$g_l^{n-1} = dt^2 + \big[(1-l)u(t) + l\cos(t)\big]^2 ds_p^2 + \big[(1-l)v(t) + l\sin(t)\big]^2 ds_{q-1}^2 \qquad (5.2)$$

between $g^{p,q-1}_{Mtorp}$ and ds^2_{n-1}, where ds^2_{q+} and ds^2_{q-1} are the upper-hemisphere and equator metrics on the round S^q.

We now consider the metric $g^{p,q}_{Mtorp,stretch}$ on S^n. Recall this metric decomposes as

$$(S^n, g^{p,q}_{Mtorp,stretch}) = (S^n_+, g^{p,q}_{Mtorp+}) \cup (S^{n-1} \times [-c,c], g^{p,q-1}_{Mtorp} + dr^2) \cup (S^n_-, g^{p,q}_{Mtorp-}).$$

We wish to construct an isotopy of this metric, turning it into $(S^n, g^{p+1,q-1}_{Mtorp})$ in a way which fixes the equator metric. We will focus on the upper-hemisphere

$$(S^n_+, g^{p,q}_{Mtorp,stretch+}) = (S^n_+, g^{p,q}_{Mtorp+}) \cup (S^{n-1} \times [0,c], g^{p,q-1}_{Mtorp})$$

given that the isotopy we constructed here will be mirrored in the obvious way on the lower hemisphere.

We make an important observation. The isotopies given in (5.1) and (5.2) determines an isotopy between $g^{p,q}_{Mtorp,stretch+}$ and the torpedo g^n_{torp}; see figure 5.9.

Precisely it is given by the formula:

$$G_l = \begin{cases} g^n_{l+} & \text{on } S^n_+ \\ g_l^{n-1} + dr^2 & \text{on } S^{n-1} \times [0,c]. \end{cases}$$

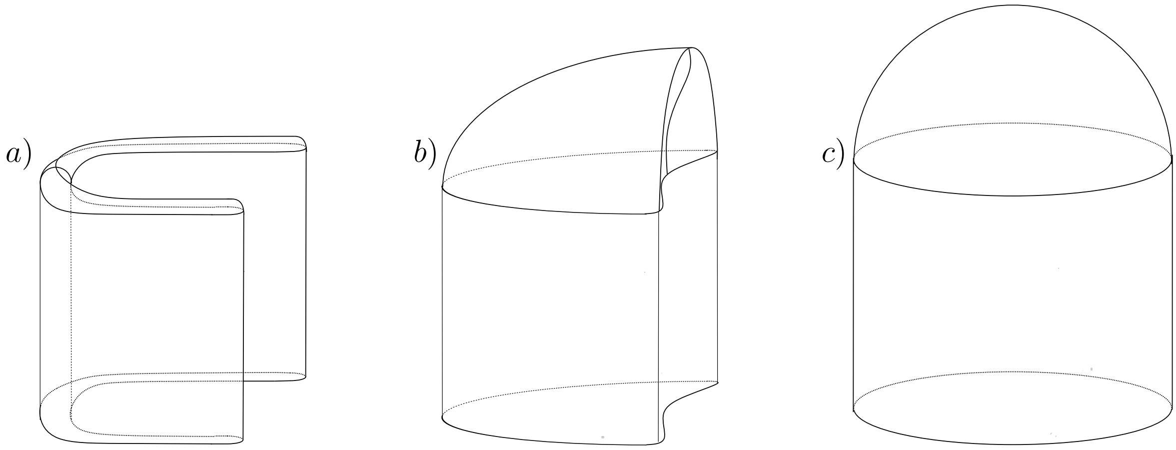

Figure 5.9: Some metrics in isotopy G_l; a) $g^{p,q}_{Mtorp,stretch+}$; b) G_l, $l \in (0,1)$; and c) g^n_{torp}.

where $G_0 = g^{p,q}_{Mtorp,stretch+}$ and $G_1 = g^n_{torp}$.

This isotopy is not sufficient for our purposes as it does not fix the metric on the boundary. However with some careful modification, it can be made to work. We will now make some adjustments to this isotopy, working only on the cylinder $(S^{n-1} \times [0,c], g_l^{n-1} + dr^2)$. For brevity we let $g_l = g_l^{n-1}$.

We define a 2-parameter family of metrics $g_{s,l}$ (schematically described in figure 5.10) as follows:

$$g_{s,l} = \begin{cases} g_l & \text{when } 0 \leq l \leq s \leq 1 \\ g_s & \text{when } 0 \leq s \leq l \leq 1. \end{cases}$$

For each $s \in [0,1]$, there is an isotopy $g_{s,l}$ over $l \in [0,1]$. By Corollary 3.3.1.1 (of Lemma 3.3.1) each such isotopy determines a concordance (on $S^{n-1} \times [0,c]$), which we denote $\overline{g}_s$ between $g_0 = g^{p,q-1}_{Mtorp}$ and g_s on S^{n-1}. The construction in Corollary 3.3.1.1 is easily seen to depend continuously on all parameters and so we can assume that we have a continuous family of concordances, $\overline{g}_s$, $s \in [0,1]$.

In particular for some $\varepsilon > 0$.

$$\overline{g}_s|_{S^{n-1}\times[0,\varepsilon]} = g^{p,q-1}_{Mtorp} + dr^2,$$

which keeps the equator metric, $g^{p,q-1}_{Mtorp}$, fixed, while

$$\overline{g}_s|_{S^{n-1}\times[c-\varepsilon,c]} = g_s^{n-1} + dr^2$$

where recall,

$$g_s^{n-1} = dt^2 + \big[(1-s)u(t) + s\cos(t)\big]^2 ds_p^2 + \big[(1-s)v(t) + s\sin(t)\big]^2 ds_{q-1}^2.$$

This is schematically illustrated in figure 5.11. This isotopy can now be smoothly

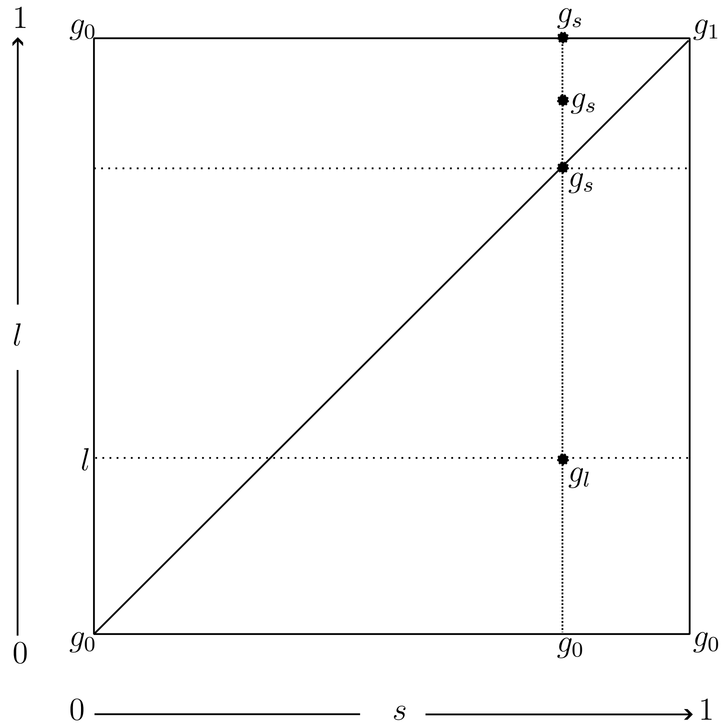

Figure 5.10: Graph of 2- parameter family of metrics, $g_{s,l}$.

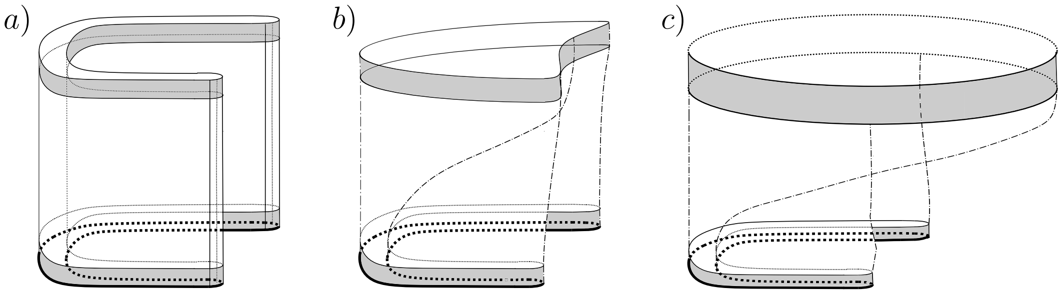

Figure 5.11: Isotopy of concordances a) $g^{n-1}_{0,0}$; b) $g^{n-1}_{0,t}$; and c) $g^{n-1}_{0,1}$.

combined, in the obvious way, with the isotopy g^n_{l+} on S^n_+ to obtain the modified version of the earlier isotopy G_s, which we denote G'_s, defined

$$G'_s = \begin{cases} g^n_{s+} & \text{on } S^n_+ \\ \bar{g}_s & \text{on } S^{n-1} \times [0, c]. \end{cases}$$

This is schematically illustrated in figure 5.12(a), (b). At this stage we have constructed an isotopy through the space of positive Ricci-(k, n) metrics from $G'_0 = g^{p,q}_{Mtorp,stretch+}$ on $S^n_+ \cup (S^{n-1} \times [0, c])$ to a metric, G'_1, which on $S^n_+ \cup (S^{n-1} \times [0, c])$ is a torpedo g^n_{torp}. Moreover, at every stage $s \in [0, 1]$

$$G'_s|_{S^{n-1}\times[0,\varepsilon]} = g^{p,q-1}_{Mtorp} + dr^2.$$

Recall that the round metric ds^2_n is isotopic through the space of positive Ricci-(k, n) metrics to $g^{p,q}_{Mtorp}$ when

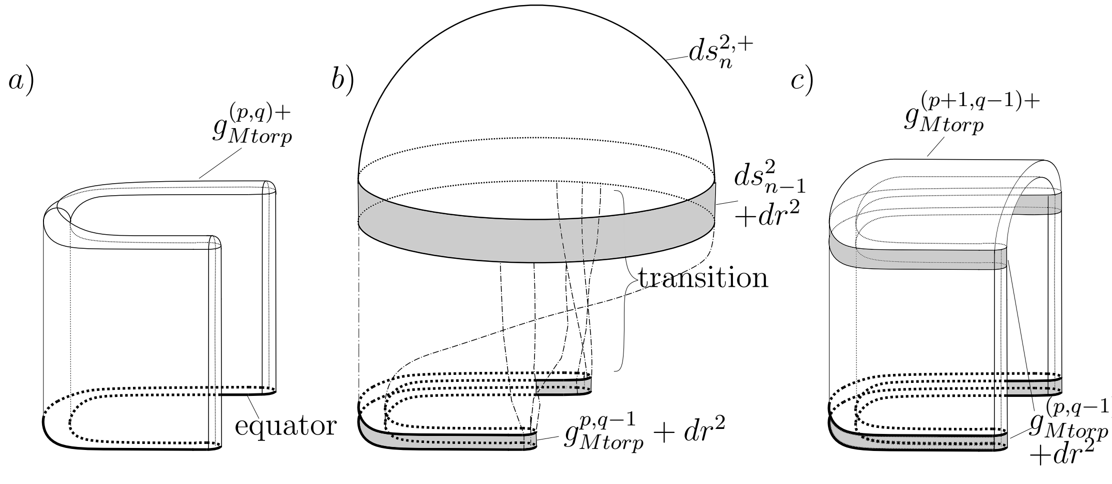

Figure 5.12: a) $g^{(p,q)+}_{Mtorp,stretch}$; b) G_1'; and c) $g^{(p+1,q-1)+}_{Mtorp,stretch}$.

- $p \geq 2$, $q \geq 2$, $n \geq 5$, then $k \geq 2$;
- $p = 1$, $q \geq 2$ or $q = 1$, $p \geq 2$ then $n \geq 4$ and $k \geq 3$; and
- $p = q = 1$, then the metric $g^{p,q}_{Mtorp}$ does not have positive scalar curvature.

Thus we may construct an isotopy between ds_n^2 and $g^{p+1,q-1}_{Mtorp}$, in a completely analogous way to the one we constructed between ds_n^2 and $g^{p,q}_{Mtorp}$ when

- $p \geq 1$, $q \geq 3$, $n \geq 5$, then $k \geq 2$;
- $p = 0$, $q \geq 3$ or $q = 2$, $p \geq 1$ then $n \geq 4$ and $k \geq 3$; and
- $p = 0$, $q = 2$, then the metric $g^{p,q}_{Mtorp}$ does not have positive scalar curvature.

Applying an almost identical construction to the above (with $p+1$ replacing p and $q-1$ replacing q) we can construct an analogous isotopy between G_1' above and $g^{(p+1,q-1)+}_{Mtorp,stretch}$ (depicted schematically in figure 5.12(c)) which, as before, is $g^{p,q-1}_{Mtorp} + dr^2$ near the boundary. Combining these isotopies in the obvious way gives rise to an isotopy between $g^{(p,q)+}_{Mtorp,stretch}$ to $g^{(p+1,q-1)+}_{Mtorp,stretch}$ on $S^n_+ \cup S^{n-1} \times [0, c]$ which fixes the metric as $g^{p,q-1}_{Mtorp} + dr^2$ near the boundary.

5.3.4 Isotopy of metric $g^{p+1,q-1}_{Mtorp,stretch}$ to $g^{p+1,q-1}_{Mtorp}$

We now describe the required isotopy of metric $g^{p+1,q-1}_{Mtorp,stretch}$ to $g^{p+1,q-1}_{Mtorp}$ which keeps the equator metric, $g^{p,q-1}_{Mtorp}$ fixed.

In section 5.3.2 we described the isotopy required on three regions of a τ-neighbourhood of the equator, $S^{n-1} \times [-\tau, \tau]$ of $g^{p,q}_{Mtorp}$ to give $g^{p,q-1}_{Mtorp} + dr^2$. This region is then stretched to give $(S^{n-1} \times [-c, c], g^{p,q-1}_{Mtorp} + dr^2)$.

We isotopy $g^{p+1,q-1}_{Mtorp,stretch}$ to $g^{p+1,q-1}_{Mtorp}$ by reversing this process as follows. We contract the region around the equator, $S^{n-1} \times [-c, c]$ to $S^{n-1} \times [-\tau, \tau]$ on which there is the metric $g^{p,q-1}_{Mtorp} + dr^2$. We now reverse the isotopy described in section 5.3.2 while replacing p in section 5.3.2 by $p+1$ and q by $q-1$.

The calculations for positive Ricci-(k,n) curvature given in section 5.3.2 may be used for this process.

Hence $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}$, $n \geq 5$, with an isotopy that keeps the equator metric fixed for $k \geq 3$ when $p \neq 1$ or $q \neq 2$. Otherwise $k \geq 4$.

This completes the proof of Theorem 5.3.1. $\square$

In Appendix A we give the value of k for which $g^{p,q}_{Mtorp}$ and $g^{p.q-1}_{Mtorp}$ are isotopic for dimensions $n \geq 4$.

Chapter 6

Positive Ricci-(k, n) curvature on the trace of a Gromov-Lawson p-surgery

Let (X, g) be a smooth, Riemannian, n-dimensional manifold in which we embed the sphere-disc product, $S^p \times D^{q+1}$, where $p + q + 1 = n$ and φ is the embedding:

$$\varphi : S^p \times D^{q+1} \hookrightarrow X.$$

We may perform a p-surgery on the embedding by removing the interior of the embedded sphere-disc product, $\varphi(S^p \times D^{q+1})^o$, and gluing the sphere-disc product $D^{p+1} \times S^q$ to the boundary, $\varphi(S^p \times S^q)$, using as gluing instructions the embedding, φ, to create the manifold X_φ:

$$X_\varphi := \overline{X \setminus (S^p \times D^{q+1})} \cup_{\varphi(S^p \times S^q)} (D^{p+1} \times S^q).$$

In Gromov-Lawson surgery, we consider the case when $q \geq 2$, and g is a metric of positive scalar curvature. The goal is to adjust the metric, g, to one which can be modified in the surgery to a positive scalar curvature metric, g_φ, on X_φ. The central result here is that it is possible to change the metric smoothly on a neighbourhood of the embedded sphere so that near the embedded sphere, the metric takes the form $ds_p^2 + g_{torp}^{q+1}$ and yet maintains positive scalar curvature throughout. Thus the metric is now a *surgery-ready metric*. It is now easy to replace $(S^p \times D^{q+1}, ds_p^2 + g_{torp}^{q+1})$ with $(D^{p+1} \times S^q, g_{torp}^{p+1} + ds_q^2)$ to complete the surgery and obtain a positive scalar curvature metric; see figure 6.1.

This technique has been strengthened and extended in various ways. For our purposes the following generalisation to positive Ricci-(k, n) metrics provides an important starting point. Before starting it, let N be the image of

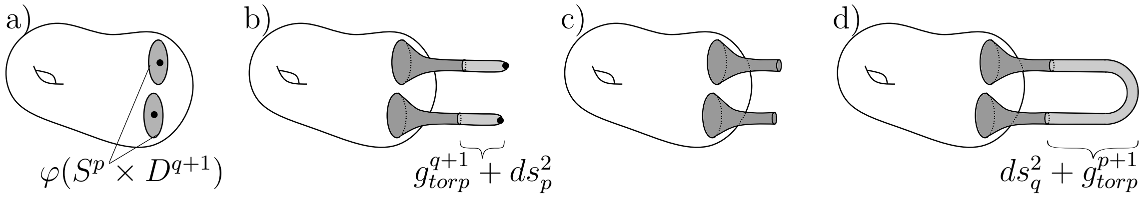

Figure 6.1: Manifold showing a) embedded sphere-disc product; b) standardised metric $ds_p^2 + g_{torp}^{q+1}$; c) preparation of manifold for surgery by removing a neighbourhood of $\varphi(ds_p^2)$; and d) attachment of handle with metric $ds_q^2 + g_{torp}^{p+1}$.

φ, that is $N := \varphi(S^p \times D^{q+1})$. We denote by $\varphi_{\frac{1}{2}}$ the restriction of this embedding to the sphere-disc product, $S^p \times D^{q+1}(\frac{1}{2})$, $\varphi_{\frac{1}{2}} : S^p \times D^{q+1}(\frac{1}{2}) \hookrightarrow X$ where $N_{\frac{1}{2}} := \varphi_{\frac{1}{2}}(S^p \times D^{q+1}(\frac{1}{2}))$. As stated above the Gromov-Lawson surgery construction requires that the metric around the neighbourhood of the embedded sphere become a surgery-ready positive Ricci-(k, n) curvature metric.

Theorem 6.0.1. *(Gromov-Lawson (1980) [16], Schoen-Yau (1979) [35], Wolfson (2009) [53])*

Let X^n be a smooth manifold and $\varphi : S^p \times D^{q+1} \hookrightarrow X$ be an embedding with $p + q + 1 = n$ and $q \geq max\{n + 1 - k, 2\}$. Then for any positive Ricci-(k, n) curvature metric, g, where $2 \leq k \leq n$, there is a positive Ricci-(k, n) curvature metric, g_{std}, so that:

(i) In the neighbourhood $N_{\frac{1}{2}} = \varphi_{\frac{1}{2}}(S^p \times D^{q+1})$, g_{std} pulls back to the metric:

$$\varphi_{\frac{1}{2}}^{\star} g_{std} = ds_p^2 + g_{torp}^{q+1}.$$

(ii) Outside $N = \varphi(S^p \times D^{q+1})$, $g_{std} = g$.

Gromov and Lawson (1980) [16] and Schoen and Yau (1979) [35]) proved this theorem in the case of positive scalar curvature, that is when $k = n$. This result was extended by Wolfson [53] to positive Ricci-(k, n) curvature when $2 \leq k \leq n$. There is another extension of the original positive scalar curvature version due to Labbi [23] to positive (l, n)-intermediate scalar curvature (normally referred to as p-curvature but in this work we have reserved the use of p for p-surgeries) where $q \geq 2 + l$.

Recall the trace of a p-surgery, W_φ, is an $(n+1)$-dimensional, compact manifold with boundary the disjoint union X and X_φ, $W_\varphi := \{W_\varphi; X, X_\varphi\}$. Recall the trace is composed of a cylinder $X \times I$ with a solid disc product, $D^{p+1} \times D^{q+1}$ attached to it using as gluing instructions the embedding $\varphi : S^p \times D^{q+1} \hookrightarrow X \times \{1\}$. Walsh in Theorem 0.2 of [44] and Gajer [15] showed that the trace of a p-surgery on a closed manifold with a positive scalar curvature metric, when $q \geq 2$, admits a positive scalar curvature metric with product structure on the boundary. This result was extended by Burkemper, Searle and Walsh [7] to positive (l,n)-intermediate scalar curvature metrics where $0 \le l \le q-2$ and $q \ge 2$. In this chapter we extend this result to certain positive Ricci-(k,n) curvature metrics.

Theorem A. *Let X be a smooth n-dimensional closed manifold with $n \ge 3$, $\varphi : S^p \times D^{q+1} \hookrightarrow X$ an embedding where $p+q+1 = n$ and let $W_\varphi := \{W_\varphi; X, X_\varphi\}$ denote the trace of a p-surgery on φ. Suppose g is a Riemannian metric on X which has positive Ricci-(k,n) curvature when $2 \le k \le n$ and $p \ne 1$, or $3 \le k \le n$ when $p = 1$. Then provided $q \ge max\{n+1-k, 2\}$, there is a metric $\overline{g}_\varphi$ on W_φ so that*

a) $\overline{g}_\varphi|_X = g$;

b) $\overline{g}_\varphi$ is a product near the boundary ∂W_φ; and

c) $\overline{g}_\varphi$ has positive Ricci-$(k+1, n+1)$ curvature.

Proof. In the proof of Theorem A, we will require a theorem of Kordass [21] that proves that certain spaces of positive Ricci-(k,n) metrics, which are standard near a compact submanifold in X, are weakly homotopy equivalent to the space of all positive Ricci-(k,n) metrics of X, the latter space denoted by $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$. Before stating the version of this theorem we require, we define by $\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)$ the space of metrics with positive Ricci-(k,n) curvature which are standard near S^p on X as

$$\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X) = \{g \in \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X) : \varphi^*_{\frac{1}{2}} g = ds_p^2 + g_{torp}^{q+1}\}.$$

Theorem 6.0.2. *(Kordass) Let X be a smooth, closed, n-dimensional manifold with a p-sphere embedded in X with trivial normal bundle. Let $p+q+1 = n$ and $q \ge max\{n+1-k, 2\}$. Then the inclusion $\mathcal{R}iem_{std}^{Ric^+_{(k,n)}}(X) \hookrightarrow \mathcal{R}iem^{Ric^+_{(k,n)}}(X)$ is a weak homotopy equivalence.*

Recall that the metric $g_{std} \in \mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)$ is defined in 6.0.1.

As the map $\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X) \hookrightarrow \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ is a weak homotopy equivalence, it induces isomorphisms $\pi_i(\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)) \to \pi_i(\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X))$, $i \in \{0, \cdots, n\}$. As $\pi_0(\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)) \to \pi_0(\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X))$ is an isomorphism, there is a bijective correspondence between the path components of $\mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)$ and $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$.

Hence for any $g \in \mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$ there is a metric, $g_{std} \in \mathcal{R}\text{iem}_{std}^{Ric^+_{(k,n)}}(X)$, which is in the same path component of $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$.

As a consequence we have the following corollary.

Corollary 6.0.2.1. *There is an isotopy between g and g_{std}, as defined in Theorem 6.0.1, in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$ provided that the codimension of the compact submanifold, S^p, is $q+1 \geq max\{n+2-k, 3\}$.*

We denote the isotopy in Corollary 6.0.2.1 as g_t, $t \in [0,1]$. By Corollary 3.3.1.1, as g and g_{std} are isotopic they are also concordant. Thus there is a concordance between g and g_{std}, which we call $\overline{g}$ on $X \times I'$. For our purposes $I' = [0, J+2]$ for some $J > 0$ so that $\overline{g} = g + dt^2$ on $X \times [0,1]$ and $\overline{g} = g_{std} + dt^2$ on $X \times [J+1, J+2]$. Figure 6.2 schematically depicts the embedding, N, in X and shows the metric g_t changing on $N \times \{t\}$, $t \in [1, J+1]$ to obtain g_{std}. The metric restricted to $X \setminus N \times \{t\}$ remains unchanged, i.e. $g_{std}|_{X\setminus N \times \{t\}} = g|_{X \setminus N}$.

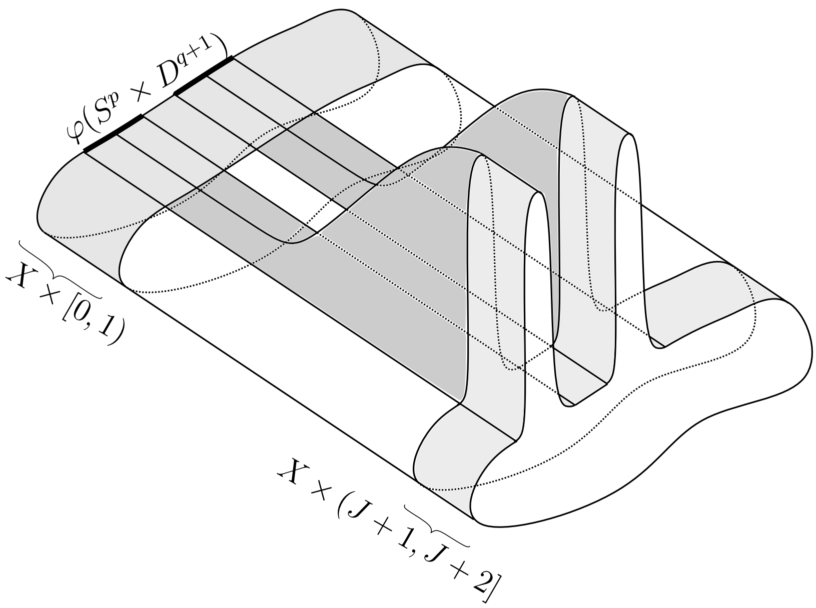

Figure 6.2: The cylinder of $X \times [0, J+2]$ with transition to torpedo metric on $N_{\frac{1}{2}} \times (J+1, J+2]$.

Recall that W_φ consists of a union of $X \times I$ and $D^{p+1} \times D^{q+1}$. Roughly, we wish to equip $X \times [0, L]$, for some $L > 0$, with a metric which is a product $g + dt^2$ near $X \times \{0\}$ and which near the other end of the cylinder takes a "standard" form so that $(D^{p+1} \times D^{q+1}, g^{p+1}_{torp} + g^{q+1}_{torp})$ can be smoothly attached. Moreover this attachment should produce a positive Ricci-$(k+1, n+1)$ metric on W_φ which is a product near the boundary.

Recall the "boot" metric, g^{n+1}_{boot}, described in section 4.4. To make the metric, $\overline{g}$, "surgery-ready" we attach at $(N_{\frac{1}{2}} \times \{J+2\}, ds^2_p + g^{q+1}_{torp})$ a product of the "boot" metric with the standard round metric, $\Big(\big((D^{q+1} \times I) \cup (S^q \times \mathcal{Q}) \cup (D^{q+2}_+)\big) \times S^p, g^{q+2}_{boot} + ds^2_p\Big)$. Figure 6.3 depicts this schematically as attaching a *"pair of boots"*. The boundary of Region 1 of the "boots", $(D^n, g^{q+1}_{torp}(\delta)_{\lambda_1} + ds^2_p)$, section 4.4.1, is attached smoothly to the boundary $(N_{\frac{1}{2}} \times \{J+2\}, g^{q+1}_{torp}(\delta)_{\lambda_1} + ds^2_p)$. This gives the cylinder a length of

$L = J{+}2{+}\lambda_2{+}\lambda_3$. The metric on $X \backslash N_{\frac{1}{2}} \times \{J{+}2\}$, $g_{std}|_{X \backslash N_{\frac{1}{2}}}$, continues unchanged for $(X \backslash N_{\frac{1}{2}}) \times \{t\}$, $t \in (J{+}2, J{+}2{+}\lambda_2{+}\lambda_3]$. The cylinder $X \times [0, J{+}2{+}\lambda_2{+}\lambda_3]$ is depicted schematically in figure 6.3 showing the attachment of a "pair of boots". Note that we have shown in section 4.4 that the metric g^{q+2}_{boot} has positive Ricci-$(3, q+2)$ curvature, provided $q \geq 2$. Hence when $p \neq 1$, the metric $g^{q+2}_{boot} + ds^2_p$ has positive Ricci-$(3, n{+}1)$ curvature. When $p = 1$, the metric $g^{q+2}_{boot} + ds^2_p$ has positive Ricci-$(4, n{+}1)$ curvature.

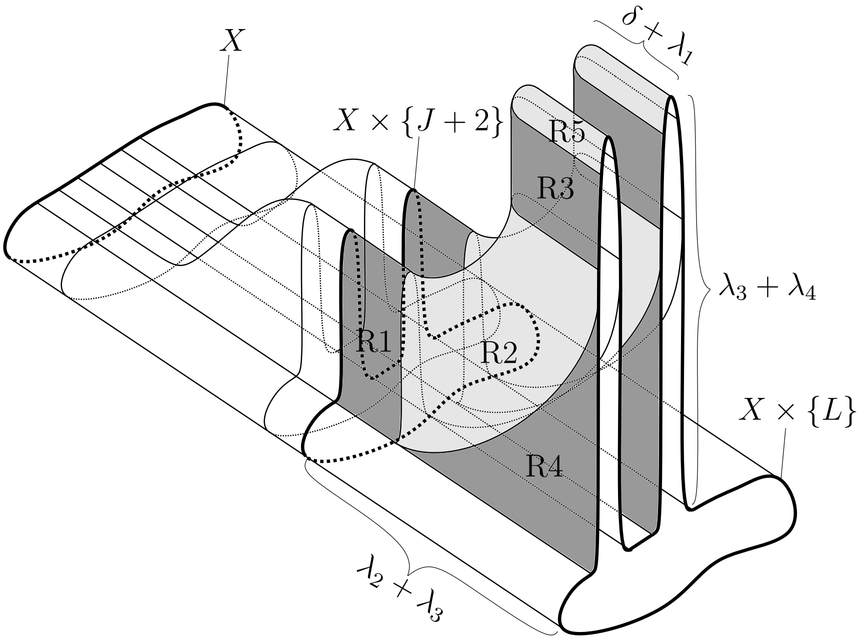

Figure 6.3: The cylinder with boots.

The surgery is performed by removing Region 5, section 4.4.5, of the "boots", $(D^{q+2}_+ \times S^p, g^{q+2}_{torp}(\delta)_{\lambda_1} + ds^2_p)$, leaving the boundary, $(D^{q+1} \times S^p, g^{q+1}_{torp}(\delta)_{\lambda_1} + ds^2_p)$, of Region 3, section 4.4.3, exposed. To this is attached a solid handle $D^{p+1} \times D^{q+1}$ with a mixed torpedo metric, $g^{p+1}_{torp} + g^{q+1}_{torp}(\delta)_{\lambda_1}$, as shown in figure 6.4. The solid handle has positive Ricci-$(3, n{+}1)$ curvature when $p \neq 1$ and positive Ricci-$(4, n{+}1)$ curvature when $p = 1$. This solid handle is attached along part of its boundary, $S^p \times D^{q+1}$, also with metric $g^{q+1}_{torp}(\delta)_{\lambda_1} + ds^2_p$. Hence this may be glued smoothly to Region 3 as shown in figure 6.5 to give the trace of the p-surgery, W_φ. Note that this gives a collar neighbourhood of X_φ of width λ_1 which has a product metric, $\overline{g}|_{X_\varphi} + dt^2$.

Provided $q \geq \max\{n{+}1{-}k, 2\}$, the metric on the trace has positive Ricci-$(3, n{+}1)$ curvature when $p \neq 1$ and positive Ricci-$(4, n+1)$ curvature when $p = 1$. $\square$

Appendix B gives a summary of the Ricci-$(k, n+1)$ curvatures of the trace.

Note that X and X_φ are cobordant. As the metric $\overline{g}_\varphi$ takes a product structure in the collar neighbourhood of X and X_φ we may then join the trace $\{W; X, X_\varphi\}$ to the trace of another surgery $\{W'; X_\varphi, (X_\varphi)_\psi\}$ using the embedding ψ.

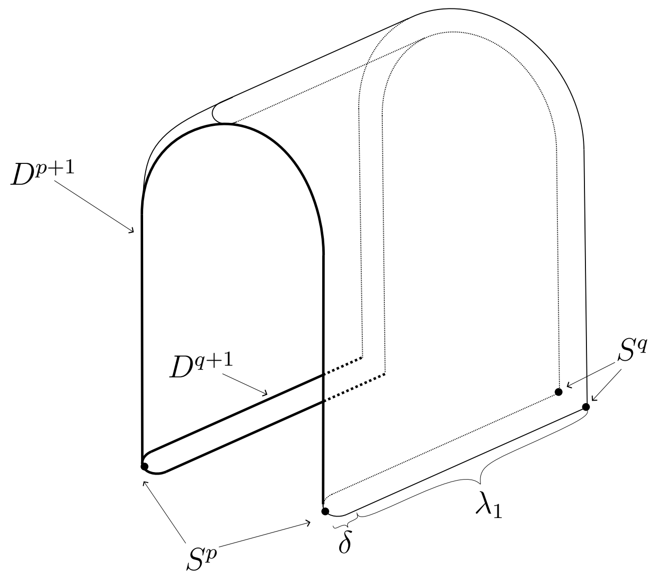

Figure 6.4: The metric $g_{torp}^{p+1} + g_{torp}^{q+1}$ on D^{n+1}

Given an arbitrary cobordism we may equip it with a Morse function with all the critical points assumed to be in the interior. Should the Morse function have a single critical point of index, $\lambda = p + 1$, then the cobordism is the trace of a p-surgery. Where the number of critical points of the Morse function is more, the cobordism may be constructed as the union of traces of surgeries.

The following definition will be used heavily throughout our main results. A *k-admissible Morse function* $f : W \to I$ is one in which all the critical points of the function have index $\lambda \leq \min\{k-1, n-2\}$. This ensures the surgeries associated to the critical points satisfy the codimension conditions of Wolfson.

Theorem 6.0.3. *Let $\{W; X_0, X_N\}$ be a compact cobordism where X_0 is a smooth n-dimensional manifold with $n \geq 3$ and $f : W \to I$ be a $k-$admissible Morse function on W. Suppose g_0 is a Riemannian metric on X_0 which has positive Ricci-(k, n) curvature. Suppose also that either f has no critical points of index 2 and $3 \leq k \leq n$, or f has a critical point of index 2 and $4 \leq k \leq n$. Then*

1. *There is a metric $\overline{g} = \overline{g}(g_0, f)$ on W so that*

   a) $\overline{g}|_{X_0} = g_0$;

   b) $\overline{g}$ is a product near the boundary ∂W; and

   c) $\overline{g}$ has positive Ricci-$(k+1, n+1)$ curvature.

2. *$\overline{g} = \overline{g}(g_0, f)$ depends continuously on g_0 and f.*

Proof. Lemma 2.8 [30] states that the Morse function f may be approximated by a Morse function F such that $F(w_i) \neq F(w_j)$ for all $i, j \in \{1, \cdots, N\}$. Hence it

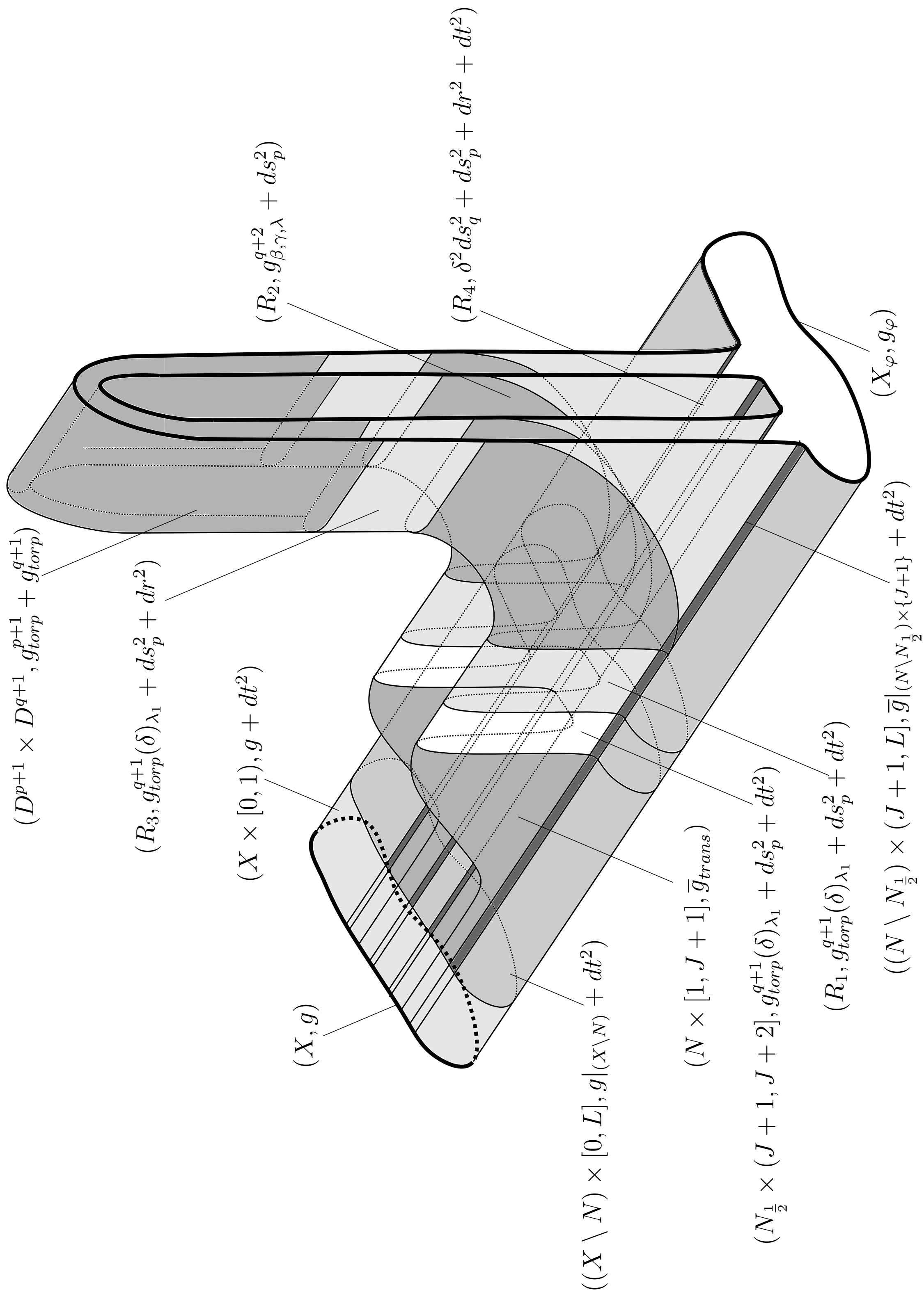

Figure 6.5: Metric, $\bar{g}_\varphi$, on the trace of a p-surgery, W_φ.

is possible to ensure that the Morse function does not have more than one critical point lying in the same level set, $F^{-1}(w)$, for $w \in W$. By Corollary 2.10 [30] any cobordism may be expressed as a composition of elementary cobordisms, $W = \{W_1; X_0, X_1\} \cup \{W_2; X_1, X_2\} \cup \cdots \cup \{W_N; X_{N-1}, X_N\}$; that is each W_i being the trace of a p_i-surgery.

Let the elementary cobordism, $\{W_1; X_0, X_1\}$, be the trace of a p_1-surgery. Recall X_0 has a metric, g_0, with positive Ricci-(k, n) curvature. From Theorem 6.0.1, the metric, g_0, may be extended to a metric, $\overline{g_1}$ on W_1, which restricts as g_1 on X_1 and has positive Ricci-(k, n) curvature provided that $n - p_1 \geq \max\{n + 2 - k, 3\}$. Moreover, by Theorem A the elementary cobordism, $\{W_1; X_0, X_1\}$, has a metric, $\overline{g}_1$, with positive Ricci-$(k + 1, n + 1)$ curvature, where

a) $\overline{g}_1|_{X_0} = g_0$; and

b) $\overline{g}_1$ is a product near the boundary ∂W_1.

The elementary cobordism $\{W_2; X_1, X_2\}$ is the trace of a p_2-surgery with disjoint boundary $\partial W_2 = X_1 \sqcup X_2$. We extend the positive Ricci-(k, n) curvature metric g_1 on X_1 over W_2 in accordance with Theorem A, provided $n - p_2 \geq \max\{n + 2 - k, 3\}$. This process may be repeated for each W_i provided $n - p_i \geq \max\{n + 2 - k, 3\}$, for all $i \in \{1, \cdots, N\}$, to give a metric of positive Ricci-$(k + 1, n + 1)$ curvature on W_i with a product near the boundary $X_{i-1} \sqcup X_i$. Thus W is a manifold which has a metric with positive Ricci-$(k + 1, n + 1)$ curvature. Moreover, by Theorem A the metrics in the neighbourhood of the disjoint boundary, $\partial W = X_0 \sqcup X_N$, are product metrics. This proves part (1).

Part (2) follows exactly as in the positive scalar curvature case proved in Theorem 0.5 of [44]. □

We call the metric $\overline{g} = \overline{g}(g_0, f)$ on W a *Gromov-Lawson cobordism* with respect to g_0 and f. Where W is the cylinder $X_0 \times I$, we call the metric $\overline{g} = \overline{g}(g_0, f)$ a *Gromov-Lawson concordance.*

Chapter 7

The space of positive Ricci-(k, n) curvature metrics

In chapter 6 we constructed a geometric trace of a p-surgery, W_φ, on a manifold, X, extending a positive Ricci-(k, n) curvature metric on X to a Ricci-$(k+1, n+1)$ curvature metric on W_φ with a product metric on the boundary; see Theorem A. In Theorem 6.0.3 we showed that this construction could extend over the union of traces of surgeries. It is possible that the union of traces can result in a cobordism, W, with disjoint boundary, $X \times \{0\} \sqcup X \times \{1\}$. Here the positive Ricci-(k, n) curvature metric on $X \times \{1\}$ is cobordant with the original metric g. This does not mean that the metrics are isotopic, that is they are not necessarily in the same path component in the space of $\mathcal{R}\text{iem}^{Ric^+_{k,n}}(X)$. Indeed, Carr in [8] has proved that the space of Riemannian metrics of S^{4n-1}, $n \geq 2$, with positive scalar curvature has infinitely many path components, using an index obstruction method. This result was extended to positive (l, n)-intermediate scalar curvature, a type of curvature defined by Labbi [23]. Burkemper, Searle and Walsh in [7] showed that the space of metrics on closed, spin $(4n-1)$-manifolds admitting positive $(l, 4n-1)$-intermediate scalar curvature metrics, where $0 \leq l \leq 2n-3$ and $n \geq 2$, has infinitely many path components. We use the manifolds described in [8] and the method used in [8] as adapted by [7] to extend this result to positive Ricci-$(k, 4n-1)$ curvature.

Theorem B. *Let M be a $(4n-1)$-dimensional, smooth, closed, spin manifold, $n \geq 2$, which admits a positive Ricci-(k,4n-1) curvature metric, for $k \geq 2n+1$. Then the space of Riemannian metrics of M, with positive Ricci-(k,4n-1) curvature, $\mathcal{R}iem^{Ric^+_{(k,4n-1)}}(M)$ has infinitely many path components.*

Proof. Carr in [8] describes how to construct a family of $4n$-dimensional manifolds with boundary, $X^{4n}_{r_i}$, which for all $n \geq 2$ and $r_i \geq 1$, have the following properties

- Each $X^{4n}_{r_i}$ is a compact, simply connected, smooth, spin manifold;

- The boundary, $\partial X^{4n}_{r_i}$ is diffeomorphic to S^{4n-1};
- The homology groups of such a manifold are as follows:

$$H_i(X^{4n}_{r_i}) = \begin{cases} \mathbb{Z} & \text{when } i \in \{0, 4n\} \\ \oplus\mathbb{Z} & \text{when } i = 2n \\ 0 & \text{when } i \notin \{0, 2n, 4n\}. \end{cases}$$

- The closed manifold, $W_{r_0,r_1} = X_{r_0} \cup (S^{4n-1} \times I) \cup X_{r_1}$, has $\hat{A}$ genus:

$$\hat{A}(W_{r_0,r_1}) = c(r_0 - r_1),$$

  where c is some constant not equal to zero.

As $\hat{A}(W_{r_0,r_1}) \neq 0$, when $r_0 \neq r_1$, W_{r_0,r_1} does not admit a positive scalar curvature metric, and hence does not admit a positive Ricci-$(k, 4n)$ curvature metric for any k.

Let $W_{r_i} = X_{r_i} \setminus D^{4n}$ be the manifold X_{r_i} with an interior disc, D^{4n}, removed. Thus the disjoint boundary of W_{r_i}, $\partial W_{r_i} = \partial D^{4n} \sqcup \partial X_{r_i}$, has two components both diffeomorphic to S^{4n-1}. The homology of W_{r_i} is concentrated in degree $2n$, which corresponds to index $2n$ critical points, λ_i, of a Morse function on the cobordism, W_{r_i}. In [53] the conditions for positive Ricci-(k, n) curvature to be a surgery stable condition were given as $q \geq \max\{4n - k, 2\}$, where $p + q + 1 = 4n - 1$. Note that $\lambda_i = p + 1$. Thus a $k-$admissible Morse function is one in which k satisfies $2n + 1 \leq k \leq 4n - 1$. Using the Morse function we may decompose W_{r_i} into a finite union of elementary cobordisms:

$$\{W_{r_i}; \partial D^{4n}, \partial X_{r_i}\} = \{Y_1; S^{4n-1}_0, S^{4n-1}_1\} \cup \{Y_2; S^{4n-1}_1, S^{4n-1}_2\} \cup \cdots \cup \{Y_m; S^{4n-1}_{m-1}, S^{4n-1}_m\},$$

where $S^{4n-1}_0 = \partial D^{4n}$ and $S^{4n-1}_m = \partial X_{r_i}$. Each of the elementary cobordisms, Y_i, is a trace of a $(2n - 1)$-surgery. The initial surgery takes place on $\partial D^{4n} = S^{4n-1}$, which we equip with a round metric, ds^2_{4n-1}. Using Theorem 6.0.3 we may extend the positive Ricci-$(k, 4n-1)$ curvature metric, where $k \geq 2n+1$, through the elementary cobordisms to obtain a positive Ricci-$(k+1, 4n)$ curvature metric, $\overline{g}_{r_i}$, on W_{r_i}. From Theorem 6.0.3, W_{r_i} has a product structure on its boundary and $\overline{g}_{r_i}|_{\partial X_{r_i}}$ is a positive Ricci-$(k, 4n - 1)$ curvature metric on S^{4n-1}, where $2n + 1 \leq k \leq 4n - 1$.

Using the construction above we equip two manifolds, W_{r_0} and W_{r_1}, $r_0 \neq r_1$, with positive Ricci-$(k+1, 4n)$ curvature metrics, $\overline{g}_{r_0}$ and $\overline{g}_{r_1}$ respectively, where $k \geq 2n+1$. As W_{r_i} have product metrics on the boundary we can smoothly cap the boundary ∂D^{4n}, where the boundary metric is the standard round one, with a disc equipped with a standard torpedo metric, g^{4n}_{torp}. The two manifolds, $W_{r_i} \cup D^{4n}$, each have a product boundary, S^{4n-1}_i, with metric $g_i = \overline{g}_{r_i}|_{S^{4n-1}_i}$. If these metrics are isotopic then by Corollary 3.3.1.1 they are also concordant. Therefore a concordance, $\overline{h}_{0,1}$, on $S^{4n-1} \times I$ would exist where $\overline{h}_{0,1}$ is a positive Ricci-$(k+1, 4n)$ curvature metric, such that the metric on the boundaries are g_i, $i \in \{0, 1\}$. Hence it would be possible to connect the two manifolds, $W_{r_0} \cup D^{4n}$ and $W_{r_1} \cup D^{4n}$, using the concordance $\overline{h}_{0,1}$ to give a closed manifold W_{r_0,r_1}, equipped with a positive Ricci-$(k+1, 4n)$ curvature metric where $k \geq 2n+1$. As W_{r_0,r_1} has $\hat{A}$ genus not equal to zero it does not admit a positive scalar curvature metric and therefore does not admit a positive Ricci-$(k+1, 4n)$ curvature metric. Hence the concordance does not exist and, therefore, by Lemma 3.3.1, there does not exist an isotopy between g_0 and g_1, that is, g_0 and g_1 lie in different path components of $\mathcal{R}iem^{Ric^+_{k,4n-1}}(S^{4n-1})$. This is the case for any g_i and g_j, where $i \neq j$ and therefore $\mathcal{R}\text{iem}^{Ric^+_{k,4n-1}}(S^{4n-1})$, has infinitely many path components.

We now wish to show that this result can be extended from $\mathcal{R}\text{iem}^{Ric^+_{(k,4n-1)}}(S^{4n-1})$ to the space of positive Ricci-$(k, 4n-1)$ metrics on M, $\mathcal{R}\text{iem}^{Ric^+_{(k,4n-1)}}(M)$. We use the method of connected sums which is given in more detail by Burkemper, Searle and Walsh in [7]. From Theorem 6.0.1 of [53], Wolfson has shown that the connected sum of two manifolds, (M_1, g_1), (M_2, g_2), where g_1 and g_2 are positive Ricci-(k, n) curvature metrics, also admits a positive Ricci-(k, n) curvature metric, $g_1 \# g_2$, for $2 \leq k \leq n$ and $n \geq 3$. The proof uses the method of Gromov and Lawson in [16]. We note that the connected sum of two such manifolds may be achieved by connecting them using a 1-handle and a 1-handle is the result of 0-surgery.

We note that the connected sum $S^{4n-1} \# M$ is diffeomorphic to M. We equip both S^{4n-1} and M with positive Ricci-$(k, 4n-1)$ curvature metrics, g_i and g_M respectively, where $2n+1 \leq k \leq 4n-1$. We show below that where $r_i \neq r_j$, each $g_M \# g_i$ lie in different path components of $\mathcal{R}\text{iem}^{Ric^+_{k,4n-1}}(M)$.

Let W_{r_i}, $i \in \{0, 1\}$, be the manifolds previously constructed as the unions of traces of $(2n-1)$ surgeries with disjoint boundaries, $\partial D^{4n}_i \sqcup \partial X_{r_i}$. Theorem A shows that there are disjoint collar embeddings $\tau_{i,j} : S^{4n-1} \times [0, \varepsilon] \hookrightarrow W_{r_i}$, $i, j \in \{0, 1\}$ on which there are product metrics. The boundaries, ∂D^{4n}_i and ∂X_{r_i} have positive Ricci-$(k, 4n-1)$ curvature metrics, ds^2_{4n-1} and g_i, respectively. Let $\gamma_i(t)$, $t \in [0, 1]$ be an embedded path in W_{r_i} such that $\gamma_i(0) \in \partial D^{4n}_i \subset \partial W_{r_i}$ and $\gamma_i(1) \in \partial X_{r_i} \subset \partial W_{r_i}$. Moreover $\tau^{-1}_{i,0} \circ \gamma_i(t) = t$, when t is near 0, and $\tau^{-1}_{i,1} \circ \gamma_i(t) = 1-t$, when t is near 1.

Let $\gamma(t)$, $t \in [0, 1]$, be a path embedded in the cylinder $M \times I$ such that $\gamma(0) = (x_0, t) \in M \times [0, 1]$ where x_0 is a fixed point. We now remove tubular neighbourhoods around the paths γ_i and γ. We then perform a slicewise connected sum $Z_{r_i} = W_{r_i} \# (M \times I)$ where ∂D^{4n}_i is identified with $M \times \{0\}$ and $\partial X_{r_i} = S^{4n-1}_i$ is identified with $M \times \{1\}$; see figure 7.1. Thus $\partial Z_{r_i} = \partial D^{4n}_i \# M \sqcup \partial X_{r_i} \# M$, with both $\partial D^{4n}_i \# M$ and $\partial X_{r_i} \# M$ being diffeomorphic to M.

The metric on Z_{r_i}, g_{Zri} is a positive Ricci-$(k+1, 4n)$ curvature metric where

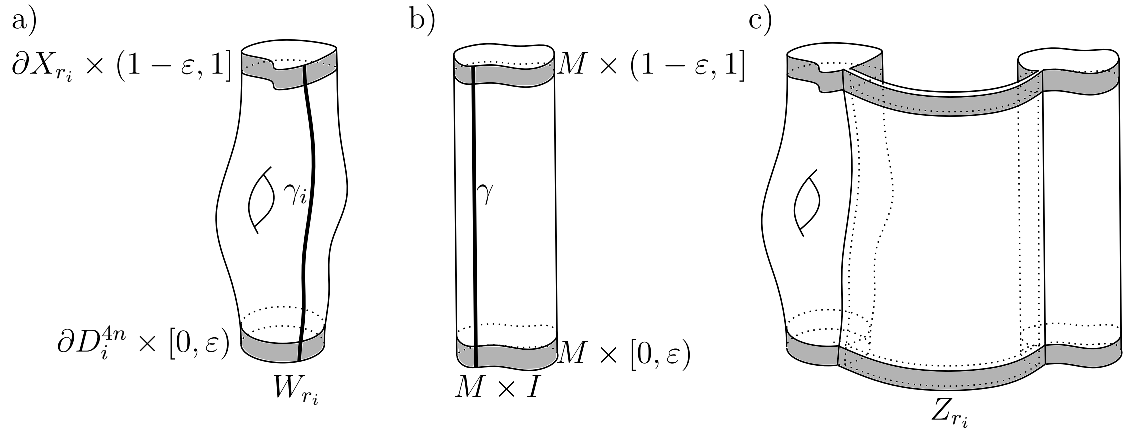

Figure 7.1: a) Union of traces of surgery W_{r_i}; and b) Cylinder $M \times I$; and c) Connected sum Z_{r_i}.

$k \geq 2n+1$ with a product structure on the boundary. The positive Ricci-$(k, 4n-1)$ curvature metric on the boundary $\partial D_i^{4n} \# M$ and $\partial X_{r_i} \# M$ being $ds^2_{4n-1} \# g_M$ and $g_i \# g_M$, respectively.

Let $Z_{r_0,r_1} = Z_{r_0} \cup Z_{r_1}$, be the manifold formed by gluing $(\partial D_0^{4n} \# M) \times \{0\}$ and $(\partial D_1^{4n} \# M) \times \{0\}$ together. Hence $\partial Z_{r_0,r_1} = \partial X_{r_0} \# M \sqcup \partial X_{r_1} \# M$ with metrics $g_{0M} = g_0 \# g_M$ and $g_{1M} = g_1 \# g_M$, respectively.

We now continue the proof by contradiction. We assume that the metrics g_{0M} and g_{1M} are isotopic and that therefore there exists a positive Ricci-$(k+1, 4n)$ concordance between the two metrics, $\overline{h}_{0M,1M}$, on the cylinder diffeomorphic to $M \times I$. We make the closed manifold $Z_{r_0,r_1,h_{0M,1M}}$ by attaching each of the boundaries of the concordance to the boundaries of Z_{r_0,r_1}. We now have a closed positive Ricci-$(k+1, 4n)$ curvature metric manifold,

$$Z_{r_0,r_1,h_{0M,1M}} = Z_{r_0,r_1} \cup \overline{h}_{0M,1M},$$

with $Z_{r_0,r_1,h_{0M,1M}}$ diffeomorphic to $W_{r_0,r_1} \# (M \times S^1)$. The $\hat{A}$ genus of $M \times S^1$ is zero as $M \times S^1$ admits a positive Ricci-$(k+1, 4n)$ curvature metric, $k \geq 2n+1$ and hence a positive scalar curvature metric. From the additive property of the $\hat{A}$-genus on connected sums

$$\hat{A}(Z_{r_0,r_1,h_{0M,1M}}) = \hat{A}(W_{r_0,r_1}) + \hat{A}(M \times S^1) = \hat{A}(W_{r_0,r_1}).$$

As $r_0 \neq r_1$, then $\hat{A}(Z_{r_0,r_1,h_{0M,1M}}) \neq 0$. Therefore $Z_{r_0,r_1,h_{0M,1M}}$ does not have a metric of positive scalar curvature and therefore not a metric of positive Ricci-$(k+1, 4n)$ curvature for any k. We therefore have a contradiction and the concordance $\overline{h}_{iM,jM}$ between the metrics, g_i and g_j, $i \neq j$, does not exist. Thus the metrics, g_i, $i \in \mathbb{Z}$, lie in different path components of $\mathcal{R}\text{iem}^{Ric^+_{(k,4n-1)}}(S_i^{4n-1} \# M)$. As $S_i^{4n-1} \# M$ is diffeomorphic to M, there are countably infinite number of path components of $\mathcal{R}\text{iem}^{Ric^+_{(k,4n-1)}}(M)$. $\square$

Chapter 8

Gromov-Lawson Concordance and Isotopy

In this chapter we will prove the main theorem, Theorem C. This is a generalisation to positive Ricci-(k, n) curvature metrics of Walsh's Theorem 5.1 in [44]. He proved that for some positive scalar curvature metric Gromov-Lawson concordances, the positive scalar curvature metrics on the disjoint boundaries are isotopic. The proof makes use of "cancelling surgeries" and so we begin with a brief reveiw.

8.1 Cancelling surgeries

We will use Theorems 5.4 and 6.4 for cancelling surgeries in [30].

Theorem 2.3.1 ((Morse) Strong Cancellation Theorem). *Let $\{W; X_0, X_1\}$ be a simply connected, smooth, compact cobordism. Let $f = (f, \mathfrak{m}, V)$ be a Morse triple, where $f : W \to I$, satisfies the following:*

(a) The function f has exactly two critical points w_1 and w_2 and $0 < f(w_1) < c < f(w_2) < 1$.

(b) The critical points w_1 and w_2 have Morse index $p + 1$ and $p + 2$, respectively, where $1 \leq p \leq n - 4$.

(c) For each $t \in (f(w_1), f(w_2))$, the trajectory sphere $S^q_{t,+}(w_1)$ emerging from the critical point w_1 and the trajectory sphere $S^{p+1}_{t,-}(w_2)$ converging towards the critical point w_2 have intersection number ± 1.

Then

(i) W is diffeomorphic to $X_0 \times I$.

(ii) *The gradient-like vector field V may be altered near $f^{-1}(c)$ so that $S^q_{c,+}(w_1)$ and $S^{p+1}_{c,-}(w_2)$ intersect transversely at a single point in $f^{-1}(c)$. The union of these intersection points forms a trajectory arc from w_1 to w_2.*

(iii) *The gradient-like vector field V may be further perturbed in a small neighbourhood of the trajectory arc from critical point w_1 to w_2 giving a nowhere zero vector field, V'. All the trajectories now commence at X_0 and end at X_1.*

(iv) *The gradient-like vector field V' is now the gradient of a function f' without critical points which agrees with f outside a neighbourhood of the aforementioned trajectory arc between the critical points.*

Let X_0 be a closed, smooth, simply-connected n-dimensional manifold with embedding

$$\varphi : S^p \times D^{q+1} \hookrightarrow X_0,$$

where $p+q+1 = n$. Let N be the image of φ, that is $N := \varphi(S^p \times D^{q+1})$. We denote by $\varphi_{\frac{1}{2}}$ the restriction of this embedding to the sphere-disc product, $S^p \times D^{q+1}(\frac{1}{2})$, $\varphi_{\frac{1}{2}} : S^p \times D^{q+1}(\frac{1}{2}) \hookrightarrow X$ where $N_{\frac{1}{2}} := \varphi_{\frac{1}{2}}(S^p \times D^{q+1}(\frac{1}{2}))$. Let $f : W \to I$ be a Morse function satisfying the conditions of Theorem 2.3.1 with critical points w_1 and w_2 with Morse index $p+1$ and $p+2$ respectively, where $0 < f(w_1) < f(w_2) < 1$. Level sets $f^{-1}(t)$ inside the Morse coordinate chart for the critical point w_1 are diffeomorphic to $S^p \times D^{q+1}$ when $t \in [0, f(w_1))$ and $D^{p+1} \times S^q$ when $t \in (f(w_1), f(w_2))$. Similarly inside the Morse coordinate chart for the critical point w_2, level sets are diffeomorphic to $S^{p+1} \times D^q$ when $t \in (f(w_1), f(w_2))$ and to $D^{p+2} \times S^{q-1}$ when $t \in (f(w_2), 1]$. Hence W is diffeomorphic to the union of two traces, the first being the trace of a p-surgery on X_0 to give X_φ followed by the trace of a $(p+1)$-surgery on X_φ. Let U and $U_{\frac{1}{2}}$ denote the union of the trajectory flows emanating from N and $N_{\frac{1}{2}}$ respectively. Figures 8.1(a) and (b) show graphs of the neighbourhood U of the trajectory discs of the p-surgery (see figure 2.4(b)) restricted to the level sets $f^{-1}(0+\varepsilon)$, $f^{-1}(f(w_1)-\varepsilon)$, $f^{-1}(f(w_1))$ and $f^{-1}(f(w_1)+\varepsilon)$.

We denote the incoming sphere, S^p, to the p-surgery as $S^p_-(w_1)$ and the outgoing sphere S^q as $S^q_+(w_1)$. Similarly the incoming sphere, S^{p+1} of the $(p+1)$-surgery is denoted $S^{p+1}_-(w_2)$. From Theorem 2.3.1 we know that the vector field, V, may be altered near $f^{-1}(c)$, $c \in (f(w_1), f(w_2))$, so that $S^{p+1}_-(w_2)$ and $S^q_+(w_1)$ intersect transversely at a single point, say α, in $f^{-1}(c)$. This gives rise to a single trajectory arc from w_1 to w_2 and the vector field may be perturbed on a small neighbourhood of this arc to give a nowhere zero gradient-like vector field, V', of a Morse function f'. The vector fields, V and V' differ only on the small neighbourhood and f' has no critical points. Thus W is now diffeomorphic to $X_0 \times I$.

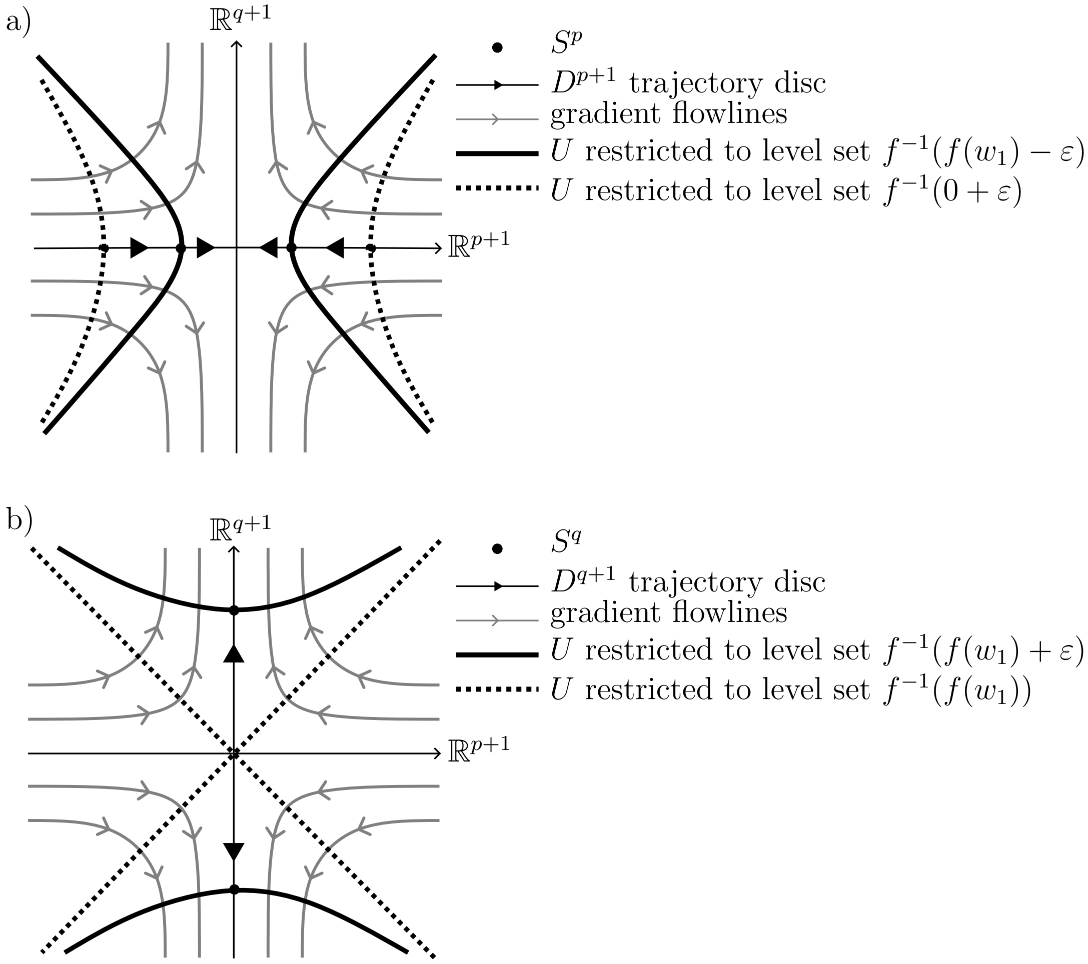

Figure 8.1: a) Graph of neighbourhood U restricted to level sets $f^{-1}(0+\varepsilon)$ and $f^{-1}(f(w_1)-\varepsilon)$; and b)) Graph of neighbourhood U restricted to level sets $f^{-1}(f(w_1))$ and $f^{-1}(f(w_1)+\varepsilon)$.

8.2 Gromov-Lawson concordance

Let X be a smooth, closed n-dimensional manifold, where $p+q+1=n$, and $W = X \times I$. Then let $f : W \to I$ be a k-admissible Morse function satisfying the conditions of Theorem 2.3.1. We decompose W as $W \cong W_p \cup W_{p+1}$ where W_p is the trace of the p-surgery associated with the critical point w_1 and W_{p+1} is the trace of the $(p+1)$-surgery associated with the critical point w_2. Let $X \times \{0\}$ be equipped with a metric of positive Ricci-(k,n) curvature, g_0. We showed in Theorem 6.0.3 that this metric can be extended over the traces of the surgeries to give a metric with positive Ricci-$(k+1,n+1)$ curvature, $\overline{g} = \overline{g}(g_0, f)$, with a product structure near the boundary, provided $q \geq \max\{n+2-k, 3\}$. Note that Wolfson's condition for $(p+1)$-surgery is $q-1 \geq \max\{n+1-k, 2\}$. We denote the positive Ricci-(k,n) curvature metric on $X \times \{1\}$, $g_1 := \overline{g}|_{X\times\{1\}}$. Hence $\overline{g}$ is a

Gromov-Lawson concordance between g_0 and g_1. Walsh in [47] showed that g_0 and g_1 are isotopic in the space of positive scalar curvature metrics, $\mathcal{R}\text{iem}^+(X)$. We wish to extend this result and show that these metrics are isotopic in the space of positive Ricci-(k,n) curvature metrics on X, $\mathcal{R}\text{iem}^{Ric^+_{k,n}}(X)$, where $k \geq 3$, when $p \neq 1$ and $k \geq 4$ when $p = 1$.

We schematically illustrate the Gromov-Lawson p-surgery associated with the critical point w_1. The sphere-disc product is embedded in the manifold, X, via some embedding $\varphi : S^p \times D^{q+1} \hookrightarrow X$; see figure 6.1(a). Note that here S^p is the incoming trajectory sphere, $S^p_-(w_1)$ on the level set $f^{-1}(0)$. The metric, g_0, on X is adjusted on $N = \varphi(S^p \times D^{q+1})$ to a surgery-ready torpedo metric on $N_{\frac{1}{2}} := \varphi_{\frac{1}{2}}(S^p \times D^{q+1}(\frac{1}{2}))$ to give the metric we denote g_{std}; see figure 8.2. In this setting, the embedded sphere S^p bounds an embedded $(p+1)$-dimensional disc in X, which when the boundary collapses to w_1 at the first critical point, emerges as $S^{p+1}_-(w_2)$, the trajectory sphere associated to the second critical point. We denote this disc D^{p+1}_-.

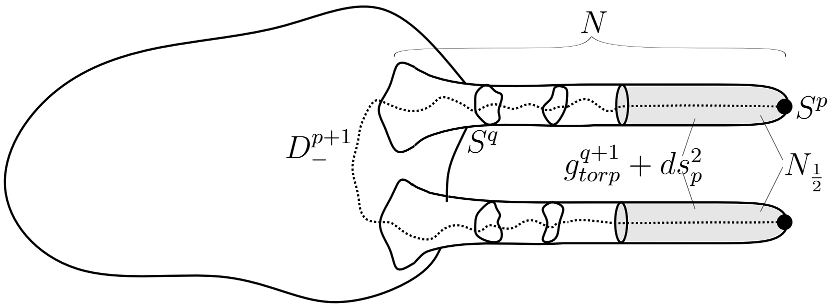

Figure 8.2: Surgery ready metric, g_{std}.

The p-surgery takes place by attaching the handle $S^q \times D^{p+1}$ via φ. The disc factor of the handle we denote D^{p+1}_+ as shown in figure 8.3. We now have the disc D^{p+1}_- in figure 8.2 joined to the disc D^{p+1}_+ in the handle $(D^{p+1}_+ \times S^q, g^{p+1}_{torp} + ds^2_q)$ to give $S^{p+1} = D^{p+1}_- \cup D^{p+1}_+$, see figure 8.3. We denote the resulting manifold (X_φ, g_φ).

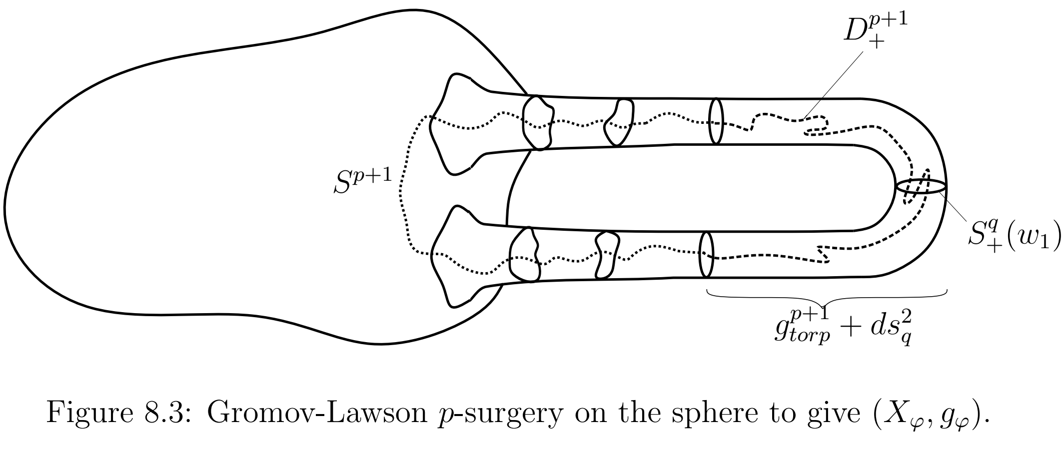

Figure 8.3: Gromov-Lawson p-surgery on the sphere to give (X_φ, g_φ).

We now consider the second $(p+1)-$ surgery corresponding to w_2. This can be characterised with an embedding $\psi : S^{p+1} \times D^q \hookrightarrow f^{-1}(c)$, $c \in (f(w_1), f(w_2))$.

In figure 8.3 we show the sphere, S^{p+1}. Recall that on the level sets $f^{-1}(c)$, $c \in (f(w_1), f(w_2))$, the incoming sphere to the critical point w_2 is denoted $S_-^{p+1}(w_2)$ and the outgoing sphere from the critical point w_1 is denoted $S_+^q(w_1)$. In section 8.1 we noted that the vector field V may be adjusted so that near $f^{-1}(c)$, $c \in (f(w_1), f(w_2))$, $S_-^{p+1}(w_2)$ and $S_+^q(w_1)$ intersect transversely at a single point, α. We show in figure 8.4(a) the effect of such an adjustment to give a transverse intersection of $S_-^{p+1}(w_2)$ and S_+^q at a single point α. We wish to adjust the Morse function f so that near α there is a standard torpedo metric g_{torp}^{p+1} on the incoming sphere, $S_-^{p+1}(w_2)$. For this we require that near α part of $S_-^{p+1}(w_2)$ in $f^{-1}(c)$ has the torpedo metric g_{torp}^{p+1} on D_+^{p+1}, as shown in figure 8.4(b).

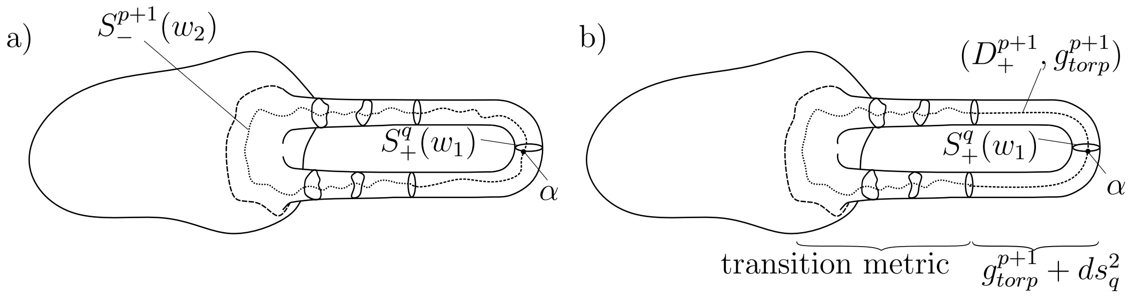

Figure 8.4: a) Single transversal intersection $S_-^{p+1}(w_2) \cap D_+^q(w_1) = \alpha$; and b) Standard torpedo metric on $S_-^{p+1)}(w_2)$ near α

Using the fact that now $S_-^{p+1}(w_2)$ intersects $S_+^q(w_1)$ transversely at a single point α, Walsh in Lemma 4.7 of chapter 4 of [44], constructs an isotopy through admissible Morse functions adjusting f near α so that locally $S_-^{p+1}(w_2)$, in $f^{-1}(c)$, coincides with the disc factor D^{p+1} of the attached handle $D^{p+1} \times S^q$, as shown in figure 8.4(b). From there f can be further isotoped so that the trajectory sphere $S_-^{p+1}(w_2)$ consists of $D_-^{p+1} \cup D_+^{p+1}$. Recall that D_-^{p+1} is the part of the trajectory sphere arising from the disc bounded by the embedded sphere $\varphi(S^p)$ associated with the first critical point, w_1; and D_+^{p+1} is the disc factor of the attached handle $D^{p+1} \times S^q$.

Note that this is a continuous adjustment of the Morse function and goes through in the positive Ricci-(k,n) curvature case exactly as in the positive scalar curvature case that Walsh is considering in [44], following Theorem 6.0.3, part 2. Hence on $S_-^{p+1}(w_2) = D_-^{p+1} \cup D_+^{p+1}$, we have D_+^{p+1} with a standard torpedo metric and D_-^{p+1} with a non-standard metric.

All metric adjustments take place on the neighbourhood of $D_-^{p+1}(w_1)$. We call this neighbourhood D and note that $D \cong D^n$. Using the Gromov-Lawson construction we obtain a standard metric, a torpedo metric, on D_-^{p+1}; see figure 8.5(a). We denote the resulting metric on X_φ as $g_{\varphi,std}$. The metric $g_{\varphi,std}$ consists of three regions:

- A region diffeomorphic to $X \setminus D$, where the restricted metric $g_{\varphi,std}|_{X\setminus D}$ is unchanged from the original metric, $g_0|_{X\setminus D}$;

- A region diffeomorphic to $S^{n-1}\times I$, the connecting cylinder, which we denote as T on which $g_{\varphi,std}|_T$ is a transition metric which has positive Ricci-(k,n) curvature by Theorem 6.0.1. One of the boundaries of T is ∂D^n and the other is a boundary with corners. T is shown in dark grey in figure 8.5(a); and

- A region diffeomorphic to $(S^{p+1}\times S^q)\setminus D^n$, denoted V, where the restricted metric $g_{\varphi,std}|_V$ has positive Ricci-(k,n) curvature by Theorem 6.0.1. As shown in figure 8.5(a) the region V may be decomposed into two further regions, $V = R\cup B_p$. The region B_p is a cobordism between S^{n-1} and $S^{p+1}\times S^{q-1}$. The region R, shown in light grey in figure 8.5 is diffeomorphic to $S^{p+1}\times D^q$ with a standard metric $g^{p+1}_{Dtorp}+g^q_{torp}$.

Note that $X_\varphi = (X\setminus D)\cup T\cup V$.

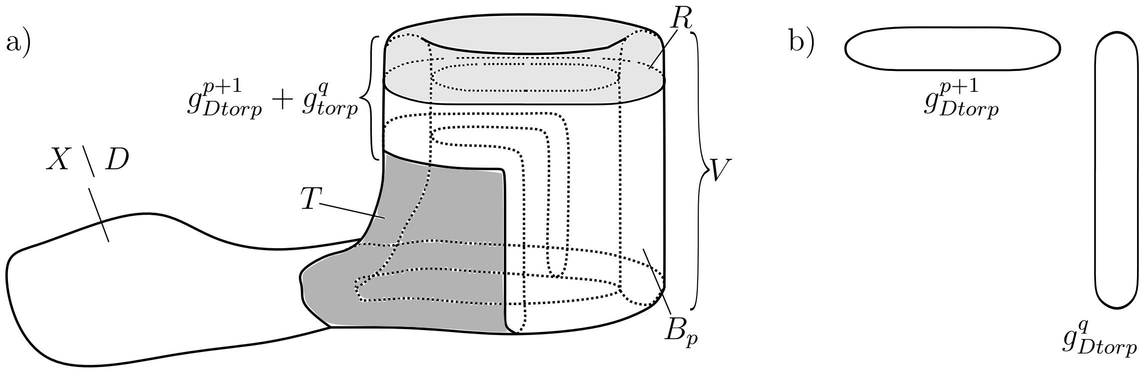

Figure 8.5: a) The manifold $(X_\varphi, g_{\varphi,std})$; and b) Decomposition of metric at top of manifold.

The manifold $(X_\varphi, g_{\varphi,std})$ is now prepared for $(p+1)$-surgery. We use the Gromov-Lawson construction for the $(p+1)$-surgery and remove $R\cong(S^{p+1}\times D^q, g^{p+1}_{Dtorp}+g^q_{torp})$. We attach the hollow handle $(D^{p+2}\times S^{q-1}, g^{p+2}_{torp}+ds^2_{q-1})$, denoted B_{p+1}, and glue on the common boundary $S^{p+1}\times S^{q-1}$.

Note that the manifold after the two surgeries is diffeomorphic to X. We denote the metric on X after the two cancelling surgeries $g_{p,p+1}$. We have given in figure 8.6 the schematic diagram of the metric on a manifold after cancelling surgeries, which we denote $(X, g_{p,p+1})$. We analyse the metric $g_{p,p+1}$ as follows:

- The metric $g_{p,p+1}$ restricted to $X\setminus D$ is $g_0|_{X\setminus D}$;

- There is a positive Ricci-(k,n) curvature transition metric, $g_{p,p+1}|_T = g_{\varphi,std}|_T$ on T provided $q-1\geq \max\{n+1-k,2\}$, in accordance with Theorem 6.0.1 in [53]. T is shown in dark grey in figure 8.6;

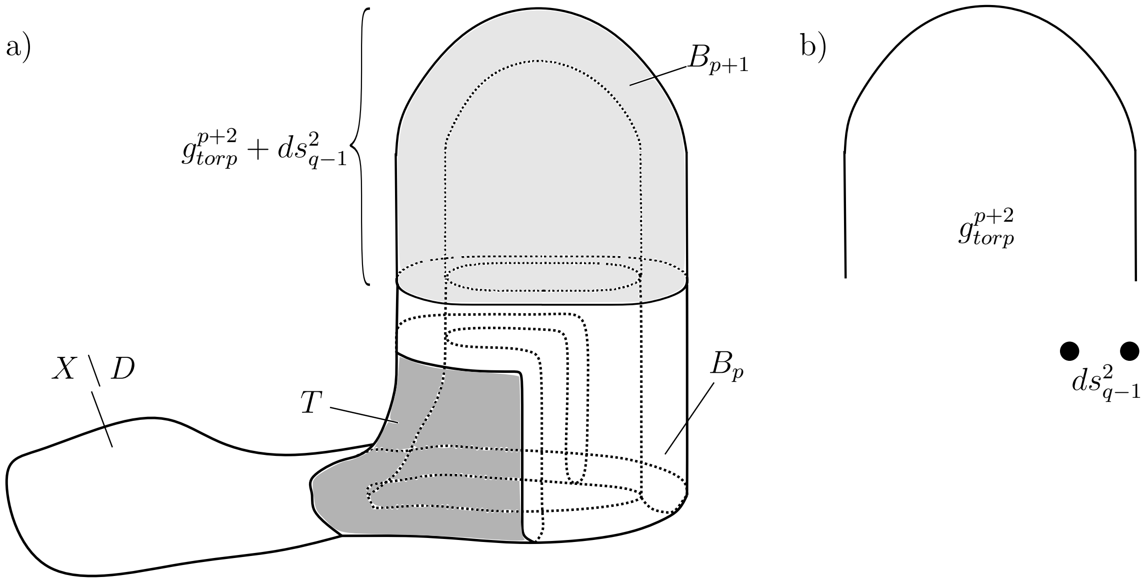

Figure 8.6: a) Manifold after $p+1$-surgery to give $(X, g_{p,p+1})$; and b) Decomposition of metric, $g_{p,p+1}$, at top of manifold.

- The metric $g_{p,p+1}$ restricted to B_p is the same as $g_{\varphi,std}|_{B_p}$; and
- The metric $g_{p,p+1}$ restricted to B_{p+1} is $g^{p+2}_{torp} + ds^2_{q-1}$. This metric has positive Ricci-(k, n) curvature for $k \geq 2$ when $q \neq 2$ and $k \geq 3$ when $q = 2$.

Walsh proved that in the case of positive scalar curvature metrics, the metrics g_0 on X and $g_{p,p+1}$ on X are isotopic. We will extend this result to positive Ricci-(k, n) curvature metrics.

8.3 Isotopy of positive Ricci-(k, n) curvature metrics in some Gromov-Lawson concordances

Theorem C. *Let X be simply connected, smooth, closed manifold of dimension $n \geq 5$. Let $f = (f, \mathfrak{m}, V)$ be a k-admissible Morse triple, $k \geq 3$, on $X \times I$, satisfying:*

1. *The conditions of Morse-Smale's Cancellation Theorems 2.3.1;*
2. *$n - \lambda_i \geq max\{n + 1 - k, 2\}$ where λ_i, $i \in \{1, 2\}$ are the indices of the two Morse critical points; and*
3. *If $\lambda_i = 2$, $k \geq 4$. In all other cases $k \geq 3$.*

Then for any positive Ricci-(k, n) curvature metric, g_0 on X and corresponding Gromov-Lawson concordance, $\overline{g} = \overline{g}(g_0, f)$, the metrics g_0 and $\overline{g}|_{X\times\{1\}}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$.

Proof. In section 8.2 we described the positive Ricci-(k,n) curvature metric, $g_{p,p+1}$, obtained after cancelling surgeries on a positive Ricci-(k,n) curvature metric manifold, X, where $q \geq \max\{n+2-k, 3\}$. We will now show how to achieve the metric, $g_{p,p+1}$, using an isotopy from g_0 to $g_{p,p+1}$ in $\mathcal{R}\text{iem}^{Ric^+_{k,n}}(X)$.

We start with the simply connected manifold, X, equipped with a positive Ricci-(k,n) curvature metric, g_0, and the embedded sphere-disc product, $\varphi(S^p \times D^{q+1})$ associated with the critical point, w_1, where $q \geq \max\{n+2-k, 3\}$. We use the Gromov-Lawson-Wolfson construction for p-surgery and Kordass' Theorem 6.0.2 to isotopy g_0 to g_{std}, a surgery-ready metric, where $g_{std}|_{N_{\frac{1}{2}}} = g^{q+1}_{torp} + ds^2_p$, a standard torpedo metric product. There is a transition region on $N \backslash N_{\frac{1}{2}}$ which from Theorems 6.0.2 [21] and 6.0.1 [53] has positive Ricci-(k,n) curvature, $k \geq 2$, provided $q \geq \max\{n+1-k, 2\}$; see figure 8.2. The metric, $g^{q+1}_{torp} + ds^2_p$, has positive Ricci-(k,n) curvature provided $k \geq 2$ and $p \neq 1$ and $k \geq 3$ when $p = 1$.

The Gromov-Lawson p-surgery is **not** performed. Recall that to make the metric, g_φ on X_φ ready for Gromov-Lawson $(p+1)$-surgery, we adjust the metric on the embedded sphere-disc product $S^{p+1} \times D^q$ smoothly to $g_{\varphi,std}$ as in figure 8.5(a). We do the same adjustment but only on $D^{p+1}_- \times D^q$ and obtain a metric, which we denote $g_{\varphi,std,-}$, schematically depicted in figure 8.7. The metric $g_{\varphi,std,-}$ may be decomposed as

- The metric restricted to $X \setminus D$ remains unchanged, $g_{\varphi,std,-}|_{X\setminus D} = g_0|_{X \setminus D}$;
- The region $D \cong D^n$ may be decomposed into three regions, $D = T \cup B_{p,-} \cup R_-$ as shown in figure 8.7:
  - The connecting cylinder, T, shown in dark grey in figure 8.7, is the same $(p+1)$-surgery ready metric restricted to T, $g_{\varphi,std,-}|_T = g_{std}|_T$, shown in figure 8.5. The metric, $g_{\varphi,std,-}$, restricted to T has positive Ricci-(k,n) curvature provided $q-1 \geq \max\{n+1-k, 2\}$, in accordance with Theorem 6.0.1 in [53];
  - $B_{p,-} \cong S^{n-1} \times I$ on which there are standard metrics which have positive Ricci-(k,n) curvature provided $q \geq \max\{n+2-k, 3\}$; and
  - Shown in light grey in figure 8.7 there is a mixed torpedo metric, $g^{(p,q)+}_{Mtorp}$ restricted to R_- which is diffeomorphic to the upper hemisphere S^n_+, decomposed as in section 4.5.
- On $B_{p,-} \cap R_- \cong S^{n-1}$, the metric restricts as the mixed torpedo metric $g^{p,q-1}_{Mtorp}$ (see section 5.3).

All of the standard metrics have positive Ricci-(k,n) curvature for $k \geq 3$ when $p \neq 1$ and $k \geq 4$ when $p = 1$.

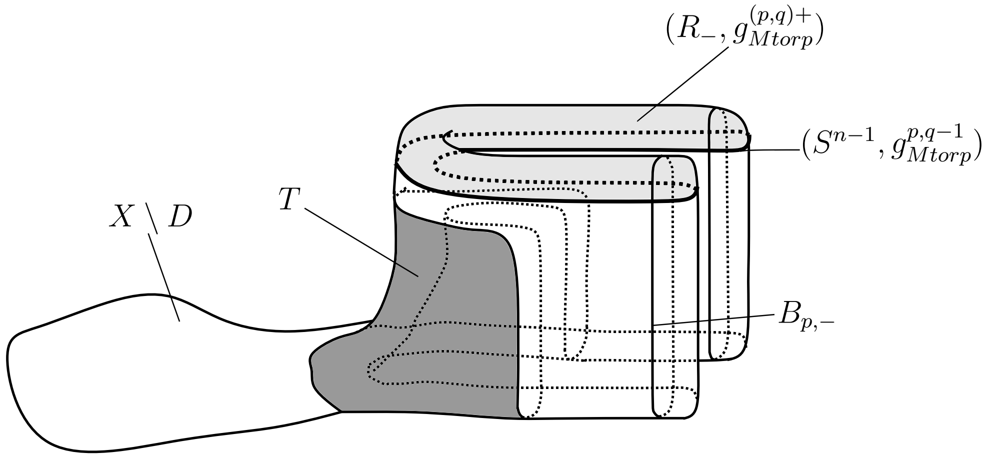

Figure 8.7: $(X, g^-_{\varphi,std})$.

Recall that in section 5.3 we proved in Theorem 5.3.1 that the metrics $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ are isotopic using a method which keeps the equator metric $g^{p,q-1}$ fixed. Thus there is an isotopy between $g^{(p,q)+}_{Mtorp}$ and $g^{(p+1,q-1)+}_{Mtorp}$ on R^- which keeps the equator metric $g^{p,q-1}$ fixed. We use this isotopy and the result is shown schematically in figure 8.8. We denote the resulting metric on X as g'_{p+1}. This may be compared to the isotopy schematically depicted in figure 5.12.

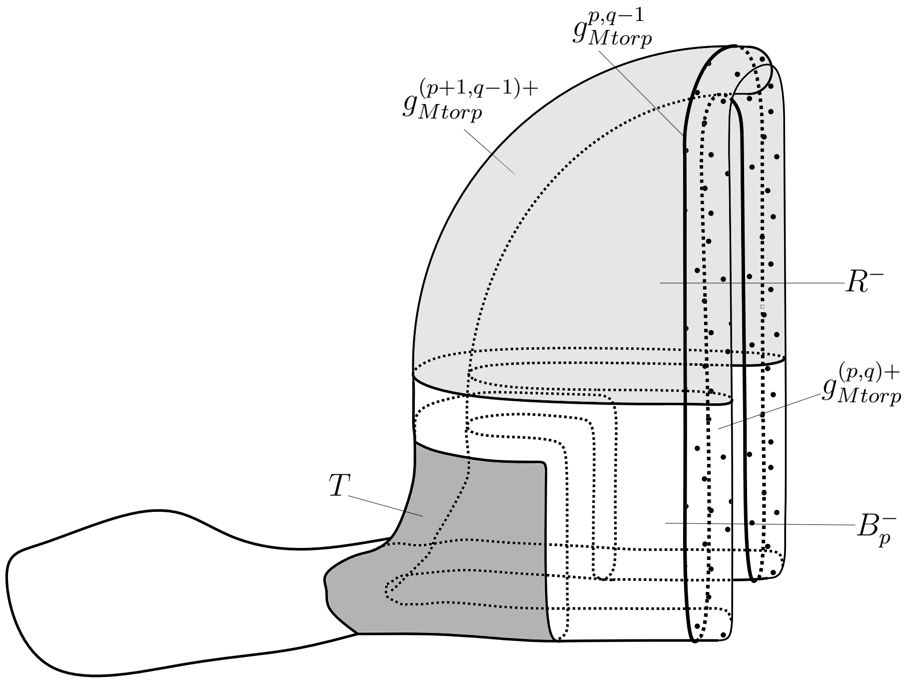

Figure 8.8: (X, g'_{p+1}) with metric $g^{(p+1,q-1)+}_{Mtorp}$ on R^-.

Note that shown in the spotted region in figure 8.8 part of the metric on R^- and $B_{p,-}$ is, again, $g^{(p,q)+}_{Mtorp}$. The equator metric $g^{p,q-1}_{Mtorp}$ on S^{n-1} now spans R^- and $B_{p,-}$ as shown in figure 8.8. We may repeat the isotopy given in Theorem 5.3.1 from $g^{(p,q)+}_{Mtorp}$ to $g^{(p+1,q-1)+}_{Mtorp}$ while keeping the equator metric, $g^{p,q-1}_{Mtorp}$, fixed to obtain the manifold (X, g''_{p+1}) schematically depicted in figure 8.9.

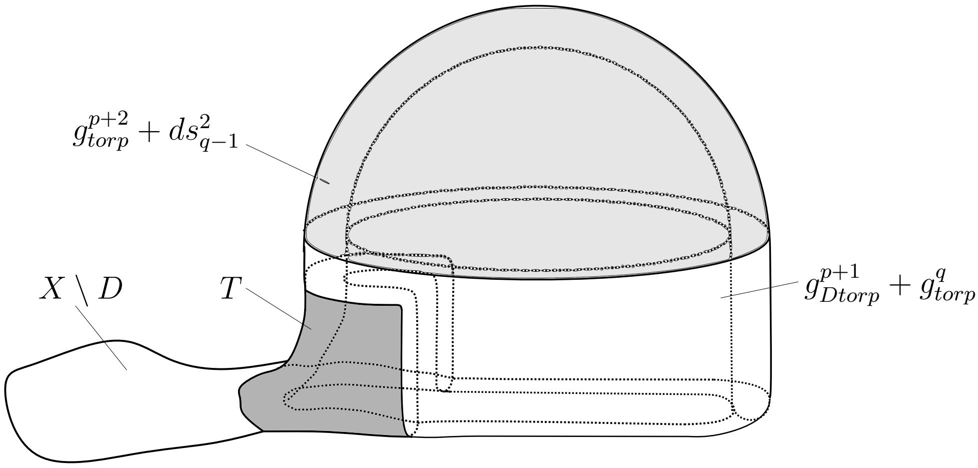

Figure 8.9: (X, g''_{p+1}).

Note that the metric on the manifold after cancelling surgeries, $g_{p,p+1}$, is isotopic to g''_{p+1} via a minor isotopy on the g^{p+1}_{Dtorp} factor away from T; see figure 8.6. We have achieved this metric without performing any surgeries by an isotopy from the original metric g_0 to g''_{p+1}. In order for this isotopy to occur we require:

1. The critical points have Morse index $p+1$ and $p+2$ where $1 \leq p \leq n-4$, from Theorem 2.3.1.

2. $q \geq \max\{n+2-k, 3\}$ from Theorems 6.0.2 and 6.0.1.

3. From metric calculations Lemmas 4.2.1 and 4.6.2 and Theorem 5.3.1 where $p = 1$, $k \geq 4$. In all other cases $k \geq 3$.

Given these conditions, the metric g_0 is isotopic to g''_{p+1} in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$. $\square$

Remark. *Note that when the Morse function f is such that on $f^{-1}(c)$, the trajectory sphere $S^q_+(w_1)$ emerging from the critical point w_1 and $S^{p+1}_-(w_2)$ converging towards the critical point w_2 intersect transversely at a single point then*

- *W is not required to be simply-connected nor $n \geq 5$ as there is no need to contract loops of intersection points of the trajectory spheres; and*

- *The critical points may have index 1 as simple-connectivity is not required.*

Let $W = X \times I$ be a smooth compact cobordism where W and X are simply connected and X has dimension $n \geq 5$. Then as in chapter 6 we may equip W with many possible Morse functions, f, with many critical points. Recall that such Morse functions can be *well-indexed*, that is on the same level set $f^{-1}(c)$, critical points have the same index, λ_i. Moreover for critical points, w_i and w_j, with indices λ_i and λ_j respectively, such that $f(w_i) < f(w_j)$, then the indices, $\lambda_i < \lambda_j$. Using such a Morse function, the cobordism W may be decomposed into cobordisms, C_i, which contain only one level set of critical points. By pairing critical points on different levels with index λ_i and λ_{i+1} we obtain pairs of cancelling critical points as in Theorem C; and all the critical points can be so paired. Walsh in [44] proved that where g_0 is a positive scalar curvature metric on $X \times \{0\}$, the metrics g_0 and g_1 on $X \times \{1\}$ are isotopic in $\mathcal{R}\mathrm{iem}^+(X)$. We extend this proof to the case of positive Ricci-(k, n) curvature metrics.

Theorem D. *Let X be a simply connected, smooth, closed manifold of dimension $n \geq 5$, equipped with a metric of positive Ricci-(k, n) curvature, g_0. Let f be a k-admissible Morse function $f : X \times I \to I$, giving a Gromov-Lawson concordance, $\overline{g}(g_0, f)$ on the cylinder $X \times I$. Then the metrics g_0 and $g_1 = \overline{g}|_{X\times\{1\}}$ are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}(X)$ if $4 \leq k \leq n$ or for k satisfying $3 \leq k \leq n$ if f has no critical points of Morse index 2.*

Proof. This proof follows that of Walsh in [47] who makes use of [30].

Recall that a well-indexed Morse function, $f|_{C_i}$, has only critical points of index λ_i and all the critical points of $f|_{C_i}$ are on the same level set, $f^{-1}(c_i)$. By Theorem 4.8 of [30] and Theorem 6.0.3(2) f can be isotopied to a k-admissible well-indexed Morse function through a path in the space of Morse functions. By Theorem 6.0.3, the metric remains a positive Ricci-$(k + 1, n + 1)$ curvature metric throughout the isotopy as the isotopy is effectively a rearrangement of critical points with index, $\lambda_i \leq \min\{k - 1, n - 2\}$. We use this well-indexed Morse function on $W = X \times I$ to obtain a decomposition of C_i:

$$W = C_0 \cup C_1 \cup \cdots \cup C_{n+1}.$$

We denote the critical points of C_i as $w_{i,j}$, where i is the index of the critical point and $j \in \{1, \cdots , N_i\}$, where N_i is the number of critical points in C_i. The value $f(w_{i,j}) = c_i$ is called the critical value. The critical values of W are thus ordered as follows: $0 < c_0 < c_1 \cdots c_n < c_{n+1} < 1$; see figure 8.10(a).

We note that it is possible at this stage that the critical points of the well-indexed Morse function on adjacent level sets may not have indices which differ by 1. Therefore some adjustment to the Morse function may be required in order to create critical points on adjacent level sets which do differ by 1. As the cobordism is topologically a cylinder it is possible to create cancelling pairs of critical points. This is described later in the proof.

Consider the case where the number of critical points in the level set $f^{-1}(c_0)$ equal the number of critical points in the level set $f^{-1}(c_1)$. Then each critical point, $w_{0,j}$ can be connected by arcs which are the unions of trajectory spheres to $w_{1,j}$, as X is simply-connected. Recall that figure 2.7 schematically depicts the problems of contracting a loop if the manifold is not simply connected. It is possible to isotopy the Morse function so that the arcs are disjoint. Then we can apply the construction in Theorem C to each pair of critical points to obtain j pairs of cancelling surgeries. Then the metric g_0 on X and $\overline{g}|_{X\times\{c_1+\varepsilon\}}$ are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$, where $p = 1$, $k \geq 4$. In all other cases $k \geq 3$.

This process may be repeated if the critical points on the level set $f^{-1}(c_2)$ equal the number of critical points in the level set $f^{-1}(c_3)$ and so on until all the critical points are in cancelling pairs. Repeating the process in Theorem C for each cancelling pair, the metric g_0 is extended to $\overline{g}|_{X\times\{1\}}$. Then the metrics g_0 and $\overline{g}|_{X\times\{1\}}$ are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$, where $p = 1$ or $q = 2$, $k \geq 4$. In all other cases $k \geq 3$.

We now consider the case where the numbers of critical points on each level set differ to the numbers on the next level set. Let the number of critical points of index i be more than that of index $i + 1$. After cancelling we have surplus index i critical points. By an isotopy of the Morse function, f to f' it is possible to create more critical points in pairs with index $i + 1$ and $i + 2$. Enough of these new pairs are created to pair all the remaining index i critical points with index $i + 1$ critical points to form cancelling surgeries. This process is now continued on the index $i+2$ critical points until all of the critical points of f' are paired into cancelling surgeries. Figure 8.10 illustrates this process. Then, as before, we can apply the construction

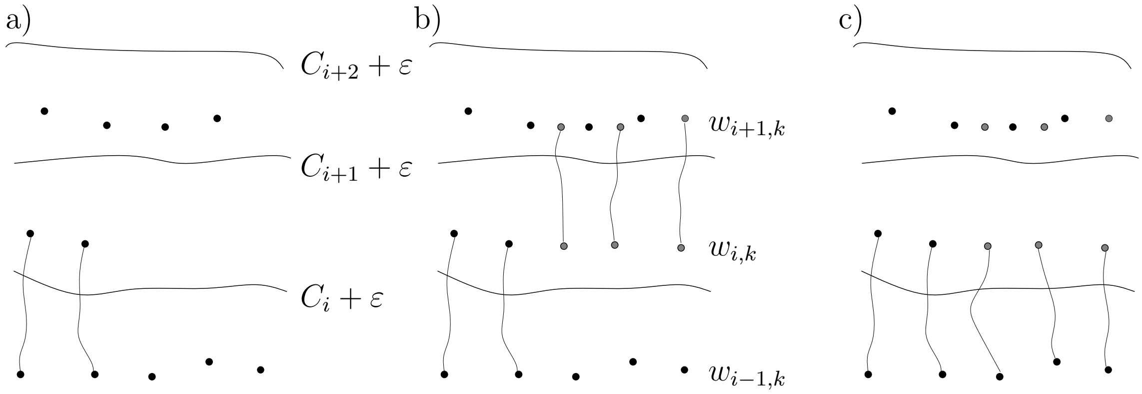

Figure 8.10: a) Two pairs of critical points with indices $i - 1$ and i; b) Creating three pairs of critical points with indices i and $i+1$; and c) pairing all of the critical points with indices $i - 1$ and i.

in Theorem C to each pair of critical points to obtain pairs of cancelling surgeries. Then the metric g_0 and $\overline{g}|_{X\times\{1\}}$ are isotopic in $\mathcal{R}\text{iem}^{Ric^+_{(k,n)}}(X)$. □

Chapter 9

Conclusion

In this thesis we have applied the Gromov-Lawson Surgery Theorem, originally used in psc metrics to positive Ricci-(k, n) curvature metrics. We use Theorems proved by Wolfson [53] and Kordass [21] and a "boot" metric construction developed by Walsh [48], to construct a positive scalar curvature metric on the trace of a p-surgery on a manifold. If the manifold X is equipped with a positive Ricci-(k, n) curvature metric and the co-dimension of the surgery, $q+1 \geq \max\{n+2-k, 3\}$, the trace has a positive Ricci-$(k+1, n+1)$ curvature metric extending the metric on X with a product near the boundary. This applies to values of k as follows: when $p \neq 1$ then $k \geq 2$ and when $p = 1$, $k \geq 3$.

We have applied this to give information on the topology of the space of positive Ricci-$(k, 4n-1)$ metrics on certain manifolds in dimension $4n-1$, $n \geq 2$. Carr [8] originally showed that the space of psc metrics on S^{4n-1} has infinitely many path components. Burkemper, Searle and Walsh [7] also showed that the space of positive (l, n)-intermediate curvature metrics on M^{4n-1}, where M are closed, spin, manifolds, has infinitely many path components. We show this to be the case for the space of positive Ricci-(k, n) curvature metrics, $\mathcal{R}iem^{Ric^{+}_{(k,4n-1)}}(M^{4n-1})$, where $k \geq 2n+1$, by using a connected sum method.

The above results have been proved in the case of psc metrics [44], positive intermediate (l, n)-curvature [7] and now positive Ricci-(k, n) curvature. It would be interesting to see if this could be further extended to other metrics such as those listed in section 1.1.

In Theorem D we showed that for certain Gromov-Lawson concordances of cancelling surgeries the metrics on the boundary, $X \times \{0\}$ and $X \times \{1\}$ are isotopic in the space of positive Ricci-(k, n) curvature metrics on X, where the dimension of X is $n \geq 5$. Walsh [44] has shown this in the case of positive scalar curvature metrics. However the analogous result for positive intermediate scalar (l, n)-curvature as defined by Labbi [23], has not yet been proved, so this could be an area of further research.

Other questions of interest that could prove fruitful to investigate include the following:

- A comparison of the topology of the spaces of positive Ricci-(k, n) metrics for a manifold for different $k \in \{1, \cdots, n\}$. For example it is possible to imagine spaces which are disconnected for some value of k with the different components eventually becoming path-connected for higher values of k, see figure 9.1 (a). Do more path components come into existence the less restrictive the metric, that is as k increases, figure 9.1 (b). Why?

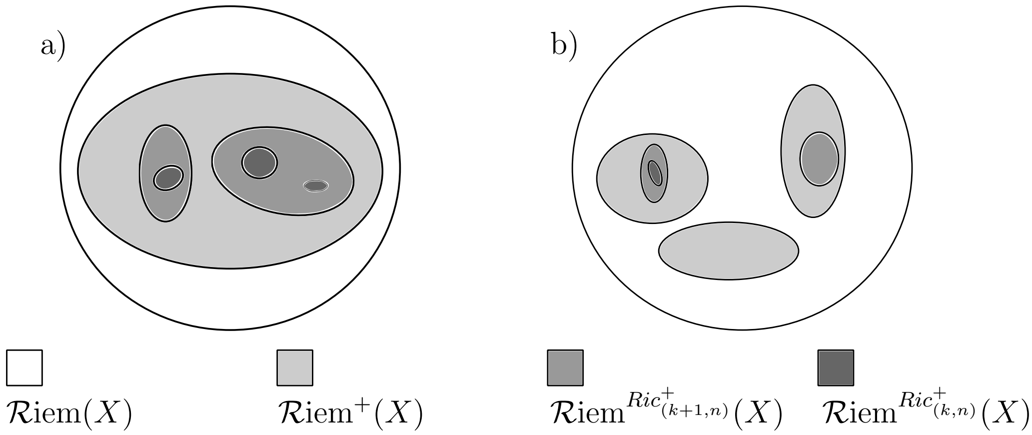

Figure 9.1: Spaces of Riemannian metrics with a) $\mathcal{R}\text{iem}^{+}(X)$ path-connected; and b) $\mathcal{R}\text{iem}^{+}(X)$ disconnected.

- A comparison of the topology of spaces of positive Ricci-(k, n) curvature and positive intermediate scalar (l, n) curvature on a manifold.
- Similarly, a comparison of the topology of spaces of positive Ricci-(k, n) curvature and other intermediate curvatures.
- There is a relationship between Ricci-(k, n) curvature, m-intermediate curvature, $\mathcal{C}_m$ and intermediate (l, n) curvature. It would be interesting to see how this is reflected in the topologies of the three curvatures when they are all positive.

We have only considered the topological characteristic of path-connectedness in our discussion, but, of course, we have also yet to explore other topological characteristics (for example, higher homotopy groups), of the space of positive Ricci-(k, n) curvature metrics for a manifold and the comparison with those for the spaces of other curvatures. Some work has been done on this. For example in [14], the authors extend results of [3] to demonstrate that, for many manifolds there is much non-triviality in the homotopy groups of the corresponding spaces of positive Ricci-(k, n) metrics for many k.

Appendices

Appendix A

Summary of isotopic mixed torpedo metrics

Theorem A.0.1. *For any non-negative integers, p and q, with $p+q+1=n$, let $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ be mixed torpedo metrics on the sphere S^n both of which have the same equator metric, $g^{p,q}_{Mtorpeq}$. Then there exists an isotopy in $\mathcal{R}iem^{Ric^+_{(k,n)}}$ between $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ which keeps $g^{p,q}_{Mtorpeq}$ fixed.*

We give in the table below the value of k for which $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ are positive Ricci-(k,n) curvature metrics with fixed equator metric, $g^{p,q}_{Mtorpeq}$, and are isotopic in $\mathcal{R}iem^{Ric^+_{(k,n)}}$.

Note 1: The metric, $g^{1,1}_{torp}$, does not have psc and does therefore not have positive Ricci-(k,n) curvature for any $k\in\{1,2,3\}$. Therefore $g^{0,2}_{torp}$ and $g^{1,1}_{torp}$ can not be isotopic in $\mathcal{R}iem^+_{k,3}$ for any k.

Note 2: We cannot show whether the metrics, $g^{1,2}_{torp}$ and $g^{2,1}_{torp}$, are isotopic or not in $\mathcal{R}iem^+_{k,4}$ for any k using this method.

n	$g^{p,q}_{Mtorp}$			$g^{p+1,q-1}_{Mtorp}$			$g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$ isotopic in $\mathcal{R}\text{iem}^{+}_{k,n}$?	Isotopic in $\mathcal{R}\text{iem}^{+}_{k,n}$, for $k \geq$
	p	q	$k \geq$	$p+1$	$q-1$	$k \geq$		
3	0	2	2	1	1	Note 1	No	N/A
4	0	3	2	1	2	3	Yes	3
4	1	2	3	2	1	3	Note 2	N/A
4	2	1	3	3	0	2	Yes	3
≥ 5	0	≥ 4	2	1	≥ 3	3	Yes	3
≥ 5	1	≥ 3	3	2	≥ 2	2	Yes	4
≥ 5	≥ 2	2	2	≥ 3	1	3	Yes	4
≥ 5	≥ 3	1	3	≥ 4	0	2	Yes	3
≥ 6	≥ 2	≥ 3	2	≥ 3	≥ 2	2	Yes	3

Table A.1: Values of k for Ricci-positive curvature and for isotopy of mixed torpedo metrics, $g^{p,q}_{Mtorp}$ and $g^{p+1,q-1}_{Mtorp}$.

Appendix B

Summary of the Ricci-(k, n) curvature properties of the metric, g_φ, on the various subspaces of the trace, W_φ

In section 6, we proved the following Theorem.

Theorem A. *Let X be a smooth n-dimensional closed manifold with $n \geq 3$, $\varphi : S^p \times D^{q+1} \hookrightarrow X$ an embedding where $p + q + 1 = n$ and let $W_\varphi := \{W_\varphi; X, X_\varphi\}$ denote the trace of a p-surgery on φ. Suppose g is a Riemannian metric on X which has positive Ricci-(k, n) curvature when $2 \leq k \leq n$ and $p \neq 1$, or $3 \leq k \leq n$ when $p = 1$. Then provided $q \geq max\{n + 1 - k, 2\}$, there is a metric $\overline{g}_\varphi$ on W_φ so that*

a) $\overline{g}_\varphi|_X = g$;

b) $\overline{g}_\varphi$ is a product near the boundary ∂W_φ; and

c) $\overline{g}_\varphi$ has positive Ricci-$(k + 1, n + 1)$ curvature.

Below we provide a summary of the Ricci-(k, n) curvature properties of the metric, g_φ, on the various subspaces of the trace, W_φ; see figure B.1.

(i) We have a collar neighbourhood of X in the cylinder, $\psi : X \times [0, 1) \hookrightarrow X \times [0, J + 2]$ (see figure 6.2) with ψ an embedding and $\psi(x, 0) = x$, $x \in X$, part of the boundary of $X \times \{0\}$. The collar neighbourhood of X is equipped with a product metric, such that the pullback of $\overline{g}|_{X\times[0,1)}$ by the embedding ψ is $g + dt^2$. By assumption, X is equipped with a metric of positive Ricci-(k, n) curvature.

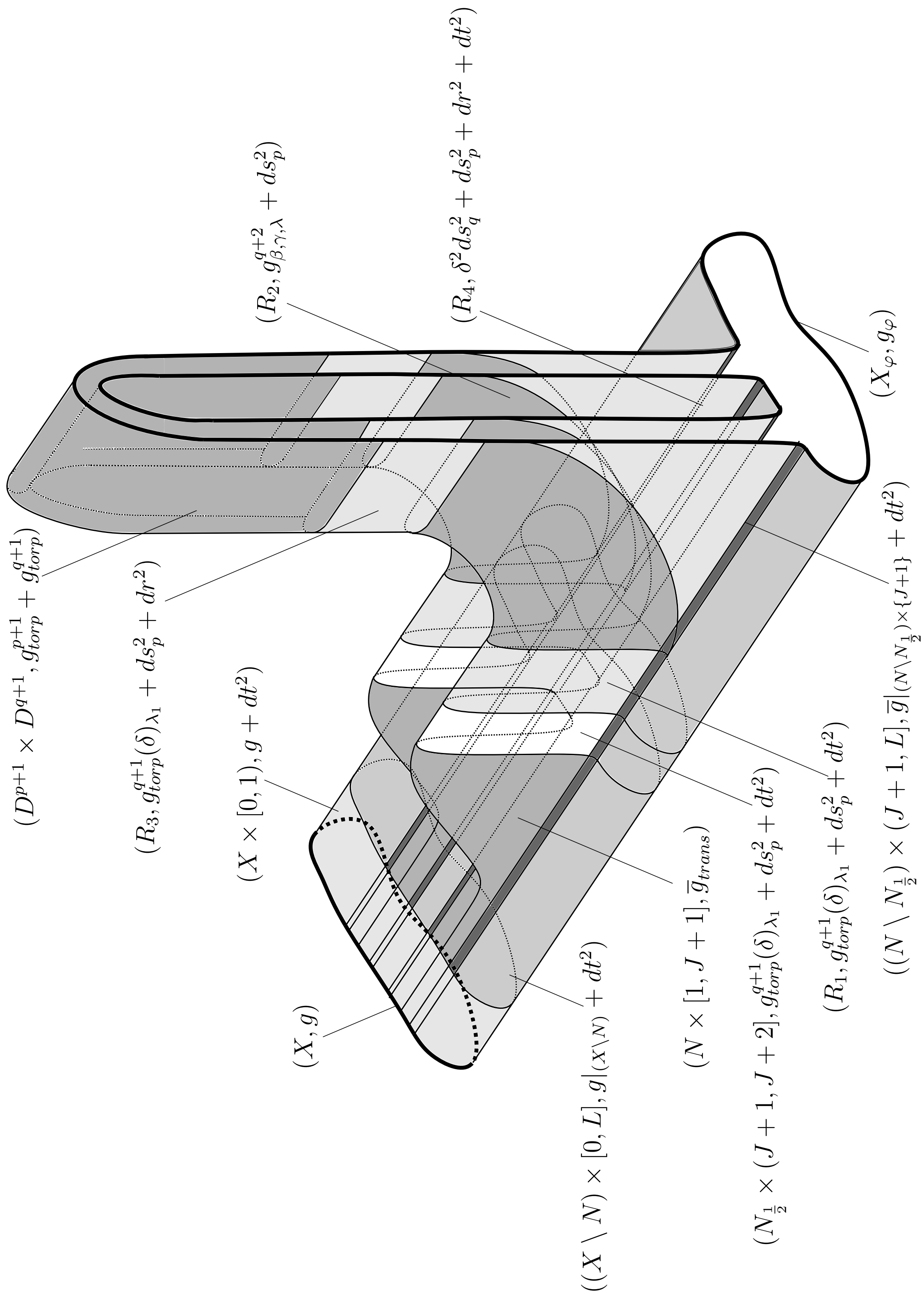

Figure B.1: Metric, $\bar{g}_\varphi$, on the trace of a p-surgery, W_φ.

Let $\{\partial_1, \cdots, \partial_n, \partial_{n+1} = \partial_t\}$ be the local coordinate vector fields for the tangent space of the collar neighbourhood of X. The Ricci curvatures of this region of the trace, $X \times [0,1)$, $\overline{Ric}(\partial_i)$, equal the Ricci curvatures, $Ric(\partial_i)$, on the original manifold, X. However $\overline{Ric}(\partial_t) = 0$ and so the Ricci-$(k+1, n+1)$ curvature on $X \times [0,1)$ is equal to $\overline{Ric}(\partial_t) + \sum_{i=1}^{k} \overline{Ric}(\partial_i) = \sum_{i=1}^{k} Ric(\partial_i)$. Therefore when X has a metric with positive Ricci-(k,n) curvature then $X \times [0,1)$ has a metric with positive Ricci-$(k+1, n+1)$ curvature.

(ii) We now consider the metric on the region $X \setminus N \times [0, L]$. In Theorem 6.0.1, adjustments to the metric take place on N, to make the manifold "surgery-ready". Following this construction, we make no adjustment to the metric on $X \setminus N \times [0, L]$. Thus the metric on $X \setminus N \times [0, L]$, is the product metric, $g|_{(X\setminus N)} + dt^2$. We follow a similar argument to (i) above. As the metric, g, by assumption, has positive Ricci-(k,n) curvature, the metric on the trace restricted to $(X \setminus N) \times [0, L]$, $\overline{g}|_{(X\setminus N)\times[0,L]}$, has positive Ricci-$(k+1, n+1)$ curvature.

(iii) We denote the metric $\overline{g}_{\varphi}$ restricted to the region $N \times [1, J+1]$ as the transition metric, $\overline{g}_{trans}$. In order to perform surgery on the embedding $\varphi : S^p \times D^{q+1} \hookrightarrow X$ we need to transition the metric to a surgery-ready metric, $ds_p^2 + g_{torp}^{q+1}$, on $N_{\frac{1}{2}}$, as shown in figure 6.2. The metric on $N \times \{t\}$, $t \in [1, J+1]$, transitions from the original arbitrary positive Ricci-(k,n) metric restricted to N, $g|_N$, to an appropriate standard form metric, $g_{torp}^{q+1} + ds_p^2$, on $N_{\frac{1}{2}} \times \{J+1\}$ with a transition metric on $(N \setminus N_{\frac{1}{2}}) \times [1, J+1]$.

From Lemma 3.3.1, given the conditions stated therein, there exists a smooth path, $g_r, r \in I$, $I = [0,1]$, in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(X)$ with a smooth function $h : \mathbb{R} \to [0,1]$ such that the metric $G = g_{h(t)} + dt^2$ on $X \times \mathbb{R}$ has positive Ricci-$(k+1, n+1)$ curvature. We need to show that there is a path that goes from $g_0 = g$ to $g_1 = \overline{g}|_{X\times\{J+1\}} = g_{std}$. We use Kordass' Theorem 6.0.2 which states that, provided $q \geq \max\{n+1-k, 2\}$, $\mathcal{R}\mathrm{iem}_{std}^{Ric^+_{(k,n)}}(X) \hookrightarrow \mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(X)$. Then $g \in \mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(X)$ is connected by a path in $\mathcal{R}\mathrm{iem}^{Ric^+_{(k,n)}}(X)$ to an element of $\mathcal{R}\mathrm{iem}_{std}^{Ric^+_{(k,n)}}(X)$, where $g|_{(X\setminus N)\times\{t\}} = g|_{(X\setminus N)\times\{0\}}$, for all $t \in [0, L]$. Thus there is an isotopy g_t from $g_0 = g$ on $X \times \{0\}$ to $g_1 = \overline{g}|_{X\times\{J+1\}}$. Therefore by Corollary 3.3.1.1 there exists a metric $G = g_{f(t)} + dt^2$ on $X \times [0, J+2]$, with J sufficiently large, which has positive Ricci-$(k+1, n+1)$ curvature, when $k \geq 2$ and $n \geq 3$, which has product structure near the boundary.

(iv) In particular near $X \times \{0\}$ and $X \times \{J+1\}$, there are product metrics, $g + dt^2$ and $\overline{g}_{\varphi}|_{X\times\{J+1\}} + dt^2$, respectively; figure 6.2, satisfying the following properties:

(a) The metric, $g|_{(X \setminus N)} + dt^2$, on $(X \setminus N) \times (J+1, J+2]$. This is a continuation of the product metric, $g|_{(X \setminus N)} + dt^2$, on the collar neighbourhood $(X \setminus N) \times [0, 1)$. By assumption, the metric, g, has positive Ricci-(k, n) curvature. From (i) above, therefore the product metric has positive Ricci-$(k+1, n+1)$ curvature.

(b) Using Kordass' Theorem 6.0.2, the metric, $\overline{g}_{trans}|_{(N \setminus N_{\frac{1}{2}}) \times \{J+1\}}$, on $(N \setminus N_{\frac{1}{2}}) \times \{t\}$, $t \in (J+1, J+2]$, has positive Ricci-(k, n) curvature, provided $q \geq \max\{n+1-k, 2\}$. Using the argument from (i), the metric $\overline{g}_{trans}|_{(N \setminus N_{\frac{1}{2}}) \times \{J+1\}} + dt^2$ has positive Ricci-$(k+1, n+1)$ curvature.

(c) There is a product torpedo metric, $g^{q+1}_{torp}(\delta)_{\lambda_1} + ds_p^2 + dt^2$, on $N_{\frac{1}{2}} \times (J+1, J+2]$. From section 4.2 the torpedo metric has positive Ricci-(k, n) curvature when $k \geq 2$ and $n \geq 3$. Hence, using the argument in (iv), $g^{q+1}_{torp}(\delta)_{\lambda_1} + ds_p^2 + dt^2$ has positive Ricci-$(k+1, n+1)$ curvature, when $q \geq 2$, $k \geq 2$ and $p \neq 1$. When $p = 1$, the metric has positive Ricci-(k, n) curvature when $q \geq 2$ and $k \geq 3$.

(v) The product metric in (iv) is continued for the regions of the trace:

(a) The region, $(X \setminus N) \times (J+2, L]$ has metric $g|_{(X \setminus N)} + dt^2$. As discussed in (iv)(a) this metric has positive Ricci-$(k+1, n+1)$ curvature.

(b) The region $(N \setminus N_{\frac{1}{2}}) \times (J+1, J+2]$ has metric $\overline{g}_{trans}|_{(N \setminus N_{\frac{1}{2}}) \times \{J+1\}} + dt^2$. As discussed in (iv)(b) the metric $\overline{g}_{trans}|_{(N \setminus N_{\frac{1}{2}}) \times \{J+1\}} + dt^2$ has positive Ricci-$(k+1, n+1)$ curvature.

(vi) We continue to adjust the cylinder to make it "surgery-ready". We initially change the region of the cylinder, $N_{\frac{1}{2}} \times (J+2, L]$, to the boot metric described in [48]; see figure 4.3. In Lemma 4.4.1 we show that the boot metric has positive Ricci-(k, n) curvature for $k \geq 3$ when $p \neq 1$ and $k \geq 4$ when $p = 1$ provided $q \geq 2$ and Λ is large enough.

We list the Regions of the boot metric on an $(n+1)$-dimensional manifold:

(a) $(R_1 = D^n \times [0, \lambda_2], g^n_{torp}(\delta)_{\lambda_1} + dt^2)$;

(b) $(R_2 = D^n \times [0, \frac{\pi}{2}\lambda_3], g^{n+1}_{\beta,\gamma,\Lambda})$;

(c) $(R_3 = D^n \times [0, \lambda_4], g^n_{torp}(\delta)_{\lambda_1} + dr^2)$;

(d) $(R_4 = S^{n-1} \times \mathcal{Q}(\lambda_3), \delta^2 ds^2_{n-1} + dr^2 + dt^2)$; and

(e) $(R_5 = D^{n+1}_{+}, g^{n+1}_{torp}(\delta)_{\lambda_1})$. In surgery this region is removed.

On the cylinder $N_{\frac{1}{2}} \times [J+2, J+2+\lambda_2+\lambda_3]$, the metric is a product metric, $g_{boot} + ds_p^2$, where g_{boot} is the metric on the submanifold

$(D^{q+1} \times I) \cup (S^q \times \mathcal{Q}(\lambda_3)) \cup D_+^{q+2}$. The boot metric commences with the region, R_1 with the product metric $g_{torp}^{q+1} + ds_p^2 + dt^2$ which is the same metric as that on $N_{\frac{1}{2}} \times (J+1, J+2]$. The variables of the boot metric, δ and λ, agree on the two regions to give a smooth metric on the cylinder $X \times [0, L]$.

Note also that the length of the cylinder is $L := J + 2 + \lambda_2 + \lambda_3$ and that there is a collar neighbourhood near $X \times \{L\}$ of width λ_1 on $(L - \lambda_1, L]$ on which there is a standard product metric.

(vii) As stated before the surgery is performed by removing R_5 of the boot metric and attaching a solid handle $(D^{p+1} \times D^{q+1}, g_{torp}^{p+1} + g_{torp}^{q+1})$. Let $\{\partial_1, \cdots, \partial_p, \partial_{p+1} = \partial_t, \partial_{p+2}, \cdots \partial_{p+q+1}, \partial_{n+1} = \partial_r\}$ be the local coordinate vector fields for the tangent space of $D^{p+1} \times D^{q+1}$. From section 4.2, the torpedo metric has positive Ricci-$(2, n+1)$ curvature. However both $Ric(\partial_t)$ and $Ric(\partial_r)$ are not strictly positive. Hence the metric on the solid handle has positive Ricci-$(3, n)$ curvature when $p \neq 1$ and $q \neq 1$ and positive Ricci-$(4, n)$ curvature when either p or q equals 1.

(viii) Using Wolfson's Theorem 6.0.1, X_φ, the manifold resulting from X after p-surgery, has a metric with positive Ricci-(k, n) curvature provided $2 \leq k \leq n$ and $q \geq \max\{n + 1 - k, 2\}$.

In the table below we give the conditions required for the metrics on the trace to have positive Ricci-$(k+1, n+1)$ curvature provided $p \neq 1$. These were calculated in subsection 4.

Region	Metric	$k+1$	q
$N \times [1, J+1)$	$g_{f(t)} + dt^2$	$k \geq 2$	$\geq max\{n+1-k, 2\}$
R_1	$g_{tor}^{q+1}(\delta)_{\lambda_1} + ds_p^2 + dt^2$	≥ 3	≥ 2
R_2	$g_{\beta,\gamma,\lambda}^{q+2} + ds_p^2$	≥ 2	≥ 2
R_3	$\delta^2 ds_q^2 + ds_p^2 + dr^2 + dt^2$	≥ 3	≥ 2
R_4	$g_{tor}^{q+1}(\delta)_{\lambda_1} + ds_p^2 + dr^2$	≥ 3	≥ 2
R_5	$g_{tor}^{q+2}(\delta)_{\lambda_1} + ds_p^2$	≥ 2	≥ 2
Surgery handle	$g_{tor}^{p+1} + g_{tor}^{q+1}$	≥ 3	≥ 2

We note that $\mathcal{R}\text{iem}^+_{(2,n+1)} \subset \mathcal{R}\text{iem}^+_{(3,n+1)}$. Hence, provided $q \geq max\{n+1-k, 2\}$, the metric on the trace has positive Ricci-$(3, n+1)$ curvature when $p \neq 1$ and positive Ricci-$(4, n+1)$ curvature when $p = 1$.

Bibliography

[1] Renan Assimos, Andreas Savas-Halilaj, and Knut Smoczyk. Graphical mean curvature flow with bounded bi-Ricci curvature. *Calculus of Variations and Partial Differential Equations*, 62.1, 2022.

[2] Thierry Aubin. Métriques riemanniennes et courbure. *Journal of Differential Geometry*, 4(4):383–424, 1970.

[3] Boris Botvinnik, Johannes Ebert, and Oscar Randal-Williams. Infinite loop spaces and positive scalar curvature. *Inventiones Mathematicae*, 209:749–835, 2017.

[4] Simon Brendle, Sven Hirsch, and Florian Johne. A generalization of Geroch's conjecture. *Communications on Pure and Applied Mathematics*, 77:441–456, 2024. arXiv:2207.08617.

[5] Bradley Lewis Burdick. Metrics of positive ricci curvature on the connected sums of products with arbitrarily many spheres. *Annals of Global Analysis and Geometry*, 58:1–44, 11 2020.

[6] Bradley Lewis Burdick. The space of positive Ricci curvature metrics on spin manifolds, 2020. Preprint, arxiv.org/abs/2009.06199.

[7] Matthew Burkemper, Catherine Searle, and Mark Walsh. Positive (p,n)-intermediate scalar curvature and cobordism. *Journal of Geometry and Physics*, 181:104625, 2022.

[8] Rodney Carr. Construction of manifolds of positive scalar curvature. *Transactions of the American Mathematical Society*, 307(1):63–74, 1988.

[9] Vladislav Chernysh. On the homotopy type of the space $\mathcal{R}^+(M)$. *Preprint, arxiv:math/0405235*, 2004.

[10] Diarmuid Crowley and David J. Wraith. Positive Ricci curvature on highly connected manifolds. *Journal of Differential Geometry*, 106(2):187–243, 2017.

[11] Johannes Ebert and Georg Frenck. The Gromov–Lawson–Chernysh surgery theorem. *Boletín de la Sociedad Matemática Mexicana*, 27, 2021.

[12] Johannes Ebert and Oscar Randal-Williams. The positive scalar curvature cobordism category. *Duke Mathematical Journal*, 171:2275–2406, 2022.

[13] Johannes Ebert and Michael Wiemeler. On the homotopy type of the space of metrics of positive scalar curvature. *Journal of the European Mathematical Society*, 26:3327–3363, 2024.

[14] Georg Frenck and Jan-Bernhard Kordass. Spaces of positive intermediate curvature metrics. *Geometriae Dedicata*, 214-1(2):767–800, 2021.

[15] Pawel Gajer. Riemannian metrics of positive scalar curvature on compact manifolds with boundary. *Annals of Global Analysis and Geometry*, 5(3):179–191, 1987.

[16] Mikhael Gromov and H Blaine Lawson Jr. The classification of simply connected manifolds of positive scalar curvature. *Annals of Mathematics*, pages 423–434, 1980.

[17] Mikhael Gromov and H. Blaine Lawson Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 58:83–196, 1983.

[18] Morris W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer-Verlag New York, 1976.

[19] Nigel Hitchin. Harmonic spinors. *Advances in Mathematics (New York. 1965)*, 14(1):1–55, 1974.

[20] Sebastian Hoelzel. Surgery stable curvature conditions. *Mathematische Annalen*, 365(1-2):13–47, 2016.

[21] Jan-Bernhard Kordaß. On the space of Riemannian metrics satisfying surgery stable curvature conditions. *Mathematische Annalen*, pages 1841–1878, 2023.

[22] A. A. Kosinski. *Differential manifolds*. Academic Press, Boston, 1993.

[23] Mohammed-Larbi Labbi. Stability of the p-curvature positivity under surgeries and manifolds with positive Einstein tensor. *Annals of Global Analysis and Geometry*, 15:299–312, 1997.

[24] H. Blaine Lawson and Marie-Louise Michelsohn. *Spin geometry*. Princeton University Press, Princeton, N.J, 1989.

[25] John M. Lee. *Riemannian manifolds: an introduction to curvature*. Springer, New York, 1997.

[26] A. Lichnerowicz. Spineurs harmoniques. *C. R. Acad. Sci., Paris*, 257:7–9, 1963.

[27] Joachim Lohkamp. Metrics of negative Ricci curvature. *Annals of Mathematics*, 140(3):655–683, 1994.

[28] Fernando Codá Marques. Deforming three-manifolds with positive scalar curvature. *Annals of Mathematics*, 176(2):815–863, 2012.

[29] J. Milnor. *Morse Theory*. Annals of Mathematics Studies. Princeton University Press, 1963.

[30] J. Milnor. *Lectures on the H-Cobordism Theorem*. Princeton Legacy Library. Princeton University Press, 1965.

[31] J. Milnor. Remarks concerning spin manifolds. In *Differential and Combinatorial Topology, A Symposium in Honor of Marston Morse*, pages 55–62. Princeton University Press, 1965.

[32] Peter Petersen. *Riemannian Geometry*. Springer-Verlag, 1998.

[33] Jonathan Rosenberg and Stephan Stolz. Metrics of positive scalar curvature and connections with surgery. In *Surveys on surgery theory: Volume 2 : papers dedicated to C.T.C. Wall*, volume 149, pages 353–386. Princeton University Press, 2001.

[34] Daniel Ruberman. Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants. *Geometry & Topology*, 5(2):895–924, 2001.

[35] Richard Schoen and Shing-Tung Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Mathematica*, 28(1):159–183, 1979.

[36] Ji-Ping Sha and DaGang Yang. Positive Ricci curvature on the connected sums of $S^n \times S^m$. *Journal of Differential Geometry*, 33(1):127 – 137, 1991.

[37] Ying Shen and Rugang Ye. On stable minimal surfaces in manifolds of positive bi-Ricci curvatures. *Duke Mathematical Journal*, 85(1):109 – 116, 1996.

[38] Zhongmin Shen. A sphere theorem for manifolds of positive Ricci curvature. *Indiana Univ. Math. J.*, 38(1):229 – 233, 1989.

[39] Stephan Stolz. Simply connected manifolds of positive scalar curvature. *Annals of Mathematics*, 136(3):511–540, 1992.

[40] Terry Tao. 254a, notes 3a: Eigenvalues and sums of hermitian matrices. `https://terrytao.wordpress.com/2010/01/12/254a-notes-3a-eigenvalues-and-sums-of-hermitian-matrices/`, 2010.

[41] Wilderich Tuschmann. Spaces and moduli spaces of Riemannian metrics. *Frontiers of mathematics in China*, 11(5):1335–1343, 2016.

[42] Wilderich Tuschmann and David J. Wraith. *Moduli spaces of Riemannian metrics*. Oberwolfach seminars ; 46. Birkhauser, Basel, 2015.

[43] Mark Walsh. *Metrics of Positive Scalar Curvature and Generalised Morse Functions*. PhD thesis, University of Oregon, 2009.

[44] Mark Walsh. Metrics of positive scalar curvature and generalised morse functions, part i. *Memoirs of the American Mathematical Society*, 2011.

[45] Mark Walsh. Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics. *Proceedings of the American Mathematical Society*, 141(7):2475–2484, 2013.

[46] Mark Walsh. H-spaces, loop spaces and the space of positive scalar curvature metrics on the sphere. *Geometry and Topology*, 18(4):2189–2243, 2014.

[47] Mark Walsh. Metrics of positive scalar curvature and generalised morse functions, part ii. *Transactions of the American Mathematical Society*, 366(1):1–50, 2014.

[48] Mark Walsh. The space of positive scalar curvature metrics on a manifold with boundary. *New York J. Math*, 26:853–930, 2020.

[49] Mark Walsh and David J. Wraith. H-space and loop space structures for intermediate curvatures. *Communications in Contemporary Mathematics*, 25(06), 2023.

[50] Eric W Weisstein. Matrix norm. `https://mathworld.wolfram.com/MatrixNorm.html`.

[51] Hubert Weyl. Über die Bestimmung einer geschlossenen konvexen Fläche durch ihr Linienelement. *Vierteljahrsschr Naturforsch Gesellschaft in Zürich*, 61:40–72, 1916.

[52] Frederick Wilhelm. On intermediate Ricci curvature and fundamental groups. *Illinois Journal of Mathematics*, 41:488–494, 1997.

[53] Jon Wolfson. Manifolds with k-positive Ricci curvature. *LMS Lecture Notes Series 394, Cambridge University Press*, 2012.

[54] David Wraith. Exotic spheres with positive Ricci curvature. *Journal of Differential Geometry*, 45(3):638 – 649, 1997.

[55] David Wraith. New connected sums with positive Ricci curvature. *Annals of Global Analysis and Geometry*, 32:343 – 360, 2007.

[56] David Wraith. On the moduli space of positive Ricci curvature metrics on homotopy spheres. *Geometry & Topology*, 15:1983–2015, 2011.

[57] H. Wu. Manifolds of partially positive curvature. *Indiana Univ. Math. J.*, 36(3):525 – 548, 1987.