A generalization of Carr's Theorem to k-positive Ricci curvature and applications

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Thesis submitted to Maynooth University for the degree of Doctor of Philosophy.

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June 2025

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To Luke.

Declaration

I declare that this thesis, submitted as part of my studies for the Doctor of Philosophy degree, is entirely my own work. I have taken reasonable care to ensure its originality, and to the best of my knowledge, it does not infringe any copyright laws. Any ideas or content from other sources have been properly cited and acknowledged within the text.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. David Wraith, for his invaluable guidance, support, and patience throughout this PhD journey. His insight and encouragement have been crucial to both my academic development and personal growth.

I am also grateful to Dr. Mark Walsh; his teaching helped build a strong foundation for my research and deepened my understanding of important concepts.

I want to thank my friends and family for their constant encouragement and support throughout this journey. Your kindness and belief in me was extremely meaningful.

I'd also like to acknowledge Maynooth University for providing me with the John and Pat Hume Doctoral Award, whose financial support made this research possible.

Finally, I want to express my deepest gratitude to my husband, Luke McEvoy, who pushed me to pursue this PhD and stood by me every step of the way. Without your love and encouragement, this achievement would not have been possible.

Abstract

The principal aim of this thesis is to establish a generalization of a classical result in positive scalar curvature due to Carr. This result asserts that given any smooth subcomplex of a Riemannian manifold with codimension at least three, there is a tubular neighbourhood whose boundary has an induced metric with positive scalar curvature. The aim is to generalize this to a range of stronger curvature conditions, namely k-positive Ricci curvature for k at least 2, Sc_k. The proposed theorem claims that for a dimension d subcomplex, there is a tubular neighbourhood boundary with $(d+1)$-positive Ricci curvature. This implies the Carr result when d is at most $n-3$, where n is the dimension of the manifold. (While Carr's result is not in doubt, his argument is problematic for various reasons, and offering a clear re-proof is of independent interest.)

We illustrate the above theorem in two different ways. Firstly, we study the boundaries of plumbed manifolds, leading to a k-positive Ricci curvature generalization of a result of Crowley-Wraith for positive Ricci curvature. Secondly, we generalize a positive scalar curvature result of Carr concerning the fundamental group. Carr's result claims that any finitely presented group is the fundamental group of a closed n-manifold with positive scalar curvature, for $n \geq 4$. We show that the same statement holds if positive scalar curvature is replaced by $\mathrm{Sc}_3 > 0$. This result is also of interest in relation to a conjecture of Wolfson.

The final chapter contains a separate project. A theorem of H. H. Wang shows that for a compact Riemannian manifold with boundary having $\mathrm{Ric} > 0$ globally and non-negative sectional curvature at the boundary, if the boundary convexity is sufficiently high, the manifold must be contractible. We develop an alternative approach which allows an explicit estimate of the required boundary convexity to guarantee contractibility.

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Chapter 1

Introduction

Riemannian geometry provides a framework for studying the shape and structure of smooth manifolds using tools from differential geometry. By introducing a Riemannian metric on a manifold, that is, a smoothly varying inner product on each tangent space, we provide a means of measuring distances and angles. In other words, a Riemannian metric gives the manifold geometric structure. The key geometric feature that we will be concerned with in this thesis is *curvature*.

Every smooth manifold can be equipped with many different Riemannian metrics, and hence many different geometric structures. Among these, those that satisfy specific curvature conditions are especially interesting, not only for their geometric meaning but also because they often reflect deep topological features of the manifold.

In two dimensions, the link between curvature and topology is particularly clear. The Uniformisation Theorem for surfaces states that every closed compact surface admits a metric with constant Gaussian curvature. This together with the Gauss–Bonnet Theorem (applied to the oriented double cover if the surface is not orientable), shows that closed compact surfaces can be classified by the sign of their constant curvature metric. For example, only the 2-sphere and the real projective plane support metrics with strictly positive Gaussian curvature, while the torus and Klein bottle admit a metric of identically zero Gaussian curvature. All other closed compact surfaces carry metrics with negative Gaussian curvature. This clear relationship is one of the most important results in low-dimensional geometry and it does not carry over to higher dimensions.

In higher dimensions the picture becomes more complicated. Curvature takes on several forms, each capturing different types of geometric behavior. These forms include the classical notions of sectional curvature, Ricci curvature, and scalar curvature.

This thesis focuses on the role of these curvature notions, especially in how they interact with the topology of manifolds. While the connection between curvature and topology is well understood in low dimensions, the higher-dimensional setting is complicated and only partially understood. It is exactly this complexity, and the many details that come with it, that motivates the results presented here.

Curvature, intuitively, measures how a Riemannian manifold differs from the flat geometry of Euclidean space. As noted above, there are several ways to measure the curvature of a Riemannian manifold, each encoding a different level of geometric detail. Among these, the scalar curvature is considered the weakest notion of curvature in terms of the information it provides. It is defined as a smooth real-valued function on the manifold,

$$scal : M \to \mathbb{R}.$$

Intuitively, scalar curvature gives an average measure of how the manifold bends at a point, but without reference to specific directions within the tangent space. Despite its limited precision, scalar curvature remains an important global invariant and plays a role in both geometric analysis and differential topology.

To understand curvature in more detail, one can consider how it behaves in specific tangent directions. This leads to the notion of Ricci curvature. Given a vector $v \in T_pM$, the Ricci curvature in the direction of v, denoted $\mathrm{Ric}(v)$, quantifies how the manifold curves in that direction. Algebraically, the Ricci curvature is a quadratic form on the tangent space, meaning it depends quadratically on the chosen vector. As the Ricci curvature is direction-specific, it retains more geometric information than scalar curvature.

Associated to the Ricci curvature is a symmetric bilinear form called the Ricci tensor, denoted $\mathrm{Ric}(v, w)$. The Ricci tensor and the Ricci curvature are related by the identity $\mathrm{Ric}(v) = \mathrm{Ric}(v, v)$, where the left-hand side represents the Ricci curvature of v, and the right-hand side evaluates the Ricci tensor on the pair (v, v). The scalar curvature at a point is then simply the sum of Ricci curvatures over an orthonormal basis for the tangent space.

The sign of curvature has long been of interest in geometry. A result of Lohkamp [21] shows that any manifold of dimension at least 3 admits a (complete) Riemannian metric with negative Ricci curvature (and hence negative scalar curvature). On the other hand, the existence of positive scalar curvature and positive Ricci curvature is subject to certain obstructions. For instance, a famous result due to Stolz [30] asserts that a closed, simply-connected spin manifold of dimension greater than or equal to 5 admits a positive scalar curvature metric if and only if its so-called α-invariant, (a topological invariant), vanishes. A different obstruction arises in the case of positive Ricci curvature. According to S. B. Myers' classical theorem [23], a compact manifold with positive Ricci curvature must have a finite fundamental group. This is not true for positive scalar curvature: for example the standard product metric on $S^1 \times S^2$ has positive scalar curvature but the fundamental group is isomorphic to the integers.

In this thesis we focus on so-called intermediate curvature conditions, specifically k-positive Ricci curvature. We now define what is meant by k-positive Ricci curvature.

Definition 1.0.1. *A manifold M has k-positive Ricci curvature ($Sc_k > 0$) if at each point $p \in M$, the sum of the k smallest eigenvalues of the Ricci tensor at p is positive.*

It is useful to note how the concept of k-positive Ricci curvature relates to more familiar curvature conditions. When $k = 1$, k-positive Ricci curvature requires that the Ricci tensor is positive in every direction at each point of the manifold. This corresponds to the classical notion of positive Ricci curvature, expressed as $\mathrm{Ric} > 0$. On the other hand, when $k = n$, where n is the dimension of the manifold, k-positive Ricci curvature implies that the scalar curvature is strictly positive. This follows from the fact that the scalar curvature is equal to the sum of the eigenvalues of the Ricci tensor. Hence, k-positive Ricci curvature provides a framework that interpolates between pointwise positivity of the Ricci tensor and positivity of the scalar curvature.

Not much is currently known about the existence of manifolds with $\mathrm{Sc}_k > 0$, where $k < n$. There are limited existence results due to Wolfson [33] and Crowley-Wraith [9].

Wolfson presents a surgery theorem, which creates new k-positive Ricci curvature manifolds from old. Surgery involves removing a copy of $S^{n-c} \times D^c \subset M^n$, and smoothly gluing in $D^{n-c+1} \times S^{c-1}$ in its place. This has a calculable effect on the topology. Here, c is said to be the codimension of the surgery, and $n - c$ the dimension. The relevance of surgery to curvature problems was first identified by Gromov and Lawson, who proved a surgery result for positive scalar curvature [16]. This states that performing surgery of codimension at least three on a positive scalar curvature manifold produces a new positive scalar curvature manifold. This result turned out to be a vital ingredient in much of the work which came afterwards in which the topological implications of positive scalar curvature were studied. Notably, this includes the classification of simply-connected manifolds with positive scalar curvature in dimensions at least 5, (see [30]). In the light of this, it makes sense to ask if surgery results exist for other curvature conditions. Adapting Gromov and Lawson's method, Wolfson went on to show that on an n-manifold with k-positive Ricci curvature, ($2 \leq k \leq n$), performing surgery with codimension $c \geq \max\{n+2-k, 3\}$, produces a new manifold which also admits k-positive Ricci curavture. For example, this means that for 2-positive Ricci curvature, connected sums (i.e. 0-surgeries) are possible. This should be contrasted with the case of positive Ricci curvature, in which connected sums are not in general possible as a consequence of Myers' Theorem, ([23]). So far, the tighter codimension requirement in Wolfson's theorem has meant that topological results like those for positive scalar curvature have not been forthcoming.

The Crowley-Wraith result ([10]) studies highly-connected manifolds in dimensions $4n-1$, ($n \geq 1$). In this context, a manifold is said to be highly-connected if it is $(2n-1)$-connected, i.e. the homotopy groups $\pi_0(M) = \pi_1(M) = \cdots = \pi_{2n-1}(M) = 0$. A previous result, [10], had shown that for highly-connected manifolds M in dimensions $4n-1$, (meaning $(2n-2)$-connected), assuming further that M is $(2n-1)$-parallelisable if $n \equiv 0$ modulo 4, there is a homotopy sphere Σ such that $M \sharp \Sigma$ admits a Ricci positive metric. (For a more precise statement, see Theorem 1.0.3 below.) Note that a manifold is said to be p-parallelisable if the tangent bundle restricted to some p-skeleton is trivial. The key to this result is showing that such manifolds can be constructed via the *plumbing* of disc bundles (see below, and Chapter 4).

Unfortunately the plumbing technique does not produce any interesting examples in dimensions $4n+1$, so a different approach is required in these dimensions. The key topological result in [9] is to show that any such manifold can be expressed as a connected sum of certain building blocks. As noted above, connected sums are always possible in 2-positive Ricci curvature, but not in general in positive Ricci curvature. Thus 2-positive Ricci curvature is the best one could hope to establish. The building blocks themselves are formed by performing surgery on a pair of linked $2n$-spheres within S^{4n+1}. Establishing positive Ricci curvature on these objects is out of reach of current techniques, but it was possible to establish 2-positive Ricci curvature. The main theorem of [9] is as follows: for every closed highly-connected $2n$-parallelisable manifold M^{4n+1}, there exists a homotopy sphere Σ such that the connected sum $M\sharp\Sigma$ admits a metric with 2-positive Ricci curvature. This implies for example that all simply-connected 5-manifolds admit such metrics.

Aside from existence questions, there are also some results concerning the topological complexity of the *space* of k-positive Ricci curvature metrics on certain manifolds. For example, see the papers of Walsh-Wraith [31], which studies H-space structures, and Frenck-Kordass [13] which shows non-triviality in certain homotopy groups. In a slightly different direction, spaces of concordances of such metrics are studied by Botivinnik-Wraith in [3], with certain moduli spaces of concordances shown to have non-trivial rational homotopy.

The following theorem about the existence of positive scalar curvature metrics provides the motivation for the central result in this thesis.

Theorem 1.0.1 (Carr, [6])**.** *Let M be an n-dimensional Riemannian manifold with a fixed smooth cell decomposition and K a dimension $q \geqslant 3$ subcomplex of M with codimension $n-q \geqslant 3$. Then there is a regular neighborhood U of K in M so that the induced metric on the boundary ∂U has positive scalar curvature.*

In this thesis, we generalize Carr's result to a broader setting. The following result is our main theorem.

Theorem A. *Let $(M^n, \bar{g})$ be a Riemannian manifold, $n \geqslant 4$, and let K be a smoothly embedded q-complex with codimension $n-q \geqslant 3$. Then K has a regular neighbourhood $U \subset M$ with smooth boundary ∂U, such that the restriction of $\bar{g}$ to ∂U has $Sc_{q+1} > 0$.*

Remark 1.0.2. In the case $n-q=3$, the theorem asserts that ∂U admits a metric with $\mathrm{Sc}_{n-2} > 0$. But note that $\mathrm{Sc}_{n-2} > 0$ implies $\mathrm{Sc}_{n-1} > 0$, which in turn implies that $\mathrm{Sc}_n > 0$, i.e. the scalar curvature is positive. Thus we recover Carr's result. However observe that the theorem above is strictly stronger than Carr's theorem even in the weakest case where $n-q=3$.

Let us comment on the proof of both of the above theorems. The basic idea behind the Carr result is to proceed by induction on the various skeleta of the complex. Beginning with the 0-skeleton, one constructs small spheres around each 0-simplex. If the radii are sufficiently small, these will all have positive sectional curvature. Collectively, these form a (generally disconnected) hypersurface H_0. Then consider each 1-cell in turn. The idea is to 'pull-out' the part of hypersurface H_0 surrounding the 1-cell's endpoints to form a tube surrounding the 1-cell. Doing this for each 1-cell then creates a new hypersurface H_1. Continuing in this way over all higher skeleta results in a hypersurface H enclosing the entrire complex K. The problem is to do this in such a way that the scalar curvature of H, induced by the ambient metric, is positive. The idea is that if H is sufficiently close in distance to the complex, then this can be achieved. Away from the boundaries of cells, H takes the form of a (normal) sphere bundle over the cells. If the spheres have sufficiently small radius, the scalar curvature will be positive. The difficulty is to control what happens near the cell junctions. It turns out that an argument closely modelled on part of the Gromov-Lawson surgery construction ([16]) can be utilised to achieve positive scalar curvature here.

Theorem A follows this strategy closely. The main problem now is to create a construction framework in which the intermediate curvatures can be computed and controlled. This turns out to be quite delicate.

We must also comment that the Carr result presents the reader with certain difficulties. In many places, the argument is merely a sketch, and offers the reader little clue as to how to proceed. This paper arose from Carr's PhD thesis, but the relevant part of the thesis agrees with the paper word for word, and provides no further details. A side benefit of the proof of Theorem A presented in this thesis is that it implies Carr's theorem, and hence places Carr's result on a firm and explicit footing.

We now turn our attention to the consequences of Theorem A. Firstly, we consider families of manifolds which arise from the *plumbing* construction. Plumbing is a method for gluing together disc bundles, in an arrangement determined by a graph. See Chapter 4 for details. Combining Theorems A and C in [10] we obtain the following statement:

Theorem 1.0.3. *Let M^{4n-1} (for $n \geq 2$) be a $(2n-2)$-connected manifold. If $n \equiv 1$ mod 4, then further assume that M is $(2n-1)$-parallelisable. Then there exists a homotopy sphere Σ^{4n-1} such that $M\sharp\Sigma$ is the boundary of a manifold obtained from plumbing D^{2n}-bundles over S^{2n} according to a simply-connected graph. Moreover, $M\sharp\Sigma$ admits a metric of positive Ricci curvature.*

Theorem A now allows us to broaden the scope of the above positive Ricci curvature result to the situation where we have arbitrary base manifolds, where the graph is not necessarily simply-connected, and dimension of the base and discs is not necessarily even. This comes at the expense of weakening the curvature condition satisfied by the resulting manifolds. The precise statement is:

Theorem B. *Given a graph G with m nodes (labelled $1, \ldots, m$), and D^n-bundles $E_1, \ldots, E_m$ with base spaces $M_1^n, \ldots, M_m^n$ respectively, for $n \geqslant 2$. Let P^{2n} be the manifold obtained by placing E_i on node i and plumbing according to the graph G. Then ∂P admits a metric with $Sc_{n+1} > 0$.*

It is well known that the curvature conditions on a manifold M can impose significant restrictions on its topology, for instance on the structure of its fundamental group, $\pi_1(M)$.

For example Myers' Theorem (mentioned earlier) asserts that a closed manifold with positive Ricci curvature has a finite fundamental group. More generally, we have the following consequence of the Splitting Theorem [8]:

Theorem 1.0.4. *If M has $Ric \geqslant 0$, then $\pi_1(M)$ is virtually abelian, i.e. $\pi_1(M)$ has an abelian subgroup of finite index.*

Further weakening the curvature condition to 2-positive Ricci curvature, we have the following conjecture due to Wolfson [33]:

Conjecture 1.0.5. *If M is a closed n-manifold that admits a metric with 2-positive Ricci curvature then its fundamental group, $\pi_1(M)$, is virtually free, i.e. $\pi_1(M)$ contains a free subgroup of finite index.*

Wolfson also remarks that the condition $\mathrm{Sc}_3 > 0$ appears to impose significantly weaker restrictions on the fundamental group $\pi_1(M)$ compared to $\mathrm{Sc}_2 > 0$ [33].

On the other hand, Carr [6] established a complementary result regarding positive scalar curvature and fundamental groups, which is a corollary of his Theorem 1.0.1.

Corollary 1.0.5.1. *Let π be a finitely presented group. Then, there exists a compact 4-manifold M of positive scalar curvature with $\pi_1(M) = \pi$.*

Building on Carr's result, and using the main theorem presented earlier, we can extend this to a more general setting:

Theorem C. *Let π be a finitely presented group. Then, there exists a closed manifold M of any given dimension $\geqslant 4$, which admits a metric with 3-positive Ricci curvature ($Sc_3 > 0$), such that $\pi_1(M) = \pi$.*

We immediately obtain:

Corollary 1.0.5.2. *Sc_3 imposes no restrictions on $\pi_1(M)$.*

In conclusion, we should remark that k-positive Ricci curvature is not the only notion of intermediate curvature. We mention two further types of intermediate curvature which have been much studied in recent years.

Firstly, there is the notion of the k^{th}-intermediate Ricci curvature. As k ranges from 1 to $\dim(M) - 1$, these interpolate between the sectional curvature and the Ricci curvature. Given a tangent vector $u \in T_pM$ and a k-dimensional subspace V of $u^\perp \subset T_pM$, we define

$$\mathrm{Ric}_k(u, V) = \sum_{i=1}^{k} \sec(u, z_i)$$

where sec denotes the sectional curvature and $\{z_1, \dots, z_k\}$ is an orthonormal basis for V. We say that M has positive k^{th} intermediate Ricci curvature if the above quantity is positive for all u, V at all points $p \in M$. Taken together with the k-positive Ricci curvatures, these form an interpolating family of positive curvature conditions from positive scalar curvature, through positive Ricci curvature, to positive sectional curvature. For some recent developments concerning these curvatures, see for example [27], [29], [26], [28], and for a comprehensive list of references see the website [22].

The other notion of intermediate curvature that we should mention is that of p-curvature. These were originally conceived by Gromov, and initially championed in the work of Labbi. The idea is to take a p-dimensional subspace $W \subset T_M$, and consider $W^{\perp}$. Let $\{e_1, \ldots, e_{n-p}\}$ be a basis for $W^{\perp}$. Then we define the p-curvature

$$s_p(W) = \sum_{i,j=1}^{n-p} \sec(e_i, e_j).$$

Like the k-positive Ricci curvatures, positive p-curvatures have been shown to satisfy a surgery theorem, see [19]. This has led to topological consequences, see [2], but with weaker conclusions than for positive scalar curvature. It is worth noting that the paper *Positive (p,n)-intermediate scalar curvature and Cobordism* by Burkemper, Searle and Walsh [5] establishes an analogue of Theorem C for positive p-curvature. Moreover, an analogue of Theorem B can be derived from the main result in [5].

This thesis is laid out as follows. The main theorem, Theorem A, is proved over the next two sections. The argument is broken into a "basic case" (presented in the next chapter), which establishes the theorem in the situation where the embedded complex K is 1-dimensional, and a "general case" in the subsequent chapter in which K has dimension at least two. The general case is much more complicated than the basic case, but the basic case sets out the strategy and presents many of the arguments that the general case uses. Theorem B and Theorem C are established in Chapter 4.

In addition to the main body of the thesis, a fifth chapter is included (Chapter 5), presenting work that is independent of the rest of the thesis. This chapter documents a preliminary project undertaken prior to commencing the main thesis topic. This project explored the relationship between contractibility and boundary convexity for manifolds with boundary and positive Ricci curvature.

Chapter 2

The Basic Case Of An Embedded 1-Complex

In this chapter, we establish the following special case of Theorem A.

Theorem 2.0.1. *Let* $(M^n, \bar{g})$ *be a Riemannian manifold, and* K *be a smoothly embedded* 1*-complex. Then,* K *has a regular neighbourhood* $U \subset M$ *with smooth boundary* ∂U*, such that the restriction of* $\bar{g}$ *to* ∂U *has* $Sc_2 > 0$*.*

The proof of Theorem 2.0.1 occupies the rest of this chapter. We start by considering the 0-skeleton of K, then we extend our construction over the 1-cells.

2.1 The set-up

In this section we outline the basic construction needed to establish Theorem 2.0.1. In particular, we set up coordinate systems and describe various metrics with respect to these systems. We will primarily focus on the case when K consists of a single 1-cell. We begin, however, by examining a 0-complex.

Note that a 0-complex is just a finite collection of points. About each of them, consider an ε-sphere for some small $\varepsilon > 0$, as highlighted in Figure 2.1.

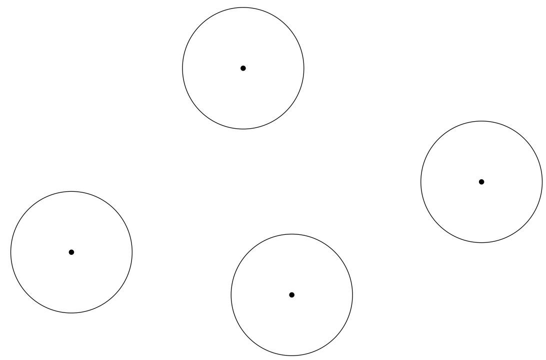

Figure 2.1: Some ε-balls around a 0-skeleton.

If ε is sufficiently small, the induced metric on these spheres has positive sectional curvature since the metric induced from the embedding $j : S^{n-1} \to M$ is C^2-close to the canonical metric on $S^n(\varepsilon)$, which has positive sectional curvature.

Now, consider a 1-complex, consisting of two 0-cells and one 1-cell; this complex is visualised in Figure 2.2. Also, two ε-spheres surrounding the 0-cells are shown, along with a tube joining these spheres, which we will construct.

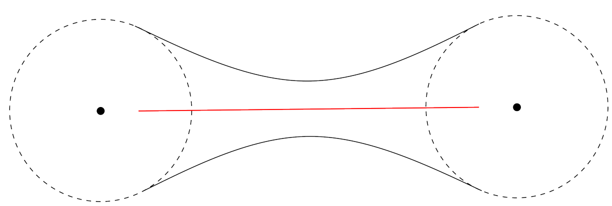

Figure 2.2: A composition of two 0-cells and a 1-cell.

We want to show that we can "pull out" this tube surrounding the 1-complex keeping $\mathrm{Sc}_2 > 0$, i.e. so that the sum of the lowest two eigenvalues of the Ricci tensor is positive. By the symmetry in the above set-up, it suffices to construct half the tube.

We construct a coordinate system about the central 1-cell in M^n, where M is our ambient manifold.

Let D be a small disc embedded in one of the ε-spheres, as illustrated in Figure 2.3.

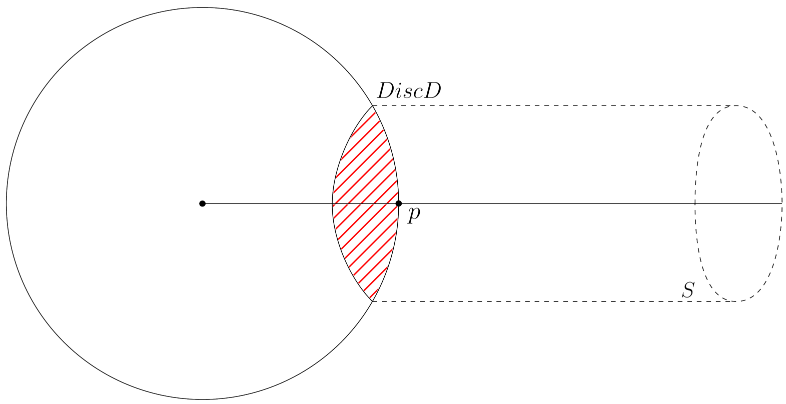

Figure 2.3: Disc D in the ambient manifold M.

On the disc D, we can choose an orthonormal frame of vectors $e_1, \dots, e_{n-1}$ at the point p, the centre of D. Restricting the metric $\bar{g}$ on M to D, $\bar{g}|_D$, we can give D a normal coordinate system determined by $\{e_i\}$.

Notice that we are using the exponential map of $\bar{g}|_D$ to push coordinates determined by $\{e_i\}$ in T_pD onto D.

We want to extend this to a coordinate system in a solid tube $S \cong D \times [0,1] \subset M$ illustrated on the right on Figure 2.3. Pushing D out along normal geodesics a length t gives the equidistant hypersurface at distance t from D, D_t.

Doing this for all $t \in [0, \delta]$, some $\delta > 0$, we get a decomposition of the solid tube S into discs (Figure 2.4).

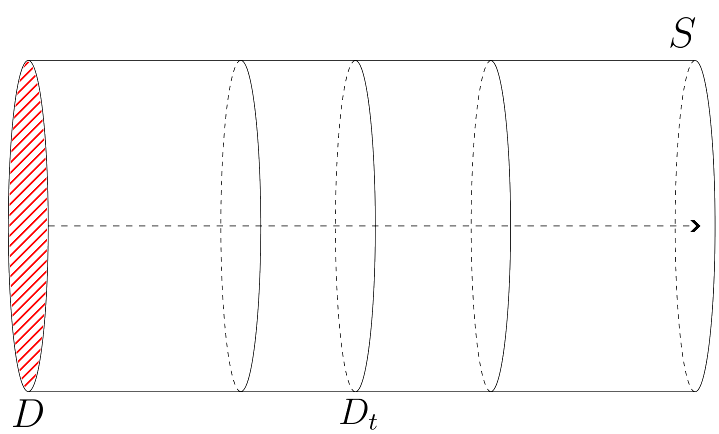

Figure 2.4: Descomposition of S.

Now, we take our orthonormal frame $\{e_i\}$ at $p \in D$ and extend it to an orthonormal frame at all other points along the central geodesic in a smooth way by parallel translation.

So, we have an orthonormal frame tangent to each D_t at each point of the normal geodesic through p.

Restricting $\bar{g}$ to each D_t, construct a normal coordinate system on $(D_t, \bar{g}|_{D_t})$ determined by the frame at p_t, where p_t is the central point of D_t. Therefore, now we have a coordinate system on the solid tube $(x_1, \ldots, x_{n-1}, t)$. With respect of this coordinate system $\bar{g}|_{D_t}$ has the form

$$\left(\bar{g}|_{D_t}\right)_{ij} = \delta_{ij} + \sum_{k,l} a_{ij}^{kl}(t) x_k x_l + \text{H.O.T.}.$$

At every point in the solid tube, $\partial/\partial t$ is orthogonal to D_t. Therefore, with respect to this coordinate system, the metric takes the form

$$\bar{g} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \delta_{ij} + \sum_{k,l} a_{ij}^{kl}(t) x_k x_l + \text{H.O.T.} \end{array}\right).$$

Following the Gromov-Lawson and Wolfson constructions, consider the curve γ illustrated in Figure 2.5, where $\gamma(s) = (t(s), r(s))$ and r is the distance from the centre of D_t.

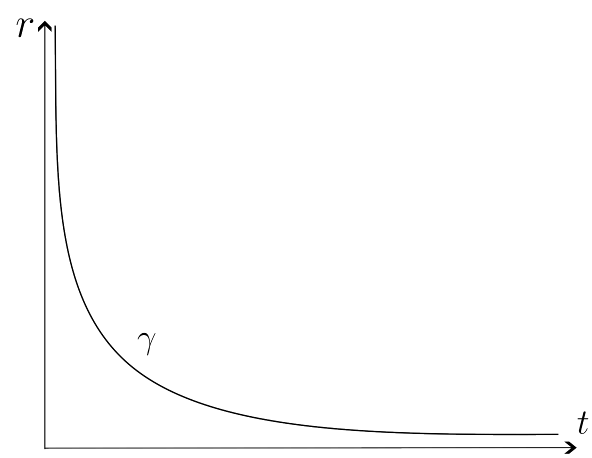

Figure 2.5: Curve γ.

Using γ we are going to construct a (hollow) tube N inside the solid tube (Figure 2.6), where

$$N := \bigcup_s \{\text{points in } D_{t(s)} \text{ a distance } r(s) \text{ from centre}\}.$$

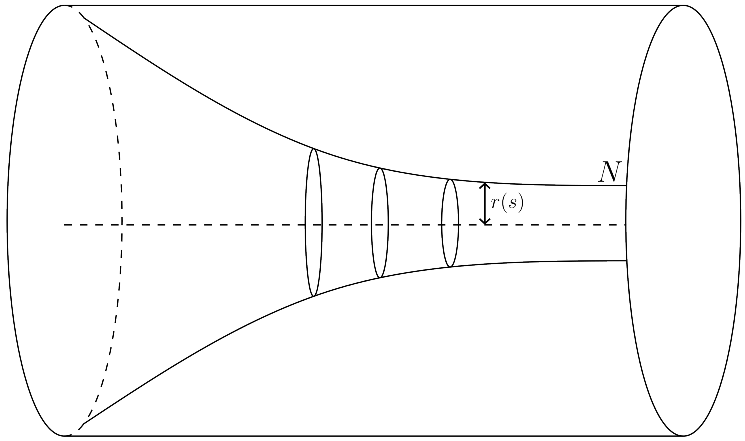

Figure 2.6: Tube N inside the solid tube.

In order to write down the metric on N and compute curvatures, we will pull the metric back to a fixed tube T defined as follows.

Let $(x_1, \ldots, x_n)$ be standard coordinates in $\mathbb{R}^n$ and define

$$T = \{(x_1, \ldots, x_n) | \sum_{i=1}^{n-1} x_i^2 = 1, x_n \in [0, s_0]\},$$

for some s_0 (Figure 2.7).

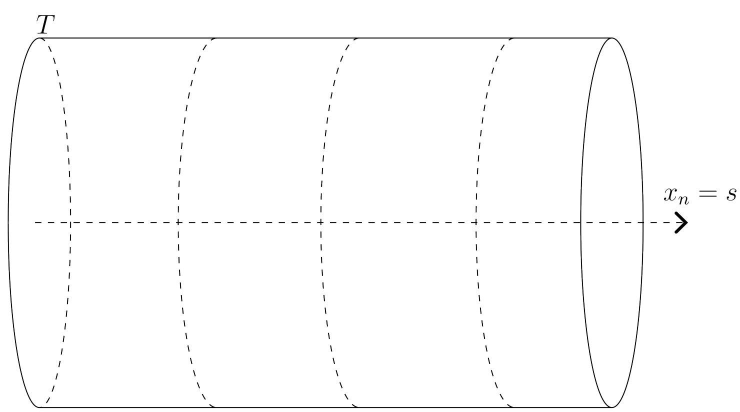

Figure 2.7: Fixed tube T.

Our strategy is to construct a map from a domain in $\mathbb{R}^n$ to the solid tube S, such that T maps diffeomorphically onto N. We then pull back the metric on S to our domain contained in $\mathbb{R}^n$, and restrict the pull-back to T. Using standard coordinates on T will allow us to express the metric on N in a straightforward way.

We have a coordinate system $(x_1, \ldots, x_{n-1}, t)$ on S, and we can introduce standard Euclidean coordinates $(x_1, \ldots, x_{n-1}, t)$ on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. By identifying coordinates in the obvious way, we obtain a smooth injective map $\iota : S \to \mathbb{R}^n$. We now scale ι on the slices $D_t \subset S$ as follows: define $f : S \to \mathbb{R}^n$ by

$$f(x_1, \ldots, x_{n-1}, t) = \left(\frac{1}{r}x_1, \ldots, \frac{1}{r}x_{n-1}, t\right),$$

where $r = r(s(t))$.

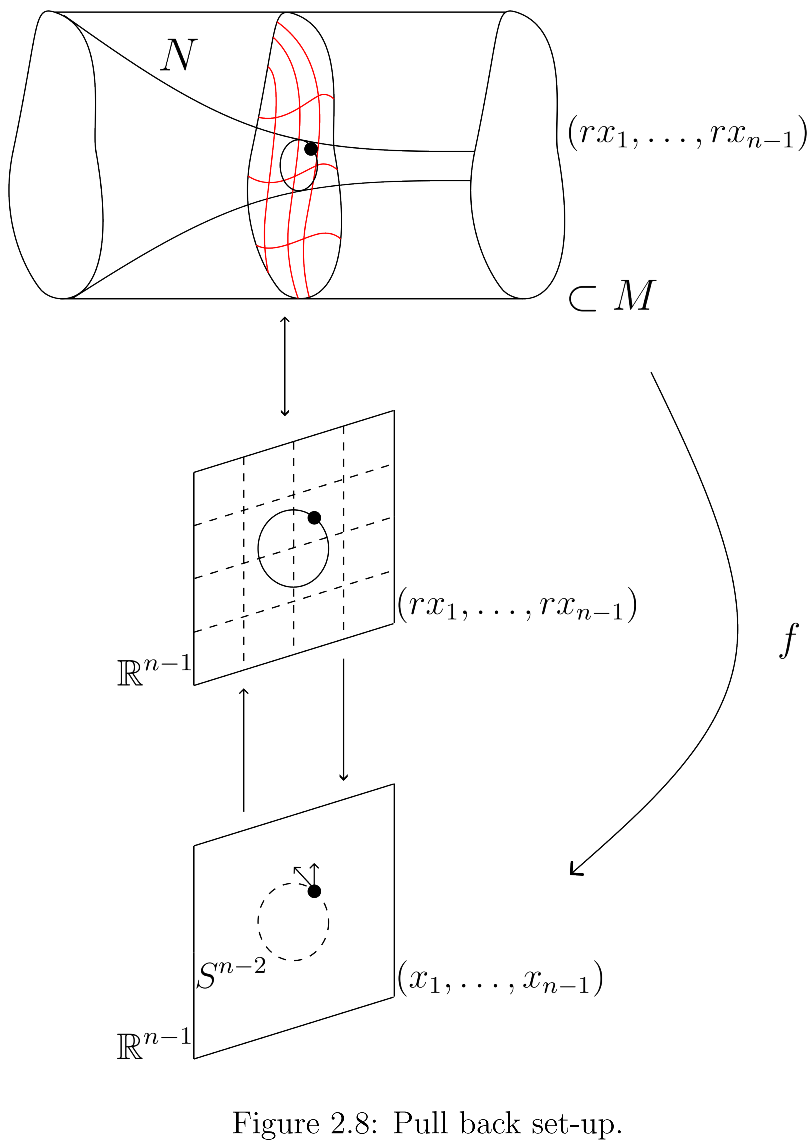

Figure 2.8: Pull back set-up.

Notice that, by construction, $f(N) = T$, as required. Clearly, f is a diffeomorphism if we restrict its target space to Imf. Let $\bar{f}$ denote this restricted map, i.e. $\bar{f} : S \to \text{Im}f \subset \mathbb{R}^n$. The inverse $\bar{f}^{-1} : \text{Im}f \to S$ exists and we want to pull-back the metric $\bar{g}$ via this map. We have

$$
\begin{aligned}
(\bar{f}^{-1})^*\bar{g}\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) &= \bar{g}\left(r\frac{\partial}{\partial x_i},r\frac{\partial}{\partial x_j}\right)_{(rx_1,\ldots,rx_{n-1},t)} \\
&= r^2\bar{g}\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)_{(rx_1,\ldots,rx_{n-1},t)} \\
&= r^2\left(\delta_{ij}+\sum_{k,l}a_{ij}^{kl}(rx_k)(rx_l)+\text{H.O.T.}\right) \\
&= r^2\delta_{ij}+r^4\sum_{k,l}a_{ij}x_kx_l+O(r^5).
\end{aligned}
$$

Now, we are going to restrict $(\bar{f}^{-1})^*\bar{g}$ to T. To study this, consider $S^{n-2}\times\{s\}\subset T$, for some $s\in[0,s_0]$. For convenience, we will suppress the second factor in $S^{n-2}\times\{s\}$ in the argument below.

We introduce normal coordinates $(y_1,\ldots,y_{n-2})$ on S^{n-2}. Restricted to S^{n-2}, the standard metric on $\mathbb{R}^{n-1}$ (with respect to standard coordinates), $r_0^2\delta_{ij}$, restricts to give $r_0^2 ds_{n-2}^2$ on S^{n-2}. Also, notice that the δ_{ij} are precisely the metric components in a normal coordinate system when we are evaluating at the central point.

Then, we have

$$
(\bar{f}^{-1})^*\bar{g}\big|_{S^{n-2}}\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right) = r^2ds_{n-2}^2\left(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j}\right)+r^4F_{ij},
$$

for some $F_{ij}=F_{ij}(y_1,\ldots,y_{n-2})$.

The function F_{ij} arises from the higher order terms in the normal coordinate metric expression. Consider $\sum_{k,l}a_{ij}^{kl}x_kx_l$ restricted to S^{n-2}. Each x_k is a function $S^{n-2}\to\mathbb{R}$, and hence we can write $x_k=\phi_k(y_1,\ldots,y_{n-2})$ for some function ϕ_k. In this way, we can re-write $\sum_{k,l}a_{ij}^{kl}x_kx_l$ in terms of the coordinates $(y_1,\ldots,y_{n-2})$, and taking the other higher order terms into consideration in a similar way, we thus obtain the function F_{ij}. Notice that F_{ij} is independent of r.

Now, pull back metric on N to $T\cong S^{n-2}\times I$ via the map $\bar{f}^{-1}$, i.e. by pulling back each individual $(n-2)$-sphere. Doing this for each $s\in[0,s_0]$, we obtain a metric g on T with matrix given by

$$g_{ij}(y_1,\dots,y_{n-2},s) = \left(\begin{array}{c:c} r^2(s)(ds^2_{n-2})_{ij} + r^4(s)F_{ij} & 0 \\ \hdashline 0 & 1 \end{array}\right). \tag{2.1}$$

2.2 Computation of Christoffel symbols

In order to compute the Christoffel symbols of g with respect to coordinates $(y_1,\dots,y_{n-2},s)$ we need to invert the above metric matrix (Equation 2.1). In other words, we want to know $\left(r^2(ds^2_{n-2})_{ij} + r^4F_{ij}\right)^{-1}$.

We can rewrite this as $1/r^2\left((ds^2_{n-2})_{ij} + r^2F_{ij}\right)^{-1}$ and consider it as

$$\frac{1}{r^2}(B + r^2C)^{-1},$$

where $C = F_{ij}$ and B^{-1} exists since $B = (ds^2_{n-2})_{ij}$.

Also, $(B + r^2C) = BB^{-1}(B + r^2C) = B(I + r^2B^{-1}C)$. Therefore,

$$\begin{aligned} (B + r^2C)^{-1} &= [B(I + r^2B^{-1}C)]^{-1} = (I + r^2B^{-1}C)^{-1}B^{-1} \\ &= (I - r^2B^{-1}C + r^4(B^{-1}C)^2 + \dots)B^{-1} \\ &= B^{-1} - r^2B^{-1}CB^{-1} + r^4(B^{-1}C)^2B^{-1} - r^6(B^{-1}C)^3B^{-1} + \dots. \end{aligned}$$

We can see from before that after the term B^{-1}, all other terms are $O(r^2)$. So, the desired inverse is

$$(g_{ij})^{-1}(y_1,\dots,y_{n-2},s) = \left(\begin{array}{c:c} \frac{1}{r^2}\left((ds^2_{n-2})^{-1} + O(r^2)\right) & 0 \\ \hdashline 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{c:c} \frac{1}{r^2}(ds^2_{n-2})^{-1} + O(1) & 0 \\ \hdashline 0 & 1 \end{array}\right).$$

Now, we calculate the Christoffel symbols. We have, in summary,

$$g_{ij} = r^2(ds^2_{n-2})_{ij} + O(r^4); \qquad g_{ss} = 1; \qquad g^{ss} = 1;$$

$$g^{ij} = \frac{1}{r^2}(ds^2_{n-2})^{ij} + O(1); \qquad g_{sj} = 0; \qquad g^{sj} = 0.$$

We need to consider Γ^s_{si}, Γ^i_{ss}, Γ^k_{si}, Γ^s_{ss}, Γ^s_{ij} and Γ^k_{ij} for $i,\ j,\ k \in \{1, ..., n-2\}$.

So,

$$\Gamma^s_{si} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{ss}}{\partial y_i} + \frac{\partial g_{si}}{\partial s} - \frac{\partial g_{si}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2} g^{sl}\left(\frac{\partial g_{sl}}{\partial y_i} + \frac{\partial g_{il}}{\partial s} - \frac{\partial g_{si}}{\partial y_l}\right)$$

$$= 0 + \frac{1}{2}\sum_{l=1}^{n-2}\delta_{sl}\left(\frac{\partial g_{il}}{\partial s}\right) = \frac{1}{2}\left(\frac{\partial g_{is}}{\partial s}\right) = 0.$$

$$\Gamma^i_{ss} = \frac{1}{2}g^{is}\left(\frac{\partial g_{ss}}{\partial s} + \frac{\partial g_{ss}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2} g^{il}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial y_l}\right)$$

$$= 0 + \frac{1}{2}\sum_{l=1}^{n-2} g^{il}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial y_l}\right) = 0.$$

$$\Gamma^k_{si} = \frac{1}{2}g^{ks}\left(\frac{\partial g_{ss}}{\partial y_i} + \frac{\partial g_{si}}{\partial s} - \frac{\partial g_{si}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2} g^{kl}\left(\frac{\partial g_{sl}}{\partial y_i} + \frac{\partial g_{il}}{\partial s} - \frac{\partial g_{si}}{\partial y_l}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2} g^{kl}\left(\frac{\partial g_{il}}{\partial s}\right) = \frac{1}{2}\sum_{l=1}^{n-2} g^{kl}\left(\frac{\partial (r^2(ds^2_{n-2})_{il} + O(r^4))}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2} g^{kl}\left(2rr'(ds^2_{n-2})_{il} + O(4r^3r') + O(r^4)\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{1}{r^2}(ds^2_{n-2})^{kl} + O(1)\right)\left(2rr'(ds^2_{n-2})_{il} + r'O(r^3) + O(r^4)\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{2r'}{r}(ds^2_{n-2})^{kl}(ds^2_{n-2})_{il} + r'O(r) + r'O(r) + r'O(r^3) + O(r^2) + O(r^4)\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{2r'}{r}(ds^2_{n-2})^{kl}(ds^2_{n-2})_{li} + r'O(r) + O(r^2)\right), \text{ as } (ds^2_{n-2})_{ij} \text{ is symmetric;}$$

$$= \frac{1}{2}\left(\frac{2r'}{r}\delta_{ki} + r'O(r) + O(r^2)\right) = \frac{r'}{r}\delta_{ki} + \frac{r'O(r)}{2} + O(r^2)$$

$$= r'\left(\frac{\delta_{ki}}{r} + O(r)\right) + O(r^2).$$

$$\Gamma^s_{ss} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{ss}}{\partial s} + \frac{\partial g_{ss}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2}g^{sl}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) = 0.$$

$$\Gamma^s_{ij} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{si}}{\partial y_j} + \frac{\partial g_{sj}}{\partial y_i} - \frac{\partial g_{ij}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2}g^{sl}\left(\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right)$$

$$= \frac{1}{2}g^{ss}\left(\frac{\partial g_{si}}{\partial y_j} + \frac{\partial g_{sj}}{\partial y_i} - \frac{\partial g_{ij}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2}\delta_{sl}\left(\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{is}}{\partial y_j} + \frac{\partial g_{js}}{\partial y_i} - \frac{\partial g_{ij}}{\partial s}\right) = \frac{1}{2}\left(-\frac{\partial g_{ij}}{\partial s}\right)$$

$$= \frac{1}{2}\left(-\frac{\partial(r^2(ds^2_{n-2})_{ij} + O(r^4))}{\partial s}\right)$$

$$= \frac{1}{2}\left(-2rr'(ds^2_{n-2})_{ij} + r'O(r^3) + O(r^4)\right)$$

$$= r'\left(-r(ds^2_{n-2})_{ij} + O(r^3)\right) + O(r^4).$$

Now, recall that

$$\Gamma^k_{ij}(S^{n-2}(1)) = \frac{1}{2}\sum_{l=1}^{n-2}(ds^2_{n-2})^{kl}\left(\frac{\partial(ds^2_{n-2})_{il}}{\partial y_j} + \frac{\partial(ds^2_{n-2})_{jl}}{\partial y_i} - \frac{\partial(ds^2_{n-2})_{ij}}{\partial y_l}\right).$$

Therefore,

$$\Gamma^k_{ij} = \frac{1}{2}g^{ks}\left(\frac{\partial g_{si}}{\partial y_j} + \frac{\partial g_{sj}}{\partial y_i} - \frac{\partial g_{ij}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-2}g^{kl}\left(\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2}g^{kl}\left(\frac{\partial(r^2(ds^2_{n-2})_{il})}{\partial y_j} + \frac{\partial O(r^4)}{\partial y_j} + \frac{\partial(r^2(ds^2_{n-2})_{jl})}{\partial y_i} + \frac{\partial O(r^4)}{\partial y_i}\right.$$

$$\left.-\frac{\partial(r^2(ds^2_{n-2})_{ij})}{\partial y_l} - \frac{\partial O(r^4)}{\partial y_l}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{1}{r^2}(ds^2_{n-2})^{kl} + O(1)\right)\left(\frac{r^2\partial(ds^2_{n-2})_{il}}{\partial y_j} + \frac{\partial O(r^4)}{\partial y_j}\right.$$

$$\left.+\frac{r^2\partial(ds^2_{n-2})_{jl}}{\partial y_i} + \frac{\partial O(r^4)}{\partial y_i} - \frac{r^2\partial(ds^2_{n-2})_{ij}}{\partial y_l} - \frac{\partial O(r^4)}{\partial y_l}\right)$$

$$= \Gamma^k_{ij}(S^{n-2}(1)) + \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{1}{r^2}(ds^2_{n-2})^{kl} + O(1)\right)\left(\frac{\partial O(r^4)}{\partial y_j} + \frac{\partial O(r^4)}{\partial y_i}\right.$$

$$\left.- \frac{\partial O(r^4)}{\partial y_l}\right)$$

$$= \Gamma^k_{ij}(S^{n-2}(1)) + \frac{1}{2}\sum_{l=1}^{n-2}\left(\frac{1}{r^2}(ds^2_{n-2})^{kl} + O(1)\right)\left(O(r^4)\right)$$

$$= \Gamma^k_{ij}(S^{n-2}(1)) + \frac{1}{2}\sum_{l=1}^{n-2}\left((ds^2_{n-2})^{kl}O(r^2) + O(r^4)\right)$$

$$= \Gamma^k_{ij}(S^{n-2}(1)) + O(r^2).$$

Notice that we are expressing the terms of the Christoffel symbols to show their dependence on r. So, even if $O(r^4)$ has some dependence on y_z ($z \in \{i, j, l\}$) and s, this dependence is bounded and its derivative will still depend on r.

In summary, we have proved:

Lemma 2.2.1. *The Christoffel symbols of* g *with respect to the local coordinates* $(y_1, \ldots, y_{n-2}, s)$ *are given by*

- $\Gamma^s_{si} = \Gamma^s_{ss} = \Gamma^i_{ss} = 0$;
- $\Gamma^k_{si} = r'(\frac{\delta_{ki}}{r} + O(r)) + O(r^2)$;
- $\Gamma^k_{ij} = \Gamma^k_{ij}(S^{n-2}) + O(r^2)$;
- $\Gamma^s_{ij} = r'(-r(ds^2_{n-2})_{ij} + O(r^3)) + O(r^4)$.

2.3 Curvature computations

In this section we work through the computation of the curvature tensor components for the metric $g = (g_{ij})$, and ultimately compute the matrix for the Ricci tensor.

Using the curvature tensor convention

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k + \nabla_{[\partial_i,\partial_j]}\partial_k$$

and setting

$$R(\partial_i, \partial_j)\partial_k = \sum_{l=1}^{n} R^l_{ijk}\partial_l$$

we have

$$R^l_{ijk} = \partial_j(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{jk}) + \sum_{m=1}^{n}(\Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}).$$

The components of the Ricci tensor, $\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j)$, in terms of the R^l_{ijk} above, are given by

$$\mathrm{Ric}_{ij} = \sum_{k=1}^{n} R^k_{ikj}.$$

We are going to abbreviate $\partial/\partial y_i$ by ∂_i and $\partial/\partial s$ by ∂_s. We intend to compute the terms Ric_{ij} by first computing the R^l_{ijk}. In our case, we have

$$\mathrm{Ric}_{ik} = R^s_{isk} + \sum_{l=1}^{n-2} R^l_{ilk},$$

where

$$R^l_{ilk} = \partial_l(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{lk}) + \sum_{m=1}^{n}(\Gamma^m_{ik}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{im})$$

$$= \partial_l(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{lk}) + \sum_{m=1}^{n-2}(\Gamma^m_{ik}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{im}) + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{lk}\Gamma^l_{is}.$$

We need to consider Ric_{ss}, Ric_{sk} and Ric_{ik}, $i, j, k \in \{1, ..., n-2\}$. Recall that,

- $\Gamma^s_{si} = \Gamma^s_{ss} = \Gamma^i_{ss} = 0$;
- $\Gamma^k_{si} = r'(\frac{\delta_{ki}}{r} + O(r)) + O(r^2)$;
- $\Gamma^k_{ij} = \Gamma^k_{ij}(S^{n-2}) + O(r^2)$;
- $\Gamma^s_{ij} = r'(-r(ds^2_{n-2})_{ij} + O(r^3)) + O(r^4)$.

We have $\mathrm{Ric}_{ss} = R^s_{sss} + \sum_{l=1}^{n-2} R^l_{sls}$.

$$R^s_{sss} = \partial_s(\Gamma^s_{ss}) - \partial_s(\Gamma^s_{ss}) + \sum_{m=1}^{n-2}(\Gamma^m_{ss}\Gamma^s_{sm} - \Gamma^m_{ss}\Gamma^s_{sm}) + \Gamma^s_{ss}\Gamma^s_{ss} - \Gamma^s_{ss}\Gamma^s_{ss} = 0.$$

$$R^l_{sls} = \partial_l(\Gamma^l_{ss}) - \partial_s(\Gamma^l_{ls}) + \sum_{m=1}^{n-2}(\Gamma^m_{ss}\Gamma^l_{lm} - \Gamma^m_{ls}\Gamma^l_{sm}) + \Gamma^s_{ss}\Gamma^l_{ls} - \Gamma^s_{ls}\Gamma^l_{ss}$$

$$= 0 - \partial_s\left(r'\left(\frac{\delta_{ll}}{r} + O(r)\right) + O(r^2)\right)$$
$$- \sum_{m=1}^{n-2}\left(r'\left(\frac{\delta_{ml}}{r} + O(r)\right) + O(r^2)\right)\left(r'\left(\frac{\delta_{lm}}{r} + O(r)\right) + O(r^2)\right)$$

$$= -\partial_s\left(r'\left(\frac{1}{r} + O(r)\right) + O(r^2)\right) - (r')^2\left(\frac{1}{r} + O(r)\right)^2 + r'O(r)$$

$$+ O(r^4) - \sum_{\substack{m=1\\m\neq l}}^{n-2}((r')^2O(r^2) + O(r^4))$$

$$= -\left(\frac{r''r - r'r'}{r^2} + r''O(r) + (r')^2O(1) + r'O(r) + r'O(r) + O(r^2)\right)$$

$$- \frac{(r')^2}{r^2} - \frac{2(r')^2O(r)}{r} - (r')^2O(r^2) + r'O(r) + O(r^4)$$

$$- (n-1)((r')^2O(r^2) + O(r^4))$$

$$= -\frac{r''}{r} + \frac{(r')^2}{r^2} + r''O(r) + (r')^2O(1) + r'O(r) + O(r^2) - \frac{(r')^2}{r^2}$$

$$- 2(r')^2O(1) - (r')^2O(r^2) + (r')^2O(r^2) + O(r^4)$$

$$= -\frac{r''}{r} + r''O(r) - (r')^2O(1) + r'O(r) + O(r^2).$$

Therefore,

$$\mathrm{Ric}_{ss} = 0 + \sum_{l=1}^{n-2}\left(-\frac{r''}{r} + r''O(r) - (r')^2O(1) + r'O(r) + O(r^2)\right)$$

$$= -(n-2)\frac{r''}{r} + r''O(r) - (r')^2O(1) + r'O(r) + O(r^2)$$

$$= -(n-2)\frac{r''}{r} + r''O(r) + O(1) + O(r) + O(r^2)$$

$$= -(n-2)\frac{r''}{r} + r''O(r) + O(1).$$

Notice that $|r'| \in [0,1]$, since $\gamma(s) = (t(s), r(s))$ is unit speed. So, we can approximate terms like $r'O(r)$, since it is at worst, $O(r)$.

Now, we have $\mathrm{Ric}_{sk} = R^s_{ssk} + \sum_{l=1}^{n-2} R^l_{slk}$. We compute:

$$R^s_{ssk} = \partial_s(\Gamma^s_{sk}) - \partial_s(\Gamma^s_{sk}) + \sum_{m=1}^{n-2}(\Gamma^m_{sk}\Gamma^s_{sm} - \Gamma^m_{sk}\Gamma^s_{sm}) + \Gamma^s_{sk}\Gamma^s_{ss} - \Gamma^s_{sk}\Gamma^s_{ss} = 0.$$

$$R^l_{slk} = \partial_l(\Gamma^l_{sk}) - \partial_s(\Gamma^l_{lk}) + \sum_{m=1}^{n-2}(\Gamma^m_{sk}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{sm}) + \Gamma^s_{sk}\Gamma^l_{ls} - \Gamma^s_{lk}\Gamma^l_{ss}$$

$$= \partial_l\left(r'\left(\frac{\delta_{lk}}{r} + O(r)\right) + O(r^2)\right) - \partial_s(\Gamma^l_{lk}(S^{n-2})) - \partial_s(O(r^2))$$

$$+ \sum_{m=1}^{n-2}\left[\left(r'\left(\frac{\delta_{mk}}{r} + O(r)\right) + O(r^2)\right)\left(\Gamma^l_{lm}(S^{n-2}) + O(r^2)\right)\right.$$

$$\left. - (\Gamma^m_{lk}(S^{n-2}) + O(r^2))\left(r'\left(\frac{\delta_{ml}}{r} + O(r)\right) + O(r^2)\right)\right]$$

$$= 0 + r'O(r) + O(r^2) - \partial_s(\Gamma^l_{lk}(S^{n-2})) + r'O(r) + O(r^2)$$

$$+ \sum_{\substack{m=1\\ m\neq k}}^{n-2}(r'O(r) + O(r^2))\left(\Gamma^l_{lm}(S^{n-2}) + O(r^2)\right)$$

$$- \sum_{\substack{m=1\\ m\neq l}}^{n-2}(r'O(r) + O(r^2))\left(\Gamma^m_{lk}(S^{n-2}) + O(r^2)\right)$$

$$+ \left(\left(\frac{r'}{r} + r'O(r)\right) + O(r^2)\right)\left(\Gamma^l_{lk}(S^{n-2}) + O(r^2)\right), \text{ when } m = k;$$

$$- (\Gamma^l_{lk}(S^{n-2}) + O(r^2))\left(\left(\frac{r'}{r} + r'O(r)\right) + O(r^2)\right), \text{ when } m = l;$$

$$= r'O(r) - \partial_s(\Gamma^l_{lk}(S^{n-2})) + O(r^2)$$

$$+ \sum_{\substack{m=1\\ m\neq k}}^{n-2}(r'O(r) + O(r^2)) - \sum_{\substack{m=1\\ m\neq l}}^{n-2}(r'O(r) + O(r^2))$$

$$= r'O(r) + O(r^2).$$

Notice that $\partial_s(\Gamma^l_{lk}(S^{n-2}))$ vanishes since it does not depend on s.

Therefore,

$$\text{Ric}_{sk} = 0 + \sum_{l=1}^{n-2} (r'O(r) + O(r^2)) = r'O(r) + O(r^2)$$

$$= O(r) + O(r^2) = O(r).$$

Finally, we have $\text{Ric}_{ik} = R^s_{isk} + \sum_{l=1}^{n-2} R^l_{ilk}$. We compute:

$$R^s_{isk} = \partial_s(\Gamma^s_{ik}) - \partial_i(\Gamma^s_{sk}) + \sum_{m=1}^{n-2} (\Gamma^m_{ik}\Gamma^s_{sm} - \Gamma^m_{sk}\Gamma^s_{im}) + \Gamma^s_{ik}\Gamma^s_{ss} - \Gamma^s_{sk}\Gamma^s_{is}$$

$$= \partial_s(r'(-r(ds^2_{n-2})_{ik} + O(r^3)) + O(r^4)) + 0$$

$$- \sum_{m=1}^{n-2} \left(r' \left(\frac{\delta_{mk}}{r} + O(r) \right) + O(r^2) \right) (r'(-r(ds^2_{n-2})_{im} + O(r^3)) + O(r^4))$$

$$= -((r')^2 + rr'')(ds^2_{n-2})_{ik} + r''O(r^3) + (r')^2 O(r^2) + r'O(r^3) + O(r^4)$$

$$- \left(\frac{r'}{r} + r'O(r) + O(r^2) \right) (r'(-r(ds^2_{n-2})_{ik} + O(r^3)) + O(r^4)), \text{ when } m = k$$

$$- \sum_{\substack{m=1 \\ m \neq k}}^{n-2} (r'O(r) + O(r^2))(r'(-r(ds^2_{n-2})_{im} + O(r^3)) + O(r^4))$$

$$= \big(-(r')^2(ds^2_{n-2})_{ik} - rr''(ds^2_{n-2})_{ik} + r''O(r^3) + (r')^2 O(r^2) + r'O(r^3)$$

$$+ O(r^4) \big) + \big((r')^2(ds^2_{n-2})_{ik} + (r')^2 O(r^2) + r'O(r^3) + (r')^2 O(r^2)$$

$$+ (r')^2 O(r^4) + r'O(r^5) + r'O(r^3) + r'O(r^5) + O(r^6) \big)$$

$$- \big((r')^2 O(r^2) + (r')^2 O(r^4) + r'O(r^5) + r'O(r^3) + r'O(r^5) + O(r^6) \big)$$

$$= -rr''(ds^2_{n-2})_{ik} + (r')^2 O(r^2) + r''O(r^3) + r'O(r^3) + O(r^4)$$

$$= -rr''(ds^2_{n-2})_{ik} + O(r^2) + r''O(r^3),$$

where we have combined $(r')^2O(r^2)$, $r'O(r^3)$ and $O(r^4)$ terms into a single $O(r^2)$ term, using the boundedness of r'. We also have:

$$R^l_{ilk} = \partial_l(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{lk}) + \sum_{m=1}^{n-2}(\Gamma^m_{ik}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{im}) + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{lk}\Gamma^l_{is}$$

$$= \partial_l(\Gamma^l_{ik}(S^{n-2}) + O(r^2)) - \partial_i(\Gamma^l_{lk}(S^{n-2}) + O(r^2))$$

$$+ \sum_{m=1}^{n-2} \big[(\Gamma^m_{ik}(S^{n-2}) + O(r^2))(\Gamma^l_{lm}(S^{n-2}) + O(r^2))$$

$$- (\Gamma^m_{lk}(S^{n-2}) + O(r^2))(\Gamma^l_{im}(S^{n-2}) + O(r^2))\big]$$

$$+ (r'(-r(ds^2_{n-2})_{ik} + O(r^3)) + O(r^4)) \left(r'\left(\frac{\delta_{ll}}{r} + O(r)\right) + O(r^2)\right)$$

$$- (r'(-r(ds^2_{n-2})_{lk} + O(r^3)) + O(r^4)) \left(r'\left(\frac{\delta_{li}}{r} + O(r)\right) + O(r^2)\right)$$

$$= \partial_l(\Gamma^l_{ik}(S^{n-2})) - \partial_i(\Gamma^l_{lk}(S^{n-2})) + O(r^2) - O(r^2)$$

$$+ \sum_{m=1}^{n-2} \big[(\Gamma^m_{ik}(S^{n-2}))(\Gamma^l_{lm}(S^{n-2})) - (\Gamma^m_{lk}(S^{n-2}))(\Gamma^l_{im}(S^{n-2}))$$

$$+ O(r^2)\left(\Gamma^m_{ik}(S^{n-2}) + \Gamma^l_{lm}(S^{n-2}) - \Gamma^m_{lk}(S^{n-2}) - \Gamma^l_{im}(S^{n-2})\right)$$

$$+ O(r^4) - O(r^4)\big] + \big(-(r')^2(ds^2_{n-2})_{ik} + (r')^2O(r^2)$$

$$+r'O(r^3) + O(r^6) + (r')^2\delta_{li}(ds^2_{n-2})_{lk}\big).$$

To simplify this expression further, let us assume that we wish to evaluate it at the central point of the normal coordinate system on S^{n-2}. At this point we have $(r')^2(ds^2_{n-2})_{ik} = -(r')^2\delta_{ik}$ and $(r')^2\delta_{li}(ds^2_{n-2})_{lk} = (r')^2\delta_{li}\delta_{lk}$. So at this point, we have

$$
\begin{aligned}
R^l_{ilk} &= \partial_l(\Gamma^l_{ik}(S^{n-2})) - \partial_i(\Gamma^l_{lk}(S^{n-2})) \\
&+ \sum_{m=1}^{n-2} \big[(\Gamma^m_{ik}(S^{n-2}))(\Gamma^l_{lm}(S^{n-2})) - (\Gamma^m_{lk}(S^{n-2}))(\Gamma^l_{im}(S^{n-2})) \\
&+ O(r^2)\left(\Gamma^m_{ik}(S^{n-2}) + \Gamma^l_{lm}(S^{n-2}) - \Gamma^m_{lk}(S^{n-2}) - \Gamma^l_{im}(S^{n-2})\right)\big] \\
&+ \left(-(r')^2\delta_{ik} + (r')^2 O(r^2) + r' O(r^3) + O(r^6) + (r')^2\delta_{li}\delta_{lk}\right) \\
&= \partial_l(\Gamma^l_{ik}(S^{n-2})) - \partial_i(\Gamma^l_{lk}(S^{n-2})) \\
&+ \sum_{m=1}^{n-2} \big[(\Gamma^m_{ik}(S^{n-2}))(\Gamma^l_{lm}(S^{n-2})) - (\Gamma^m_{lk}(S^{n-2}))(\Gamma^l_{im}(S^{n-2})) \\
&+ O(r^2)\left(\Gamma^m_{ik}(S^{n-2}) + \Gamma^l_{lm}(S^{n-2}) - \Gamma^m_{lk}(S^{n-2}) - \Gamma^l_{im}(S^{n-2})\right)\big] \\
&+ \left(-(r')^2\delta_{ik} + (r')^2 O(r^2) + r' O(r^3) + O(r^6) + (r')^2\delta_{li}\delta_{lk}\right) \\
&= R^l_{ilk}(S^{n-2}) + O(r^2) + (r')^2 O(r^2) + r' O(r^3) - (r')^2(\delta_{ik} - \delta_{lk}\delta_{li}).
\end{aligned}
$$

Therefore, at the central point of the normal coordinate system on S^{n-2},

$$
\begin{aligned}
\mathrm{Ric}_{ik} &= -rr''(ds^2_{n-2})_{ik} + O(r^2) + r'' O(r^3) + \sum_{l=1}^{n-2} \big[R^l_{ilk}(S^{n-2}) \\
&+ O(r^2) + (r')^2 O(r^2) + r' O(r^3) - (r')^2(\delta_{ik} - \delta_{lk}\delta_{li})\big] \\
&= -rr''(ds^2_{n-2})_{ik} + (r')^2 O(r^2) + r'' O(r^3) + r' O(r^3) + O(r^2) \\
&+ \mathrm{Ric}_{ik}(S^{n-2}) - (n-3)(r')^2\delta_{ik}.
\end{aligned}
$$

Notice that

$$
\sum_{l=1}^{n-2}(\delta_{ik} - \delta_{lk}\delta_{li}) = (n-2)\delta_{ik} - \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k (= l) \end{cases} = \begin{cases} 0 & \text{if } i \neq k; \\ n-3 & \text{if } i = k(= l). \end{cases}
$$

So, since $\mathrm{Ric}_{ik}(S^{n-2}) = (n-3)\delta_{ik}$, we have that when $i \neq k$,

$$\begin{aligned}\mathrm{Ric}_{ii} &= -rr'' + (n-3)(1-(r')^2) + (r')^2 O(r^2) + r'' O(r^3) + O(r^2) + r' O(r^3) \\ &= -rr'' + (n-3)(1-(r')^2) + O(r^2) + r'' O(r^3) + O(r^2) + O(r^3) \\ &= -rr'' + (n-3)(1-(r')^2) + r'' O(r^3) + O(r^2). \\ \mathrm{Ric}_{ik} &= (r')^2 O(r^2) + r'' O(r^3) + O(r^2) + r' O(r^3) \\ &= O(r^2) + r'' O(r^3) + O(r^2) + O(r^3) \\ &= r'' O(r^3) + O(r^2).\end{aligned}$$

The computations above have now established:

Lemma 2.3.1. *The matrix Ric_{ik} for $i, k \in \{1, \ldots, n-2, s\}$ is*

$$(Ric_{ik}) = \left[\begin{array}{c|c} A & \begin{matrix} O(r) \\ \vdots \\ O(r) \end{matrix} \\ \hline O(r) \quad \cdots \quad O(r) & -(n-2)\frac{r''}{r} + r'' O(r) + O(1) \end{array}\right],$$

where A is a matrix such that

$$A_{ii} = -rr'' + (n-3)(1-(r')^2) + r'' O(r^3) + O(r^2);$$

$$A_{ik} = r'' O(r^3) + O(r^2).$$

This is the Ricci tensor matrix with respect to $\{\partial/\partial y_1, \ldots, \partial/\partial y_{n-2}, \partial/\partial s\}$, but we are going to re-express it with respect to an approximately orthonormal basis $\{1/r \cdot \partial/\partial y_1, \ldots, 1/r \cdot \partial/\partial y_{n-2}, \partial/\partial s\}$. Notice that this is orthonormal at $y_1 = \ldots = y_{n-2} = 0$, and more generally the deviation from orthonormality is $O(|y|^2)$.

Since $\mathrm{Ric}(\mu e_i, \mu e_j) = \mu^2 \mathrm{Ric}(e_i, e_j)$, we have, for $i, j \in \{1, \ldots n-2\}$,

- $\mathrm{Ric}(1/r \cdot \partial/\partial y_i, 1/r \cdot \partial/\partial y_j) = \left(\frac{1}{r^2}\right) \mathrm{Ric}(\partial/\partial y_i, \partial/\partial y_j)$;

- $\mathrm{Ric}(1/r \cdot \partial/\partial y_i, \partial/\partial s) = \left(\frac{1}{r}\right) \mathrm{Ric}(\partial/\partial y_i, \partial/\partial s)$.

So, if $i \neq j$,

$$\mathrm{Ric}(1/r \cdot \partial/\partial y_i, 1/r \cdot \partial/\partial y_j) = \frac{1}{r^2}\left(r'' O(r^3) + O(r^2)\right) = r'' O(r) + O(1).$$

$$\begin{aligned}\mathrm{Ric}(1/r \cdot \partial/\partial y_i, 1/r \cdot \partial/\partial y_i) &= \frac{1}{r^2}\left(-rr'' + (n-3)(1-(r')^2) + r'' O(r^3) + O(r^2)\right) \\ &= -\frac{r''}{r} + (n-3)\left(\frac{1-(r')^2}{r^2}\right) + r'' O(r) + O(1).\end{aligned}$$

$$\mathrm{Ric}(1/r \cdot \partial/\partial y_i, \partial/\partial s) = \left(\frac{1}{r}\right) O(r) = O(1).$$

Let $\widetilde{\mathrm{Ric}}_{ik}$ denote the matrix of the Ricci tensor with respect to this rescaled basis.

Corollary 2.3.1.1. *The new matrix $\widetilde{Ric}_{ik}$ for $i, k \in \{1, \ldots, n-2, s\}$ is*

$$(\widetilde{Ric}_{ik}) = \left(\begin{array}{ccc|c} & & & O(1) \\ & \widetilde{A} & & \vdots \\ & & & O(1) \\ \hline O(1) & \cdots & O(1) & -(n-2)\frac{r''}{r} + r'' O(r) + O(1) \end{array}\right),$$

where $\widetilde{A}$ is a matrix such that

$$\widetilde{A}_{ii} = -\frac{r''}{r} + (n-3)\left(\frac{1-(r')^2}{r^2}\right) + r'' O(r) + O(1);$$

$$\widetilde{A}_{ik} = r'' O(r) + O(1).$$

2.4 Explicit tube construction

We will now show how to construct the tube. This reduces to constructing the curve γ, since the tube will be formed by rotating γ around the t-axis.

It is easy to see that for our desired tube, we initially (i.e. for small s) need γ to run down the r-axis, and for s large we want γ to be parallel to the t-axis. (See Figure 2.9). Of course our aim is to define γ in such a way that the resulting curvature of the tube satisfies $\mathrm{Sc}_2 > 0$.

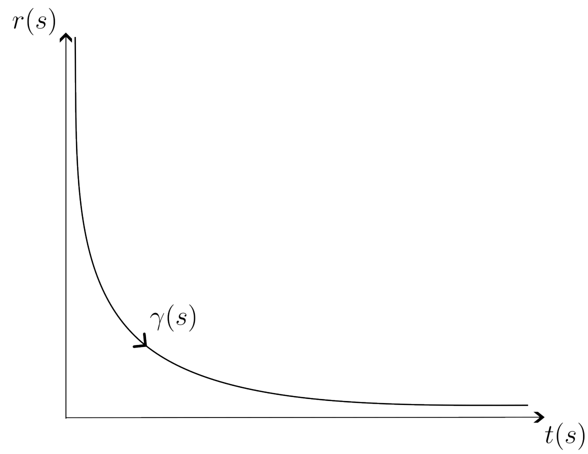

Figure 2.9: Curve γ.

We assume $\gamma(s)$ is unit speed (Figure 2.9). Then,

$$\gamma'(s) = (t'(s), r'(s)), \text{ so } |\gamma'(s)| = \sqrt{(t')^2 + (r')^2} = 1.$$

So, $(r')^2 + (t')^2 = 1$.

2.4.1 First bending

We start with γ running down the r-axis. Then, "peel" γ away from the axis, turning it through some very small angle θ_0.

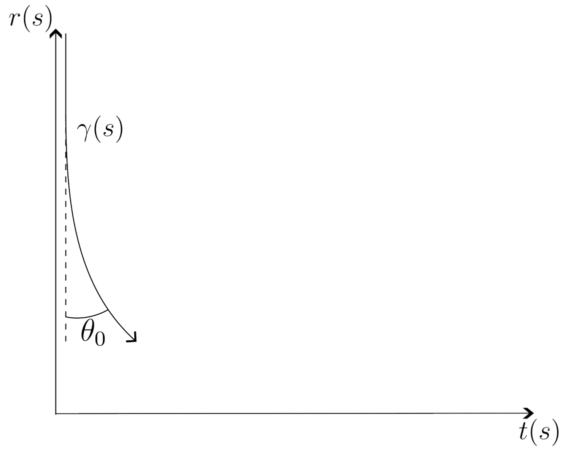

Figure 2.10: Curve γ "peeling" away from the axis (first bend).

Since initially, when γ is running down the r-axis, the tube has $\mathrm{Sc}_2 > 0$ by assumption, by the openness of positivity there exists $\Theta > 0$ such that for all $0 < \theta_0 \leqslant \Theta$, we can bend γ over a small interval through an angle of θ_0 away from the r-axis, and continue γ as a straight line as far as (but not meeting) the t-axis, retaining $\mathrm{Sc}_2 > 0$.

Having chosen θ_0, we can extend γ as a straight line as far as some very small height r_0 (to be determined).

2.4.2 Second bending

Our next aim is to make a further bend to γ starting at height r_0. The choice of θ_0 allows us to choose r_0 arbitrarily small.

Let s_0 be the value of s corresponding to r_0, and therefore $t(s_0)$ is the corresponding point on the t-axis.

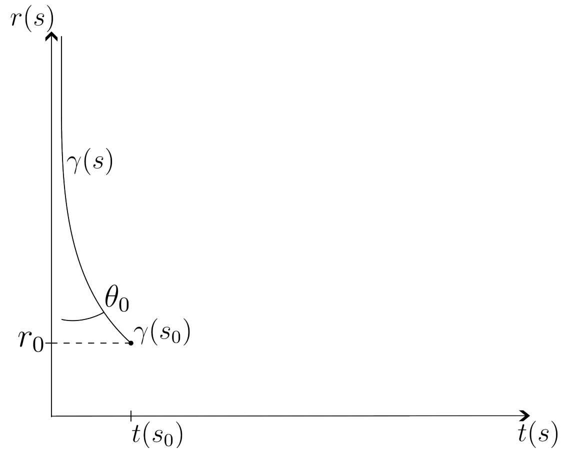

Figure 2.11: Curve γ after first bending.

By our choice of θ_0, we have $|r'|$ bounded above strictly less than 1 for the rest of the construction.

Consider the curvature $\kappa(s)$ of $\gamma(s)$. This is given by $\kappa = t'r'' - t''r'$, and since $(r')^2 + (t')^2 = 1$, we can replace t above (provided γ is not vertical) to get

$$\kappa = \frac{r''}{\sqrt{1-(r')^2}}.$$

This comes from the fact that $(r')^2 + (t')^2 = 1$, so $t' = \sqrt{1-(r')^2}$. Thus,

$$t'' = \left(\sqrt{1-(r')^2}\right)' = \frac{(0 - 2r'r'')}{2\sqrt{1-(r')^2}} = \frac{-r'r''}{\sqrt{1-(r')^2}}.$$

Since $\kappa = t'r'' - t''r'$,

$$\kappa = (\sqrt{1-(r')^2})r'' + \frac{(r')^2 r''}{\sqrt{1-(r')^2}} = \frac{(1-(r')^2)r'' + (r')^2 r''}{\sqrt{1-(r')^2}}$$

$$= \frac{r'' - (r')^2 r'' + (r')^2 r''}{\sqrt{1-(r')^2}} = \frac{r''}{\sqrt{1-(r')^2}}.$$

Now, we have that $\cos\theta = \dfrac{|r'|}{|\gamma'|} = \dfrac{|r'|}{1} = -r'$, since $r' \leqslant 0$.

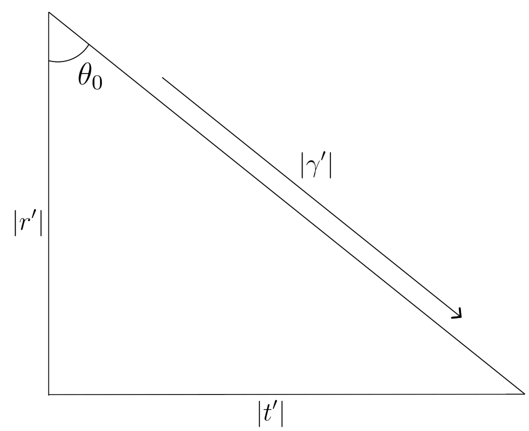

Figure 2.12: Velocity vector triangle.

Then, $\sqrt{1-(r')^2} = \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta$. So, $\kappa = r''/\sin\theta$, i.e. $r'' = \kappa.\sin\theta$.

In the following construction we will determine γ by fixing the function κ. Let $\kappa(s)$ be the function illustrated in Figure 2.13.

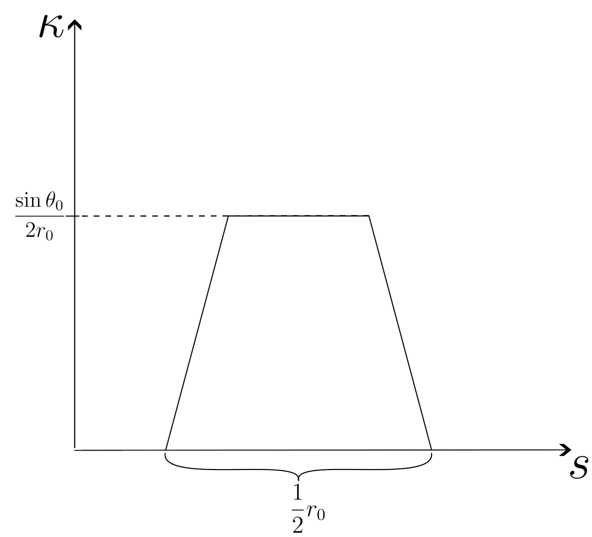

Figure 2.13: Function $\kappa(s)$ for first bend.

Let us consider the effect of defining γ in this way. First notice that the bend takes place over an s-interval of length $(1/2)r_0$. Since the height of γ above the t-axis at the start of this bend is r_0, and $|r'| < 1$, it follows that γ does not meet the t-axis during this bend. Indeed, the end height of γ, r_1, satisfies $r_1 > r_0/2$.

Next, consider the angle $\theta(s)$ between (the tangent to) γ and the r-axis, during this bend. Since κ measures the rate of change of this angle, we see that the change in angle is

$$\triangle\theta \approx \left(\frac{1}{2}r_0\right)(\kappa_{\max}) = \frac{r_0}{2}\left(\frac{\sin\theta_0}{2r_0}\right) = \frac{1}{4}\sin\theta_0.$$

Finally, we consider the curvature implications of this bend.

Recall that we have two types of diagonal entries in our matrix $\widetilde{\mathrm{Ric}}$:

$$\widetilde{\mathrm{Ric}_{ii}} = -\left(\frac{r''}{r}\right) + (n-3)\left(\frac{1-(r')^2}{r^2}\right) + r''O(r) + (r')^2O(1) + O(1),$$

for $i \in \{1, \ldots, n-2\}$;

$$\widetilde{\mathrm{Ric}_{ss}} = -(n-2)\left(\frac{r''}{r}\right) + r''O(r) - (r')^2O(1).$$

We want to check the positivity of the sum of the lowest two diagonal entries when $r \longrightarrow 0$.

Since $r'' = \kappa\sin\theta$, having chosen γ such that $\kappa_{\max} = \dfrac{\sin\theta_0}{4r_0}$, we have

$$r'' \leqslant \frac{\sin\theta_0}{4r_0}\sin\theta,$$

where $\theta = \theta(s)$.

But $\theta \geqslant \theta_0$ during the bend, as θ is increasing. In the first bend $\theta \geqslant \theta_0$, so $\sin\theta \geqslant \sin\theta_0$, and

$$\kappa_{\max} = \frac{\sin\theta_0}{4r_0} \leqslant \frac{\sin\theta}{4r_0}.$$

Therefore,

$$r'' \leqslant \frac{\sin\theta}{4r_0}\sin\theta = \frac{\sin^2\theta}{4r_0}.$$

Recall that we have $\sin^2\theta = 1-(r')^2$ and $\cos^2\theta = (r')^2$; and we are also considering $\theta \in [\theta_0, \theta_0 + \triangle\theta]$, where $\triangle\theta$ represents the change in θ over the curvature bend.

At worst, the curvature is

$$\widetilde{\mathrm{Ric}_{ii}} = -\left(\frac{\sin^2\theta}{4r_0 r}\right) + (n-3)\left(\frac{\sin^2\theta}{r^2}\right) + \left(\frac{1}{r_0}\right)O(r) + O(1)$$

$$= \left(\frac{\sin^2\theta}{r}\right)\left[\left(\frac{n-3}{r}\right) - \left(\frac{1}{4r_0}\right)\right] + O(1).$$

We know $r \leqslant r_0$, so the term between brackets will be non-negative if

$$\left(\frac{n-3}{r}\right) > \left(\frac{1}{4r_0}\right) \iff n-3 > \frac{1}{4},$$

i.e. $n > 3 + 1/4$; so $n \geqslant 4$.

Then, provided r_0 is sufficiently small we can ignore the other terms in the expression for $\widetilde{\mathrm{Ric}}_{ii}$. We conclude $\widetilde{\mathrm{Ric}}_{ii} > 0$ throughout the bend.

Moreover, since $\left[\left(\frac{n-3}{r}\right) - \left(\frac{1}{4r_0}\right)\right]$ is bounded below by a positive constant

$$\left(\frac{n-3}{r_0} - \frac{1}{4r_0}\right) = \frac{1}{r_0}\left(n - \frac{13}{4}\right),$$

then by choosing r_0 smaller, we can make this curvature term arbitrarily large during the bend.

Now, for the sum of the lowest two diagonal entries to be positive, provided r_0 sufficiently small, it suffices to consider $\widetilde{\mathrm{Ric}}_{ii} + \widetilde{\mathrm{Ric}}_{ss}$ and check that

$$\left(-\left(\frac{r''}{r}\right) + (n-3)\frac{(1-(r')^2)}{r^2}\right) + \left(-(n-2)\left(\frac{r''}{r}\right)\right)$$

can be made arbitrarily large, i.e.

$$-(n-1)\left(\frac{r''}{r}\right) + (n-3)\frac{(1-(r')^2)}{r^2} \gg 0.$$

In other words, we need to check that

$$\left(\frac{\sin^2\theta}{r}\right)\left[\frac{n-3}{r} - \frac{n-1}{4r_0}\right] > 0,$$

since if this is positive, we can clearly make it arbitrarily positive by choosing r_0 (and therefore r) smaller.

We have

$$\left(\frac{\sin^2\theta}{r}\right)\left[\frac{n-3}{r} - \frac{n-1}{4r_0}\right] > 0 \iff \left(\frac{n-3}{r_0}\right) > \left(\frac{n-1}{4r_0}\right)$$

$$\iff n-3 > \left(\frac{n-1}{4}\right)$$

$$\iff 4n-12 > n-1$$

$$\iff 3n > 11$$

$$\iff n > \frac{11}{3} \iff n \geqslant 4.$$

We have now established:

Lemma 2.4.1. *By choosing θ_0 and r_0 sufficiently small, we can ensure that the sum of the smallest two diagonal entries of $\widetilde{Ric}$ for the tube created by the initial and second bendings of γ can be made arbitrarily large, provided $n \geqslant 4$.*

Corollary 2.4.1.1. *$Sc_2 > 0$ for r_0 sufficiently small.*

Proof. This will follow provided the eigenvalues of $\widetilde{\mathrm{Ric}}$ are sufficiently close to the diagonal entries. By the above calculations, along the bend we have that the off-diagonal entries of the matrix are all $O(1)$. Thus for the scaled matrix $\varepsilon\widetilde{\mathrm{Ric}}$ the off-diagonal terms are all $O(\varepsilon)$. We also saw above that by choosing r_0 sufficiently small, we can make the sum of the smallest two diagonal entries in $\widetilde{\mathrm{Ric}}$, and therefore (for given ε) in $\varepsilon\widetilde{\mathrm{Ric}}$, arbitrarily large.

So, for ε sufficiently small, the scaled matrix is very close to being diagonal and the sum of the lowest two diagonal entries is bigger than 1 say. Since the eigenvalues of any matrix vary continuously with the entries, we see that provided our scaled matrix is sufficiently close to being diagonal, the sum of the lowest two eigenvalues will be bigger than 1. So, for ε sufficiently small, $\varepsilon\widetilde{\mathrm{Ric}}$ is 2-positive, and therefore $\widetilde{\mathrm{Ric}}$ is also 2-positive , i.e. $\mathrm{Sc}_2 > 0$ (for r_0 sufficiently small). □

Currently, the curve γ has the form illustrated in Figure 2.14 with the new angle labelled θ_1 and new height r_1.

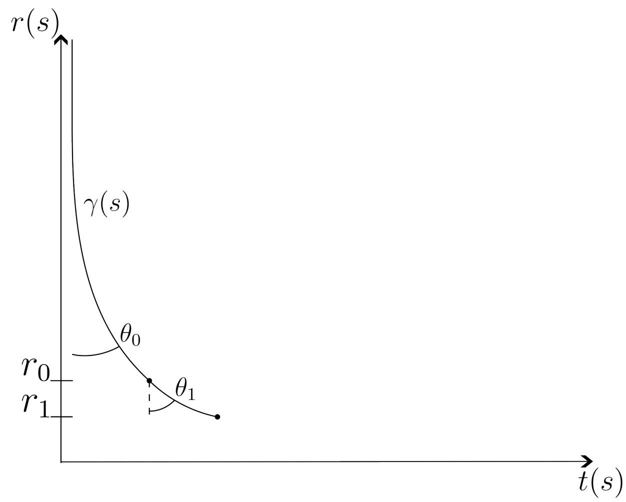

Figure 2.14: Curve γ with angle θ_1 and height r_1.

2.4.3 Further bends

We now wish to make a further bend which we again achieve by specifying the curvature function. This is given in the diagram below.

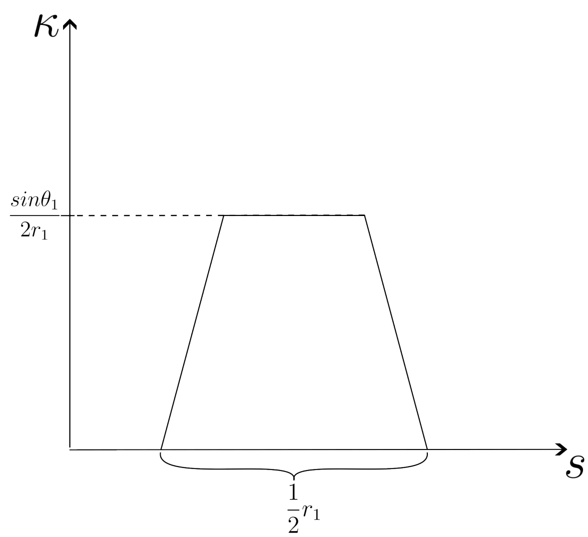

Figure 2.15: Function $\kappa(s)$ for second bend.

For the second bend,

$$\triangle\theta \approx \left(\frac{\sin\theta_1}{2r_1}\right)\left(\frac{1}{2}r_1\right) = \frac{\sin\theta_1}{4}.$$

Notice that $\theta_1 > \theta_0$, so $\sin\theta_1 > \sin\theta_0$. So, $\triangle\theta$ for the second bend is bigger than $\triangle\theta$ for first bend. The height of γ at the end of this bend is $> r_1/2 > 0$.

For the curvature, we need to check that the positivity of the previous formulas involving r_0 remain positive when r_0 is replaced by r_1, (and also for $\theta \in [\theta_1, \theta_1 + \triangle\theta]$ with $\triangle\theta$ as above). But this is immediate since $r_1 < r_0$. (We are also free to choose a smaller r_1, if desired, by extending γ downwards as a straight line as in the previous step, since this clearly preserves $\mathrm{Sc}_2 > 0$). Thus, exactly the same argument as for the first bend shows that $\mathrm{Sc}_2 > 0$ throughout the second bend, and indeed throughout each subsequent bend which follows the same procedure.

Continuing in this way, we can bend γ to the horizontal in a finite number of steps.

The number of bends required is less than

$$\left\lceil \frac{\pi/2}{\sin\theta_0/4} \right\rceil = \left\lceil \frac{2\pi}{\sin\theta_0} \right\rceil < \infty.$$

Lemma 2.4.2. *The total s-length and t-length of the bend tends to 0 as $r_0 \to 0$.*

Proof. First, we can see that the total s-length of the first bend is $r_0/2$ and for the second bend is $r_1/2$. Following this we have that the s-length of the n^{th} bend is $r_{n-1}/2$. Also, since $r_0 > r_1 > \ldots > r_m$, where $m =$ number of bends, we have

$$\frac{1}{2}r_0 + \frac{1}{2}r_1 + \ldots \frac{1}{2}r_m < \frac{mr_0}{2}.$$

So, our total s-length is $\sum_{i=0}^{m} r_i < \dfrac{mr_0}{2} \xrightarrow[r_0 \to 0]{} 0$.

Now, γ under projection has length less than or equal to the length of γ, i.e. its s-length.

Therefore, the t-length is $< \dfrac{mr_0}{2}$. □

Having chosen θ_0 first, we can then choose r_0 such that this length can be as small as needed.

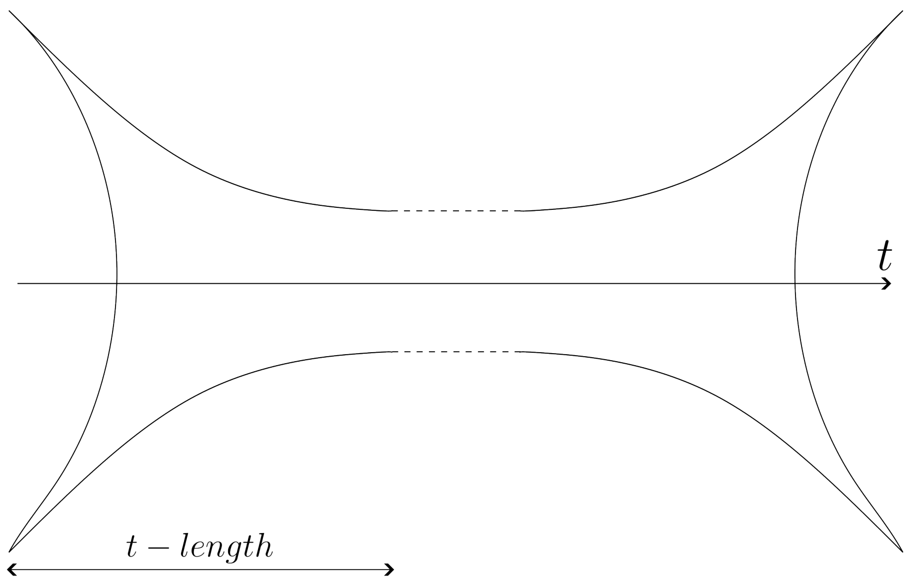

Figure 2.16: Joining the half-tubes from two different 1-cells.

2.5 Proof of Theorem 2.0.1

Proof. Let us consider a collection of 0-complexes with an ε-sphere around each of them for some small $\varepsilon > 0$. By choosing ε sufficiently small, we can not only ensure that the ε-spheres around each 0-cell have positive sectional curvature ($K > 0$), but also that each ε-sphere intersects each 1-cell in at most one point.

Now, consider a 1-complex formed by two 0-cells and one 1-cell (Figure 2.2). We start with a hypersurface consisting of ε-spheres around each of the 0-cells. Recall that we want to show that we can "pull out" a tube from these two ε-spheres as illustrated, to create a new connected hypersurface keeping $\mathrm{Sc}_2 > 0$.

Observe that, during the tube construction in Section 2.4, we can make the first bend for each tube in such a way that the radius at the wide end is so small that all tubes leaving a given ε-sphere are disjoint. Also, by Corollary 2.4.1.1, we are able to preserve $\mathrm{Sc}_2 > 0$ during the first bend, as well as each subsequent bend.

Our aim is to construct a tube from each ε-sphere for each 1-cell which intersects the sphere. By Lemma 2.4.2, this tube can be made as short as we like or need, by choosing r_0

sufficiently small.

By controlling r_0 at each end, we can arrange for each pair of half-tubes for a given 1-cell to have same radius, so they can be joined, as in Figure 2.16, to create the desired $\mathrm{Sc}_2 > 0$ hypersurface within M.

Repeating this procedure for every 1-cell completes the construction.

□

Chapter 3

The Proof Of The Main Theorem

Recall the statement of the Main Theorem.

Theorem 3.0.1. *Let $(M^n, \bar{g})$ be a Riemannian manifold, and let K be a smoothly embedded q-complex with codimension $n - q \geqslant 3$. Then, K has a regular neighbourhood $U \subset M$ with smooth boundary ∂U, such that the restriction of $\bar{g}$ to ∂U has $Sc_{q+1} > 0$.*

The rest of this chapter is occupied by the proof of Theorem 3.0.1. We will follow Carr's general idea in the sense that we can perform induction on the dimension of the cells in the complex.

The "basic case" covered in the previous chapter is the base case, so for the inductive step assume we have constructed an hypersurface, H, with $\mathrm{Sc}_d > 0$ bounding a regular neighborhood of the $(d-1)$-skeleton of the complex K. We therefore need to focus on the problem of extending the hypersurface across the d-cells in K, preserving $\mathrm{Sc}_{d+1} > 0$. We address this problem one d-cell, σ^d, at a time.

3.1 The set up

We are going to follow Carr's set up:

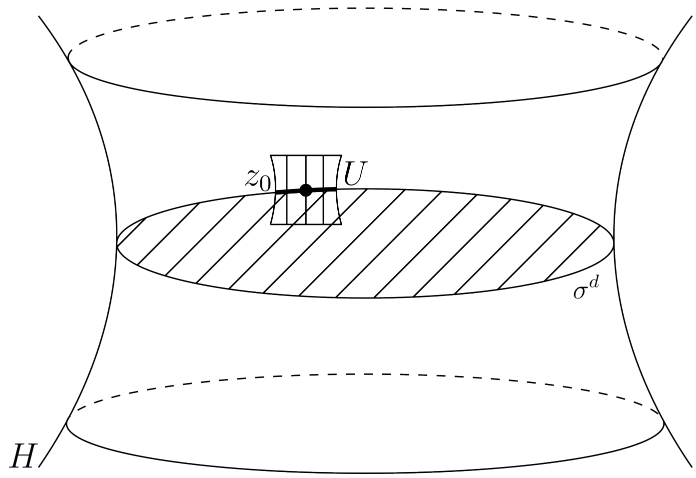

Figure 3.1: σ^d meeting H transversally in S^{d-1}.

Assume, by making a very small adjustment to the d-cell (σ^d) if necessary, that σ^d meets H transversally in a sphere S^{d-1} (Figure 3.1), where H is the hypersurface. By making a further small adjustment if required, we can assume that the normal geodesics issuing from $\sigma^d \cap H$ lie in σ^d for small time $t \in (-\delta, \delta)$, some $\delta > 0$.

Consider a local neighborhood $U \subset S^{d-1}$ ($\subset H$). We pick a point $\bar{z}_0 \in U$ and choose a normal coordinate system centered on $\bar{z}_0$ for U with respect to the restricted metric $\bar{g}|_U$.

Locally, H has structure of a disc bundle over U, in a tubular neighborhood V of U in H.

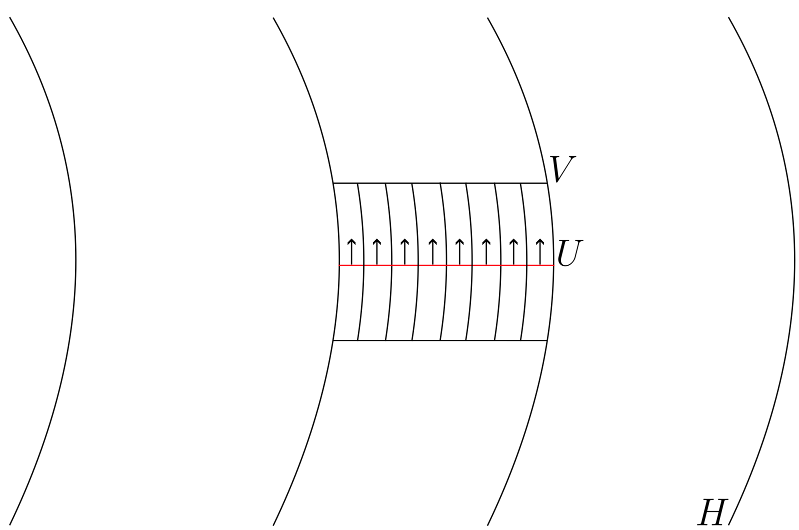

Figure 3.2: Disc bundle structure of H over U.

We then extend our coordinate system on U to a coordinate system on V by choosing a smoothly varying orthonormal frame tangent to each disc fibre for every $z \in U$, as highlighted in Figure 3.2. Using these frames, we can generate a smoothly varying family of normal coordinate systems with respect to the induced metric on each disc.

Next, we are going to push V out along normal (unit speed) geodesics to H parametrized by t, in the direction of the centre of σ^d. Let us call the normal geodesic flow ϕ_t and set $V_t = \phi_t(V)$ (Figure 3.3). So, $t =$ distance from V, and V_t is the equidistant surface to V at distance t. We can use the flow ϕ_t to "push" the disc bundle structure from V to each V_t. Setting $U_t = \phi_t(U)$, we have that V_t is a disc bundle over U_t.

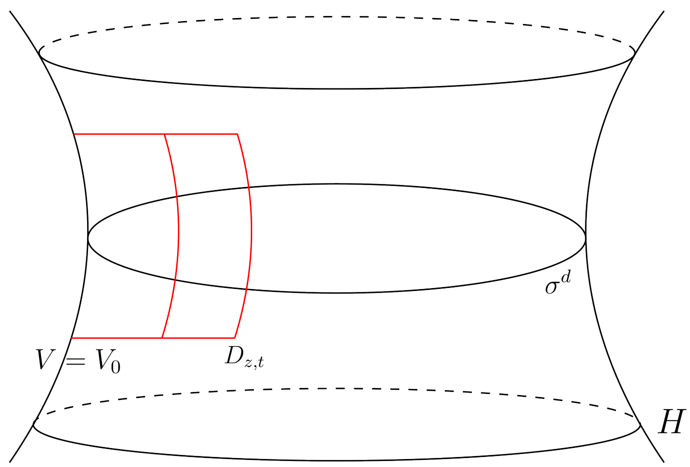

Figure 3.3: Pushing V out along normal geodesics to H.

Collectively, $\bigcup_{t\in[0,\epsilon]} V_t$ gives a local neighborhood in M (a "solid tube"), as shown in Figure 3.4.

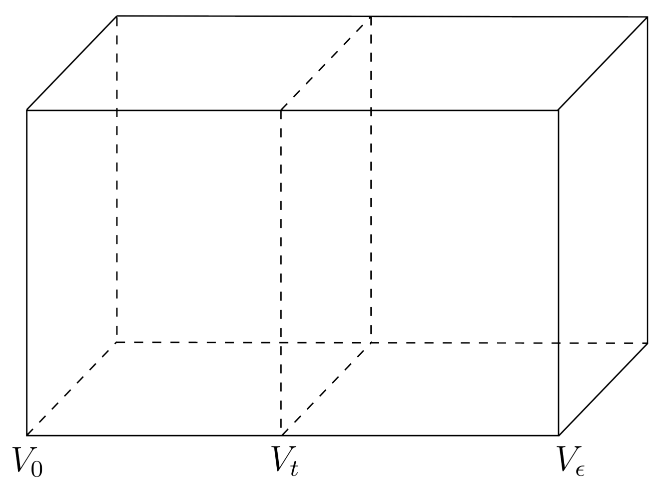

Figure 3.4: Solid tube $\bigcup_{t\in[0,\epsilon]} V_t$.

Now, our aim is to introduce coordinates into each V_t. Of course, we already have coordinates in $V = V_0$.

Let $D_{z,0}$ be the disc fibre over $z \in U \subset S^{d-1}$. For $t > 0$, consider the normal disc bundle to U_t in V_t, and use the normal exponential map to U_t in $(V_t, \bar{g}|_{V_t})$ to project each normal disc to a disc $D_{z,t} \subset V_t$. Then, for fixed t, the $D_{z,t}$ are the fibres of a bundle over U_t, meeting U_t orthogonally, with total space V_t. This can be visualised in Figure 3.5.

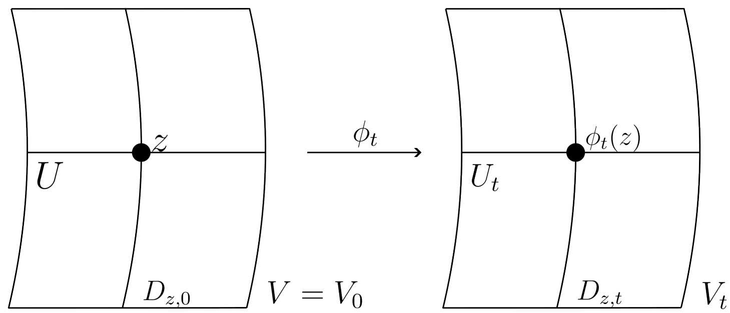

Figure 3.5: $D_{z,0}$ and $D_{z,t}$.

We have a "central" point $\bar{z}_0$ in U. Notice that this point is central in the normal coordinate system set previously on U.

Denote the normal geodesic to V at z by $\mu_z(t)$.

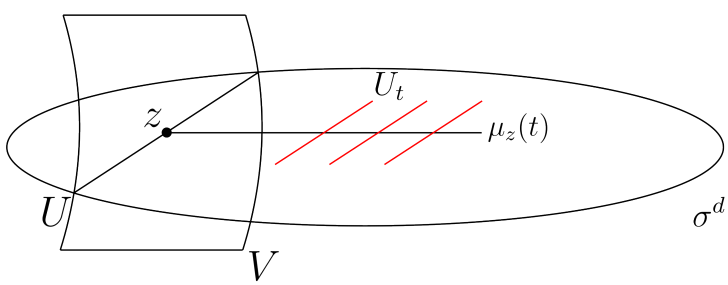

Figure 3.6: Construction of U_t.

At each point of $\mu_{\bar{z}_0}(t)$, we choose an orthonormal frame $v_1, \dots, v_{d-1}$ for TU_t in a way that varies smoothly with t. By restricting the ambient metric to each U_t, we can then use $\{v_1, \dots, v_{d-1}\}$ to construct a normal coordinate system in each U_t, centred on $\mu_{\bar{z}_0}(t)$.

Similarly, for each $z \in U$ and $\mu_z(t)$, pick an orthonormal frame of vectors $w_1, \dots, w_{n-d}$, which are tangent to each $D_{z,t}$ at $\mu_z(t)$, and therefore perpendicular to U_t, in a way that varies smoothly with t and z.

Restricting the ambient metric to each $D_{z,t}$, we will use the frame $\{w_1, \ldots, w_{n-d}\}$ to generate a normal coordinate system on each disc.

Overall, we obtain coordinates $(t, x_1, \ldots, x_{n-d}, z_1, \ldots, z_{d-1})$ in our solid tube, where $(z_1, \ldots, z_{d-1}) \in U_t$, $(x_1, \ldots, x_{n-d}) \in D_{z,t}$ and t is the normal parameter.

Let us investigate the form taken by the ambient metric in M with respect to this coordinate system.

Along U_t we have the following:

- $\bar{g}(\partial/\partial t, \partial/\partial x_i) = 0 = \bar{g}(\partial/\partial t, \partial/\partial z_i)$, by construction;
- $\bar{g}(\partial/\partial z_i, \partial/\partial x_i) = 0$, by construction;
- $\bar{g}(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}$;
- $\bar{g}(\partial/\partial z_i, \partial/\partial z_j) = \delta_{ij} + \sum_{k,l} b_{ij}^{kl} z_k z_l +$ H.O.T. .

So, the matrix which describes the metric at points of U_t takes the form

$$\bar{g}|_{U_t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_{ij} & 0 \\ 0 & 0 & \delta_{ij} + \sum_{k,l} b_{ij}^{kl} z_k z_l + \text{ H.O.T.} \end{pmatrix}.$$

Now, we want to examine the metric's form away from U_t. Consider moving from $\mu_z(t)$ through the disc fibre $D_{z,t}$. There is no guarantee that the inner product of some $\partial/\partial x_i$ and $\partial/\partial z_j$ will always be 0, as it is at the centre. In the following, assume that t is fixed.

Expanding $\bar{g}(\partial/\partial x_i, \partial/\partial z_j)$ about $\mu_z(t)$ in a Taylor series gives $c_{ij} + \sum_k c_i^k x_k + \sum_l d_j^l z_l +$ H.O.T., where

- $c_{ij} = 0$ as $\bar{g}(\partial/\partial x_i, \partial/\partial z_j) = 0$ at $\mu_z(t)$;
- $\sum_l d_j^l z_l = 0$ as $\bar{g}(\partial/\partial x_i, \partial/\partial z_j) = 0$ all along U_t (Figure 3.7).

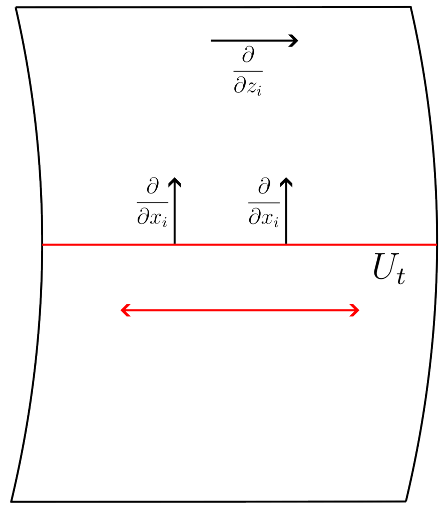

Figure 3.7: $\partial/\partial x_i$ and $\partial/\partial z_i$ along U_t.

Therefore, $\bar{g}(\partial/\partial x_i, \partial/\partial z_j) = O(|x|)$, close to U_t as it holds in each fibre disc individually. In particular, we cannot get pure z terms (z^2, z^3, etc.) in the Taylor expansion, since the inner product vanishes on $\mu_z(t)$, (i.e. when $x_1 = \ldots = x_{n-d} = 0$).

Note that by $O(|x|)$ we mean a function for which the absolute value is bounded by $C(\sum_{i=1}^{n-d} |x_i|)$, where $C > 0$ depends only on $(M, \bar{g})$ and the choice of coordinate system. Similar remarks apply to $O(|z|)$, $O(|z|^2)$, etc., below.

Similarly, we have

$$\bar{g}(\partial/\partial z_i, \partial/\partial z_j) = b_{ij} + \sum_k b_{ij}^k z_k + \sum_l a_{ij}^l x_l + \sum_{k,l} b_{ij}^{kl} z_k z_l + \ldots$$

$$= b_{ij} + \sum_{k,l} b_{ij}^{kl} z_k z_l + \text{ H.O.T. (pure } z \text{ terms)}$$

$$+ \sum_l a_{ij}^l x_l + \text{ H.O.T. (pure } x \text{ and mixed } x - z \text{ terms)}$$

$$= \delta_{ij} + O(|z|^2) + O(|x|),$$

since $b_{ij} = \delta_{ij}$ and $\sum_k b_{ij}^k z_k = 0$ in a normal coordinate system.

Finally, we have that in each $D_{z,t}$, $\bar{g}(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij} + \sum_{k,l} a_{ij}^{kl} x_k x_l + \text{H.O.T.}$.

Therefore, the form of the metric $\bar{g}$ on M with respect to the coordinates $(t, x_1, \ldots, x_{n-d}, z_1, \ldots, z_{d-1})$ is

$$\bar{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbb{1} + O(|x|^2) & O(|x|) \\ 0 & O(|x|) & \mathbb{1} + O(|x|) + O(|z|^2) \end{pmatrix}.$$

We also introduce an r-coordinate (as in the basic case), with r measuring distance from the centre of each disc $D_{z,t}$ (from the point $\mu(t)$) within the disc, i.e. with respect to the metric $\bar{g}$ on M restricted to $D_{z,t}$.

With a view to "pulling out" the hypersurface H so as to enclose σ^d, we wish to define a curve γ in $r-t$ plane as in the basic case (see Section 2.4). We will use this to define a bundle of "tubes" coming out of H, which will allow us to extend H over σ^d in an appropriate way.

Consider the local hypersurface $\mathcal{H}_z$ which is defined by fixing a point $z = (z_1, \ldots, z_{d-1}) \in U$ and allowing the other coordinates in our local coordinate system to vary.

As in the basic case, we will construct a tube $N_z \subset \mathcal{H}_z$ via a curve $\gamma(s) = (t(s), r(s))$. Doing this for all $z \in U$ using the same curve γ, we obtain a bundle of tubes N_z parametrized by $z \in U$. Let N denote the totality of this bundle of tubes. Restrict $\bar{g}$ to N to obtain a metric g, and then pull-back to $U \times T$, where $U \subset S^{d-1}$ and $T = S^{n-d-1} \times \mathbb{R}$ is the standard tube, along the map $f : U \times T \to N$ given by $f(z, x, s) = (t(s), r(s)x, z)$. Here x on the left can be viewed as an $(n-d)$-tuple of the standard $\mathbb{R}^{n-d}$ coordinates of the point $x \in S^{n-d-1} \in \mathbb{R}^{n-d}$, and on the right as the same $(n-d)$-tuple viewed as coordinates within our local coordinate system on M.

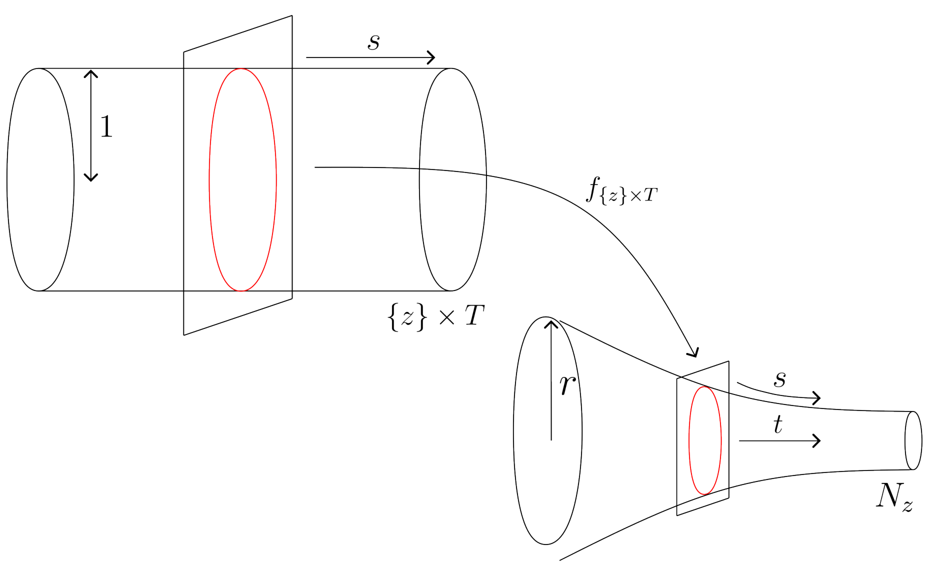

Figure 3.8: Pull back set-up.

We are going to consider some metric components of f^*g. As in the basic case, we will introduce normal coordinates around a fixed choice of point in each copy of S^{n-d-1} within the tube T. Call these $\{y_1, \ldots, y_{n-d-1}\}$. We obtain:

- $f^*g(\partial/\partial y_i, \partial/\partial y_j) = r^2 ds^2_{n-d-1}(\partial/\partial y_i, \partial/\partial y_j) + r^4 F_{ij}$. This term is analogous to the basic case.

- $f^*g(\partial/\partial y_i, \partial/\partial z_k)$. In the previous diagram, $\partial/\partial y_i = \sum_j \lambda_{ij}(\partial/\partial x_j)$, where λ_{ij} are functions of y_i. We can push forward via f, obtaining $\partial/\partial x_i \mapsto r(\partial/\partial x_i)$. Similarly, $\partial/\partial y_i \mapsto r \sum_j \lambda_{ij}(\partial/\partial x_j)$. So,

$$f^*g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_k}\right) = g\left(r \sum_j \lambda_{ij} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial z_k}\right)$$

$$= r\bar{g}\left(\sum_j \lambda_{ij} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial z_k}\right)$$

$$= rO(|x|) = O(r^2).$$

  Notice that $O(|x|) = O(r)$, since $r = |x|$ by construction, and $O(|x|)$ describes the metric behaviour since $\bar{g}(\partial/\partial x_j, \partial/\partial z_k) = 0$ when $|x| = r = 0$.

- $f^*g(\partial/\partial y_i, \partial/\partial s)$. In M, we have that $f_*(\partial/\partial s) = -\cos\theta(\partial/\partial r) + \sin\theta(\partial/\partial t)$.

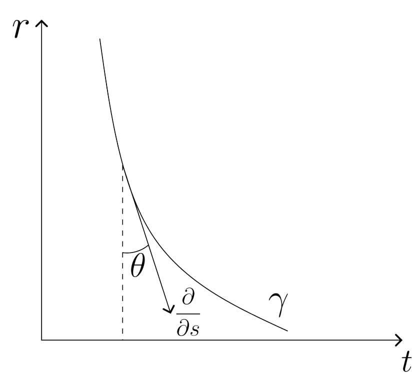

Figure 3.9: $\partial/\partial s$ in γ.

Therefore,

$$f^*g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial s}\right) = g\left(f_*\left(\frac{\partial}{\partial y_i}\right), f_*\left(\frac{\partial}{\partial s}\right)\right)$$

$$= -\cos\theta . g\left(f_*\left(\frac{\partial}{\partial y_i}\right), \frac{\partial}{\partial r}\right)$$

$$+ \sin\theta . g\left(f_*\left(\frac{\partial}{\partial y_i}\right), \frac{\partial}{\partial t}\right) = 0,$$

since $\partial/\partial r$ and $\partial/\partial t$ are both perpendicular to the sphere directions in N.

- $f^* g(\partial/\partial z_k, \partial/\partial s)$. Similarly, as before

$$f^* g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial s}\right) = g\left(f_*\left(\frac{\partial}{\partial z_k}\right), f_*\left(\frac{\partial}{\partial s}\right)\right)$$

$$= -\cos\theta . \bar{g}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial r}\right)$$

$$+ \sin\theta . \bar{g}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial t}\right)$$

$$= -\cos\theta . \bar{g}\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial r}\right).$$

At any point, $\partial/\partial r = \sum_j \mu_j(\partial/\partial x_j)$, for some functions μ_j. We know that $\bar{g}(\partial/\partial x_j, \partial/\partial z_k) = O(|x|) = O(r)$. Therefore,

$$f^* g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial s}\right) = -\cos\theta(O(r)) = O(r)\cos\theta.$$

Recall that $\cos\theta = -r'$, with θ depending on s. If we write everything together, we have a matrix g with respect to coordinates $(s, y_1, \ldots, y_{n-d-1}, z_1, \ldots, z_{d-1})$, given by

$$g = \begin{pmatrix} 1 & 0 & r'O(r) \\ 0 & r^2(ds^2_{n-d-1})_{ij} + O(r^4) & O(r^2) \\ r'O(r) & O(r^2) & \mathbb{1} + O(r) + O(|z|^2) \end{pmatrix}. \tag{3.1}$$

3.2 Computation of Christoffel symbols

In order to compute the Christoffel symbols of g, we need to invert the matrix metric in Equation 3.1.

Lemma 3.2.1. *The matrix g^{-1} is*

$$g^{-1} = \begin{pmatrix} 1+(r')^2O(r^2) & r'O(r) & r'O(r) \\ r'O(r) & \frac{1}{r^2}(ds^2_{n-d-1})^{-1}_{ij} + O(1) + (r')^2O(r^2) & O(1)+(r')^2O(r^2)+O(|z|^2) \\ r'O(r) & O(1)+(r')^2O(r^2)+O(|z|^2) & \mathbb{1}+O(r)+O(|z|^2)+(r')^2O(r^2) \end{pmatrix}.$$

Proof. We are going to compute g^{-1} using the Block Inversion Formula. First, we want to calculate the inverse of the matrix:

$$X = \begin{pmatrix} r^2(ds^2_{n-d-1})_{ij} + O(r^4) & O(r^2) \\ O(r^2) & \mathbb{1}+O(r)+O(|z|^2) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Using the Block Inversion Formula, our new matrix X^{-1} will take the form:

$$X^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -(A-BD^{-1}C)^{-1}BD^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}.$$

We have:

$$D^{-1} = \mathbb{1}+O(r)+O(|z|^2);$$

$$D^{-1}C = (\mathbb{1}+O(r)+O(|z|^2))O(r^2) = O(r^2);$$

$$BD^{-1}C = O(r^4);$$

$$A - BD^{-1}C = r^2(ds^2_{n-d-1})_{ij} + O(r^4);$$

$$
\begin{aligned}
(A - BD^{-1}C)^{-1} &= \frac{1}{r^2}\left((ds^2_{n-d-1})_{ij} + O(r^2)\right)^{-1} \\
&= \frac{1}{r^2}\left(((ds^2_{n-d-1})_{ij})(\mathbb{1} + O(r^2))\right)^{-1} \\
&= \frac{1}{r^2}\left(\mathbb{1} + O(r^2)\right)((ds^2_{n-d-1})_{ij})^{-1} \\
&= \frac{1}{r^2}((ds^2_{n-d-1})_{ij})^{-1} + O(1),
\end{aligned}
$$

since $(ds^2_{n-d-1})_{ij}$ is independent of r;

$$
\begin{aligned}
-(A - BD^{-1}C)^{-1}BD^{-1} &= \left(\frac{1}{r^2}((ds^2_{n-d-1})_{ij})^{-1} + O(1)\right) O(r^2)\,(\mathbb{1} \\
&\quad + O(r) + O(|z|^2)) \\
&= (((ds^2_{n-d-1})_{ij})^{-1} + O(r^2))(\mathbb{1} + O(r) \\
&\quad + O(|z|^2)) \\
&= ((ds^2_{n-d-1})_{ij})^{-1} + O(r) + O(|z|^2) \\
&= O(1) + O(|z|^2),
\end{aligned}
$$

as $(ds^2_{n-d-1})_{ij}$ is $O(1)$;

$$
\begin{aligned}
D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} &= \mathbb{1} + O(r) + O(|z|^2) \\
&\quad + O(r^2)(O(1) + O(|z|^2)) \\
&= \mathbb{1} + O(r) + O(|z|^2).
\end{aligned}
$$

Therefore,

$$X^{-1} = \begin{pmatrix} \frac{1}{r^2}((ds^2_{n-d-1})_{ij})^{-1} + O(1) & O(1) + O(|z|^2) \\ O(1) + O(|z|^2) & \mathbb{1} + O(r) + O(|z|^2) \end{pmatrix}.$$

We are going to use the Block Inversion Formula again to calculate g^{-1}. We have:

$$g = \begin{pmatrix} 1 & 0 & r'O(r) \\ 0 & r^2(ds^2_{n-d-1})_{ij} + O(r^4) & O(r^2) \\ r'O(r) & O(r^2) & \mathbb{1} + O(r) + O(|z|^2) \end{pmatrix}.$$

We will define: $A = 1$, $B = C = (0 \;\; r'O(r))$ and $D = X$. So, as before,

$$D^{-1}C = \begin{pmatrix} r'O(r) \\ r'O(r) \end{pmatrix} \text{ and so } BD^{-1}C = (r')^2 O(r^2);$$

$$A - BD^{-1}C = 1 - (r')^2 O(r^2);$$

$$(A - BD^{-1}C)^{-1} = 1 + (r')^2 O(r^2);$$

$$\begin{aligned} -(A - BD^{-1}C)^{-1}BD^{-1} &= (1 + (r')^2 O(r^2))(r'O(r) \;\; r'O(r)) \\ &= (r'O(r) \;\; r'O(r)); \end{aligned}$$

$$\begin{aligned} D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} &= X^{-1} + \begin{pmatrix} r'O(r) \\ r'O(r) \end{pmatrix} (r'O(r) \;\; r'O(r)) \\ &= X^{-1} + \begin{pmatrix} (r')^2 O(r^2) & (r')^2 O(r^2) \\ (r')^2 O(r^2) & (r')^2 O(r^2) \end{pmatrix}, \end{aligned}$$

which is equal to

$$\begin{pmatrix} \frac{1}{r^2}(ds^2_{n-d-1})^{-1}_{ij} + O(1) + (r')^2 O(r^2) & O(1) + (r')^2 O(r^2) + O(|z|^2) \\ O(1) + (r')^2 O(r^2) + O(|z|^2) & \mathbb{1} + O(r) + O(|z|^2) + (r')^2 O(r^2) \end{pmatrix}.$$

Therefore,

$$g^{-1} = \begin{pmatrix} 1 + (r')^2 O(r^2) & r' O(r) & r' O(r) \\ r' O(r) & \frac{1}{r^2}(ds^2_{n-d-1})^{-1}_{ij} + O(1) + (r')^2 O(r^2) & O(1) + (r')^2 O(r^2) + O(|z|^2) \\ r' O(r) & O(1) + (r')^2 O(r^2) + O(|z|^2) & \mathbb{1} + O(r) + O(|z|^2) + (r')^2 O(r^2) \end{pmatrix}.$$

□

In summary, we have with respect to coordinates $(s, y_1, \ldots, y_{n-d-1}, z_1, \ldots, z_{d-1})$,

$$g = \begin{pmatrix} 1 & 0 & r' O(r) \\ 0 & r^2 (ds^2_{n-d-1})_{ij} + O(r^4) & O(r^2) \\ r' O(r) & O(r^2) & \mathbb{1} + O(r) + O(|z|^2) \end{pmatrix},$$

$$g^{-1} = \begin{pmatrix} 1 + (r')^2 O(r^2) & r' O(r) & r' O(r) \\ r' O(r) & \frac{1}{r^2}(ds^2_{n-d-1})^{-1}_{ij} + O(1) + (r')^2 O(r^2) & O(1) + (r')^2 O(r^2) + O(|z|^2) \\ r' O(r) & O(1) + (r')^2 O(r^2) + O(|z|^2) & \mathbb{1} + O(r) + O(|z|^2) + (r')^2 O(r^2) \end{pmatrix}.$$

For our notation, we are using $i, j, k \in \{1, \ldots, n-d-1\}$ for the fibre-sphere directions with l as a generic index and $x, v, w \in \{1, \ldots, d-1\}$ for the sphere S^{d-1} directions with p as their index.

Thus, we have

$$g_{ss} = 1; \qquad g_{si} = 0; \qquad g_{sx} = r' O(r);$$

$$g^{ss} = 1 + (r')^2 O(r^2); \qquad g^{si} = r' O(r); \qquad g^{sx} = r' O(r);$$

$$g_{ij} = r^2(ds^2_{n-d-1})_{ij} + O(r^4); \qquad g^{ix} = O(1) + (r')^2 O(r^2) + O(|z|^2);$$

$$g^{ij} = \frac{1}{r^2}(ds^2_{n-d-1})^{ij} + O(1) + (r')^2 O(r^2); \qquad g_{xv} = \delta_{xv} + O(r) + O(|z|^2);$$

$$g_{ix} = O(r^2); \qquad g^{xv} = \delta_{xv} + O(r) + O(|z|^2) + (r')^2 O(r^2).$$

We are going to calculate Γ^s_{ss}, Γ^i_{ss}, Γ^x_{ss}, Γ^s_{si}, Γ^s_{sx}, Γ^j_{si}, Γ^x_{si}, Γ^i_{sx}, Γ^v_{sx}, Γ^s_{ij}, Γ^s_{xv}, Γ^s_{ix}, Γ^k_{ij}, Γ^x_{ij}, Γ^j_{ix}, Γ^v_{ix}, Γ^i_{xv} and Γ^w_{xv}; where

$$\Gamma^c_{ab} = \frac{1}{2} g^{cs}\left(\frac{\partial g_{as}}{\partial_b} + \frac{\partial g_{bs}}{\partial_a} - \frac{\partial g_{ab}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{cl}\left(\frac{\partial g_{al}}{\partial_b} + \frac{\partial g_{bl}}{\partial_a} - \frac{\partial g_{ab}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{cp}\left(\frac{\partial g_{ap}}{\partial_b} + \frac{\partial g_{bp}}{\partial_a} - \frac{\partial g_{ab}}{\partial z_p}\right).$$

Here we have adopted the convention that

- if $a = s$, then $\partial_a = \partial s$;
- if $a \in \{1, \ldots, n-d-1\}$ (fibre-sphere direction), then $\partial_a = \partial y_a$;
- if $a \in \{1, \ldots, d-1\}$ (S^{d-1} direction), then $\partial_a = \partial z_a$.

Recall that even if an $O(r)$ term, say, has some dependence on s, y_l and z_p, this dependence is bounded, and any derivatives with respect to s, y_l or z_p will still depend on r. In particular, we will assume (as the worst case scenario) that differentiating an $O(1)$ term leaves an $O(1)$ term.

Also, as well as in the basic case, r' is bounded, since $|r'| \in [0, 1]$. Therefore, by approximation, we can combine $(r')^2 O(r^2)$, $r'O(r^3)$ and $O(r^4)$ terms into a single $O(r^2)$ term, for example.

Notice that differentiating $O(|z|^2)$ with respect to anything other than a z-direction will yield $O(|z|^2)$ again. Although the z-coordinates themselves are independent of s and y_l, the coefficients of terms involving some z_p may depend on s and y_l. After differentiating with respect to s or y_l, the overall expression is still $O(|z|^2)$. On the other hand, differentiating $O(|z|^2)$ with respect to any term z_p will give $O(|z|)$.

So,

$$\Gamma^s_{ss} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{ss}}{\partial s} + \frac{\partial g_{ss}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{sl}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{sp}\left(\frac{\partial g_{sp}}{\partial s} + \frac{\partial g_{sp}}{\partial s} - \frac{\partial g_{ss}}{\partial z_p}\right)$$

$$= 0 + 0 + \frac{1}{2}\sum_{p=1}^{d-1}(r'O(r))\left(\frac{2\partial(r'O(r)}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{p=1}^{d-1}(r'O(r))(r''O(r) + (r')^2O(1) + r'O(r))$$

$$= r''r'O(r^2) + (r')^3O(r) + (r')^2O(r^2)$$

$$= r'O(r)(r''O(r) + (r')^2 + r'O(r))$$

$$= r''O(r^2) + O(r).$$

$$\Gamma^i_{ss} = \frac{1}{2}g^{is}\left(\frac{\partial g_{ss}}{\partial s} + \frac{\partial g_{ss}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{il}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{ip}\left(\frac{\partial g_{sp}}{\partial s} + \frac{\partial g_{sp}}{\partial s} - \frac{\partial g_{ss}}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2O(r^2) + O(|z|^2))\left(\frac{2\partial(r'O(r))}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2 O(r^2) + O(|z|^2))(r'' O(r) + (r')^2 O(1) + r' O(r))$$

$$= r'' O(r) + (r')^2 O(1) + r' O(r) + r''(r')^2 O(r^3) + (r')^4 O(r^2)$$

$$+ (r')^3 O(r^3) + r'' O(r) O(|z|^2) + (r')^2 O(|z|^2) + r' O(r) O(|z|^2)$$

$$= r'' O(r) + (r')^2 O(1) + r' O(r) + r''(r')^2 O(r^3) + (r')^2 O(|z|^2)$$

$$= r'' O(r) + (r')^2 O(1) + r' O(r) + r'' O(r^3) + O(|z|^2)$$

$$= r'' O(r) + (r')^2 O(1) + r' O(r) + O(|z|^2)$$

$$= r'' O(r) + O(1) + O(|z|^2).$$

$$\Gamma^x_{ss} = \frac{1}{2} g^{xs}\left(\frac{\partial g_{ss}}{\partial s} + \frac{\partial g_{ss}}{\partial s} - \frac{\partial g_{ss}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{xl}\left(\frac{\partial g_{sl}}{\partial s} + \frac{\partial g_{sl}}{\partial s} - \frac{\partial g_{ss}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{xp}\left(\frac{\partial g_{sp}}{\partial s} + \frac{\partial g_{sp}}{\partial s} - \frac{\partial g_{ss}}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{p=1}^{d-1}\left(\delta_{xp} + O(r) + O(|z|^2) + (r')^2 O(r^2)\right)\left(\frac{2\partial(r' O(r))}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{p=1}^{d-1}\left(\delta_{xp} + O(r) + O(|z|^2) + (r')^2 O(r^2)\right)(r'' O(r)$$

$$+ (r')^2 O(1) + r' O(r))$$

$$= r'' O(r) + (r')^2 O(1) + r' O(r) + r'' O(r^2) + (r')^2 O(r) + r' O(r^2)$$

$$+ r'' O(r) O(|z|^2) + (r')^2 O(|z|^2) + r' O(r) O(|z|^2) + r''(r')^2 O(r^3)$$

$$+ (r')^4 O(r^2) + (r')^3 O(r^3)$$

$$= (r''O(r) + (r')^2O(1) + r'O(r))(O(1) + O(|z|^2)) + r''(r')^2O(r^3)$$

$$= r''O(r) + (r')^2O(1) + r'O(r) + r''(r')^2O(r^3)$$

$$= r''O(r) + (r')^2O(1) + r'O(r) + r''O(r^3)$$

$$= r''O(r) + (r')^2O(1) + r'O(r)$$

$$= r''O(r) + O(1).$$

$$\Gamma^s_{si} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{ss}}{\partial y_i} + \frac{\partial g_{is}}{\partial s} - \frac{\partial g_{is}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{sl}\left(\frac{\partial g_{sl}}{\partial y_i} + \frac{\partial g_{il}}{\partial s} - \frac{\partial g_{is}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{sp}\left(\frac{\partial g_{sp}}{\partial y_i} + \frac{\partial g_{ip}}{\partial s} - \frac{\partial g_{is}}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}(r'O(r))\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il} + O(r^4))}{\partial s}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(r'O(r))\left(\frac{\partial(r'O(r))}{\partial y_i} + \frac{O(r^2)}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}(r'O(r))(2r'r(ds^2_{n-d-1})_{il} + r'O(r^3) + O(r^4))$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(r'O(r))(r'O(r) + r'O(r) + O(r^2))$$

$$= (r')^2O(r^2) + (r')^2O(r^4) + r'O(r^5) + (r')^2O(r^2)$$

$$+ (r')^2O(r^2) + r'O(r^3) = (r')^2O(r^2) + r'O(r^3)$$

$$= O(r^2).$$

$$\Gamma^s_{sx} = \frac{1}{2} g^{ss} \left(\frac{\partial g_{ss}}{\partial z_x} + \frac{\partial g_{xs}}{\partial s} - \frac{\partial g_{sx}}{\partial s} \right) + \frac{1}{2} \sum_{l=1}^{n-d-1} g^{sl} \left(\frac{\partial g_{sl}}{\partial z_x} + \frac{\partial g_{xl}}{\partial s} - \frac{\partial g_{sx}}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} g^{sp} \left(\frac{\partial g_{sp}}{\partial z_x} + \frac{\partial g_{xp}}{\partial s} - \frac{\partial g_{sx}}{\partial z_p} \right)$$

$$= \frac{1}{2} \sum_{l=1}^{n-d-1} (r' O(r)) \left(\frac{\partial (O(r^2))}{\partial s} - \frac{\partial (r' O(r))}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (r' O(r)) \left(\frac{\partial (r' O(r))}{\partial z_x} + \frac{\partial (\delta_{xp} + O(r) + O(|z|^2))}{\partial s} \right.$$

$$\left. - \frac{\partial (r' O(r))}{\partial z_p} \right)$$

$$= \frac{1}{2} \sum_{l=1}^{n-d-1} (r' O(r))(r' O(r) + O(r^2) + r' O(r))$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (r' O(r)) \left(r' O(r) + r' O(1) + O(r) + O(|z|^2) + r' O(r) \right)$$

$$= (r')^2 O(r^2) + r' O(r^3) + (r')^2 O(r) + (r')^2 O(1) + r' O(r^2)$$

$$+ r' O(r) O(|z|^2) = (r')^2 O(1) + r' O(r^2) + r' O(r) O(|z|^2)$$

$$= O(1) + O(r) O(|z|^2).$$

$$\Gamma^j_{si} = \frac{1}{2} g^{js} \left(\frac{\partial g_{ss}}{\partial y_i} + \frac{\partial g_{is}}{\partial s} - \frac{\partial g_{si}}{\partial s} \right) + \frac{1}{2} \sum_{l=1}^{n-d-1} g^{jl} \left(\frac{\partial g_{sl}}{\partial y_i} + \frac{\partial g_{il}}{\partial s} - \frac{\partial g_{si}}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} g^{jp} \left(\frac{\partial g_{sp}}{\partial y_i} + \frac{\partial g_{ip}}{\partial s} - \frac{\partial g_{si}}{\partial z_p} \right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{jl} + O(1) + (r')^2O(r^2))\right)\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il} + O(r^4))}{\partial s}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2O(r^2) + O(|z|^2))\left(\frac{\partial(r'O(r))}{\partial y_i} + \frac{\partial(O(r^2))}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{jl} + O(1)) + (r')^2O(r^2)\right)\big(2r'r(ds^2_{n-d-1})_{il} + r'O(r^3)$$

$$+O(r^4)\big) + \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2O(r^2) + O(|z|^2))(r'O(r) + r'O(r) + O(r^2))$$

$$= \frac{1}{2}\bigg(\frac{2r'}{r}\delta_{ij} + r'O(r) + O(r^2) + r'O(r) + r'O(r^3) + O(r^4)$$

$$+ (r')^3O(r^3) + (r')^3O(r^5) + (r')^2O(r^6)\big) + \big(r'O(r) + O(r^2)$$

$$+(r')^3O(r^3) + (r')^2O(r^2) + r'O(r)O(|z|^2) + O(r^2)O(|z|^2)\big)$$

$$= \frac{r'}{r}\delta_{ij} + r'O(r) + O(r^2)$$

$$= \frac{r'}{r}\delta_{ij} + O(r).$$

Similarly to our basic case, we have that $(ds^2_{n-d-1})^{jl}(ds^2_{n-d-1})_{il} = \delta_{ij}$, since (ds^2_{n-d-1}) is symmetric.

$$\Gamma^x_{si} = \frac{1}{2}g^{xs}\left(\frac{\partial g_{ss}}{\partial y_i} + \frac{\partial g_{is}}{\partial s} - \frac{\partial g_{si}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{xl}\left(\frac{\partial g_{sl}}{\partial y_i} + \frac{\partial g_{il}}{\partial s} - \frac{\partial g_{si}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{xp}\left(\frac{\partial g_{sp}}{\partial y_i} + \frac{\partial g_{ip}}{\partial s} - \frac{\partial g_{si}}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il}+O(r^4))}{\partial s}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}\left(\delta_{xp}+O(r)+O(|z|^2+(r')^2O(r^2))\right)\left(\frac{\partial(r'O(r))}{\partial y_i}\right.$$

$$\left.+\frac{\partial(O(r^2))}{\partial s}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))(2r'r(ds^2_{n-d-1})_{il}+r'O(r^3)+O(r^4))$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}\left(\delta^{xp}+O(r)+O(|z|^2)+(r')^2O(r^2)\right)(r'O(r)+O(r^2))$$

$$= r'O(r)+r'O(r^3)+O(r^4)+(r')^3O(r^3)+(r')^3O(r^5)+(r')^2O(r^6)$$

$$+r'rO(|z|^2)+r'O(r^3)O(|z|^2)+O(r^4)O(|z|^2)+r'O(r)+O(r^2)$$

$$+r'O(r^2)+O(r^3)+r'O(r)O(|z|^2)+O(r^2)O(|z|^2)+(r')^3O(r^3)$$

$$+(r')^2O(r^2)$$

$$= r'O(r)+O(r^2)+r'rO(|z|^2)$$

$$= O(r)+rO(|z|^2)$$

$$= O(r).$$

$$\Gamma^i_{sx} = \frac{1}{2}g^{is}\left(\frac{\partial g_{ss}}{\partial z_x}+\frac{\partial g_{xs}}{\partial s}-\frac{\partial g_{sx}}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1}g^{il}\left(\frac{\partial g_{sl}}{\partial z_x}+\frac{\partial g_{xl}}{\partial s}-\frac{\partial g_{sx}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}g^{ip}\left(\frac{\partial g_{sp}}{\partial z_x}+\frac{\partial g_{xp}}{\partial s}-\frac{\partial g_{sx}}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{il} + O(1) + (r')^2 O(r^2)\right)\left(\frac{\partial(O(r^2))}{\partial s} - \frac{\partial(r'O(r))}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2 O(r^2) + O(|z|^2))\left(\frac{\partial(r'O(r))}{\partial z_x}\right.$$

$$\left.+\frac{\partial(\delta_{xp} + O(r) + O(|z^2|))}{\partial s} - \frac{\partial(r'O(r))}{\partial z_p}\right)$$

$$= \frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{il} + O(1) + (r')^2 O(r^2)\right)(r'O(r) + O(r^2))$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(O(1) + (r')^2 O(r^2) + O(|z|^2))(r'O(r) + r'O(1) + O(r) + O(|z|^2))$$

$$= r'O\left(\frac{1}{r}\right) + O(1) + r'O(r) + O(r^2) + (r')^3 O(r^3) + (r')^2 O(r^4)$$

$$+ r'O(r) + r'O(1) + O(r) + O(|z|^2) + (r')^3 O(r^3) + (r')^2 O(r^2)$$

$$+ (r')^2 O(r^3) + (r')^2 O(r^2) O(|z|^2) + r'O(r)O(|z|^2) + r'O(|z|^2)$$

$$+ O(r)O(|z|^2) + O(|z|^4)$$

$$= r'O\left(\frac{1}{r}\right) + O(1) + O(|z|^2)$$

$$= O\left(\frac{1}{r}\right) + O(|z|^2).$$

$$\Gamma^v_{sx} = \frac{1}{2}g^{vs}\left(\frac{\partial g_{ss}}{\partial z_x} + \frac{\partial g_{xs}}{\partial s} - \frac{\partial g_{sx}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{vl}\left(\frac{\partial g_{sl}}{\partial z_x} + \frac{\partial g_{xl}}{\partial s} - \frac{\partial g_{sx}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1} g^{vp}\left(\frac{\partial g_{sp}}{\partial z_x}+\frac{\partial g_{xp}}{\partial s}-\frac{\partial g_{sx}}{\partial z_p}\right)$$

$$=\frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(\frac{\partial(O(r^2))}{\partial s}-\frac{\partial(r'O(r))}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(\delta_{vp}+O(r)+O(|z|^2)+(r')^2O(r^2))\left(\frac{\partial(r'O(r))}{\partial z_x}\right.$$

$$\left.+\ \frac{\partial(\delta_{xp}+O(r)+O(|z|^2))}{\partial s}-\frac{\partial(r'O(r))}{\partial z_p}\right)$$

$$=\frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))(r'O(r)+O(r^2))$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(\delta_{vp}+O(r)+O(|z|^2)+(r')^2O(r^2))\left(r'O(r)+r'O(1)\right.$$

$$\left.+O(r)+O(|z|^2)\right)$$

$$=r'O(r)+O(r^2)+(r')^3O(r^3)+(r')^2O(r^4)+r'O(r)O(|z|^2)$$

$$+O(r^2)O(|z|^2)+r'O(r)+r'O(1)+O(r)+O(|z|^2)+r'O(r^2)+r'O(r)$$

$$+O(r^2)+O(r)O(|z|^2)+r'O(r)O(|z|^2)+r'O(|z|^2)+O(r)O(|z|^2)$$

$$+(O(|z|^2))^2+(r')^3O(r^3)+(r')^3O(r^2)+(r')^2O(r^3)+(r')^2O(r^2)O(|z|^2)$$

$$=r'O(1)+O(r)+O(|z|^2)$$

$$=O(1)+O(|z|^2).$$

$$\Gamma^s_{ij}=\frac{1}{2}g^{ss}\left(\frac{\partial g_{is}}{\partial y_j}+\frac{\partial g_{js}}{\partial y_i}-\frac{\partial g_{ij}}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1} g^{sl}\left(\frac{\partial g_{il}}{\partial y_j}+\frac{\partial g_{jl}}{\partial y_i}-\frac{\partial g_{ij}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1} g^{sp}\left(\frac{\partial g_{ip}}{\partial y_j}+\frac{\partial g_{jp}}{\partial y_i}-\frac{\partial g_{ij}}{\partial z_p}\right)$$

$$=\frac{1}{2}(1+(r')^2O(r^2))\left(-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4))}{\partial s}\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il}+O(r^4))}{\partial y_j}+\frac{\partial(r^2(ds^2_{n-d-1})_{jl}+O(r^4))}{\partial y_i}\right.$$

$$\left.-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4))}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(\frac{\partial(O(r^2))}{\partial y_j}+\frac{\partial(O(r^2))}{\partial y_i}-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4))}{\partial z_p}\right)$$

$$=\frac{1}{2}(1+(r')^2O(r^2))(-2r'r(ds^2_{n-d-1})_{ij}+r'O(r^3)+O(r^4))$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(r^2\left(\frac{\partial(ds^2_{n-d-1})_{il}}{\partial y_j}+\frac{\partial(ds^2_{n-d-1})_{jl}}{\partial y_i}-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}}{\partial y_l}\right)\right.$$

$$\left.+O(r^4)\right)+\frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(O(r^2)-\frac{r^2\partial(ds^2_{n-d-1})_{ij}}{\partial z_p}+O(r^4)\right)$$

$$=-r'r(ds^2_{n-d-1})_{ij}+r'O(r^3)+O(r^4)+(r')^3O(r^3)+(r')^3O(r^5)$$

$$+(r')^2O(r^6)+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(r'O(r^3)+r'O(r^5)\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(r'O(r^3)+r'O(r^3)+r'O(r^5))$$

$$=-r'r(ds^2_{n-d-1})_{ij}+r'O(r^3)+O(r^4)$$

$$= -r'r(ds^2_{n-d-1})_{ij} + O(r^3).$$

$$\Gamma^s_{xv} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{xs}}{\partial z_v} + \frac{\partial g_{vs}}{\partial z_x} - \frac{\partial g_{xv}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{sl}\left(\frac{\partial g_{xl}}{\partial z_v} + \frac{\partial g_{vl}}{\partial z_x} - \frac{\partial g_{xv}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{sp}\left(\frac{\partial g_{xp}}{\partial z_v} + \frac{\partial g_{vp}}{\partial z_x} - \frac{\partial g_{xv}}{\partial z_p}\right)$$

$$= \frac{1}{2}(1 + (r')^2 O(r^2))\left(\frac{\partial(r'O(r))}{\partial z_v} + \frac{\partial(r'O(r))}{\partial z_x}\right.$$

$$\left. - \frac{\partial(\delta_{xv} + O(r) + O(|z|^2))}{\partial s}\right)$$

$$+ \frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(\frac{\partial(O(r^2))}{\partial z_v} + \frac{\partial(O(r^2))}{\partial z_x}\right.$$

$$\left. - \frac{\partial(\delta_{xv} + O(r) + O(|z|^2))}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(\frac{\partial(\delta_{xp} + O(r) + O(|z|^2))}{\partial z_v}\right.$$

$$\left. + \frac{\partial(\delta_{vp} + O(r) + O(|z|^2))}{\partial z_x} - \frac{\partial(\delta_{xv} + O(r) + O(|z|^2))}{\partial z_p}\right)$$

$$= \frac{1}{2}(1 + (r')^2 O(r^2))(r'O(r) + r'O(r) + r'O(1) + O(r) + O(|z|^2))$$

$$+ \frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(O(r^2) - \frac{\partial(\delta_{xv})}{\partial y_l} + O(r) + O(|z|^2)\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(\frac{\partial(\delta_{xp})}{\partial z_v} + \frac{\partial(\delta_{vp})}{\partial z_x} - \frac{\partial(\delta_{xv})}{\partial z_p} + O(r) + O(|z|)\right)$$

$$= \frac{1}{2}\left(r'O(1) + O(r) + O(|z|^2) + (r')^3O(r^2) + (r')^2O(r^3)\right.$$

$$\left. + \ (r')^2O(r^2)O(|z|^2)\right)$$

$$+ \frac{1}{2}\sum_{l=1}^{n-d-1}(r'O(r^3) + r'O(r^2) + r'O(r)O(|z|^2))$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(r'O(r^2) + r'O(r)O(|z|)) = r'O(1) + O(r) + O(|z|^2)$$

$$= O(1) + O(|z|^2).$$

$$\Gamma^s_{ix} = \frac{1}{2}g^{ss}\left(\frac{\partial g_{is}}{\partial z_x} + \frac{\partial g_{xs}}{\partial y_i} - \frac{\partial g_{ix}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{sl}\left(\frac{\partial g_{il}}{\partial z_x} + \frac{\partial g_{xl}}{\partial y_i} - \frac{\partial g_{ix}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{sp}\left(\frac{\partial g_{ip}}{\partial z_x} + \frac{\partial g_{xp}}{\partial y_i} - \frac{\partial g_{ix}}{\partial z_p}\right)$$

$$= \frac{1}{2}(1 + (r')^2O(r^2))\left(\frac{\partial(r'O(r))}{\partial y_i} - \frac{\partial(O(r^2))}{\partial s}\right)$$

$$+ \frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il} + O(r^4))}{\partial z_x} + \frac{\partial(O(r^2))}{\partial y_i}\right.$$

$$\left. - \ \frac{\partial(O(r^2))}{\partial y_l}\right) + \frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(\frac{\partial(O(r^2))}{\partial z_x}\right.$$

$$\left. + \ \frac{\partial(\delta_{xp} + O(r) + O(|z|^2)))}{\partial y_i} - \frac{\partial(O(r^2))}{\partial z_p}\right)$$

$$= \frac{1}{2}(1 + (r')^2O(r^2))(r'O(r) + r'O(r) + O(r^2))$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1} r'O(r)\left(r^2\frac{\partial(ds^2_{n-d-1})_{il}}{\partial z_x}+O(r^4)+O(r^2)\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1} r'O(r)\left(O(r^2)+\frac{\partial(\delta_{xp})}{\partial y_i}+O(r)+O(|z|^2)\right)$$

$$=r'O(r)+O(r^2)+(r')^3O(r^3)+(r')^2O(r^4)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}(r'O(r^3)+r'O(r^5)+r'O(r^3))$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(r'O(r^3)+r'O(r^2)+r'O(r)O(|z|^2))$$

$$=r'O(r)+O(r^2)=O(r).$$

$$\Gamma^k_{ij}=\frac{1}{2}g^{ks}\left(\frac{\partial g_{is}}{\partial y_j}+\frac{\partial g_{js}}{\partial y_i}-\frac{\partial g_{ij}}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1} g^{kl}\left(\frac{\partial g_{il}}{\partial y_j}+\frac{\partial g_{jl}}{\partial y_i}-\frac{\partial g_{ij}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1} g^{kp}\left(\frac{\partial g_{ip}}{\partial y_j}+\frac{\partial g_{jp}}{\partial y_i}-\frac{\partial g_{ij}}{\partial z_p}\right)$$

$$=\frac{1}{2}r'O(r)\left(-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4))}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{kl}\right.$$

$$\left.+O(1)+(r')^2O(r^2)\right)\left(\frac{\partial(r^2(ds^2_{n-d-1})_{il}+O(r^4)))}{\partial y_j}\right.$$

$$\left.+\frac{\partial(r^2(ds^2_{n-d-1})_{jl}+O(r^4)))}{\partial y_i}-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4)))}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(\frac{\partial(O(r^2))}{\partial y_j}+\frac{\partial(O(r^2))}{\partial y_i}\right.$$

$$-\frac{\partial(r^2(ds^2_{n-d-1})_{ij}+O(r^4))}{\partial z_p}\Bigg)$$

$$=\frac{1}{2}r'O(r)\left(-2r'r\frac{\partial(ds^2_{n-d-1})_{ij}}{\partial s}+r'O(r^3)+O(r^4)\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}(ds^2_{n-d-1})^{kl}\left(\frac{\partial(ds^2_{n-d-1})_{il}}{\partial y_j}+\frac{\partial(ds^2_{n-d-1})_{jl}}{\partial y_i}-\frac{\partial(ds^2_{n-d-1})_{ij}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}O(1)\left(r^2\left(\frac{\partial(ds^2_{n-d-1})_{il}}{\partial y_j}+\frac{\partial(ds^2_{n-d-1})_{jl}}{\partial y_i}-\frac{\partial(ds^2_{n-d-1})_{ij}}{\partial y_l}\right)\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}(r')^2O(r^2)\left(r^2\left(\frac{\partial(ds^2_{n-d-1})_{il}}{\partial y_j}+\frac{\partial(ds^2_{n-d-1})_{jl}}{\partial y_i}-\frac{\partial(ds^2_{n-d-1})_{ij}}{\partial y_l}\right)\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{kl}+O(1)+(r')^2O(r^2)\right)(O(r^4))$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\Bigg(O(r^2)-r^2\frac{\partial(ds^2_{n-d-1})_{ij}}{\partial z_p}$$

$$+O(r^4)\Bigg)$$

$$=((r')^2O(r^2)+(r')^2O(r^4)+r'O(r^5))+\Gamma^k_{ij}(S^{n-d-1}(1))+O(r^2)$$

$$+(r')^2O(r^4)+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(O(r^2)+O(r^4)+(r')^2O(r^6)\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(r^2)+O(r^4)+(r')^2O(r^4)+(r')^2O(r^6)+O(r^2)O(|z|^2)$$

$$+r^2O(|z|^2)+O(r^4)O(|z|^2))$$

$$= (r')^2 O(r^2) + r' O(r^5) + O(r^2) + \Gamma_{ij}^k(S^{n-d-1}(1)) + r^2 O(|z|^2)$$

$$= O(r^2) + O(r^5) + O(r^2) + r^2 O(|z|^2) + \Gamma_{ij}^k(S^{n-d-1}(1))$$

$$= O(r^2) + r^2 O(|z|^2) + \Gamma_{ij}^k(S^{n-d-1}(1))$$

$$= O(r^2) + \Gamma_{ij}^k(S^{n-d-1}(1)).$$

$$\Gamma_{ij}^x = \frac{1}{2} g^{xs} \left(\frac{\partial g_{is}}{\partial y_j} + \frac{\partial g_{js}}{\partial y_i} - \frac{\partial g_{ij}}{\partial s} \right) + \frac{1}{2} \sum_{l=1}^{n-d-1} g^{xl} \left(\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} g^{xp} \left(\frac{\partial g_{ip}}{\partial y_j} + \frac{\partial g_{jp}}{\partial y_i} - \frac{\partial g_{ij}}{\partial z_p} \right)$$

$$= \frac{1}{2} (r' O(r)) \left(- \frac{\partial (r^2 (ds^2_{n-d-1})_{ij} + O(r^4))}{\partial s} \right)$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) \left(\frac{\partial (r^2 (ds^2_{n-d-1})_{il} + O(r^4))}{\partial y_j} \right.$$

$$\left. + \frac{\partial (r^2 (ds^2_{n-d-1})_{jl} + O(r^4))}{\partial y_i} - \frac{\partial (r^2 (ds^2_{n-d-1})_{ij} + O(r^4))}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (\delta_{xp} + O(r) + O(|z|^2) + (r')^2 O(r^2)) \left(\frac{\partial (O(r^2))}{\partial y_j} + \frac{\partial O(r^2)}{\partial y_i} \right.$$

$$\left. - \frac{\partial (r^2 (ds^2_{n-d-1})_{ij} + O(r^4))}{\partial z_p} \right)$$

$$= \frac{1}{2} (r' O(r)) \left(-2r'r \frac{\partial (ds^2_{n-d-1})_{ij}}{\partial s} + r' O(r^3) + O(r^4) \right)$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) \left(r^2 \left(\frac{\partial (ds^2_{n-d-1})_{il}}{\partial y_j} \right. \right.$$

$$+ \frac{\partial(ds^2_{n-d-1})_{jl}}{\partial y_i} - \frac{\partial(ds^2_{n-d-1})_{ij}}{\partial y_l}\Big) + O(r^4)\Big)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}(\delta_{xp} + O(r) + O(|z|^2) + (r')^2 O(r^2))\left(O(r^2)\right)$$

$$= (r')^2 O(r^2) + (r')^2 O(r^4) + r' O(r^5) + \frac{1}{2}\sum_{l=1}^{n-d-1}\big(O(r^2) + O(r^4)$$

$$+ (r')^2 O(r^4) + (r')^2 O(r^6) + r^2 O(|z|^2) + O(r^4) O(|z|^2)\big)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1}\left(O(r^2) + O(r^3) + O(r^2)O(|z|^2) + (r')^2 O(r^4)\right)$$

$$= (r')^2 O(r^2) + r' O(r^5) + O(r^2) + O(r^2) O(|z|^2)$$

$$= (r')^2 O(r^2) + O(r^2) + r' O(r^5) = O(r^2) + O(r^2) + O(r^5)$$

$$= O(r^2).$$

$$\Gamma^j_{ix} = \frac{1}{2} g^{js}\left(\frac{\partial g_{is}}{\partial z_x} + \frac{\partial g_{xs}}{\partial y_i} - \frac{\partial g_{ix}}{\partial s}\right) + \frac{1}{2}\sum_{l=1}^{n-d-1} g^{jl}\left(\frac{\partial g_{il}}{\partial z_x} + \frac{\partial g_{xl}}{\partial y_i} - \frac{\partial g_{ix}}{\partial y_l}\right)$$

$$+ \frac{1}{2}\sum_{p=1}^{d-1} g^{jp}\left(\frac{\partial g_{ip}}{\partial z_x} + \frac{\partial g_{xp}}{\partial y_i} - \frac{\partial g_{ix}}{\partial z_p}\right)$$

$$= \frac{1}{2}(r' O(r))\left(\frac{\partial(r' O(r))}{\partial y_i} - \frac{\partial(O(r^2))}{\partial s}\right)$$

$$+ \frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{jl} + O(1) + (r')^2 O(r^2)\right)\Bigg(\frac{\partial(O(r^2))}{\partial y_i}$$

$$+ \frac{\partial(r^2(ds^2_{n-d-1})_{il} + O(r^4))}{\partial z_x} - \frac{\partial(O(r^2))}{\partial y_l}\Bigg)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(\frac{\partial(O(r^2))}{\partial z_x}\right.$$

$$\left.+\ \frac{\partial(\delta_{xp}+O(r)+O(|z|^2)))}{\partial y_i}-\frac{\partial(O(r^2))}{\partial z_p}\right)$$

$$=\frac{1}{2}(r'O(r))(r'O(r)+r'O(r)+O(r^2))$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(\frac{1}{r^2}(ds^2_{n-d-1})^{jl}+O(1)+(r')^2O(r^2)\right)O(r^2)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(O(r)+O(|z|^2)\right)$$

$$=(r')^2O(r^2)+r'O(r^3)+(O(1)+O(r^2)+(r')^2O(r^4)))$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}\left(O(r)+O(|z|^2)+(r')^2O(r^3)+(r')^2O(r^2)O(|z|^2)\right.$$

$$\left.+O(r)O(|z|^2)+O(|z|^4)\right)$$

$$=O(1)+(r')^2O(r^2)+r'O(r^3)+O(|z|^2))$$

$$=O(1)+O(r^2)+O(r^3)+O(|z|^2))$$

$$=O(1)+O(|z|^2)).$$

$$\Gamma^v_{ix}=\frac{1}{2}g^{vs}\left(\frac{\partial g_{is}}{\partial z_x}+\frac{\partial g_{xs}}{\partial y_i}-\frac{\partial g_{ix}}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1}g^{vl}\left(\frac{\partial g_{il}}{\partial z_x}+\frac{\partial g_{xl}}{\partial y_i}-\frac{\partial g_{ix}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}g^{vp}\left(\frac{\partial g_{ip}}{\partial z_x}+\frac{\partial g_{xp}}{\partial y_i}-\frac{\partial g_{ix}}{\partial z_p}\right)$$

$$= \frac{1}{2}(r'O(r)) \left(\frac{\partial (r'O(r))}{\partial y_i} - \frac{\partial (O(r^2))}{\partial s} \right)$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) \left(\frac{\partial (r^2 (ds^2_{n-d-1})_{il} + O(r^4))}{\partial z_x} \right.$$

$$\left. + \frac{\partial (O(r^2))}{\partial y_i} - \frac{\partial (O(r^2))}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (\delta_{vp} + O(r) + O(|z|^2) + (r')^2 O(r^2)) \left(\frac{\partial (O(r^2))}{\partial z_x} \right.$$

$$\left. + \frac{\partial (\delta_{xp} + O(r) + O(|z|^2)))}{\partial y_i} - \frac{\partial (O(r^2))}{\partial z_p} \right)$$

$$= \frac{1}{2}(r'O(r))(r'O(r) + r'O(r) + O(r^2))$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) O(r^2)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (\delta_{vp} + O(r) + O(|z|^2) + (r')^2 O(r^2)) \left(O(r) + O(|z|^2) \right)$$

$$= (r')^2 O(r^2) + r'O(r^3) + (O(r^2) + (r')^2 O(r^4) + O(r^2) O(|z|^2))$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} \left(O(r) + O(|z|^2) + O(r^2) + O(r) O(|z|^2) + (r')^2 O(r^3) \right.$$

$$\left. + (r')^2 O(r^2) O(|z|^2) \right)$$

$$= O(r) + (r')^2 O(r^2) + r'O(r^3) + O(|z|^2)$$

$$= O(r) + O(r^2) + O(r^3) + O(|z|^2)$$

$$= O(r) + O(|z|^2).$$

$$\Gamma^i_{xv} = \frac{1}{2} g^{is} \left(\frac{\partial g_{xs}}{\partial z_v} + \frac{\partial g_{vs}}{\partial z_x} - \frac{\partial g_{xv}}{\partial s} \right) + \frac{1}{2} \sum_{l=1}^{n-d-1} g^{il} \left(\frac{\partial g_{xl}}{\partial z_v} + \frac{\partial g_{vl}}{\partial z_x} - \frac{\partial g_{xv}}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} g^{ip} \left(\frac{\partial g_{xp}}{\partial z_v} + \frac{\partial g_{vp}}{\partial z_x} - \frac{\partial g_{xv}}{\partial z_p} \right)$$

$$= \frac{1}{2} (r' O(r)) \left(\frac{\partial (r' O(r))}{\partial z_v} + \frac{\partial (r' O(r))}{\partial z_x} - \frac{\partial (\delta_{xv} + O(r) + O(|z|^2))}{\partial s} \right)$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} \left(\frac{1}{r^2} (ds^2_{n-d-1})^{il} + O(1) + (r')^2 O(r^2) \right) \left(\frac{\partial (O(r^2))}{\partial z_v} + \frac{\partial (O(r^2))}{\partial z_x} \right.$$

$$\left. - \frac{\partial (\delta_{xv} + O(r) + O(|z|^2))}{\partial y_l} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) \left(\frac{\partial (\delta_{xp} + O(r) + O(|z|^2))}{\partial z_v} \right.$$

$$\left. + \frac{\partial (\delta_{vp} + O(r) + O(|z|^2))}{\partial z_x} - \frac{(\delta_{xv} + O(r) + O(|z|^2))}{\partial z_p} \right)$$

$$= \frac{1}{2} (r' O(r)) \left(r' O(r) + O(r) + O(|z|^2) \right)$$

$$+ \frac{1}{2} \sum_{l=1}^{n-d-1} \left(\frac{1}{r^2} (ds^2_{n-d-1})^{il} + O(1) + (r')^2 O(r^2) \right) \left(O(r^2) + O(r) \right.$$

$$\left. + O(|z|^2) \right) + \frac{1}{2} \sum_{p=1}^{d-1} (O(1) + (r')^2 O(r^2) + O(|z|^2)) \left(O(r) + O(|z|) \right)$$

$$= (r')^2 O(r^2) + r' O(r^2) + r' O(r) O(|z|^2)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}\left(O(1)+O\left(\frac{1}{r}\right)+\frac{1}{r^2}O(|z|^2)+O(r^2)+O(r)+O(|z|^2)\right.$$

$$\left.+(r')^2O(r^4)+(r')^2O(r^3)+(r')^2O(r^2)O(|z|)\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}(O(r)+O(|z|)+(r')^2O(r^3)+(r')^2O(r^2)O(|z|)$$

$$+O(r)O(|z|^2)+O(|z|^3))$$

$$=O\left(\frac{1}{r}\right)+r'O(r)+O(|z|)$$

$$=O\left(\frac{1}{r}\right)+O(|z|).$$

$$\Gamma^w_{xv}=\frac{1}{2}g^{ws}\left(\frac{\partial g_{xs}}{\partial z_v}+\frac{\partial g_{vs}}{\partial z_x}-\frac{\partial g_{xv}}{\partial s}\right)+\frac{1}{2}\sum_{l=1}^{n-d-1}g^{wl}\left(\frac{\partial g_{xl}}{\partial z_v}+\frac{\partial g_{vl}}{\partial z_x}-\frac{\partial g_{xv}}{\partial y_l}\right)$$

$$+\frac{1}{2}\sum_{p=1}^{d-1}g^{wp}\left(\frac{\partial g_{xp}}{\partial z_v}+\frac{\partial g_{vp}}{\partial z_x}-\frac{\partial g_{xv}}{\partial z_p}\right)$$

$$=\frac{1}{2}(r'O(r))\left(\frac{\partial(r'O(r))}{\partial z_v}+\frac{\partial(r'O(r))}{\partial z_x}-\frac{\partial(\delta_{xv}+O(r)+O(|z|^2))}{\partial s}\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(\frac{\partial(O(r^2))}{\partial z_v}+\frac{\partial(O(r^2))}{\partial z_x}\right.$$

$$\left.-\frac{\partial(\delta_{xv}+O(r)+O(|z|^2))}{\partial y_l}\right)+\Gamma^w_{xv}(S^{d-1}(1))$$

$$=\frac{1}{2}(r'O(r))\left(r'O(r)+r'O(1)+O(r)+O(|z|^2)\right)$$

$$+\frac{1}{2}\sum_{l=1}^{n-d-1}(O(1)+(r')^2O(r^2)+O(|z|^2))\left(O(r^2)+O(r)+O(|z|^2)\right)$$

$$+\Gamma_{xv}^{w}(S^{d-1}(1))$$

$$=((r')^2O(r^2)+(r')^2O(r)+r'O(r^2)+r'O(r)O(|z|^2))+O(r^2)+O(r)$$

$$+O(|z|^2)+(r')^2O(r^4)+(r')^2O(r^3)+(r')^2O(r^2)O(|z|^2)+O(r^2)O(|z|^2)$$

$$+O(r)O(|z|^2)+O(|z|^4)+\Gamma_{xv}^{w}(S^{d-1}(1))$$

$$=r'O(r)+(r')^2O(r)+O(r)+O(|z|^2)+\Gamma_{xv}^{w}(S^{d-1}(1))$$

$$=O(r)+O(r)+O(r)+O(|z|^2)+\Gamma_{xv}^{w}(S^{d-1}(1))$$

$$=O(r)+O(|z|^2)+\Gamma_{xv}^{w}(S^{d-1}(1)).$$

In summary, we have proved:

Lemma 3.2.2. *The Christoffel symbols of g with respect to the coordinates $(s, y_1, \ldots, y_{n-d-1}, z_1, \ldots, z_{d-1})$ are given by*

- $\Gamma_{ss}^{s}=r''O(r^2)+O(r);$
- $\Gamma_{ss}^{i}=r''O(r)+O(1)+O(|z|^2);$
- $\Gamma_{ss}^{x}=r''O(r)+O(1);$
- $\Gamma_{si}^{s}=O(r^2);$
- $\Gamma_{sx}^{s}=O(1)+O(r)O(|z|^2);$
- $\Gamma_{si}^{j}=\left(\frac{r'}{r}\right)\delta_{ij}+O(r);$
- $\Gamma_{si}^{x}=O(r);$
- $\Gamma_{sx}^{i}=O\left(\frac{1}{r}\right)+O(|z|^2);$
- $\Gamma_{sx}^{v}=O(1)+O(|z|^2);$
- $\Gamma_{ij}^{s}=-r'r(ds_{n-d-1}^2)_{ij}+O(r^3);$
- $\Gamma_{xv}^{s}=O(1)+O(|z|^2);$
- $\Gamma_{ix}^{s}=O(r);$
- $\Gamma_{ij}^{k}=O(r^2)+\Gamma_{ij}^{k}(S^{n-d-1});$
- $\Gamma_{ij}^{x}=O(r^2);$

- $\Gamma^j_{ix} = O(1) + O(|z|^2)$;
- $\Gamma^v_{ix} = O(r) + O(|z|^2)$;
- $\Gamma^i_{xv} = O\left(\frac{1}{r}\right) + O(|z|)$;
- $\Gamma^w_{xv} = O(r) + O(|z|^2) + \Gamma^w_{xv}(S^{d-1})$.

3.3 Curvature computations

In this section, we are going to calculate the Ricci tensor components for the metric g and, at the end, compute the matrix for the Ricci tensor (Ric_{ab}).

Recall that we have

$$R^l_{ilk} = \partial_l(\Gamma^l_{ik}) - \partial_i(\Gamma^l_{lk}) + \sum_{m=1}^{n-2}(\Gamma^m_{ik}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{im}) + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{lk}\Gamma^l_{is}$$

$$\mathrm{Ric}_{ik} = R^s_{isk} + \sum_{l=1}^{n-2} R^l_{ilk}.$$

Therefore, in the general case, we would have

$$R^b_{abc} = \partial_b(\Gamma^b_{ac}) - \partial_a(\Gamma^b_{bc}) + \Gamma^s_{ac}\Gamma^b_{bs} - \Gamma^s_{bc}\Gamma^b_{as}$$

$$+ \sum_{l=1}^{n-d-1}(\Gamma^l_{ac}\Gamma^b_{bl} - \Gamma^l_{bc}\Gamma^b_{al}) + \sum_{p=1}^{d-1}(\Gamma^p_{ac}\Gamma^b_{bp} - \Gamma^p_{bc}\Gamma^b_{ap});$$

$$\mathrm{Ric}_{ab} = R^s_{asb} + \sum_{\xi=1}^{n-d-1} R^\xi_{a\xi b} + \sum_{\varrho=1}^{d-1} R^\varrho_{a\varrho b};$$

where $\xi \in \{1, \dots, n-d-1\}$ (fibre-sphere direction) and $\varrho \in \{1, \dots, d-1\}$ (S^{d-1} direction).

So, we need to compute Ric_{ss}, Ric_{si}, Ric_{sx}, Ric_{ij}, Ric_{ix} and Ric_{xv}.

We have $\mathrm{Ric}_{ss} = R^s_{sss} + \sum_{\xi=1}^{n-d-1} R^\xi_{s\xi s} + \sum_{\varrho=1}^{d-1} R^\varrho_{s\varrho s}$.

$$R^s_{sss} = \partial_s(\Gamma^s_{ss}) - \partial_s(\Gamma^s_{ss}) + \Gamma^s_{ss}\Gamma^s_{ss} - \Gamma^s_{ss}\Gamma^s_{ss}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{ss}\Gamma^s_{sl}-\Gamma^l_{ss}\Gamma^s_{sl})+\sum_{p=1}^{d-1}(\Gamma^p_{ss}\Gamma^s_{sp}-\Gamma^p_{ss}\Gamma^s_{sp})=0.$$

$$R^{\xi}_{s\xi s}=\partial_\xi(\Gamma^\xi_{ss})-\partial_s(\Gamma^\xi_{\xi s})+\Gamma^s_{ss}\Gamma^\xi_{\xi s}-\Gamma^s_{\xi s}\Gamma^\xi_{ss}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{ss}\Gamma^\xi_{\xi l}-\Gamma^l_{\xi s}\Gamma^\xi_{sl})+\sum_{p=1}^{d-1}(\Gamma^p_{ss}\Gamma^\xi_{\xi p}-\Gamma^p_{\xi s}\Gamma^\xi_{sp})$$

$$=\partial_\xi(r''O(r)+O(1)+O(|z|^2))-\partial_s\left(\frac{r'}{r}1+O(r)\right)$$

$$+(r''O(r^2)+O(r))\left(\frac{r'}{r}+O(r)\right)-O(r^2)\left(r''O(r)+O(1)+O(|z|^2)\right)$$

$$+\sum_{l=1}^{n-d-1}\Bigg[(r''O(r)+O(1)+O(|z|^2))(O(r^2)+\Gamma^\xi_{\xi l}(S^{n-d-1}))$$

$$-\left(\frac{r'}{r}\delta_{\xi l}+O(r)\right)\left(\frac{r'}{r}\delta_{\xi l}+O(r)\right)\Bigg]$$

$$+\sum_{p=1}^{d-1}\big[(r''O(r)+O(1))(O(1)+O(|z|^2))$$

$$-O(r)\left(O\left(\frac{1}{r}\right)+O(|z|^2)\right)\bigg]$$

$$=r''O(r)+O(1)+O(|z|^2)-\left(\frac{r''r-r'r'}{r^2}+r'O(1)+O(r)\right)$$

$$+r''r'O(r)+r''O(r^3)+r'O(1)+O(r^2)+r''O(r^3)+O(r^2)$$

$$+O(r^2)O(|z|^2)+\sum_{l=1}^{n-d-1}\big[r''O(r^3)+r''O(r)+O(r^2)+O(1)$$

$$+ O(r^2)O(|z|^2) + O(|z|^2) - \left(\frac{r'}{r}\right)^2 \delta_{\xi l} + r'O(1) + r'O(1) + O(r^2)\Big]$$

$$+ \sum_{p=1}^{d-1} \big[r''O(r) + r''O(r)O(|z|^2) + O(1) + O(|z|^2) + O(1)$$

$$+O(r)O(|z|^2)\big]$$

$$= r''O(r) + O(1) - \left(\frac{r''r - r'r'}{r^2}\right) + r'O(1) + r''r'O(r)$$

$$- \left(\frac{r'}{r}\right)^2 \delta_{\xi\xi} + O(|z|^2)$$

$$= r''O(r) + O(1) - \left(\frac{r''}{r}\right) + \left(\frac{r'}{r}\right)^2 + O(1) + r''O(r)$$

$$- \left(\frac{r'}{r}\right)^2 + O(|z|^2)$$

$$= O(1) + r''O(r) + O(|z|^2) - \left(\frac{r''}{r}\right).$$

$$R^{\varrho}_{s\varrho s} = \partial_{\varrho}(\Gamma^{\varrho}_{ss}) - \partial_s(\Gamma^{\varrho}_{\varrho s}) + \Gamma^{s}_{ss}\Gamma^{\varrho}_{\varrho s} - \Gamma^{s}_{\varrho s}\Gamma^{\varrho}_{ss}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^{l}_{ss}\Gamma^{\varrho}_{\varrho l} - \Gamma^{l}_{\varrho s}\Gamma^{\varrho}_{sl}) + \sum_{p=1}^{d-1}(\Gamma^{p}_{ss}\Gamma^{\varrho}_{\varrho p} - \Gamma^{p}_{\varrho s}\Gamma^{\varrho}_{sp})$$

$$= \partial_{\varrho}(r''O(r) + O(1)) - \partial_s(O(1) + O(|z|^2))$$

$$+ (r''O(r^2) + O(r))(O(1) + O(|z|^2))$$

$$- (O(1) + O(r)O(|z|^2))(r''O(r) + O(1))$$

$$+\sum_{l=1}^{n-d-1}\Bigg[(r''O(r)+O(1)+O(|z|^2))(O(r)+O(|z|^2))$$

$$-\left(O\left(\frac{1}{r}\right)+O(|z|^2)\right)O(r)\Bigg]$$

$$+\sum_{p=1}^{d-1}\big[(r''O(r)+O(1))(O(r)+O(|z|^2)+\Gamma^{\varrho}_{\varrho p}(S^{d-1}))$$

$$-(O(1)+O(|z|^2))(O(1)+O(|z|^2))\big]$$

$$=r''O(r)+O(1)-(O(1)+O(|z|^2))+\big(r''O(r^2)$$

$$+r''O(r^2)O(|z|^2)+O(r)+O(r)O(|z|^2)\big)$$

$$-(r''O(r)+O(1)+r''O(r^2)O(|z|^2)+O(r)O(|z|^2))$$

$$+\sum_{l=1}^{n-d-1}\big[r''O(r^2)+r''O(r)O(|z|^2)+O(r)+O(|z|^2)$$

$$+O(r)O(|z|^2)+O(|z|^4)-(O(1)+O(r)O(|z|^2))\big]$$

$$+\sum_{p=1}^{d-1}\big[r''O(r^2)+r''O(r)O(|z|^2)+r''O(r)+O(r)$$

$$+O(|z|^2)+O(1)-(O(1)+O(|z|^2)+O(|z|^2)+O(|z|^4))\big]$$

$$=O(1)+r''O(r)+O(|z|^2).$$

Therefore,

$$\mathrm{Ric}_{ss}=0+\sum_{\xi=1}^{n-d-1}\left(O(1)+r''O(r)+O(|z|^2)-\left(\frac{r''}{r}\right)\right)$$

$$+\sum_{\varrho=1}^{d-1}(O(1)+r''O(r)+O(|z|^2))$$

$$=O(1)+r''O(r)+O(|z|^2)-(n-d-1)\left(\frac{r''}{r}\right)$$

$$=O(1)+O(|z|^2)-(n-d-1)\left(\frac{r''}{r}\right),$$

since $\left|\frac{r''}{r}\right| \gg r''O(r)$ for r sufficiently small.

Now, $\mathrm{Ric}_{si} = R^s_{ssi} + \sum_{\xi=1}^{n-d-1} R^\xi_{s\xi i} + \sum_{\varrho=1}^{d-1} R^\varrho_{s\varrho i}$.

$$R^s_{ssi} = \partial_s(\Gamma^s_{si}) - \partial_s(\Gamma^s_{si}) + \Gamma^s_{si}\Gamma^s_{ss} - \Gamma^s_{si}\Gamma^s_{ss}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{si}\Gamma^s_{sl} - \Gamma^l_{si}\Gamma^s_{sl}) + \sum_{p=1}^{d-1}(\Gamma^p_{si}\Gamma^s_{sp} - \Gamma^p_{si}\Gamma^s_{sp}) = 0.$$

$$R^\xi_{s\xi i} = \partial_\xi(\Gamma^\xi_{si}) - \partial_s(\Gamma^\xi_{\xi i}) + \Gamma^s_{si}\Gamma^\xi_{\xi s} - \Gamma^s_{\xi i}\Gamma^\xi_{ss}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{si}\Gamma^\xi_{\xi l} - \Gamma^l_{\xi i}\Gamma^\xi_{sl}) + \sum_{p=1}^{d-1}(\Gamma^p_{si}\Gamma^\xi_{\xi p} - \Gamma^p_{\xi i}\Gamma^\xi_{sp})$$

$$=\partial_\xi\left(\frac{r'}{r}\delta_{i\xi}+O(r)\right) - \partial_s(O(r^2)+\Gamma^\xi_{\xi i}(S^{n-d-1}))$$

$$+O(r^2)\left(\frac{r'}{r}\delta_{\xi\xi}+O(r)\right)$$

$$-(-r'r(ds^2_{n-d-1})_{\xi i}+O(r^3))(r''O(r)+O(1)+O(|z|^2))$$

$$+\sum_{l=1}^{n-d-1}\left[\left(\frac{r'}{r}\delta_{il}+O(r)\right)(O(r^2)+\Gamma^\xi_{\xi l}(S^{n-d-1}))\right.$$

$$- \left(O(r^2) + \Gamma^l_{\xi i}(S^{n-d-1})\right)\left(\frac{r'}{r}\delta_{\xi l} + O(r)\right)\Bigg]$$

$$+ \sum_{p=1}^{d-1}\Bigg[O(r)(O(1) + O(|z|^2)) - O(r^2)\left(O\left(\frac{1}{r}\right) + O(|z|^2)\right)\Bigg]$$

$$= 0 + O(r) - (r'O(r) + O(r^2) + 0) + r'O(r)$$

$$+ O(r^3) + \big(r'r''O(r^2) + r'O(r) + r''O(r^4) + O(r^3)$$

$$+ r'rO(|z|^2) + O(r^3)O(|z|^2)\big)$$

$$+ \sum_{l=1}^{n-d-1}\Bigg[r'O(r)\delta_{il} + \left(\frac{r'}{r}\right)\delta_{il}\Gamma^{\xi}_{\xi l}(S^{n-d-1}) + O(r^3) + O(r)$$

$$- \left(r'O(r)\delta_{il} + O(r^3) + \left(\frac{r'}{r}\right)\delta_{\xi l}\Gamma^l_{\xi i}(S^{n-d-1}) + O(r)\right)\Bigg]$$

$$+ \sum_{p=1}^{d-1}\left[O(r) + O(r)O(|z|^2) - (O(r) + O(r^2)O(|z|^2))\right]$$

$$= O(r) + r'O(r) + r''O(r^2) + \left(\frac{r'}{r}\right)\Gamma^{\xi}_{\xi i}(S^{n-d-1})$$

$$- \left(\frac{r'}{r}\right)\Gamma^{\xi}_{\xi i}(S^{n-d-1}) + O(r)O(|z|^2) + r'rO(|z|^2)$$

$$= O(r) + r''O(r^2) + O(r) + O(r)O(|z|^2) + rO(|z|^2)$$

$$= O(r) + r''O(r^2).$$

$$R^{\varrho}_{s\varrho i} = \partial_{\varrho}(\Gamma^{\varrho}_{si}) - \partial_s(\Gamma^{\varrho}_{\varrho i}) + \Gamma^s_{si}\Gamma^{\varrho}_{\varrho s} - \Gamma^s_{\varrho i}\Gamma^{\varrho}_{ss}$$

$$+ \sum_{l=1}^{n-d-1}(\Gamma^l_{si}\Gamma^{\varrho}_{\varrho l} - \Gamma^l_{\varrho i}\Gamma^{\varrho}_{sl}) + \sum_{p=1}^{d-1}(\Gamma^p_{si}\Gamma^{\varrho}_{\varrho p} - \Gamma^p_{\varrho i}\Gamma^{\varrho}_{sp})$$

$$= \partial_\varrho(O(r)) - \partial_s(O(r) + O(|z|^2))$$

$$+ O(r^2)(O(1) + O(|z|^2)) - O(r)(r''O(r) + O(1))$$

$$+ \sum_{l=1}^{n-d-1} \left[\left(\left(\frac{r'}{r} \right) \delta_{il} + O(r) \right) (O(r) + O(|z|^2)) \right.$$

$$\left. - (O(1) + O(|z|^2))O(r) \right]$$

$$+ \sum_{p=1}^{d-1} \left[O(r)(O(r) + O(|z|^2) + \Gamma^{\varrho}_{\varrho p}(S^{d-1})) \right.$$

$$\left. - (O(r) + O(|z|^2))(O(1) + O(|z|^2)) \right]$$

$$= O(r) - (r'O(1) + O(r) + O(|z|^2)) + O(r^2) + O(r^2)O(|z|^2)$$

$$- (r''O(r^2) + O(r)) + \sum_{l=1}^{n-d-1} \left[r'O(1)\delta_{il} + \left(\frac{r'}{r} \right) \delta_{il} O(|z|^2) \right.$$

$$\left. + O(r^2) + O(r)O(|z|^2) - (O(r) + O(r)O(|z|^2)) \right]$$

$$+ \sum_{p=1}^{d-1} \left[O(r^2) + O(r)O(|z|^2) + O(r) - \big(O(r) + O(r)O(|z|^2) \right.$$

$$\left. + O(|z|^2) + O(|z|^4) \big) \right]$$

$$= O(r) + r'O(1) + O(|z|^2) + r''O(r^2) + \left(\frac{r'}{r} \right) O(|z|^2)$$

$$= O(1) + r''O(r^2) + \left(1 + \left(\frac{r'}{r} \right) \right) O(|z|^2).$$

So,

$$\mathrm{Ric}_{si} = 0 + \sum_{\xi=1}^{n-d-1} \big(O(r) + r''O(r^2)\big)$$

$$+ \sum_{\varrho=1}^{d-1} \left(O(1) + r''O(r^2) + \left(1 + \left(\frac{r'}{r}\right)\right) O(|z|^2) \right)$$

$$= O(1) + r''O(r^2) + O(|z|^2) + \left(\frac{r'}{r}\right) O(|z|^2).$$

We have $\mathrm{Ric}_{sx} = R^s_{ssx} + \sum_{\xi=1}^{n-d-1} R^\xi_{s\xi x} + \sum_{\varrho=1}^{d-1} R^\varrho_{s\varrho x}$.

$$R^s_{ssx} = \partial_s(\Gamma^s_{sx}) - \partial_s(\Gamma^s_{sx}) + \Gamma^s_{sx}\Gamma^s_{ss} - \Gamma^s_{sx}\Gamma^s_{ss}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^l_{sx}\Gamma^s_{sl} - \Gamma^l_{sx}\Gamma^s_{sl}) + \sum_{p=1}^{d-1} (\Gamma^p_{sx}\Gamma^s_{sp} - \Gamma^p_{sx}\Gamma^s_{sp}) = 0.$$

$$R^\xi_{s\xi x} = \partial_\xi(\Gamma^\xi_{sx}) - \partial_s(\Gamma^\xi_{\xi x}) + \Gamma^s_{sx}\Gamma^\xi_{\xi s} - \Gamma^s_{\xi x}\Gamma^\xi_{ss}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^l_{sx}\Gamma^\xi_{\xi l} - \Gamma^l_{\xi x}\Gamma^\xi_{sl}) + \sum_{p=1}^{d-1} (\Gamma^p_{sx}\Gamma^\xi_{\xi p} - \Gamma^p_{\xi x}\Gamma^\xi_{sp})$$

$$= \partial_\xi \left(O\left(\frac{1}{r}\right) + O(|z|^2) \right) - \partial_s(O(1) + O(|z|^2))$$

$$+ (O(1) + O(r)O(|z|^2)) \left(\left(\frac{r'}{r}\right) \delta_{\xi\xi} + O(r) \right)$$

$$- O(r)(r''O(r) + O(1) + O(|z|^2))$$

$$+ \sum_{l=1}^{n-d-1} \left[\left(O\left(\frac{1}{r}\right) + O(|z|^2) \right) (O(r^2) + \Gamma^\xi_{\xi l}(S^{n-d-1})) \right.$$

$$\left. -(O(1) + O(|z|^2)) \left(\left(\frac{r'}{r}\right) \delta_{\xi l} + O(r) \right) \right]$$

$$+\sum_{p=1}^{d-1}\Bigg[(O(1)+O(|z|^2))(O(1)+O(|z|^2))$$

$$-(O(r)+O(|z|^2))\left(O\left(\frac{1}{r}\right)+O(|z|^2)\right)\Bigg]$$

$$=O\left(\frac{1}{r}\right)+O(|z|^2)-(O(1)+O(|z|^2))+r'O\left(\frac{1}{r}\right)+O(r)$$

$$+r'O(|z|^2)+O(r^2)O(|z|^2)-(r''O(r^2)+O(r)+O(r)O(|z|^2))$$

$$+\sum_{l=1}^{n-d-1}\Bigg[O(r)+O\left(\frac{1}{r}\right)+O(r^2)O(|z|^2)+O(|z|^2)$$

$$-\left(r'O\left(\frac{1}{r}\right)\delta_{\xi l}+O(r)+\left(\frac{r'}{r}\right)O(|z|^2)\delta_{\xi l}+O(r)O(|z|^2)\right)\Bigg]$$

$$+\sum_{p=1}^{d-1}\Bigg[O(1)+O(|z|^2)+O(|z|^2)+O(|z|^4)-\Bigg(O(1)$$

$$+O(r)O(|z|^2)+O\left(\frac{1}{r}\right)O(|z|^2)+O(|z|^4)\Bigg)\Bigg]$$

$$=O\left(\frac{1}{r}\right)+O(|z|^2)+r'O\left(\frac{1}{r}\right)+r'O(|z|^2)+r''O(r^2)$$

$$+\left(\frac{r'}{r}\right)O(|z|^2)$$

$$=O\left(\frac{1}{r}\right)+O(|z|^2)+r''O(r^2)+\left(\frac{r'}{r}\right)O(|z|^2)$$

$$=O\left(\frac{1}{r}\right)+O(|z|^2)+r''O(r^2).$$

$$R^{\varrho}_{s\varrho x} = \partial_{\varrho}(\Gamma^{\varrho}_{sx}) - \partial_s(\Gamma^{\varrho}_{\varrho x}) + \Gamma^{s}_{sx}\Gamma^{\varrho}_{\varrho s} - \Gamma^{s}_{\varrho x}\Gamma^{\varrho}_{ss}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^{l}_{sx}\Gamma^{\varrho}_{\varrho l} - \Gamma^{l}_{\varrho x}\Gamma^{\varrho}_{sl}) + \sum_{p=1}^{d-1}(\Gamma^{p}_{sx}\Gamma^{\varrho}_{\varrho p} - \Gamma^{p}_{\varrho x}\Gamma^{\varrho}_{sp})$$

$$= \partial_{\varrho}(O(1) + O(|z|^2)) - \partial_s \left(O(r) + O(|z|^2) + \Gamma^{\varrho}_{\varrho x}(S^{d-1})\right)$$

$$+ (O(1) + O(r)O(|z|^2))(O(1) + O(|z|^2))$$

$$- (O(1) + O(|z|^2))(r'' O(r) + O(1))$$

$$+ \sum_{l=1}^{n-d-1} \left[\left(O\left(\frac{1}{r}\right) + O(|z|^2)\right)(O(r) + O(|z|^2))\right.$$

$$\left.- \left(O\left(\frac{1}{r}\right) + O(|z|)\right) O(r)\right]$$

$$+ \sum_{p=1}^{d-1} \left[(O(1) + O(|z|^2)) \left(O(r) + O(|z|^2) + \Gamma^{\varrho}_{\varrho p}(S^{d-1})\right)\right.$$

$$\left.- \left(O(r) + O(|z|^2) + \Gamma^{p}_{\varrho x}(S^{d-1})\right)(O(1) + O(|z|^2))\right]$$

$$= O(1) + O(|z|) - (r' O(1) + O(|z|^2)) + O(1) + O(|z|^2)$$

$$+ O(r)O(|z|^2) + O(r)O(|z|^4) - (r'' O(r) + O(1)$$

$$+ r'' O(r)O(|z|^2) + O(|z|^2)) + \sum_{l=1}^{n-d-1} \left[O(1) + O\left(\frac{1}{r}\right) O(|z|^2)\right.$$

$$\left.+ O(r)O(|z|^2) + O(|z|^4) - (O(1) + O(r)O(|z|))\right]$$

$$+ \sum_{p=1}^{d-1} \left[O(r) + O(|z|^2) + O(1) + O(r)O(|z|^2) + O(|z|^4)\right.$$

$$+ O(|z|^2) - \big(O(r) + O(r)O(|z|^2) + O(|z|^2) + O(|z|^4)$$

$$+O(1) + O(|z|^2)\big)\big]$$

$$= O(1) + O(|z|) + r'O(1) + r''O(r) + r''O(r)O(|z|^2)$$

$$+ O\left(\frac{1}{r}\right) O(|z|^2)$$

$$= O(1) + O(|z|) + r''O(r)(1 + O(|z|^2)) + O\left(\frac{1}{r}\right) O(|z|^2)$$

$$= O(1) + O(|z|) + r''O(r) + O\left(\frac{1}{r}\right) O(|z|^2).$$

Then,

$$\mathrm{Ric}_{sx} = 0 + \sum_{\xi=1}^{n-d-1} \left(O\left(\frac{1}{r}\right) + O(|z|^2) + r''O(r^2) + \left(\frac{r'}{r}\right) O(|z|^2) \right)$$

$$+ \sum_{\varrho=1}^{d-1} \left(O(1) + O(|z|) + r''O(r) + O\left(\frac{1}{r}\right) O(|z|^2) \right)$$

$$= O\left(\frac{1}{r}\right) + O(|z|) + r''O(r) + \left(\frac{r'}{r}\right) O(|z|^2) + O\left(\frac{1}{r}\right) O(|z|^2)$$

$$= O\left(\frac{1}{r}\right) (1 + O(|z|^2)) + O(|z|) + r''O(r)$$

$$= O\left(\frac{1}{r}\right) + O(|z|) + r''O(r).$$

We have $\mathrm{Ric}_{ij} = R^s_{isj} + \sum_{\xi=1}^{n-d-1} R^\xi_{i\xi j} + \sum_{\varrho=1}^{d-1} R^\varrho_{i\varrho j}$.

$$R^s_{isj} = \partial_s(\Gamma^s_{ij}) - \partial_i(\Gamma^s_{sj}) + \Gamma^s_{ij}\Gamma^s_{ss} - \Gamma^s_{sj}\Gamma^s_{is}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^l_{ij}\Gamma^s_{sl} - \Gamma^l_{sj}\Gamma^s_{il}) + \sum_{p=1}^{d-1}(\Gamma^p_{ij}\Gamma^s_{sp} - \Gamma^p_{sj}\Gamma^s_{ip})$$

$$= \partial_s(-r'r(ds^2_{n-d-1})_{ij} + O(r^3)) - \partial_i(O(r^2))$$

$$+ (-r'r(ds^2_{n-d-1})_{ij} + O(r^3))(r''O(r^2) + O(r)) - O(r^2)O(r^2)$$

$$+ \sum_{l=1}^{n-d-1} \big[(O(r^2) + \Gamma^l_{ij}(S^{n-d-1}))O(r^2)$$

$$- \left(\left(\frac{r'}{r}\right)\delta_{jl} + O(r)\right)(-r'r(ds^2_{n-d-1})_{il} + O(r^3))\Big]$$

$$+ \sum_{p=1}^{d-1} \big[O(r^2)(O(1) + O(r)O(|z|^2)) - O(r)O(r)\big]$$

$$= -r''r(ds^2_{n-d-1})_{ij} - (r')^2(ds^2_{n-d-1})_{ij} + r'O(r^2) + O(r^3)$$

$$- O(r^2) + r''r'O(r^3) + r'O(r^2) + r''O(r^5) + O(r^4) - O(r^4)$$

$$+ \sum_{l=1}^{n-d-1} \big[O(r^4) + O(r^2) + (r')^2\delta_{jl}(ds^2_{n-d-1})_{il} + r'O(r^2)\delta_{jl}$$

$$+r'O(r^2) + O(r^4)\big] + \sum_{p=1}^{d-1} \big[O(r^2) + O(r^3)O(|z|^2) + O(r^2)\big]$$

$$= -r''r(ds^2_{n-d-1})_{ij} - (r')^2(ds^2_{n-d-1})_{ij} + r'O(r^2) + O(r^2)$$

$$+ r''r'O(r^3) + r''O(r^5) + (r')^2(ds^2_{n-d-1})_{ij}\delta_{jj} + O(r^3)O(|z|^2)$$

$$= -r''r(ds^2_{n-d-1})_{ij} + O(r^2) + r''O(r^3).$$

$$R^\xi_{i\xi j} = \partial_\xi(\Gamma^\xi_{ij}) - \partial_i(\Gamma^\xi_{\xi j}) + \Gamma^s_{ij}\Gamma^\xi_{\xi s} - \Gamma^s_{\xi j}\Gamma^\xi_{is}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{ij}\Gamma^\xi_{\xi l}-\Gamma^l_{\xi j}\Gamma^\xi_{il})+\sum_{p=1}^{d-1}(\Gamma^p_{ij}\Gamma^\xi_{\xi p}-\Gamma^p_{\xi j}\Gamma^\xi_{ip})$$

$$=\partial_\xi(O(r^2)+\Gamma^\xi_{ij}(S^{n-d-1}))-\partial_i(O(r^2)+\Gamma^\xi_{\xi j}(S^{n-d-1}))$$

$$+(-r'r(ds^2_{n-d-1})_{ij}+O(r^3))\left(\left(\frac{r'}{r}\right)\delta_{\xi\xi}+O(r)\right)$$

$$-(-r'r(ds^2_{n-d-1})_{\xi j}+O(r^3))\left(\left(\frac{r'}{r}\right)\delta_{i\xi}+O(r)\right)$$

$$+\sum_{l=1}^{n-d-1}\Big[(O(r^2)+\Gamma^l_{ij}(S^{n-d-1}))(O(r^2)+\Gamma^\xi_{\xi l}(S^{n-d-1}))$$

$$-(O(r^2)+\Gamma^l_{\xi j}(S^{n-d-1}))(O(r^2)+\Gamma^\xi_{il}(S^{n-d-1}))\Big]$$

$$+\sum_{p=1}^{d-1}\left[O(r^2)(O(1)+O(|z|^2))-O(r^2)(O(1)+O(|z|^2))\right]$$

$$=O(r^2)+\partial_\xi(\Gamma^\xi_{ij}(S^{n-d-1}))+O(r^2)-\partial_i(\Gamma^\xi_{\xi j}(S^{n-d-1}))$$

$$-(r')^2(ds^2_{n-d-1})_{ij}+r'O(r^2)+r'O(r^2)+O(r^4)$$

$$+(r')^2(ds^2_{n-d-1})_{\xi j}\delta_{i\xi}+r'O(r^2)+r'O(r^2\delta_{i\xi})+O(r^4)$$

$$+\sum_{l=1}^{n-d-1}\Big[O(r^4)+O(r^2)+O(r^2)+\Gamma^l_{ij}(S^{n-d-1})\Gamma^\xi_{\xi l}(S^{n-d-1})$$

$$-(O(r^4)+O(r^2)+O(r^2)+\Gamma^l_{\xi j}(S^{n-d-1})\Gamma^\xi_{il}(S^{n-d-1}))\Big]$$

$$+\sum_{p=1}^{d-1}\left[O(r^2)+O(r^2)O(|z|^2)+O(r^2)+O(r^2)O(|z|^2)\right]$$

$$= O(r^2) + \partial_\xi(\Gamma^\xi_{ij}(S^{n-d-1})) - \partial_i(\Gamma^\xi_{\xi j}(S^{n-d-1})) - (r')^2(ds^2_{n-d-1})_{ij}$$

$$+ (r')^2(ds^2_{n-d-1})_{\xi j}\delta_{i\xi} + \sum_{l=1}^{n-d-1} \Big[\Gamma^l_{ij}(S^{n-d-1})\Gamma^\xi_{\xi l}(S^{n-d-1})$$

$$-\Gamma^l_{\xi j}(S^{n-d-1})\Gamma^\xi_{il}(S^{n-d-1})\Big]$$

$$= O(r^2) + (r')^2((ds^2_{n-d-1})_{\xi j}\delta_{i\xi} - (ds^2_{n-d-1})_{ij}) + R^\xi_{i\xi j}(S^{n-d-1}).$$

$$R^\varrho_{i\varrho j} = \partial_\varrho(\Gamma^\varrho_{ij}) - \partial_i(\Gamma^\varrho_{\varrho j}) + \Gamma^s_{ij}\Gamma^\varrho_{\varrho s} - \Gamma^s_{\varrho j}\Gamma^\varrho_{is}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^l_{ij}\Gamma^\varrho_{\varrho l} - \Gamma^l_{\varrho j}\Gamma^\varrho_{il}) + \sum_{p=1}^{d-1}(\Gamma^p_{ij}\Gamma^\varrho_{\varrho p} - \Gamma^p_{\varrho j}\Gamma^\varrho_{ip})$$

$$= \partial_\varrho(O(r^2)) - \partial_i(O(r) + O(|z|^2))$$

$$+ (-r'r(ds^2_{n-d-1})_{ij} + O(r^3))(O(1) + O(|z|^2)) - O(r)O(r)$$

$$+ \sum_{l=1}^{n-d-1} \big[(O(r^2) + \Gamma^l_{ij}(S^{n-d-1}))(O(r) + O(|z|^2))$$

$$-(O(1) + O(|z|^2))O(r^2)\big]$$

$$+ \sum_{p=1}^{d-1} \big[O(r^2)(O(r) + O(|z|^2) + \Gamma^\varrho_{\varrho p}(S^{d-1}))$$

$$-(O(r) + O(|z|^2))(O(r) + O(|z|^2))\big]$$

$$= O(r^2) - (O(r) + O(|z|^2)) + r'O(r) + r'rO(|z|^2) + O(r^3)$$

$$+ O(r^3)O(|z|^2) + O(r^2) + \sum_{l=1}^{n-d-1} \big[O(r^3) + O(r^2)O(|z|^2) + O(r)$$

$$+O(|z|^2) - (O(r^2) + O(r^2)O(|z|^2))\big] + \sum_{p=1}^{d-1} \big[O(r^3) + O(r^2)O(|z|^2)$$

$$+O(r^2) - (O(r^2) + O(r)O(|z|^2) + O(r)O(|z|^2) + O(|z|^4))\big]$$

$$= O(r) + r'O(r) + O(|z|^2) + rO(|z|^2)$$

$$= O(r) + O(|z|^2).$$

So,

$$\mathrm{Ric}_{ij} = (-r''r(ds^2_{n-d-1})_{ij} + O(r^2) + r''O(r^3))$$

$$+ \sum_{\xi=1}^{n-d-1} \Big[O(r^2) + (r')^2((ds^2_{n-d-1})_{\xi j}\delta_{i\xi} - (ds^2_{n-d-1})_{ij}) + R^{\xi}_{i\xi j}(S^{n-d-1})\Big]$$

$$+ \sum_{\varrho=1}^{d-1} \big[O(r) + O(|z|^2)\big]$$

$$= O(r) + O(|z|^2) + r''O(r^3) - r''r(ds^2_{n-d-1})_{ij} + \mathrm{Ric}_{ij}(S^{n-d-1})$$

$$-(r')^2(n-d-2)(ds^2_{n-d-1})_{ij}$$

$$= O(r) + O(|z|^2) + r''O(r^3) + \mathrm{Ric}_{ij}(S^{n-d-1})$$

$$-(r')^2(n-d-2)(ds^2_{n-d-1})_{ij} - r''r(ds^2_{n-d-1})_{ij}.$$

We have $\mathrm{Ric}_{ix} = R^s_{isx} + \sum_{\xi=1}^{n-d-1} R^{\xi}_{i\xi x} + \sum_{\varrho=1}^{d-1} R^{\varrho}_{i\varrho x}$.

$$R^s_{isx} = \partial_s(\Gamma^s_{ix}) - \partial_i(\Gamma^s_{sx}) + \Gamma^s_{ix}\Gamma^s_{ss} - \Gamma^s_{sx}\Gamma^s_{is}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^l_{ix}\Gamma^s_{sl} - \Gamma^l_{sx}\Gamma^s_{il}) + \sum_{p=1}^{d-1}(\Gamma^p_{ix}\Gamma^s_{sp} - \Gamma^p_{sx}\Gamma^s_{ip})$$

$$= \partial_s(O(r)) - \partial_i(O(1) + O(r)O(|z|^2)) + O(r)(r''O(r^2) + O(r))$$

$$- (O(1) + O(r)O(|z|^2))O(r^2) + \sum_{l=1}^{n-d-1} \Big[(O(1) + O(|z|^2))O(r^2)$$

$$- \left(O\left(\frac{1}{r}\right) + O(|z|^2)\right)(-r'r(ds^2_{n-d-1})_{il} + O(r^3))\Big]$$

$$+ \sum_{p=1}^{d-1} \big[(O(r) + O(|z|^2))(O(1) + O(r)O(|z|^2))$$

$$-(O(1) + O(|z|^2))O(r)\big]$$

$$= r'O(1) + O(r) + O(1) + O(r)O(|z|^2) + r''O(r^3) + O(r^2)$$

$$+ O(r^2) + O(r^3)O(|z|^2) + \sum_{l=1}^{n-d-1} \big[O(r^2) + O(r^2)O(|z|^2) + r'O(1)$$

$$+O(r^2) + r'rO(|z|^2) + O(r^3)O(|z|^2)\big]$$

$$+ \sum_{p=1}^{d-1} \big[O(r) + O(r^2)O(|z|^2) + O(|z|^2) + O(r)O(|z|^4) + O(r)$$

$$+O(r)O(|z|^2)\big]$$

$$= r'O(1) + O(1) + O(r)O(|z|^2) + O(|z|^2) + r''O(r^3)$$

$$= O(1) + O(|z|^2) + r''O(r^3).$$

$$R^{\xi}_{i\xi x} = \partial_\xi(\Gamma^{\xi}_{ix}) - \partial_i(\Gamma^{\xi}_{\xi x}) + \Gamma^{s}_{ix}\Gamma^{\xi}_{\xi s} - \Gamma^{s}_{\xi x}\Gamma^{\xi}_{is}$$

$$+ \sum_{l=1}^{n-d-1} (\Gamma^{l}_{ix}\Gamma^{\xi}_{\xi l} - \Gamma^{l}_{\xi x}\Gamma^{\xi}_{il}) + \sum_{p=1}^{d-1}(\Gamma^{p}_{ix}\Gamma^{\xi}_{\xi p} - \Gamma^{p}_{\xi x}\Gamma^{\xi}_{ip})$$

$$= \partial_\xi(O(1) + O(|z|^2)) - \partial_i(O(1) + O(|z|^2))$$

$$+ O(r)\left(\left(\frac{r'}{r}\right)\delta\xi\xi + O(r)\right) - O(r)\left(\left(\frac{r'}{r}\right)\delta\xi i + O(r)\right)$$

$$+ \sum_{l=1}^{n-d-1}\Big[(O(1) + O(|z|^2))(O(r^2) + \Gamma^\xi_{\xi l}(S^{n-d-1}))$$

$$-(O(1) + O(|z|^2))(O(r^2) + \Gamma^\xi_{il}(S^{n-d-1}))\Big]$$

$$+ \sum_{p=1}^{d-1}\big[(O(r) + O(|z|^2))(O(1) + O(|z|^2))$$

$$-(O(1) + O(|z|^2))(O(1) + O(|z|^2))\big]$$

$$= O(1) + O(|z|^2) + O(1) + O(|z|^2) + r'O(1) + O(r^2)$$

$$+ r'O(1)\delta_{\xi i} + O(r^2) + \sum_{l=1}^{n-d-1}\big[O(r^2) + O(1) + O(r^2)O(|z|^2)$$

$$+ O(|z|^2) + O(r^2) + O(1) + O(r^2)O(|z|^2) + O(|z|^2)\big]$$

$$+ \sum_{p=1}^{d-1}\big[O(r) + O(r)O(|z|^2) + O(|z|^2) + O(|z|^4) + O(1)$$

$$+ O(|z|^2) + O(|z|^2) + O(|z|^4)\big]$$

$$= O(1) + O(|z|^2).$$

$$R^\varrho_{i\varrho x} = \partial_\varrho(\Gamma^\varrho_{ix}) - \partial_i(\Gamma^\varrho_{\varrho x}) + \Gamma^s_{ix}\Gamma^\varrho_{\varrho s} - \Gamma^s_{\varrho x}\Gamma^\varrho_{is}$$

$$+ \sum_{l=1}^{n-d-1}(\Gamma^l_{ix}\Gamma^\varrho_{\varrho l} - \Gamma^l_{\varrho x}\Gamma^\varrho_{il}) + \sum_{p=1}^{d-1}(\Gamma^p_{ix}\Gamma^\varrho_{\varrho p} - \Gamma^p_{\varrho x}\Gamma^\varrho_{ip})$$

$$= \partial_\varrho(O(r) + O(|z|^2)) - \partial_i(O(r) + O(|z|^2) + \Gamma^\varrho_{\varrho x}(S^{d-1}))$$

$$+ O(r)(O(1) + O(|z|^2)) - (O(1) + O(|z|^2))O(r)$$

$$+ \sum_{l=1}^{n-d-1} \Big[(O(1) + O(|z|^2))(O(r) + O(|z|^2))$$

$$- \left(O\left(\frac{1}{r}\right) + O(|z|)\right) O(r^2)\Big]$$

$$+ \sum_{p=1}^{d-1} \big[(O(r) + O(|z|^2))(O(r) + O(|z|^2) + \Gamma^\varrho_{\varrho p}(S^{d-1}))$$

$$- \ (O(r) + O(|z|^2) + \Gamma^p_{\varrho x}(S^{d-1}))(O(r) + O(|z|^2))\big]$$

$$= O(r) + O(|z|) + O(r) + O(|z|^2) + 0 + O(r)$$

$$+ O(r)O(|z|^2) + O(r) + O(r)O(|z|^2)$$

$$+ \sum_{l=1}^{n-d-1} \big[O(r) + O(|z|^2) + O(r)O(|z|^2) + O(|z|^4)$$

$$+O(r) + O(r^2)O(|z|)\big] + \sum_{p=1}^{d-1} \big[O(r^2) + O(r)O(|z|^2)$$

$$+ O(r) + O(r)O(|z|^2) + O(|z|^4) + O(|z|^2) + O(r^2)$$

$$+O(r)O(|z|^2) + O(r)O(|z|^2) + O(|z|^4) + O(r) + O(|z|^2)\big]$$

$$= O(r) + O(|z|).$$

Then,

$$\mathrm{Ric}_{ix} = (O(1) + O(|z|^2) + r''O(r^3)) + \sum_{\xi=1}^{n-d-1} \big[O(1) + O(|z|^2)\big]$$

$$+\sum_{\varrho=1}^{d-1}[O(r)+O(|z|)]$$

$$=O(1)+O(|z|)+r''O(r^3).$$

Finally, $\mathrm{Ric}_{xv} = R^s_{xsv} + \sum_{\xi=1}^{n-d-1} R^{\xi}_{x\xi v} + \sum_{\varrho=1}^{d-1} R^{\varrho}_{x\varrho v}$.

$$R^s_{xsv} = \partial_s(\Gamma^s_{xv}) - \partial_x(\Gamma^s_{sv}) + \Gamma^s_{xv}\Gamma^s_{ss} - \Gamma^s_{sv}\Gamma^s_{xs}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^l_{xv}\Gamma^s_{sl} - \Gamma^l_{sv}\Gamma^s_{xl}) + \sum_{p=1}^{d-1}(\Gamma^p_{xv}\Gamma^s_{sp} - \Gamma^p_{sv}\Gamma^s_{xp})$$

$$=\partial_s(O(1)+O(|z|^2)) - \partial_x(O(1)+O(r)O(|z|))$$

$$+(O(1)+O(|z|^2))(r''O(r^2)+O(r))$$

$$-(O(1)+O(r)O(|z|))(O(1)+O(r)O(|z|))$$

$$+\sum_{l=1}^{n-d-1}\left[\left(O\left(\frac{1}{r}\right)+O(|z|)\right)O(r^2) - \left(O\left(\frac{1}{r}\right)+O(|z|^2)\right)O(r)\right]$$

$$+\sum_{p=1}^{d-1}\big[(O(r)+O(|z|^2)+\Gamma^p_{xv}(S^{d-1}))(O(1)+O(r)O(|z|^2))$$

$$-(O(1)+O(|z|^2))(O(1)+O(|z|^2))\big]$$

$$=O(1)+O(|z|^2)+O(1)+O(r)O(|z|)+O(r)+r''O(r^2)+O(r)$$

$$+r''O(r^2)O(|z|^2)+O(r)O(|z|^2)+O(1)+O(r)O(|z|)+O(r)O(|z|)$$

$$+O(r^2)O(|z|^2)+\sum_{l=1}^{n-d-1}\left[O(r)+O(r^2)O(|z|)+O(1)+O(r)O(|z|^2)\right]$$

$$+\sum_{p=1}^{d-1}\big[O(r)+O(r^2)O(|z|^2)+O(|z|^2)+O(r)O(|z|^4)+O(1)$$

$$+O(r)O(|z|^2)+O(1)+O(|z|^2)+O(|z|^2)+O(|z|^4)\big]$$

$$=O(1)+r''O(r^2)+O(|z|^2)+O(r)O(|z|).$$

$$R^{\xi}_{x\xi v}=\partial_\xi(\Gamma^{\xi}_{xv})-\partial_x(\Gamma^{\xi}_{\xi v})+\Gamma^{s}_{xv}\Gamma^{\xi}_{\xi s}-\Gamma^{s}_{\xi v}\Gamma^{\xi}_{xs}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^{l}_{xv}\Gamma^{\xi}_{\xi l}-\Gamma^{l}_{\xi v}\Gamma^{\xi}_{xl})+\sum_{p=1}^{d-1}(\Gamma^{p}_{xv}\Gamma^{\xi}_{\xi p}-\Gamma^{p}_{\xi v}\Gamma^{\xi}_{xp})$$

$$=\partial_\xi\left(O\left(\frac{1}{r}\right)+O(|z|)\right)-\partial_x(O(1)+O(|z|^2))$$

$$+(O(1)+O(|z|^2))\left(\left(\frac{r'}{r}\right)\delta_{\xi\xi}+O(r)\right)$$

$$-O(r)\left(O\left(\frac{1}{r}\right)+O(|z|^2)\right)$$

$$+\sum_{l=1}^{n-d-1}\left[\left(O\left(\frac{1}{r}\right)+O(|z|)\right)(O(r^2)+\Gamma^{\xi}_{\xi l}(S^{n-d-1}))\right.$$

$$\left.-(O(1)+O(|z|^2))(O(1)+O(|z|^2))\right]$$

$$+\sum_{p=1}^{d-1}\big[(O(r)+O(|z|^2)+\Gamma^{p}_{xv}(S^{d-1}))(O(1)+O(|z|^2))$$

$$\left.-(O(r)+O(|z|^2))\left(O\left(\frac{1}{r}\right)+O(|z|)\right)\right]$$

$$=O\left(\frac{1}{r}\right)+O(|z|)+O(1)+O(|z|)+r'O\left(\frac{1}{r}\right)+O(r)$$

$$+\left(\frac{r'}{r}\right)O(|z|^2)+O(r)O(|z|^2)+O(1)+O(r)O(|z|^2)$$

$$+\sum_{l=1}^{n-d-1}\Bigg[O(r)+O\left(\frac{1}{r}\right)+O(r^2)O(|z|)+O(|z|)+O(1)+O(|z|^2)$$

$$+O(|z|^2)+O(|z|^4)\Bigg]+\sum_{p=1}^{d-1}\Bigg[O(r)+O(r)O(|z|^2)+O(|z|^2)$$

$$+O(|z|^4)+O(1)+O(|z|^2)+O(1)+O(r)O(|z|)+O\left(\frac{1}{r}\right)O(|z|^2)$$

$$+O(|z|^3)\Bigg]$$

$$=O\left(\frac{1}{r}\right)+O(|z|).$$

$$R^{\varrho}_{x\varrho v}=\partial_{\varrho}(\Gamma^{\varrho}_{xv})-\partial_x(\Gamma^{\varrho}_{\varrho v})+\Gamma^{s}_{xv}\Gamma^{\varrho}_{\varrho s}-\Gamma^{s}_{\varrho v}\Gamma^{\varrho}_{xs}$$

$$+\sum_{l=1}^{n-d-1}(\Gamma^{l}_{xv}\Gamma^{\varrho}_{\varrho l}-\Gamma^{l}_{\varrho v}\Gamma^{\varrho}_{xl})+\sum_{p=1}^{d-1}(\Gamma^{p}_{xv}\Gamma^{\varrho}_{\varrho p}-\Gamma^{p}_{\varrho v}\Gamma^{\varrho}_{xp})$$

$$=\partial_{\varrho}(O(r)+O(|z|^2)+\Gamma^{\varrho}_{xv}(S^{d-1}))$$

$$-\partial_x(O(r)+O(|z|^2)+\Gamma^{\varrho}_{\varrho v}(S^{d-1}))$$

$$+(O(1)+O(|z|^2))(O(1)+O(|z|^2))$$

$$-(O(1)+O(|z|^2))(O(1)+O(|z|^2))$$

$$+\sum_{l=1}^{n-d-1}\left[\left(O\left(\frac{1}{r}\right)+O(|z|)\right)(O(r)+O(|z|^2))\right.$$

$$- \left(O\left(\frac{1}{r}\right) + O(|z|) \right) (O(r) + O(|z|^2)) \Big]$$

$$+ \sum_{p=1}^{d-1} \big[(O(r) + O(|z|^2) + \Gamma^p_{xv}(S^{d-1}))(O(r) + O(|z|^2) + \Gamma^\varrho_{\varrho p}(S^{d-1}))$$

$$- (O(r) + O(|z|^2) + \Gamma^p_{\varrho v}(S^{d-1}))(O(r) + O(|z|^2) + \Gamma^\varrho_{xp}(S^{d-1})) \big]$$

$$= O(r) + O(|z|) + \partial_\varrho(\Gamma^\varrho_{xv}(S^{d-1})) + O(r) + O(|z|) - \partial_x(\Gamma^\varrho_{\varrho v}(S^{d-1}))$$

$$+ O(1) + O(|z|^2) + O(|z|^2) + O(|z|^4) + O(1) + O(|z|^2) + O(|z|^2)$$

$$+ O(|z|^4) + \sum_{l=1}^{n-d-1} \Big[O(1) + O\left(\frac{1}{r}\right) O(|z|^2) + O(r)O(|z|) + O(|z|^3)$$

$$+ O(1) + O\left(\frac{1}{r}\right) O(|z|^2) + O(r)O(|z|) + O(|z|^3) \Big]$$

$$+ \sum_{p=1}^{d-1} \big[O(r^2) + O(r)O(|z|^2) + O(r) + O(r)O(|z|^2) + O(|z|^2)$$

$$+ O(r) + O(|z|^2) + \Gamma^p_{xv}(S^{d-1})\Gamma^\varrho_{\varrho p}(S^{d-1}) + O(r^2) + O(r)O(|z|^2)$$

$$+ O(r) + O(r)O(|z|^2) + O(|z|^2) + O(r) + O(|z|^2)$$

$$- \Gamma^p_{\varrho v}(S^{d-1})\Gamma^\varrho_{xp}(S^{d-1}) \big]$$

$$= O(1) + O(|z|) + O\left(\frac{1}{r}\right) O(|z|^2) + \partial_\varrho(\Gamma^\varrho_{xv}(S^{d-1})) - \partial_x(\Gamma^\varrho_{\varrho v}(S^{d-1}))$$

$$+ \sum_{p=1}^{d-1} \big[\Gamma^p_{xv}(S^{d-1})\Gamma^\varrho_{\varrho p}(S^{d-1}) \big] - \Gamma^p_{\varrho v}(S^{d-1})\Gamma^\varrho_{xp}(S^{d-1})$$

$$= O(1) + O(|z|) + O\left(\frac{1}{r}\right) O(|z|^2) + R^\varrho_{x\varrho v}(S^{d-1}).$$

Therefore,

$$\begin{aligned}
\mathrm{Ric}_{xv} &= (O(1) + r''O(r^2) + O(|z|^2) + O(r)O(|z|)) \\
&+ \sum_{\xi=1}^{n-d-1} \left[O\left(\frac{1}{r}\right) + O(|z|) \right] \\
&+ \sum_{\varrho=1}^{d-1} \left[O(1) + O(|z|) + O\left(\frac{1}{r}\right) O(|z|^2) + R^{\varrho}_{x\varrho v}(S^{d-1}) \right] \\
&= O\left(\frac{1}{r}\right) + O(|z|) + r''O(r^2) + \mathrm{Ric}_{xv}(S^{d-1}).
\end{aligned}$$

Altogether, we have:

Lemma 3.3.1. *The matrix Ric_{ab} is*

$$(Ric_{ab}) = \begin{pmatrix} Ric_{ss} & Ric_{si} & Ric_{sx} \\ Ric_{si} & Ric_{ij} & Ric_{ix} \\ Ric_{sx} & Ric_{ix} & Ric_{xv} \end{pmatrix},$$

where

- $Ric_{ss} = O(1) + O(|z|^2) + r''O(r) - (n-d-1)\left(\frac{r''}{r}\right);$
- $Ric_{si} = O(1) + r''O(r^2) + O(|z|^2) + \left(\frac{r'}{r}\right) O(|z|^2);$
- $Ric_{sx} = O\left(\frac{1}{r}\right) + O(|z|) + r''O(r);$
- $Ric_{ij} = O(r) + O(|z|^2) + r''O(r^3) + Ric_{ij}(S^{n-d-1})$

  $-(r')^2(n-d-2)(ds^2_{n-d-1})_{ij} - r''r(ds^2_{n-d-1})_{ij};$

- $Ric_{ix} = O(1) + O(|z|) + r''O(r^3)$;
- $Ric_{xv} = O\left(\frac{1}{r}\right) + O(|z|) + r''O(r^2) + Ric_{xv}(S^{d-1})$.

This is our Ricci tensor matrix with respect to $\{\partial/\partial s, \partial/\partial y_1, \ldots, \partial/\partial y_{n-d-1}, \partial/\partial z_1, \ldots, \partial/\partial z_{d-1}\}$. Similarly, as in the basic case, we are going to redefine it with respect to an approximately orthonormal basis $\{\partial/\partial s, 1/r \cdot \partial/\partial y_1, \ldots, 1/r \cdot \partial/\partial y_{n-d-1}, \partial/\partial z_1, \ldots, \partial/\partial z_{d-1}\}$. We will denote the new matrix by $\widetilde{\mathrm{Ric}}$.

Let $i, j \in \{1, \ldots, n-d-1\}$.

$$
\begin{aligned}
\mathrm{Ric}\left(\frac{1}{r}\cdot\left(\frac{\partial}{\partial y_i}\right), \frac{1}{r}\cdot\left(\frac{\partial}{\partial y_j}\right)\right) &= \left(\frac{1}{r^2}\right)\big(O(r) + r''O(r^3) + \mathrm{Ric}_{ij}(S^{n-d-1}) \\
&+ O(|z|^2) - (r')^2(n-d-2)(ds^2_{n-d-1})_{ij} \\
&- r''r(ds^2_{n-d-1})_{ij}\big) \\
&= O\left(\frac{1}{r}\right) + r''O(r) + \left(\frac{\mathrm{Ric}_{ij}(S^{n-d-1})}{r^2}\right) \\
&+ \left(\frac{O(|z|^2)}{r^2}\right) - (r')^2\left(\frac{n-d-2}{r^2}\right)(ds^2_{n-d-1})_{ij} \\
&- \left(\frac{r''r}{r^2}\right)(ds^2_{n-d-1})_{ij}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}\left(\frac{1}{r}\cdot\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial s}\right)\right) &= \left(\frac{1}{r}\right)\Big(O(1) + r''O(r^2) + O(|z|^2) \\
&+ \left(\frac{r'}{r}\right)O(|z|^2)\Big) \\
&= O\left(\frac{1}{r}\right) + r''O(r) + \left(\frac{1}{r}\right)O(|z|^2) \\
&+ \left(\frac{r'}{r^2}\right)O(|z|^2).
\end{aligned}
$$

Corollary 3.3.1.1. *The matrix* $(\widetilde{Ric_{ab}})$ *with respect to our approximately orthonormal basis at the center point of the coordinate system* ($z = 0$) *is*

$$\left(\begin{array}{c|c|c}
\begin{array}{c} O(1) + r''O(r) \\ \\ -(n-d-1)\left(\frac{r''}{r}\right) \end{array} & O\left(\frac{1}{r}\right) + r''O(r) & O\left(\frac{1}{r}\right) + r''O(r) \\
\hline
O\left(\frac{1}{r}\right) + r''O(r) & \begin{array}{c} O\left(\frac{1}{r}\right) + r''O(r) \\ +(n-d-2)\left(\frac{1-(r')^2}{r^2}\right)\delta_{ij} \\ -\left(\frac{r''}{r}\right)\delta_{ij} \end{array} & O(1) + r''O(r^3) \\
\hline
O\left(\frac{1}{r}\right) + r''O(r) & O(1) + r''O(r^3) & \begin{array}{c} O\left(\frac{1}{r}\right) + r''O(r^2) \\ \\ +Ric_{xv}(S^{d-1}) \end{array}
\end{array}\right).$$

In the basic case we had, for $r \approx 0$, a Ricci tensor matrix where the eigenvalues are approximately the diagonal entries, and we had one "bad" diagonal entry and $(n-1)$ "good" ones. We argued that the sum of any good diagonal entry and the bad one is strictly positive. We concluded that we had 2-positive Ricci curvature.

In the general case, we wish to claim $(d+1)$-positive Ricci curvature, i.e. the sum of the lowest $(d+1)$ eigenvalues should be strictly positive. The above Ricci tensor matrix from the general case has blocks with the following dimensions:

$$\begin{pmatrix}
1 \times 1 & 1 \times (n-d-1) & 1 \times (d-1) \\
(n-d-1) \times 1 & (n-d-1) \times (n-d-1) & (n-d-1) \times (d-1) \\
(d-1) \times 1 & (d-1) \times (n-d-1) & (d-1) \times (d-1)
\end{pmatrix}.$$

Altogether, we have (potentially at worst) $1 + (d-1) = d$ "bad terms". Provided that the diagonal terms dominate the off-diagonal terms for r small, and any term on the diagonal in the centre block dominates the sum of "bad terms" for r small, we will have $(d+1)$-positive Ricci curvature.

3.4 Proof of Theorem 3.0.1

Proof. In order to prove the main theorem, we need to show that the hypersurface H can be extended to enclose the cell σ^d in such a way that $\mathrm{Sc}_{d+1} > 0$.

In a similar manner to the basic case, we want to check the positivity of the sum of the lowest $d+1$ diagonal entries of the matrix $\widetilde{\mathrm{Ric}}$ for the extension we construct, while ensuring that the diagonal entries dominate the off-diagonal entries. In this way, we can conclude that the sum of the $d+1$ lowest eigenvalues of $\widetilde{\mathrm{Ric}}$ is positive, as required.

Notice that summing $d+1$ diagonal entries guarantees that we have taken at least a term from the central block of $\widetilde{\mathrm{Ric}}$. So, at worst the sum of $d+1$ diagonal entries is

$$-(n-d)\left(\frac{r''}{r}\right)+(n-d-2)\left(\frac{1-(r')^2}{r^2}\right)+(d-1)(d-2)+O\left(\frac{1}{r}\right)+r''O(r).$$

We are going to scale the whole matrix $\widetilde{\mathrm{Ric}}$ by r to deal with the off-diagonal $O(1/r)$ terms. So, $r\widetilde{\mathrm{Ric}}$ is

$$\left(\begin{array}{c|c|c}
\begin{array}{c} O(r)+r''O(r^2) \\ -(n-d-1)r'' \end{array} & O(1)+r''O(r^2) & O(1)+r''O(r^2) \\
\hline
O(1)+r''O(r^2) & \begin{array}{c} O(1)+r''O(r^2) \\ +(n-d-2)\left(\dfrac{1-(r')^2}{r}\right)\delta_{ij} \\ -r''\delta_{ij} \end{array} & O(r)+r''O(r^4) \\
\hline
O(1)+r''O(r^2) & O(r)+r''O(r^4) & \begin{array}{c} O(1)+r''O(r^3) \\ +r\mathrm{Ric}_{xv}(S^{d-1}) \end{array}
\end{array}\right).$$

Now, the new worst case sum of $d+1$ diagonal entries is

$$-(n-d)r'' + (n-d-2)\left(\frac{1-(r')^2}{r}\right) + (d-1)(d-2)r + O(1) + r''O(r^2).$$

Following the basic case format for constructing γ, we have $r'' \leqslant \sin^2\theta/4r_0$ for the initial bend, where r_0 is a very small height. As $r \leqslant r_0$ for this bend, so $r''O(r^2) = O(r)$, the last three terms of the new diagonal sum are $O(1)+r''O(r^2)+(d-1)(d-2)r = O(1)$. Therefore, it suffices to show that

$$-(n-d)\left(\frac{\sin^2\theta}{4r_0}\right) + (n-d-2)\left(\frac{1-(r')^2}{r}\right) \gg 0. \tag{3.2}$$

As in the basic case, $1-(r')^2 = \sin^2\theta$, so this gives

$$\sin^2\theta\left(\left(\frac{n-d-2}{r}\right) - \left(\frac{n-d}{4r_0}\right)\right) > 0 \tag{3.3}$$

$$\iff \left(\frac{n-d-2}{r}\right) > \left(\frac{n-d}{4r_0}\right)$$

$$\iff \left(\frac{4r_0}{r}\right) > \left(\frac{n-d}{n-d-2}\right)$$

$$\iff \left(\frac{4r_0}{r}\right) > 1 + \left(\frac{2}{n-d-2}\right).$$

Notice that $\sin^2\theta$ is bounded below away from 0. Now, clearly the right hand side is less or equal to 3. But since $r_0 \geqslant r$, then the left hand side is bigger or equal to 4. Therefore, the inequality holds. Moreover, we can make the left hand side of Equation 3.3 as large as we like by choosing r_0 sufficiently small.

As observed above, the off-diagonal blocks of $r\widetilde{\mathrm{Ric}}$ are all $O(1)$ terms, in contrast with the $O(1/r)$ terms from the unscaled matrix $\widetilde{\mathrm{Ric}}$. Therefore, because we can control the size of the left-hand side of Equation 3.3, the same argument as used in the basic case can now show that the sum of the lowest $d+1$ eigenvalues of $r\widetilde{\mathrm{Ric}}$, and consequently also of $\widetilde{\mathrm{Ric}}$, is positive if r_0 is sufficiently small.

For subsequent bends, the equivalent calculation involves replacing r_0 by r_1 (or r_i for $i > 1$), and also a higher value of $\theta \in [\theta_0, \pi/2]$. But this is equivalent of lowering the value of r_0 in the above calculation, which clearly only improves the sum of the lowest $d+1$ eigenvalues. We conclude that we can bend γ to the horizontal keeping $\mathrm{Sc}_{d+1} > 0$.

Finally, we need to make an uniform choice of γ for each $z \in S^{d-1}$. By compactness, we can make an uniform choice for r_0, which works for all z. This allows us to create the desired curve γ.

We define r_∞ to be the final height of γ (when horizontal). Then, the same argument as in the basic case shows that the t-length of γ can be made arbitrarily small, and in particular, we are free to assume that the length of each tube is shorter than δ, the parameter describing the time for which normal geodesics from $\sigma^d \cap H \subset S^{d-1}$ lie in σ^d. (Thus, the cross-sections of our tubes are always centered on points in σ^d).

Assuming δ is sufficiently small, the union of tubes over all $z \in S^{d-1}$ will not "cover" all of σ^d (in the sense that the normal projection to σ^d is not surjective). Therefore, we can complete the construction by joining the union of tubes with the radius r_∞ sphere bundle over the "non-covered" points of σ^d. Notice that this is a S^{n-d-1}-bundle joining with tubes which are themselves S^{n-d-1}-bundles over an interval. This is reasonable since we have in effect been viewing a neighborhood of the boundary in σ^d as being decomposed into normal geodesic line segments from the boundary.

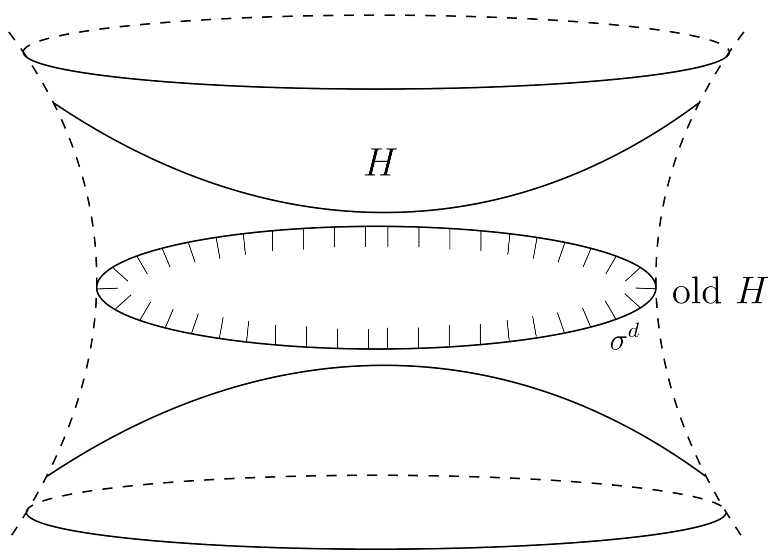

Figure 3.10: Final construction.

The new, modified hypersurface H^{n-1}, now agrees with the union of tubes/sphere bundle over the interior of σ^d, and with the previous hypersurface on the complement.

Repeating this procedure for each d-cell in the complex K and continuing the hypersurface extension inductively over higher dimensional cells completes the construction of ∂U

with $\mathrm{Sc}_{d+1} > 0$.

As argued in the basic case, by controlling the initial bend of γ we can ensure that the hypersurface extensions constructed for each d-cell are mutually disjoint.

$\square$

Chapter 4

Applications

The principal aim in this chapter is to establish Theorem B and Theorem C from the Introduction.

4.1 Plumbings and k-positive Ricci curvature

Plumbing is a procedure for joining disc bundles. The basic idea is as follows. Consider smooth disc bundles $D^n \to E_i \to B_i^n$ with $i = 1, 2$. Given any points $x_i \in B_i$, we consider closed neighbourhoods $C_i \subset B_i$ containing x_i, with $C_i \cong D^n$. Then the preimage of C_i under the bundle projection map $\pi_i : E_i \to B_i$ is diffeomorphic to $D^n \times C_i$. The idea is now to glue the bundles E_1 and E_2 by identifying the neighbourhoods $\pi_1^{-1}(C_1) \subset E_1$ with $\pi_2^{-1}(C_2) \subset E_2$ via a product diffeomorphism between $D^n \times C_1$ and $D^n \times C_2$ which switches the factors. (Precisely: given diffeomorphisms $\phi_i : C_i \to D^n$, we consider the product diffeomorphism $(\phi_2^{-1}, \phi_1) : D^n \times C_1 \to C_2 \times D^n$.) We can clearly iterate this process to glue several bundles together. (We could also use this technique to glue a bundle to itself, by choosing x_1, x_2 in the same base space. Moreover we could also glue bundles with different base and fibre dimensions, provided these dimensions match accordingly).

The result of plumbing disc bundles is clearly a new manifold with boundary. Let us consider the structure of such a manifold. The act of performing a single plumbing between E_1 and E_2 (as above) identifies the bases B_1 and B_2 at a single point. In the same way that a disc bundle retracts onto its base, the plumbed manifold clearly retracts onto this one-point union $B_1 \vee B_2$. More generally, a plumbed manifold retracts onto (and therefore has the same homotopy type as) a certain arrangement of the base spaces where each plumbing has resulted in a one-point gluing between the respective bases.

We can model such an arrangement of base spaces by a graph. Represent each base space by a node, and for each plumbing join the appropriate vertices by an arc. More generally, we can encode the construction of the entire plumbed manifold by such a graph, simply by letting each node represent the whole bundle. Notice that in the case where the base manifolds are all simply-connected, the fundamental group of the plumbed manifold is isomorphic to that of the graph.

The boundaries of plumbed manifolds are of particular interest. In many cases the topology of the boundary can be deduced from the gluing data and the bundles involved. For example, it is a classical result that if one plumbs together eight copies of the tangent disc bundle of S^{2n} according to the E_8 graph, the resulting boundary is an exotic sphere Σ^{4n-1}. (See for example [4, V.2].) In [34] it was shown that in fact every homotopy sphere which bounds a parallelisable manifold in dimensions $4n-1$ (with $n \geqslant 2$) can be obtained as the boundary of a certain explicit plumbing arrangement involving D^{2n}-bundles over S^{2n} plumbed according to a simply-connected graph.

This result about homotopy spheres raised the question of which manifolds in dimension $4n-1$ can be obtained as the boundary of a plumbing in the *classical sense* of plumbing disc bundles over S^{2n} according to a simply-connected graph. This question was resolved in [10], where it was shown that up to forming a connected sum with a homotopy sphere, the set of classical plumbing boundaries essentially coincides with the set of highly-connected manifolds. (Note that a manifold M^{4n-1} is said to be *highly-connected* if $\pi_0(M) = \pi_1(M) = \ldots = \pi_{2n-2}(M) = 0$). The precise statement is:

Theorem 4.1.1. *[10, Theorem C.] Let M^{4n-1} (for $n \geq 2$) be a $(2n-2)$-connected manifold. If $n \equiv 1 \bmod 4$, then further assume that M is $(2n-1)$-parallelisable. Then there exists a homotopy sphere Σ^{4n-1} such that $M\sharp\Sigma$ is the boundary of a plumbed manifold in the classical sense.*

As pointed out in [10], it follows for example that *all* highly connected manifolds arise as classical plumbing boundaries in dimensions 7 and 11.

From a geometric point of view, the above topological result is of geometric significance since by the main result in [35], all classical plumbing boundaries carry metrics of positive Ricci curvature. Hence the above theorem implies that essentially all highly connected manifolds (modulo forming a connected sum with a homotopy sphere) admit positive Ricci curvature metrics:

Theorem 4.1.2. *[10, Theorem A.] Let M^{4n-1} (for $n \geq 2$) be a $(2n-2)$-connected manifold. If $n \equiv 1 \bmod 4$, then further assume that M is $(2n-1)$-parallelisable. Then there exists a homotopy sphere Σ^{4n-1} such that $M\sharp\Sigma$ admits a metric of positive Ricci curvature.*

With this in mind the following questions seems natural: what can we say about the curvature properties of plumbing boundaries *away* from the classical situation?

Let us put some more detail into this question. In the classical situation, the bundles used in the plumbing all have even-dimensional spheres as base. What if we removed the even dimension restriction, and also removed the requirements that the bases be spheres and the graphs simply-connected? We then arrive at the following more refined question:

Question 4.1.3. *Given a graph G with m nodes (labelled $1, \ldots, m$), and D^n-bundles $E_1, \ldots, E_m$ with base spaces $M_1^n, \ldots, M_m^n$ respectively, let P^{2n} be the manifold obtained by placing E_i on node i and plumbing according to the graph G. Then does ∂P satisfy any kind of positive curvature condition?*

Let us recall that Theorem B stated in the Introduction provides an answer to this question, namely that the boundary ∂P admits a metric with $\mathrm{Sc}_{n+1} > 0$. Of course this leaves open the possibility that one might be able to do better. However given the degree of generality involved, this would seem to be a difficult task without a much better understanding of intermediate curvature conditions.

Proof of Theorem B. Consider the arrangement of base manifolds within P. As noted previously, this is the result of forming a one-point union between the relevant pair of base manifolds for each arc of the graph G. The key point here is that this arrangement is an embedded n-complex within P. Let us denote this complex by K. As in the Main Theorem, we can consider a regular neighbourhood U of K within P. The following observation is trivial, but crucial: $\partial P \cong \partial U$. Theorem B is now an immediate corollary of this observation together with Theorem A.

□

4.2 The fundamental group and 3-positive Ricci curvature

Recall from the Introduction the following Theorem of Carr:

Theorem 4.2.1. *[6, Corollary 2] Let π be a finitely presented group. Then, there exists a compact 4-manifold M of positive scalar curvature with $\pi_1(M) = \pi$.*

It is well known that the curvature of a manifold M can restrict the possible fundamental groups, $\pi_1(M)$. As noted in the Introduction, for M is closed we have:

- If M has $\mathrm{Ric} > 0$, then $\pi_1 M$ is finite;
- If M has $\mathrm{Ric} \geqslant 0$, then $\pi_1 M$ is virtually abelian, i.e. $\pi_1(M)$ has an abelian subgroup of finite index.

In the case of $\mathrm{Sc}_2 > 0$ we have the following conjecture due to Wolfson:

Conjecture 4.2.2. *[33] If M is a closed n-manifold that admits a metric with 2-positive Ricci curvature then its fundamental group, $\pi_1(M)$, is virtually free.*

Wolfson [33] also comments that "3-positive Ricci curvature imposes much weaker restrictions on $\pi_1(M)$ than 2-positive Ricci curvature".

Following Carr's strategy, Theorem A leads us to the following generalization of Carr's theorem:

Theorem 4.2.3. *Let π be a finitely presented group. Then, there exists a closed manifold M of any given dimension bigger or equal to 4 and 3-positive Ricci curvature ($Sc_3 > 0$) such that $\pi_1(M) = \pi$.*

We immediately obtain:

Corollary 4.2.3.1. *Sc_3 imposes no restrictions on $\pi_1(M)$.*

In particular this implies

Corollary 4.2.3.2. *The Wolfson Conjecture is false if 2-positive Ricci curvature is replaced by 3-positive Ricci curvature.*

Proof of Theorem C. Given a finitely presented group π, we begin by realizing π as the fundamental group of a CW-complex as follows. Firstly, fix a finite presentation of π. Consider a one-point union of circles, with one circle for each generator. This has an obvious CW complex structure. Next, for each relation, glue a 2-cell in the obvious way along its boundary to the circles (with appropriate direction) corresponding to the ordered list of generators in that relation. This clearly creates a finite 2-dimensional CW complex C with fundamental group π. (Note that in general, C will not be a regular CW complex).

By [18, Theorem 2C.5], every CW complex is homotopy equivalent to a simplicial complex of the same dimension. Thus there is a simplicial complex (or rather its realization) K such that $C \simeq K$. Moreover K can be chosen to be finite, given that C is finite. Clearly we have $\pi_1(K) \cong \pi$.

By [14, Theorem A], every countable and locally finite CW complex of dimension m can be embedded into $\mathbb{R}^{2m+1}$. In particular, this applies to the simplicial complex K. Let $\phi : K \to \mathbb{R}^5$ be such an an embedding. The finiteness of K means that whatever the embedding ϕ, we can assume that $\phi(K)$ is smoothly embedded, that is, that ϕ restricted to the interior of each simplex is a smooth embedding. (This follows from the classical Whitney Approximation Theorem, see for example [20, Theorem 6.21]: given a continuous embedding ϕ, fix the image of the 0-skeleton of K in $\mathbb{R}^5$, then apply the approximation Theorem one simplex at a time to the 1-skeleton, and to the 2-skeleton in turn to show that we can render ϕ smooth by arbitrarily small pertubations).

We can now apply Theorem A to the image $\phi(K)$, to deduce the existence of a regular neighbourhood U of $\phi(K)$ such that ∂U has $\mathrm{Sc}_3 > 0$. Moreover, the construction of U shows that we can choose U so small that the nearest point map $pr : \partial U \to \phi(K)$ is well-defined. It is clear that U itself is precisely the mapping cylinder of *pr*. In particular, we have $\pi_1(U) \cong \pi_1(K) \cong \pi$.

The Corollary will follow if we can show that $\pi_1(\partial U) \cong \pi$. In order to do this, we study the homotopy long exact sequence of the pair $(U, \partial U)$:

$$\ldots \to \pi_2(U, \partial U) \to \pi_1(\partial U) \to \pi_1(U) \to \pi_1(U, \partial U) \to \ldots.$$

Since U and K have the same homotopy type, by exactness, then $\pi_1(U) \cong \pi_1(K) \cong \pi$. Therefore, it suffices to show that $\pi_2(U, \partial U) \cong \pi_1(U, \partial U) \cong 0$.

Consider $\pi_2(U, \partial U)$. (The argument for $\pi_1(U, \partial U)$ is exactly the same, so will be omitted.) The elements of $\pi_2((U, \partial U)$ are homotopy classes of maps of pairs $(D^2, \partial D^2) \to (U, \partial U)$. By the Whitney Approximation Theorem we can assume that the map representing any given element of $\sigma \in \pi_2(U, \partial U)$ is smooth. For any such map f, by transversality, we can make an arbitrarily small deformation to f away from the boundary so that the image $f(D^2)$ and $\phi(K)$ are disjoint. (This follows, for example by applying the Transverse Homotopy Theorem in [17, page 70] to the interior of each simplex of $\phi(K)$ in turn).

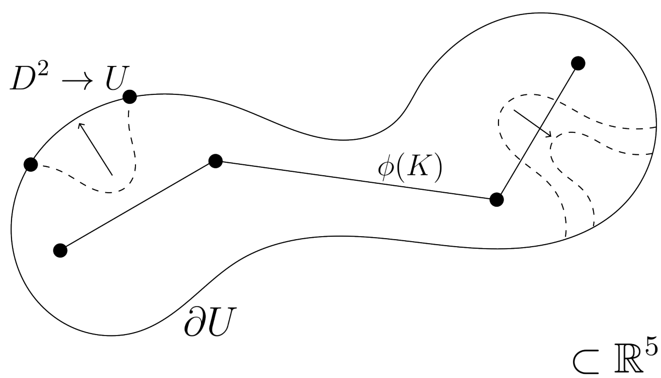

Figure 4.1: Representation of $\pi_2(U, \partial U)$ in $\mathbb{R}^5$.

In other words, we are free to assume that $f(D^2) \subset U \setminus \phi(K)$. But since U is the mapping cylinder of pr, we have $U \setminus \phi(K) \cong \partial U \times [0, 1)$, and in particular, ∂U is a deformation retract of $U \setminus \phi(K)$. Applying this retraction to f shows that f is homotopic as a map of pairs to a map $\tilde{f}$ with $\tilde{f}(S^2) \subset \partial U$. Hence $\sigma = 0$. As σ is an arbitrary element of $\pi_2(U, \partial U)$, it follows that $\pi_2(U, \partial U) = 0$, as desired.

$\square$

In the context of the above proof, notice that any free group π is the fundamental group of a 1-complex. In this case the proof shows that there exists a manifold M with 2-positive Ricci curvature and $\pi_1(M) = \pi$. This supports the Wolfson Conjecture.

Chapter 5

Contractibility and boundary convexity in positive Ricci curvature

Throughout this chapter, we will assume that M denotes a Riemannian manifold with boundary.

In [32], H.-H. Wang explores the connection between curvature, boundary convexity and topology. Theorem 1 of that paper proves that if M has $\mathrm{Ric} > 0$, the boundary is sufficiently highly convex and the sectional curvature at the boundary is non-negative, then M is contractible. In other words, if we want to have positive Ricci curvature (or some stronger curvature condition) and non-trivial topology, then the boundary cannot be too highly convex.

In order to state to state this theorem, we need some notation. Let

$$\Lambda(M^n) = \lambda \left(\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(S^{n-1})} \right)^{\frac{1}{n-1}},$$

where λ is the infimun of the eigenvalues of the boundary shape operator with respect to the outward normal.

Theorem 5.0.1 ([32, Theorem 1]). *For any integer $n \geqslant 4$, there exists $1 > \delta_n > 0$ with the following property. If M^n satisfies:*

1. *$Ric_M > 0$;*
2. *$\Lambda(M^n) > 1 - \delta_n$;*
3. *$K_M \geqslant 0$ at ∂M,*

then M^n is contractible.

Note that this is an existence result for δ_n only, and does not provide a means for estimating its value. The aim of this chapter is to establish an analogous contractibility result that gives an *explicit* estimate for the level of boundary convexity that will guarantee contractibility.

The proof of Wang's result hinges on a theorem about volume growth in Ric $\geqslant 0$ due to Perelman [24]. In this chapter we want to explore a different approach to study the connection between curvature, topology and boundary convexity by replacing the Perelman volume growth result [24] with ideas introduced by M. do Carmo and C. Xia in [12].

Let (M, g) be an n-dimensional compact Riemannian manifold with boundary and positive Ricci curvature. To this we add a collar $C = \partial M \times [0, \infty)$ along the boundary, to yield a non-compact manifold $\overline{M}$, i.e. $\overline{M} := M \cup C$. Assume we can smoothly extend the metric g to a complete metric on $\overline{M}$, which on the collar takes the form $dr^2 + f^2(r)g_{\partial M}$ for $r \geq r_0$, some $r_0 > 0$.

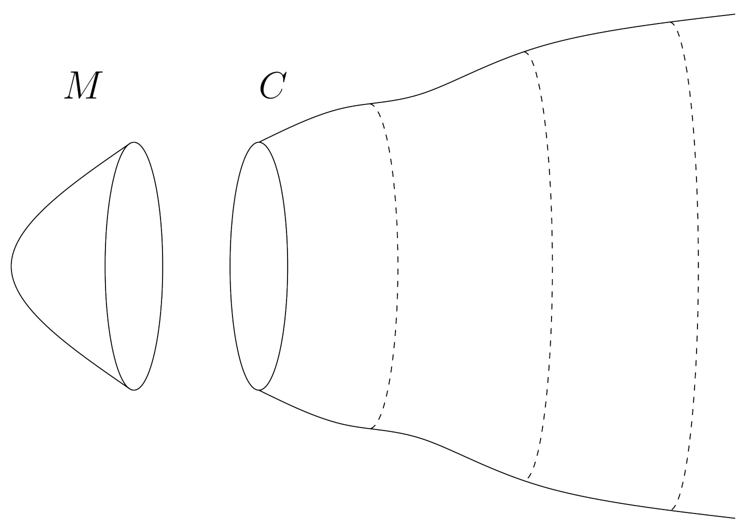

Figure 5.1: Riemannian manifold M with collar C.

A cross-section of the collar corresponds to any fixed value of r, which we will refer to as a "slice" of the collar. The volume of a slice is given by

$$\mathrm{Vol}(\mathrm{Slice}(r)) = \mathrm{Vol}(\partial M, g_{\partial M}) f^{n-1}(r).$$

So, the volume of the collar over $r \in [r_0, R]$ is

$$\mathrm{Vol}(\mathrm{Collar}) = \mathrm{Vol}(\partial M) \int_{r_0}^{R} f^{n-1}(r) dr.$$

In [12], M. do Carmo and Xia define the asymptotic volume α_N of a non-compact Riemannian manifold N by

$$\alpha_N = \lim_{r\to\infty} \frac{\mathrm{Vol}[B(p,r)]}{\omega_n r^n},$$

where $B(p,r)$ denotes the open geodesic ball around $p \in N$ with radius r, and ω_n denotes the volume of the unit ball in Euclidean space $\mathbb{R}^n$. Since α_N is independent of $p \in N$, it is a global geometric invariant which measures the volume growth. We say that N has large volume growth if $\alpha_N > 0$.

Similarly to α_N, we can define the volume growth in the collar. Let $C(R)$ be the part of the collar corresponding to $r \in [r_0, R]$. Then we set

$$\alpha = \lim_{R\to\infty} \mathrm{Vol}\ C(R) = \lim_{R\to\infty} \mathrm{Vol}(\partial M) \lim_{R\to\infty} \frac{\int_{r_0}^{R} f^{n-1}(r)dr}{\omega_n R^n}.$$

With the help of this quantity, we will prove a variant of Wang's Theorem [32] that will provide a more explicit estimate for a lower bound on Λ, which guarantees contractibility.

Lemma 5.0.2. *Let $f(r) = cr - be^{-dr}$ for $r \geqslant r_0 > 0$, and some $b, c, d > 0$. Then,*

$$\alpha = Vol(\partial M)\frac{c^{n-1}}{n\omega_n}.$$

Proof.

$$\int_{r_0}^{R} (cr - be^{-dr})^{n-1} dr = \int_{r_0}^{R} \Big(c^{n-1}r^{n-1} - (n-1)c^{n-2}r^{n-2}be^{-dr}$$

$$+ \frac{(n-1)(n-2)}{2}c^{n-3}r^{n-3}b^2e^{-2dr} + \dots$$

$$+(-1)^{n-1}b^{n-1}e^{-(n-1)dr}\Big)\, dr$$

$$= \frac{c^{n-1}R^n}{n} + \sum_{i=1}^{n-1} O(R^i).$$

We can focus on the first term in this sum since the rest of terms will tend to zero when we take the relative volume limit. Therefore,

$$\alpha = \mathrm{Vol}(\partial M) \lim_{R\to\infty} \frac{c^{n-1}R^n}{n\omega_n R^n} = \mathrm{Vol}(\partial M)\frac{c^{n-1}}{n\omega_n}.$$

□

Lemma 5.0.3. $\alpha = \alpha_{\bar{M}}$.

Proof. Let $p \in M$. For R large, $M \cup C(R - \mathrm{diam}(M)) \subseteq B(p, R - r_0) \subseteq M \cup C(R)$.

In order to see that $M \cup C(R - \mathrm{diam}(M)) \subseteq B(p, R - r_0)$, consider any point $y \in C(R - \mathrm{diam}(M))$, and let $x \in M$ and $z \in \partial M$.

Since $d(x, z)$ and $d(z, y)$ are at most $\mathrm{diam}(M)$ and $R - \mathrm{diam}(M) - r_0$ respectively, then,

$$d(x, y) \leqslant d(x, z) + d(z, y) = \mathrm{diam}(M) + R - \mathrm{diam}(M) - r_0 = R - r_0,$$

as we wanted.

Now, we claim

$$\lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R - \mathrm{diam}(M)))}{\omega_n R^n} = \lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R))}{\omega_n R^n}.$$

We can see that

$$\begin{aligned}\lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R - \mathrm{diam}(M)))}{\omega_n R^n} &= \lim_{R\to\infty} \frac{\mathrm{Vol}(M)}{\omega_n R^n} + \\ &+ \mathrm{Vol}(\partial M) \lim_{R\to\infty} \left(\frac{\int_{r_0}^{R-\mathrm{diam}(M)} f^{n-1}(r)dr}{\omega_n R^n} \right) \\ &= \mathrm{Vol}(\partial M) \frac{c^{n-1}}{n\omega_n} \\ &= \alpha.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R))}{\omega_n R^n} &= \lim_{R\to\infty} \frac{\mathrm{Vol}(M)}{\omega_n R^n} + \mathrm{Vol}(\partial M) \lim_{R\to\infty} \left(\frac{\int_{r_0}^{R} f^{n-1}(r)dr}{\omega_n R^n} \right) \\ &= \mathrm{Vol}(\partial M) \frac{c^{n-1}}{n\omega_n}\end{aligned}$$

$$= \alpha.$$

Therefore,

$$\lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R - \mathrm{diam}(M)))}{\omega_n R^n} = \lim_{R\to\infty} \frac{\mathrm{Vol}(M \cup C(R))}{\omega_n R^n}.$$

Equally,

$$\alpha_{\bar{M}} = \lim_{R\to\infty} \frac{\mathrm{Vol}B(p, R - r_0)}{\omega_n (R - r_0)^n}$$

$$= \lim_{R\to\infty} \Big[\frac{\mathrm{Vol}B(p, R - r_0)}{\omega_n R^n} \times \frac{R^n}{(R - r_0)^n}\Big]$$

$$= \lim_{R\to\infty} \frac{\mathrm{Vol}B(p, R - r_0)}{\omega_n R^n}.$$

It now follows inmediately from the Sandwich Theorem that $\alpha = \alpha_{\bar{M}}$ as claimed.

□

We now turn our attention to the curvature of $\bar{M}$.

Lemma 5.0.4. *Suppose that $K_M \geqslant 0$ at all points in ∂M, and that the principal curvatures at the boundary are all $\geqslant \lambda > 0$. Then assuming $n = \dim M \geqslant 3$, we have that $\nu := \min K_{\partial M}$ is strictly positive.*

Remark 5.0.5. Notice that for $K_{\partial M}$ to be positive, we need $\dim \partial M \geqslant 2$, and hence we need $n \geqslant 3$ in the above Lemma.

Proof. Let X, Y, Z, T be smooth vector fields tangent to ∂M. Then by [11, Proposition 3.1(a)],

$$\langle R_M(X,Y)Z,T\rangle = \langle R_{\partial M}(X,Y),Z,T\rangle - II(Y,T)II(X,Z) + II(X,T)II(Y,Z),$$

where II is the second fundamental form of ∂M with respect to the outward normal.

If X and Y are unit fields, $X \perp Y$, then

$$\langle R_M(X,Y)X,Y\rangle = K_M(X,Y) = K_{\partial M}(X,Y) - II(X,X)II(Y,Y) + II(X,Y)^2. \tag{5.1}$$

Since $K_M \geqslant 0$ at ∂M, we have that

$$K_{\partial M}(X,Y) \geqslant II(X,X)II(Y,Y) - II(X,Y)^2.$$

Now II is bilinear and symmetric, and the boundary convexity condition means that it is also positive definite. In other words, II in our situation is an inner product on each tangent space to the boundary. Thus, the Cauchy-Schwarz inequality applies, giving $II(X,X)II(Y,Y) \geqslant II(X,Y)^2$. Moreover we would have equality here if and only if X and Y were colinear, but since these vectors are orthogonal, we deduce that in our case we have $II(X,X)II(Y,Y) > II(X,Y)^2$. Thus $K_{\partial M} > 0$, and therefore by compactness there exists $\nu > 0$ such that $\nu = \min K_{\partial M}$.

□

The next result is a corollary of the proof above.

Lemma 5.0.6. *Under the hypotheses of the above Lemma, we have* $\lambda \geqslant \sqrt{\nu}$.

Proof. This is immediate from equation (5.1), since the right-hand side of that expression takes a minimum value $\lambda_1\lambda_2$ where $\lambda = \lambda_1 \leqslant \lambda_2$ are the two lowest eigenvalues. The justification of this algebraic fact is presented in the next lemma. □

Lemma 5.0.7. *The minimum value of the expression* $II(X,X)II(Y,Y) - II(X,Y)^2$, *where* X *and* Y *range over all pairs of orthonormal tangent vectors to* ∂M, *is* $\lambda_1\lambda_2$ *where* $\lambda = \lambda_1 \leqslant \lambda_2$ *are the two lowest eigenvalues of* II.

Proof. Let $\{v_1, \ldots, v_{n-1}\}$ be an orthonormal set of eigenvectors for II with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_{n-1}$. First note that the value $\lambda_1\lambda_2$ is achieved by setting $X = v_1$ and $Y = v_2$. More generally, let us write $X = \sum_i k_i v_i$ and $Y = \sum_j \ell_j v_j$. Then $II(X,X) = \sum_i \lambda_i k_i^2$, $II(Y,Y) = \sum_j \lambda_j \ell_j^2$, $II(X,Y) = \sum_i \lambda_i k_i \ell_i$ and therefore $II(X,Y)^2 = \sum_{ij} \lambda_i\lambda_j k_i\ell_i k_j \ell_j$. Thus we obtain

$$
\begin{aligned}
II(X,X)II(Y,Y) - II(X,Y)^2 &= (\sum_i \lambda_i k_i^2)(\sum_j \lambda_j \ell_j^2) - \sum_{ij} \lambda_i\lambda_j k_i\ell_i k_j\ell_j \\
&= \sum_{ij} \lambda_i\lambda_j k_i^2\ell_j^2 - \sum_{ij} \lambda_i\lambda_j k_i\ell_i k_j\ell_j \\
&= \sum_{ij} \lambda_i\lambda_j \big(k_i^2\ell_j^2 - k_i\ell_i k_j\ell_j\big) \\
&= \sum_i \lambda_i^2 (k_i^2\ell_i^2 - k_i^2\ell_i^2) + 2\sum_{i<j} \lambda_i\lambda_j \big(k_i^2\ell_j^2 + k_j^2\ell_i^2 - 2k_i\ell_i k_j\ell_j\big) \\
&= 2\sum_{i<j} \lambda_i\lambda_j (k_i\ell_j - k_j\ell_i)^2 \\
&\geqslant 2\lambda_1\lambda_2 \sum_{i<j} (k_i\ell_j - k_j\ell_i)^2,
\end{aligned}
$$

where the last line follows from the fact that all terms on the previous line are non-negative. Thus replacing all coefficients on the third line of the calculation above by $\lambda_1\lambda_2$ we obtain

$$II(X,X)II(Y,Y) - II(X,Y)^2 \geqslant \lambda_1\lambda_2 \sum_{ij} \big(k_i^2\ell_j^2 - k_i\ell_i k_j\ell_j\big). \qquad (*)$$

The orthonormality of X and Y imply that $\sum_i k_i^2 = \sum_j \ell_j^2 = 1$ and $\sum_i k_i\ell_i = 0$. We can use these facts to analyze the right-hand side of $(*)$ as follows. $\sum_{ij} k_i^2\ell_j^2 = (\sum_i k_i^2)(\sum_j \ell_j^2) = 1 \times 1 = 1$. Similarly $\sum_{ij} k_i\ell_i k_j\ell_j = (\sum_i k_i\ell_i)^2 = 0$. Therefore $(*)$ reduces to

$$II(X,X)II(Y,Y) - II(X,Y)^2 \geq \lambda_1\lambda_2(1-0)$$

$$= \lambda_1\lambda_2,$$

as required.

□

The next result allows us to control the curvature in the collar.

Proposition 5.0.8. *Assume that the Ricci curvature of (M,g) is positive, and the sectional curvature satisfies $K_M > -\kappa$ for some $\kappa > 0$, and also satisfies $K_M \geqslant 0$ at all points of ∂M. Assume further that the principal curvatures of ∂M are all $\geqslant \lambda > 0$, and set $\nu = \min K_{\partial M}$. Then for any function $f : [r_0, \infty) \to (0, \infty)$ satisfying $f(r_0) = 1$, $f''(r) < 0$ for all $r \geqslant r_0$, and $0 < f'(r_0)/f(r_0) < \min\{\lambda, \sqrt{\nu}\}$, the collar metric $dr^2 + f^2(r)g_{\partial M}$ has positive sectional curvature, and the metrics on $\bar{M} = M \cup C$ can be smoothed in a neighbourhood of $\partial M = \partial C$ (where ∂C corresponds to $r = r_0$) in such a way that $Ric_{\bar{M}} > 0$ and $K_{\bar{M}} > -\kappa$.*

Proof. Recall that the metric on the collar takes the form $dr^2 + f^2(r)g_{\partial M}$, for some positive function f. In order to investigate the sectional curvature of this metric, we begin by considering the curvature operator $\mathfrak{R}$.

Given an orthonormal basis $\{e_i\}$ for any tangent space, if $e_i \wedge e_j$ diagonalize $\mathfrak{R}$ with corresponding eigenvalues λ_{ij}, then according to [25, chapter 3, Proposition 1.1], the sectional curvatures at that point all lie in the interval $[\min \lambda_{ij}, \max \lambda_{ij}]$. If $\{e_i\}$ is in fact an orthonormal basis for $T_p\partial M$, then $\{\partial/\partial r\} \cup \{f^{-1}(r)e_i\}$ forms an orthonormal basis for $T_{r,p}(I \times \partial M)$ in the collar. Then adapting [25, chapter 3, Proposition 1.4] to our situation, we see easily that

$$\mathfrak{R}(f^{-1}e_i \wedge \partial/\partial r) = -\frac{f''}{f}(f^{-1}e_i \wedge \partial/\partial r);$$

$$\mathfrak{R}(f^{-1}e_i \wedge f^{-1}e_j) = \frac{1}{f^2}\big(K_{\partial M}(e_i,e_j) - f'^2\big)(f^{-1}e_i \wedge f^{-1}e_j).$$

We now observe that the vectors $\{f^{-1}e_i \wedge \partial/\partial r\}$ together with $\{f^{-1}e_i \wedge f^{-1}e_j\}$ form a complete set of eigenvectors for $\mathfrak{R}$, and hence by the result quoted above, we see that the sectional curvatures in the collar all lie in the range

$$\big[\min\{-f''/f, (K_{\partial M}(e_i, e_j) - f'^2)/f^2\}, \max\{-f''/f, (K_{\partial M}(e_i, e_j) - f'^2)/f^2\}\big]. \quad (5.2)$$

Suppose we choose $f : [r_0, \infty) \to (0, \infty)$ so that $f(r_0) = 1$, $0 < f'(r_0) < \min\{\lambda, \sqrt{\nu}\}$ and $f''(r) < 0$ for all $r \geqslant r_0$. Then f' is a decreasing function, and so the minimum in the expression (5.2) is positive for all $r \geqslant r_0$, i.e. the sectional curvature of the collar is everywhere strictly positive.

On the other hand, this collar metric will not in general give a smooth metric when glued to (M, g), and therefore we must smooth over the join. (Notice that the join is automatically C^0, but not in general C^1.) We claim that we can do this in such a way that the sectional curvature of $\bar{M} = M \cup C$ is globally $> -\kappa$.

In order to smooth the join, we employ a classic technique due to Perelman ([24]). Perelman only deals with the case of gluing manifolds of positive Ricci curvature with isometric boundaries $\partial N_1, \partial N_2$ which satisfy the boundary convexity condition $II_{\partial N_1} + II_{\partial N_2} > 0$. However the result also holds under stronger curvature conditions (subject to the same boundary requirements), and in particular gluing is possible preserving any given lower bound on the sectional curvature. See [26, Corollary B] (in the case $k = 1$). The claim now follows from the fact that $II_{\partial M} \geq \lambda$ and $II_{\partial C} = -f'(r_0)/f(r_0) > -\min\{\lambda, \sqrt{\nu}\}$, and thus by Lemma 5.0.6 we see that $II_{\partial M} + II_{\partial C} > 0$.

Finally, we note that both M and the collar have positive Ricci curvature, and the original Perelman result shows that this smoothing will preserve positive Ricci curvature.

□

Recall that for some $b, c, d > 0$ we want f to take the form $f(r) = cr - be^{-dr}$, with $r \geqslant r_0 > 0$. We will now make explicit choices for these constants, and demonstrate that the resulting function satisfies the requirements of Proposition 5.0.8.

Definition 5.0.1. *For any integer $m \geqslant 2$ (to be determined later), set*

$$f(r) := \lambda\Big(\frac{m-1}{m}\Big)r - \frac{1}{2m}e^{-r/r_0},$$

where

$$r_0 := \frac{m}{\lambda(m-1)}\Big(\frac{2me+1}{2me}\Big).$$

Lemma 5.0.9. *The function f defined above satisfies the requirements of Proposition 5.0.8.*

Proof. It is easily checked that $f(r_0) = 1$, as required. Equally, it is clear that $f''(r) < 0$ for all $r \geqslant r_0$. Consider then the first derivative requirement. We have

$$f'(r) = \lambda\Big(\frac{m-1}{m}\Big) + \frac{1}{2mr_0}e^{-r/r_0},$$

and so

$$\begin{aligned} f'(r_0) &= \lambda\Big(\frac{m-1}{m}\Big) + \lambda\Big(\frac{m-1}{m}\Big)\Big(\frac{2me}{2me+1}\Big)\frac{1}{2me} \\ &= \lambda\Big(\frac{m-1}{m}\Big)\Big(1 + \frac{1}{2me+1}\Big). \end{aligned}$$

To satisfy the convexity condition in Proposition 5.0.8 we need to check that the last expression above is $< \lambda$, which amounts to checking that

$$1 + \frac{1}{2me+1} < \frac{m}{m-1},$$

or equivalently that $2me > m - 2$, which is true since $2e > 4$ and $m > m - 2$.

□

Now consider the distance function $d_p(x) = d(p, x)$. This is smooth on $M \setminus \{p \cup C_p\}$, where C_p is the cut locus of p.

Definition 5.0.2. *The point $q \in M^n$ $(q \neq p)$ is a critical point of d_p if for all v in the tangent space T_qM, there is a minimal geodesic, γ, from q to p, making an angle, $\measuredangle(v, \gamma'(0)) \leqslant \pi/2$, with $\gamma'(0)$.*

From now on we just say that q is critical for p.

Suppose there exists $q \in M$ which is critical for p. Then, observe that the point q is still critical for p in $\overline{M}$.

Lemma 5.0.10. *Adding a collar does not introduce any new critical points for p.*

Proof. Recall that in the collar $C = \partial M \times [0, \infty]$, the metric is $dr^2 + f^2(r)g_{\partial M}$. This is a warped product manifold where r represents the distance along the collar direction.

We define ∂_r to be the unit vector field pointing outward along the collar, specifically, it points away from M.

Let $x \in C$ and $\gamma : [0, L] \to \bar{M}$ be a minimizing geodesic from $x = \gamma(0) \in C$ to $p = \gamma(L) \in M$. We want to prove that x is not critical for p.

Since $p \in M$ and $x \in C$, every geodesic from x to p must enter M in order to end at p, so in the collar, $\gamma'(t)$ must always have a component in the $-\partial_r$ direction, i.e. $\langle -\partial_r, \gamma'(0)\rangle > 0$. Therefore

$$\cos\sigma = \langle -\partial_r, \gamma'(0)\rangle > 0, \text{ i.e. } \sigma \in \left(0, \frac{\pi}{2}\right),$$

where $\sigma = \measuredangle(-\partial_r, \gamma'(0))$.

Setting $v = \partial_r$ in Definition 5.0.2, we see that for every minimal geodesic γ from x to p we have $\measuredangle(v, \gamma'(0)) > \pi/2$, and hence q cannot be critical for p.

□

Using ideas from the proof of [12, Theorem 2], we show that the existence of a critical point means that $\text{Vol}(B(p,r))$ is restricted for large r. This restricts the size of α for $\overline{M}$. But, on the other hand, the higher the boundary convexity Λ is for M, the larger α is. Therefore, if boundary convexity is too high, there cannot be a critical point.

Assuming the existence of a critical point q for p, our aim is to find an upper bound for α on $\bar{M}$ in terms of $d_p(q)$ and lower bound for the sectional curvature $-\kappa < 0$ on M. We will also need to assume that $\text{Ric} > 0$ on M, and that the sectional curvature K_M is non-negative at ∂M. This will lead to a maximum possible boundary convexity for M, beyond which critical points cannot exist. The absence of critical points then means that the manifold is contractible.

Lemma 5.0.11. *[15] Let q_1 be critical with respect to p and let q_2 satisfy $d(p,q_2) \geqslant \zeta d(p,q_1)$, for some $\nu > 1$. Let γ_1, γ_2 be minimal geodesics from p to q_1, q_2 respectively and put $\phi = \measuredangle(\gamma_1'(0), \gamma_2'(0))$. If $K_M \geqslant -\kappa^2$, ($\kappa > 0$) and $d(p,q_2) \leqslant \delta$, then*

$$\phi \geqslant \cos^{-1}\left(\frac{\tanh(\kappa\delta/\zeta)}{\tanh(\kappa\delta)}\right). \tag{5.3}$$

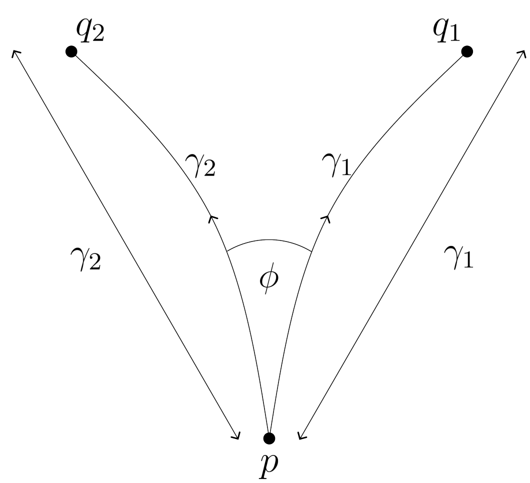

Figure 5.2: Minimal geodesics from p to q_1 and q_2.

Let $\delta_1 = d(p,q_1)$ and $\delta_2 = d(p,q_2)$. We can rewrite (5.3) as

$$\phi \geqslant \cos^{-1}\left(\frac{\tanh(\kappa\delta_1)}{\tanh(\kappa\delta_2)}\right).$$

Assuming $\kappa > 0$, as δ_2 increases, so does the lower bound for ϕ.

The following result about the non-existence of critical points will be crucial later.

Lemma 5.0.12. *[7] If there are no critical points for $p \in M$, then M is contractible.*

Let $p \in M^n$ and suppose there exists $q \in M$ which is critical for p. Let S_pM be the set of unit vectors in T_pM.

As in the proof of [12, Theorem 2], let Γ_{pq} be the set of unit vectors in T_pM corresponding to the set of normal minimal geodesics from p to q.

For any $\theta \in [0, \frac{\pi}{2}]$, let $\Gamma_{pq}(\theta) = \{u \in S_pM | \measuredangle(u, \Gamma_{pq}) \leqslant \theta\}$.

Note that Γ_{qp} is $\frac{\pi}{2}$-dense in S_qM, since q is critical for p.

Therefore, Γ_{pq} contains at least two distinct vectors and so there exists a constant $c > 0$ such that $\mathrm{Vol}(\Gamma_{pq}(\theta_0)) \geqslant v(\theta_0) + c = (1-\mu)c_{n-1} + c$, where $v(\theta_0)$ denotes the volume of a geodesic ball of radius θ_0 in an $(n-1)$-unit sphere, c_m is the volume of $S^m(1)$ and $\mu \in (1/2, 1)$.

Notice that

$$1 - \mu = \frac{v(\theta_0)}{c_{n-1}} = \frac{c_{n-2}\int_0^{\theta_0} \sin^{n-2} t dt}{c_{n-2}\int_0^{\pi} \sin^{n-2} t dt}.$$

We define r_1 to be the distance between p and q ($r_1 = d(p, q)$) along the minimal geodesic γ_1. For $u \in S_pM$ define the geodesic $\gamma_2 = exp_p(tu) \in \bar{M}$.

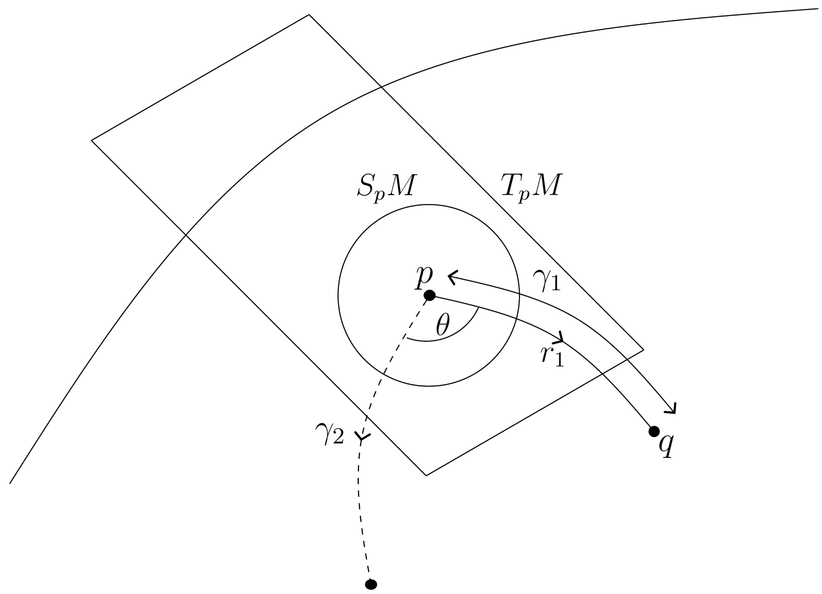

Figure 5.3: Diagram of the point p and geodesics γ_1 and γ_2.

For a geodesic γ_2 to "escape" to infinity in $\overline{M}$, and therefore to contribute to α_M, the angle of separation θ between γ_1 and γ_2 must satisfy

$$\theta \geqslant \cos^{-1}\left(\lim_{t\to\infty} \frac{\tanh(\kappa r_1)}{\tanh(\kappa d(p, \gamma_2(t)))}\right) = \cos^{-1}(\tanh(\kappa r_1)). \tag{5.4}$$

(Note that $\tanh(r)$ is increasing with limit 1.) Here we are using the fact that $K_{\bar{M}} > -\kappa$, as guaranteed by Proposition 5.0.8.

Short geodesics exist from p in all directions. If we are looking for minimal geodesics of length r_1 then

$$\theta \geqslant \cos^{-1}\left(\frac{\tanh(\kappa r_1)}{\tanh(\kappa r_1)}\right) = \cos^{-1}(1) = 0.$$

So, θ is unrestricted. But, if we look for longer minimal geodesics than γ_1, then we obtain positive lower bound on θ.

We now show that if there exists $q \in M$ critical for p, then we can compute an upper bound for volume growth in terms of κ and r_1, i.e. there exists $\alpha' = \alpha'(\kappa, r_1)$ such that $\alpha_{\bar{M}} \leqslant \alpha'$.

Lemma 5.0.13. *If there exists $q \in M$ critical for p, then there exists an upper bound $\alpha' = \alpha'(\kappa, \operatorname{diam}(M))$ for the asymptotic volume growth $\alpha_{\bar{M}}$ given by*

$$\alpha' = 1 - \frac{\int_0^{\cos^{-1}(\tanh(\kappa\,\mathrm{diam}(M)))} \sin^{n-2} t dt}{\int_0^{\pi} \sin^{n-2} t dt}.$$

Proof. Recall that the Bishop-Gromov volume comparison Theorem [1] says that for $\mathrm{Ric} \geqslant 0$, $\mathrm{Vol}[B(p,r)] \leqslant \omega_n r^n$. Then,

$$\alpha_{\bar{M}} = \lim_{r\to\infty} \frac{\mathrm{Vol}[B(p,r)]}{\omega_n r^n} \leqslant 1,$$

i.e. in $\mathrm{Ric} \geqslant 0$, $\alpha_{\bar{M}} \leqslant 1$.

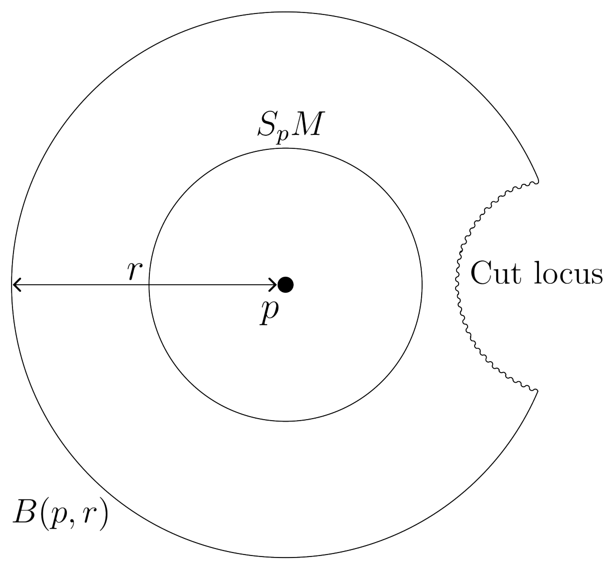

Figure 5.4: Cut locus.

We subdivide $B(p,r)$ as characterised in Figure 5.5, where the wavy line indicates the cut locus. Then,

$$\mathrm{Vol}[B(p,r)] = \mathrm{Vol}(Reg_1) + \mathrm{Vol}(Reg_2),$$

where $\mathrm{Vol}(Reg_1)$ is independent of r for r sufficiently large and $\mathrm{Vol}(Reg_2)$ depends on r.

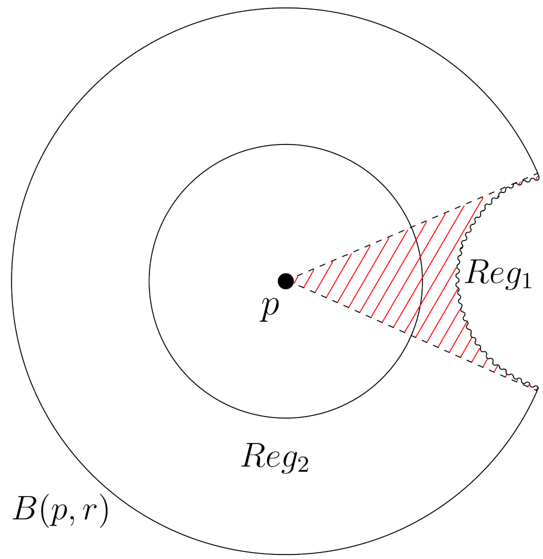

Figure 5.5: Division of $B(p,r)$ by Reg_1 and Reg_2.

Therefore, the largest possible asymptotic volume growth is

$$\alpha' = \lim_{r\to\infty} \frac{\mathrm{Vol}(Reg_1) + \mathrm{Vol}(Reg_2)}{\omega_n r^n} = \lim_{r\to\infty} \frac{\mathrm{Vol}(Reg_2)}{\omega_n r^n}.$$

Notice that the volume of $B(p,r)$ is at most $\omega_n r^n$ by Bishop-Gromov.

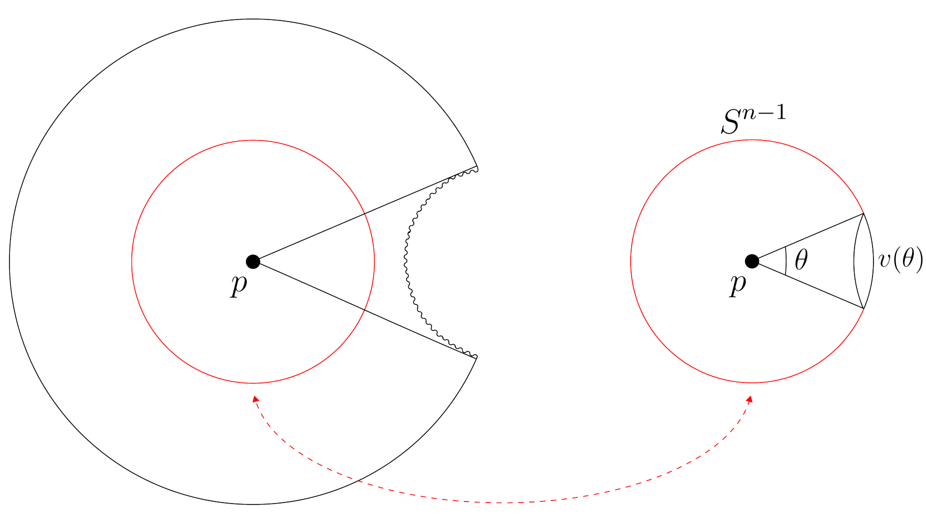

Figure 5.6: "Cone" region of S_pM.

Given a minimal geodesic from p to q with initial velocity vector $u \in S_pM$, consider the disc in S_pM consisting of the end points of those vectors based at p which make an angle of $< \theta_{min}$ with u, where θ_{min} is given by the right-hand side of Equation (5.4). This disc has volume $v(\theta_{\min})$. We can see that every geodesic whose initial vector lies in this "cone" region must hit cut locus and so does not contribute to α' (See Figure 5.6). Clearly,

$$\mathrm{Vol}(Reg_2(r)) \leqslant \frac{(c_{n-1} - v(\theta_{\min}))\omega_n r^n}{c_{n-1}}.$$

Therefore,

$$\alpha_{\bar{M}} \leqslant \lim_{r\to\infty} \frac{c_{n-1} - v(\theta_{\min})}{c_{n-1}} \frac{\omega_n r^n}{\omega_n r^n} = 1 - \frac{v(\theta_{\min})}{c_{n-1}}.$$

Explicitly, using Equation (5.4), we have

$$\frac{v(\theta_{\min})}{c_{n-1}} = \frac{\int_0^{\cos^{-1}(\tanh(\kappa r_1))} \sin^{n-2} t dt}{\int_0^{\pi} \sin^{n-2} t dt}.$$

Instead of $r_1 = d_p(q)$, we will consider $\mathrm{diam}(M) = \max_{x,y\in M} d(x,y)$. As $r_1 \leqslant \mathrm{diam}(M)$ combining the above expressions we obtain

$$\alpha_{\bar{M}} \leqslant 1 - \frac{\int_0^{\cos^{-1}(\tanh(\kappa\, \mathrm{diam}(M)))} \sin^{n-2} t dt}{\int_0^{\pi} \sin^{n-2} t dt} = \alpha'.$$

□

Corollary 5.0.13.1. *If $\alpha_{\bar{M}} > \alpha'$, then M is contractible.*

Proof. If $\alpha_{\bar{M}} > \alpha'$, then by Lemma (5.0.13), there do not exist any critical points for p. Hence, by Lemma (5.0.12), M is contractible.

□

Recall from Lemmas 5.0.2 and 5.0.3 that $\alpha_{\bar{M}} = \mathrm{Vol}(\partial M)\dfrac{c^{n-1}}{n\omega_n}$, and that by Lemma 5.0.13 the upper bound for the volume growth on $\bar{M}$ in the prescence of critical points is given by

$$\alpha' = 1 - \frac{\int_0^{cos^{-1}(tanh(\kappa\, \mathrm{diam}(M)))} sin^{n-2} t dt}{\int_0^{\pi} sin^{n-2} t dt}.$$

By Corollary (5.0.13.1), our original manifold with boundary is contractible if $\alpha_M > \alpha'$. By Definition 5.0.1, we see that this inequality amounts to

$$\frac{\mathrm{Vol}(\partial M)\lambda^{n-1}\left(\frac{m-1}{m}\right)^{n-1}}{n\omega_n} > 1 - \frac{\int_0^{cos^{-1}(tanh(\kappa\, \mathrm{diam}(M)))} sin^{n-2} t dt}{\int_0^{\pi} sin^{n-2} t dt}. \tag{5.5}$$

We are now in a position to state and prove our main theorem. Recall that Wang's convexity invariant Λ is defined by

$$\Lambda(M^n) = \lambda \left(\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(S^{n-1})} \right)^{\frac{1}{n-1}} = \lambda \left(\frac{\mathrm{Vol}(\partial M)}{n\omega_n} \right)^{\frac{1}{n-1}}.$$

Also recall that $n = \dim M \geqslant 3$, see Remark 5.0.5.

Theorem 5.0.14. *Let* $n \geqslant 3$ *and* $\psi = \dfrac{\int_0^{cos^{-1}(tanh(\kappa\,\mathrm{diam}(M)))} sin^{n-2}tdt}{\int_0^{\pi} sin^{n-2}tdt}$. *If* M^n *satisfies:*

1. $Ric \geqslant 0$;
2. $K_M \geqslant 0$ *at* ∂M;
3. $\Lambda > (1-\psi)^{\frac{1}{n-1}}$;

then M^n *is contractible.*

Proof. With ψ as in the statement of the Theorem, we can rewrite (5.5) as

$$\Lambda^{n-1} \left(\frac{m-1}{m} \right)^{n-1} > 1 - \psi.$$

But, we can make $\frac{m-1}{m}$ as close to 1 as we want by choosing m sufficiently large. Therefore, if $\Lambda^{n-1} > 1-\psi$ or equivalently $\Lambda > (1-\psi)^{\frac{1}{n-1}}$, then we can choose m sufficiently large so that $\Lambda^{n-1}\left(\frac{m-1}{m}\right)^{n-1} > 1-\psi$ is also true.

We conclude that if $\Lambda > (1-\psi)^{\frac{1}{n-1}}$, then the manifold is contractible. □

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