GROMOV-LAWSON CONCORDANCE IMPLIES ISOTOPY FOR POSITIVE INTERMEDIATE SCALAR CURVATURE

by

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A DISSERTATION

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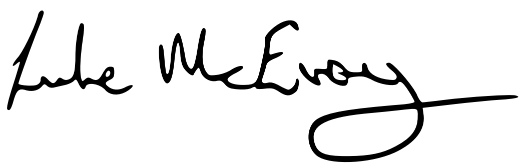
for Laura

Declaration

I hereby affirm that this thesis, submitted for evaluation as part of the Doctor of Philosophy program, is solely my own work. I have taken appropriate care to ensure its originality and, to the best of my knowledge, it does not violate any copyright laws. Any use of others' work has been properly cited and acknowledged within the text. The thesis work was conducted from October 2021 to September 2025 under the supervision of Dr. Mark Walsh, Department of Mathematics and Statistics, Maynooth University, and was funded by the Maynooth University John and Pat Hume Doctoral Scholarship.

Signed: 

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Gloria in excelsis Deo

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Abstract

This thesis generalises a result of M. Walsh by proving that Gromov–Lawson concordance implies isotopy in the space of Riemannian metrics with positive (p, n)-intermediate scalar curvature. We work in the setting of closed, simply-connected manifolds of dimension $n \geq 5$, equipped with Morse functions that satisfy suitable admissibility and cancellation conditions. Building on the surgery stability results of Gromov-Lawson and Labbi, we construct a relative isotopy between specific metrics and use it to prove that any Gromov–Lawson $(s_{p,n} > 0)$-concordant metrics are isotopic through positive (p, n)-intermediate scalar curvature metrics. The main result shows that for a closed, simply-connected manifold M of dimension at least 5, Gromov–Lawson concordance implies isotopy in $\mathcal{R}^{s_{p,n}>0}(M)$, thereby extending the known relationship between isotopy and concordance from the scalar curvature case to the intermediate curvature setting.

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Chapter 1

Introduction

GAUSS, in his 1828 work *General Remarks on Curved Surfaces* [10], studied the differential geometry of surfaces embedded in Euclidean space, notably distinguishing between the notions of *intrinsic* curvature and *extrinsic* curvature, where the latter is contingent on the nature of the embedding. The intrinsic notion, known as *Gaussian curvature*, is independent of any embedding in higher dimensional space. Given that this quantity pertains to surfaces, manifolds of dimension 2, it therefore naturally permits a generalisation to manifolds of dimension n. This generalisation was formalised by Bernhard Riemann[*], a student of Gauss, in his 1854 work *On the Hypotheses which Lie at the Foundations of Geometry* [39][†]. In this text, Riemann studied the intrinsic geometry of manifolds of some arbitrary dimension n. This more general curvature quantity, now referred to as *sectional curvature*, assigns to each 2-dimensional subspace of the tangent space at a point the Gaussian curvature of an associated locally specified 2-dimensional submanifold. Given therefore that each point on a manifold admits a multitude of sectional curvatures, it is natural to seek a derived quantity which assigns a single curvature value to a point. This quantity is known as *scalar curvature*, which is essentially an average of the collection of sectional curvatures at a point and can be realised as a scalar function

$$s : M \to \mathbb{R}.$$

In the case where $n = 2$, and the manifold is a surface, the sectional and scalar curvatures are essentially the same[‡]. However, as n grows, a large gap opens up, between the relatively weak scalar curvature and the much stronger sectional curvature, in the amount of geometric

[*]Felix Klein on page 231 of [18] notes that *"After a quiet preparation Riemann came forward like a bright meteor, only to be extinguished soon afterwards."*

[†]In the original German, *Über die Hypothesen, welche der Geometrie zu Grunde liegen.* We cite the English version, translated by William Kingdon Clifford.

[‡]In this case, the scalar is precisely twice the sectional.

information carried. There is another curvature notion lying between the two, the *Ricci* curvature, which we will discuss in the next chapter.

A central problem in this subject concerns the topological implications of placing certain constraints on the curvature of a Riemannian manifold. This is well understood in the classical $n = 2$ case. Here we have the famous Theorem of Gauss-Bonnet which states that for a compact oriented 2-dimensional Riemannian manifold M, we can express the *Euler characteristic*, $\chi(M)$ as

$$4\pi\chi(M) = \int_M s.$$

This implies that a closed orientable surface with an Euler characteristic of 0 does not admit a metric of positive, or negative, scalar curvature; the torus T^2 for example. Moreover, the sphere, S^2, which we know in its round form admits positive curvature, will not admit metrics of zero or non-positive curvature. Combining the Gauss-Bonnet Theorem with the Uniformisation Theorem, which guarantees that every closed surface admits a metric of constant curvature, a complete geometric classification of closed surfaces is known. That is, a closed surface admits positive curvature if and only if it is a sphere or a projective plane, zero curvature if and only if it is a torus or a Klein bottle and negative curvature if and only if it has genus greater than or equal to 2.

In higher dimensions, where these various curvature notions (sectional, Ricci and scalar) diverge in their relative geometric strengths, these classification problems become much more complicated. We will focus in this thesis on certain questions relating to positive curvature. In particular, we will study positivity for a collection of curvature notions, defined by Labbi in [22] which interpolate between the sectional and scalar curvatures. These are known as *intermediate scalar curvatures* and we will define them carefully in the next section. The motivation for studying these notions is the gap in geometric information captured by the scalar and sectional curvatures. This has meant that the kinds of techniques used in finding examples of metrics with positive curvature are radically different. Put simply, there are powerful techniques for constructing examples of positive scalar curvature metrics which do not work for the construction of positive sectional curvature metrics. One motivation for working with intermediary notions, is to help find the limit of applicability of these techniques and get a better sense of the difference in these curvatures.

Returning to sectional curvature for a moment, the problem of whether or not a given smooth closed manifold admits a Riemannian metric of positive sectional curvature is a very complicated and difficult one; see [14] and [57]. We will discuss this in more detail in the next chapter but it is important to understand that this condition places severe topological restrictions on the underlying manifold, for example forcing its fundamental group to be finite. In the case when the manifold does not display obvious topological obstructions to positive sectional curvature, finding examples of such metrics is an extremely challenging problem.

The condition of positive scalar curvature is much weaker. For example, any closed manifold of the form $M \times S^2$, where M is a compact manifold, admits a metric of positive scalar curvature, regardless of the topological complexity of M. Famously, the following theorem due to Gromov and Lawson and, independently, Schoen and Yau provides a powerful technique for the construction of positive scalar curvature metrics.

Theorem 1.0.1 (Gromov-Lawson [12]; Schoen-Yau [43])**.** *Let M be a compact manifold, and M' a manifold obtained from M by surgery in codimension at least* 3*. Then if M admits a metric of positive scalar curvature so too does M'.*

This theorem, and its various extensions which we will discuss in detail, massively increased the number of known examples of metrics of positive scalar curvature and lead, as we will discuss shortly, to an impressive classification result. There is no analogue of this theorem in the positive sectional curvature case.

Now we consider obstructions to positive scalar curvature. For example, the n-torus, $T^n = S^1 \times S^1 \times \cdots \times S^1$ admits no metric of positive scalar curvature; see [13]. A more subtle, and surprising, obstruction can be found among certain closed spin manifolds. The following result due to Schrödinger and Lichnerowicz shows that a particular topological index defined for closed manifolds with dimension a multiple of four, $\hat{A}$ (known as the *A-hat* genus)[§] acts as an obstruction to the existence of metrics of positive scalar curvature in the spin case.

Theorem 1.0.2 (Schrödinger; Lichnerowicz [28])**.** *Let M^{4k} be a closed, spin manifold which admits a metric of positive scalar curvature. Then $\hat{A}(M) = 0$.*

Remark 1.0.3. *A smooth orientable manifold is said to be spin if and only if its second Stiefel-Whitney class*[¶] *vanishes, which, for simply-connected manifolds of dimension at least* 5*, ensures that every embedded S^2 admits a trivial normal bundle. This fact has important implications regarding surgery on manifolds, a subject we will discuss later. We will not make any direct use of the notion of spin manifolds in this thesis; for further and more technical discussion regarding spin structures on manifolds we direct the reader to Chapter* 2 *of [25] and to the article [32].*

[§] Consult Chapter 2, Section 6, Example 6.3 of [25] for a formal definition.
[¶] See Section 4 of [34] for a definition.

The above theorem was furthered by Hitchin in [16], in which it was shown that if a manifold admits a metric of positive scalar curvature then its spin cobordism class lies in the kernel of an invariant denoted α, which roughly agrees with $\hat{A}$ in dimensions divisible by 4. The α-invariant, introduced by Atiyah and Singer, is a homomorphism from the spin cobordism ring to the real K-theory of a point, $\alpha : \Omega_*^{\text{spin}} \to KO_*$. For a closed spin Riemannian manifold M, this invariant can be understood as the index of the Dirac operator, a self-adjoint elliptic operator on the space of sections of the spinor bundle associated to M. The index, in its most basic form, is defined as the difference between the dimensions of the kernel and cokernel of this operator[‖], which is independent not just of the choice of Riemannian metric, but of the spin-bordism class of the manifold M. Following this result, a complete classification of simply-connected manifolds of dimension at least five admitting positive scalar curvature metrics was completed by Stolz (following significant work by Gromov and Lawson in [12]), in which it was shown that every spin cobordism class in the kernel of α has positive scalar curvature representatives.

Theorem 1.0.4 (Stolz [45])**.** *Let M be a compact simply connected manifold of dimension $n \geq 5$. Then M admits a metric of positive scalar curvature if and only if M is either not spin or M is spin with $\alpha([M]) = 0$.*

Thus, a non-vanishing α-invariant obstructs the existence of positive scalar curvature metrics on spin manifolds. For analogues of this question in the non simply-connected case, we direct the reader to [42], and to Chapter 3 of [56] for further study of obstruction theory in the context of positive scalar curvature.

1.1 Intermediate scalar curvatures

The intermediate scalar curvatures, defined by Labbi in [22][**], will be discussed in detail in the next chapter. It is worth giving a short description here however. Given that scalar curvature is an average of sectional curvatures, we may define partial averages. The (p,n)-intermediate scalar curvature of a Riemannian n-manifold denotes one of a collection of curvatures, $s_{p,n}$, $p \in \{0, 1, \cdots, n-2\}$ which interpolate between sectional curvature ($p = n-2$) and scalar curvature ($p = 0$). Each is defined as a real-valued function on the p-Grassmann bundle:

$$s_{p,n} : \mathrm{Gr}_p(M) \to \mathbb{R},$$

‖Both of which are always finite dimensional.

**In [22], Labbi ascribes this definition to Gromov.

and, at each point in $x \in M$, maps a p-plane $V_x \subset T_xM$ to the scalar curvature at x of the locally specified submanifold of M determined by applying the exponential map to the orthogonal complement of V_x. Thus, when $p = 0$ and $V_x = 0$, $V_x^{\perp} = T_xM$ and $s_{p,n}(V_x) = s(x)$, the scalar curvature at x. When V_x is an $(n-2)$-dimensional subspace of T_xM, we obtain the sectional curvature at x of the corresponding 2-dimensional orthogonal subspace.

In this spirit, we state a generalisation of the Gromov-Lawson-Schoen-Yau surgery result due to Labbi.

Theorem 1.1.1 (Labbi [22]). *Let M be a compact manifold of dimension n, and M' a manifold obtained from M by surgery in codimension at least $3+p$. Then if M admits a metric of positive (p, n)-intermediate scalar curvature so too does M'.*

In this context of (p, n)-intermediate scalar curvature, there is an analogue of the classification due to Gromov-Lawson and Stolz articulated in Theorem 1.0.4. This theorem, due to B. Botvinnik and M. Labbi, concerns manifolds which are spin but not *string*[††]:

Theorem 1.1.2 (Theorem A, [4]). *Let M be a compact, spin, 3-connected non-string manifold of dimension $n \geq 9$. Then M admits a Riemannian metric g with positive $(2, n)$-intermediate scalar curvature if and only if $\alpha([M]) = 0$.*

Although the focus of this work is on Labbi's intermediate scalar curvature, other notions of intermediate curvature are studied, and there exists a growing body of literature on the subject. Of particular relevance here is *k-Ricci curvature* defined by Wolfson in [53] and interpolating between the Ricci curvature (when $k = 1$) and the scalar curvature (when $k = n$). The techniques and results we will employ in this thesis have their analogues in the k-Ricci curvature context. It is important to point out that there is no simple direct correspondence between the two curvature notions in the sense of positivity of one implying positivity of the other[‡‡]. Moreover, given the relative weakness, and therefore flexibility, of the Ricci curvature over the sectional, and the fact that the Ricci curvature is tensorial, there are extra challenges in working with the intermediate scalar curvature.

1.2 Spaces of metrics, isotopy and concordance

Beyond the existence question for manifolds of positive scalar (or $s_{p,n}$) curvature, attention has shifted in recent years to a related question. Let $\mathcal{R}(M)$ denote the space of Riemannian

[††]A closed manifold is said to be *string* if it is spin and its fourth Stiefel-Whitney class vanishes.

[‡‡]Except in the obvious situations when these curvatures coincide with the scalar, Ricci and sectional curvatures.

metrics on a smooth n-manifold, M. This space has a natural C^∞ topology, defined carefully in chapter one of [56], and discussed in the next chapter. We consider the subspace[§§] $\mathcal{R}^+(M) \subset \mathcal{R}(M)$, of Riemannian metrics of positive scalar curvature on M.

Over the last few decades, a great deal of progress has been made in better understanding the topology of this space. Early results, like that of Carr in [6] for example, have shown that for many manifolds M, this space has multiple, even infinitely many, path components. We know that this space often has a great deal of non-trivial topology. For example, Botvinnik, Ebert and Randal-Williams show in [3], that when M is a closed spin manifold of dimension at least 5 admitting positive scalar curvature metrics, this space has infinitely many non-trivial higher homotopy groups.

In studying the topology of $\mathcal{R}^+(M)$, two notions of equivalence on this space are of particular importance. These are *isotopy* and *concordance*. Two metrics $g_0, g_1 \in \mathcal{R}^+(M)$ are said to be isotopic if there is a path in $\mathcal{R}^+(M)$ connecting them, and concordant if there is a positive scalar curvature metric $\bar{g}$ on the cylinder $M \times I$ ($I = [0,1]$) such that for some $\epsilon > 0$. $\bar{g}|_{M\times[0,\epsilon)} = g_0 + dt^2$ and $\bar{g}|_{M\times(1-\epsilon,1]} = g_1 + dt^2$. As we will discuss, it is well known that isotopic metrics in this space are concordant; see for example [12]. This is an enormously important fact when it comes to exhibiting distinct path components in $\mathcal{R}^+(M)$ and used extensively in the literature.

We now turn to the converse question: *are concordant metrics isotopic?* In the case of dimension 4, D. Ruberman has shown that concordance does not imply isotopy:

Theorem 1.2.1 (Theorem 5.3, [41])**.** *There are concordant, but not isotopic, metrics of positive scalar curvature on simply-connected* 4*–manifolds.*

More generally however, when $n \geq 5$, this question has been wide open for decades. It is considered formidably hard; see [40] for a discussion by Rosenberg and Stolz of this problem. An affirmative answer to this question would be hugely significant.

Under certain conditions, a partial affirmative result to the concordance-isotopy problem for positive scalar curvature exists. There is a particular type of concordance, which relates to the surgery techniques of Gromov and Lawson pioneered in [12], and known as Gromov-Lawson concordance. Essentially, it takes in a positive scalar curvature metric, g_0, on M and a certain kind of Morse function, $f : M \times I \to I$, deemed admissible, and returns a positive scalar curvature concordance on the cylinder $M \times I$, denoted $\bar{g} = \bar{g}(g_0, f)$. This admissibility condition on f essentially means that critical points of the Morse function correspond to surgeries which satisfy the codimension restriction of the Gromov-Lawson surgery technique. The following has been proven by Walsh.

[§§]This generalises to analogous subspaces of positive $s_{p,n} > 0$ metrics, which we will come to later.

Theorem 1.2.2 (Walsh [48]). *Let M be a closed simply-connected manifold of dimension $n \geq 5$ and let g_0 be a positive scalar curvature metric on M Suppose $\bar{g} = \bar{g}(g_0, f)$ is a Gromov-Lawson concordance with respect to g_0 and an admissible Morse function $f : M \times I \to I$. Then the metrics g_0 and $g_1 = \bar{g}|_{M \times \{1\}}$ are isotopic.*

The extent to which the set of Gromov-Lawson concordances "fills" the set of all concordances on a given smooth manifold M is unclear. In one sense, because of their relationship with surgery, the most powerful known technique for constructing new metrics of positive scalar curvature, Gromov-Lawson concordances are very important. It has been speculated that perhaps, in some sense, a general concordance might be approximated by Gromov-Lawson concordances, or that perhaps that Gromov-Lawson concordances are in some sense generic. However, this is not at all obvious, and may well be false. Indeed, as we will discuss later, an arbitrary concordance could be very complicated indeed.

1.3 Organisation and main results of the thesis

In this thesis we consider the collection of spaces $\mathcal{R}^{s_{p,n}>0}(M)$, of positive (p,n)-intermediate scalar curvature metrics on M, a smooth closed n-dimensional manifold. Provided p is not too close to $n-2$ (the case of sectional curvature), much of the topological non-triviality detected for $\mathcal{R}^+(M) = \mathcal{R}^{s_{0,n}>0}(M)$ has been found in these spaces. See in particular, a recent paper due to Kordass and Frenck, [19]. However, nothing had been done, to the best of our knowledge, on the concordance-isotopy problem in this more general context. This is the subject of the thesis.

After defining analogous notions of concordance and isotopy for $s_{p,n} > 0$ curvature metrics, our main goal is to generalise, as best we can, Theorem 1.2.2 to this setting. This is Main Theorem B below. Its proof involves demonstrating an explicit isotopy between two concordant metrics. Our main construction differs from that of Theorem 1.2.2 in two important respects. The first is that we must ensure that a stricter curvature condition, $s_{p,n} > 0$ curvature, is adhered to. This involves much more delicate and challenging calculations. Secondly, we make use of a notion of relative isotopy (see below) to provide a simpler construction (and hence proof) which subsumes that done in [48].

In Chapter 2, we carefully introduce various curvature notions and review relevant theory from differential topology and especially, surgery, cobordism and Morse Theory. In Chapters 3 and 4, we review a number of important metric examples, including so-called torpedo and mixed torpedo metrics, on the disc and sphere, and perform curvature calculations to determine under what conditions these metrics have $s_{p,n} > 0$ curvature.

These mixed torpedo metrics arise naturally in the surgery constructions we study. They are based on a well-known decomposition of the sphere S^n into a union of sphere-disc summands,

$$S^n \cong S^k \times D^{\ell+1} \cup_{S^k \times S^\ell} D^{k+1} \times S^\ell,$$

where $k + \ell + 1 = n$ and involve equipping these summands with certain standard product metrics which glue back smoothly together to obtain interesting metrics on S^n. A variation of these metrics on the disc D^n, and denoted $\bar{g}_{\mathrm{Mtor}}^{k,\ell+1}$, will be particularly important.

Chapter 5 makes use of these mixed torpedo metrics on the disc. In it we study a generalisation of the notion of isotopy for metrics on compact manifolds with boundary, namely a *relative isotopy* introduced in [52], translated to the context of (p, n)-intermediate scalar curvature. A relative isotopy is a path of metrics on a manifold with boundary, which fixes the metric on the boundary. The following technical result involving mixed torpedo metrics on the disc is, in a sense, the technical heart of the thesis.

Main Theorem A. *Let $n \geq 3$ and suppose k and ℓ satisfy $n = k + \ell + 1$ with $\ell \geq 2$. Then for all p satisfying $0 \leq p \leq \ell - 2$, the metrics $\bar{g}_{Mtor}^{k,\ell+1}$ and $\bar{g}_{Mtor}^{k+1,\ell}$ are $(s_{p,n+1} > 0)$ isotopic, relative to the metric $g_{Mtor}^{k,\ell}$ on S^n, in the space $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$.*

Main Theorem A, which is proven in Chapter 5 allows us to prove a restricted case of the main theorem in Chapter 7. We call this Lemma 7.1.1, and it concerns a "building block" case of a Gromov-Lawson concordance equipped with a Morse function which has exactly two "cancelling" critical points.

To prove our main theorem, Main Theorem B below, we make use of Morse-Smale theory, and certain deformations of a Morse function f, to move a general Gromov-Lawson concordance into one which is a union of simpler concordances of the "building block" case.

Main Theorem B. *Let M^n be a closed, simply-connected manifold of dimension $n \geq 5$. Let g_0 denote an $(s_{p,n} > 0)$-metric on M, $f : M \times I \to I$, a p-admissible Morse function and $\bar{g} = \bar{g}(g_0, f)$, the corresponding Gromov-Lawson concordance metric on $M \times I$. Then the metrics g_0 and $g_1 = \bar{g}|_{M\times\{1\}}$ are $(s_{p,n} > 0)$-isotopic.*

Chapter 2

Background

In this chapter we review a number of important results as well as some relevant background information in Riemannian Geometry and Differential Topology. We begin by establishing some notation concerning spheres and discs which we will make use of throughout this work.

As usual $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ and $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ denote the standard Euclidean n-dimensional sphere and disc. We will make use of various subspaces of these objects. Let $\mathbb{R}^n_+ := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ and $\mathbb{R}^n_- := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \leq 0\}$, we denote by S^n_+ and S^n_-, the upper and lower hemispheres defined:

$$S^n_\pm := S^n \cap \mathbb{R}^{n+1}_\pm.$$

These are canonically identified with the standard disc D^n via the maps

$$S^n_\pm \to D^n, (x_1, \cdots, x_n, x_{n+1}) \mapsto (x_1, \cdots, x_n).$$

We will occasionally make use of upper and lower half-discs, $D^n_\pm$, defined by:

$$D^n_\pm := D^n \cap \mathbb{R}^n_\pm.$$

2.1 Curvature

With a view to providing a definition for (p, n)-intermediate scalar curvature, we begin with a survey of preliminary curvature notions in Riemannian geometry. Let (M, g) be a Riemannian manifold and let ∇ denote the covariant derivative of the associated Levi-Civita connection. For preliminary information on the theory of smooth manifolds in general, we direct the reader to [26]. For discourse related to the Levi-Civita connection in particular, we direct the reader to Chapter 2 of [8] and Chapter 5 of [27]. Recall the *Riemann curvature endomorphism*, which measures the extent to which a manifold deviates from being Euclidean, measuring the non-commutativity of the covariant derivative.

Definition 2.1.1 (page 89, [8]). *The Riemann curvature endomorphism R of a Riemannian manifold M is a correspondence which associates to every pair $X, Y \in \Gamma TM$ a mapping $R(X,Y) : \Gamma TM \to \Gamma TM$ given by*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \tag{2.1}$$

where ΓTM denotes the space of all vector fields on M, $Z \in \Gamma TM$.

The Riemann curvature endomorphism R satisfies the following identity, known as the First Bianchi identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

This endomorphism in turn gives rise to the following definition:

Definition 2.1.2. *The Riemann curvature tensor is defined as the inner product*

$$R(X,Y,Z,W) = g(R(X,Y)Z, W),$$

where $X, Y, Z, W \in \Gamma TM$.

It is well known that the Riemann curvature tensor satisfies the following symmetries:

$$R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0; \tag{2.2}$$

$$R(X,Y,Z,W) = -R(Y,X,Z,W); \tag{2.3}$$

$$R(X,Y,Z,W) = -R(X,Y,W,Z); \tag{2.4}$$

$$R(X,Y,Z,W) = R(Z,W,X,Y). \tag{2.5}$$

In local coordinates $(x_1, \cdots, x_n)$, the covariant derivative is given by the expression

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k,$$

where $\partial_i := \frac{\partial}{\partial x_i}$ are co-ordinate vector fields and where each Γ_{ij}^k is a smooth scalar function, referred to as the *ijk-Christoffel symbol*, defined as

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{\ell} g^{kl} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_\ell g_{ij} \right), \tag{2.6}$$

where $g_{ij} = g(\partial_i, \partial_j)$ are the local co-ordinate functions of the metric g. The matrix of these g_{ij}, denoted (g_{ij}), admits an inverse, the elements of which are denoted g^{ij}. We express the Riemann curvature endomorphism in terms of local coordinates as

$$R\left(\partial_i, \partial_j\right) \partial_k = \sum_{\ell} R_{ijk}^{\ell} \partial_\ell,$$

where R_{ijk}^{ℓ} are the components of R in the local coordinate system, computed as

$$R_{ijk}^{s} = \sum_{\ell} \Gamma_{ik}^{\ell} \Gamma_{j\ell}^{s} - \sum_{\ell} \Gamma_{jk}^{\ell} \Gamma_{i\ell}^{s} + \partial_j \Gamma_{ik}^{s} - \partial_i \Gamma_{jk}^{s}.$$

We express the components of the Riemann curvature tensor in local coordinates as

$$R_{ijks} = \sum_{\ell} R_{ijk}^{\ell} g_{\ell s} = g(R(\partial_i, \partial_j)\partial_k, \partial_s), \tag{2.7}$$

where again we can express the right-hand side notationally as

$$g(R(\partial_i, \partial_j)\partial_k, \partial_s) = R(\partial_i, \partial_j, \partial_k, \partial_s).$$

Definition 2.1.3. *Let (M, g) be a Riemannian manifold and let $x \in M$. The exponential map at x, denoted*

$$\exp_x : T_x M \to M,$$

is defined by

$$\exp_x(v) = \gamma_v(1),$$

where $\gamma_v : [0, \epsilon) \to M$ is the unique geodesic with initial conditions $\gamma_v(0) = x$ and $\gamma_v'(0) = v \in T_x M$.

The point $\exp_x(v)$ is the point on the manifold M reached by following the uniquely specified geodesic starting at x in the direction v for unit time. The exponential map is smooth and, for a sufficiently small ball around the origin in $T_x M$, it is a diffeomorphism onto a neighbourhood of x in M.

We now introduce the commonly studied curvature notions derived from the Riemann Curvature tensor: the sectional, Ricci and scalar curvatures. The *sectional curvature* on a Riemannian manifold (M, g) associates to each pair (x, P), where $x \in M$ and P is a 2-dimensional subspace of the tangent space T_xM, the classical Gaussian curvature of the locally specified 2-dimensional submanifold in M obtained by applying the exponential map $\exp_x$ to P near the origin. A more computationally useful definition is given below.

Definition 2.1.4. *Let (M, g) be a Riemannian manifold of dimension at least 2, $x \in M$ and $P \subseteq T_xM$ a 2-dimensional subspace containing linearly independent vectors u and v. Then the sectional curvature of M at x for the plane P is*

$$K(P) = \frac{R(u, v, v, u)}{|u \wedge v|^2}, \tag{2.8}$$

where $|u \wedge v|^2 = |u|^2|v|^2 - g(u, v)^2$.

In local coordinates, we write $K(\partial_i, \partial_j) = K_{ij}$. Then, in the case when $|\partial_i \wedge \partial_j| = 1$ we have that $K_{ij} = R_{ijji}$, a quantity defined in equation (2.7). Importantly, the sectional curvature does not depend on the choice of vectors u and v, as shown in [8] page 94, Proposition 3.1.

Remark 2.1.5. *Strictly speaking, the vectors u and v need to be extended as vector fields locally, although the choice of extension makes no difference.*

We say that (M, g) has *positive sectional curvature* if for every point $x \in M$ and every two dimensional subspace $P \subset T_xM$, the sectional curvature $K(P) > 0$. Other curvature constraints such as negative or everywhere zero sectional curvature are defined analogously. Roughly speaking, the sectional curvature measures the extent to which geodesics emanating from a point converge or diverge. Positive sectional curvature, such as on a round sphere, causes geodesics to converge while negative sectional curvature causes divergence. In particular, as in the example of the round sphere, positive sectional curvature seems to limit topological complexity. The theorem of Bonnet-Myers for example implies that if the sectional curvature of (M, g) (assumed to be connected and complete) is bounded below by a positive constant, then M is compact and has finite fundamental group. Indeed, the number of manifolds known to admit metrics with positive sectional curvature is very small and the task of finding new examples is especially challenging; see [14] and [57].

The Ricci and scalar curvatures are progressively weaker curvature notions than the sectional in that they involve a partial and then complete averaging out of sectional curvatures. They are defined as follows.

Definition 2.1.6. *Let (M, g) be a Riemannian manifold. The* Ricci curvature tensor *is the $(0, 2)$-tensor defined as the trace of the Riemann curvature tensor:*

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\left(Z \mapsto R(Z, X)Y\right),$$

where R is the Riemann curvature tensor and X, Y, Z are vector fields on M. Let $(x_1, \cdots, x_n)$ be local coordinates near a point $x \in M$, with coordinate vector fields $(\partial_1, \cdots, \partial_n)$ chosen to be an orthonormal basis for T_xM. Interpreting the Ricci tensor as a quadratic form, the Ricci curvature at x in the direction ∂_j *is given by the formula*

$$Ric_x(\partial_j) = \sum_{i \neq j} g(R(\partial_i, \partial_j)\partial_j, \partial_i) = \sum_{i \neq j} R_{ijji} = \sum_{i \neq j} K_{ij}.$$

Definition 2.1.7. *The scalar curvature of a Riemannian manifold (M, g) is the real-valued function $s : M \to \mathbb{R}$ obtained as the trace of the Ricci tensor. In the above local coordinates around $x \in M$, it satisfies*

$$s(x) = \sum_{ij} R_{ijji} = \sum_{ij} K_{ij}. \tag{2.9}$$

Though weaker than the sectional curvature, the Ricci curvature and to a lesser extent the scalar curvature, still contain important geometric information especially relating to volume in (M, g). This is something we will return to shortly. Positivity of the Ricci and scalar curvatures are defined analogously to that of the sectional curvature with a Riemannian manifold (M, g) having *positive Ricci curvature* if and only if at every point $x \in M$, the form Ric_x is positive definite and *positive scalar curvature* if and only if the scalar function $s(x) > 0$ for all $x \in M$. Although a weaker notion than positive sectional curvature, positive Ricci curvature is still quite restrictive with respect to the topology on M. In particular, the Bonnet-Myers theorem holds here as well and, as in the case of sectional curvature, the existence of a positive Ricci curvature metric on M forces its fundamental group to be finite.

The scalar curvature is substantially weaker and, provided the manifold M has dimension at least four, positivity of the scalar curvature places no restrictions on the size of the fundamental group; see [6]. Moreover, the scalar curvature of a product metric is particularly well-behaved. In the case when (X, g_X) and (Y, g_Y) are Riemannian manifolds, the scalar curvature $s_{X \times Y}$ of the Riemannian product manifold $(X \times Y, g_X + g_Y)$ satisfies:

$$s_{X \times Y} = s_X + s_Y,$$

where s_X and s_Y are the respective scalar curvatures of (X, g_X) and (Y, g_Y). Since the round 2 dimensional sphere of radius r has scalar curvature equal to $\frac{2}{r^2}$, it follows that, provided M is compact, any manifold of the form $M \times S^2$ admits a metric of positive scalar curvature. Simply let g be any metric on M and equip $M \times S^2$ with the product $g + r^2 ds_2^2$, where ds_2^2 is the standard round metric on S^2 and r is chosen sufficiently small. As discussed in the introduction, despite its relative weakness, it is a surprising fact that there are manifolds for which no positive scalar curvature metric exists. Recall in particular that any closed spin manifold M, for which the α-invariant $\alpha(M)$ is non-zero, admits no metric of positive scalar curvature.

One of the motivations of this thesis is to better understand the differences between the sorts of topological constraints placed on a Riemannian manifold by strong conditions such as positive sectional curvature and weaker ones like positive scalar curvature. It is with this in mind that we proceed to define the (p, n)-intermediate scalar curvatures, defined by Labbi in [22]* and denoted $s_{p,n}$. This can be regarded as a collection of curvatures which interpolate between sectional curvature and scalar curvature. We denote by $\mathrm{Gr}_k(V^n)$ the Grassmann manifold of all k-dimensional subspaces of an n-dimensional vector space V. Let M^n be a smooth manifold. We denote the *k-Grassmann bundle* by $\mathrm{Gr}_k(M)$, obtained by taking the union of $\mathrm{Gr}_k(T_xM)$ over all $x \in M$. We formally define the quantity $s_{p,n}$ as follows:

Definition 2.1.8. *Let (M^n, g) be a Riemannian manifold, $x \in M$ and $P \subset T_xM$ a p-plane, $0 \leq p \leq n-2$. Let $(\partial_{p+1}, \cdots, \partial_n)$ denote any orthonormal basis for $P^\perp$. Then the (p, n)-intermediate scalar curvature of M at x with respect to P is given by*

$$s_{p,n}(x, P) = \sum_{i,j=p+1}^{n} K_x(\partial_i, \partial_j). \tag{2.10}$$

Considering all $x \in M$ and all p-planes $P \subset T_xM$, the (p, n)-intermediate scalar curvatures collectively give a map $s_{p,n} : \mathrm{Gr}_p(M) \to \mathbb{R}$. Note that $s_{p,n}$ is well-defined for any choice of orthonormal basis for $P^\perp$. It follows that the quantity $s_{p,n}(x, P)$ is in fact the scalar curvature at x of the metric induced on the locally specified $(n-p)$ dimensional submanifold determined by applying the exponential map of g at x to $P^\perp$ near the origin.

For the purpose of this text, we are interested in imposing the condition of positivity on $s_{p,n}$; we say that a Riemannian metric g on a smooth manifold M has *positive (p, n)-intermediate scalar curvature* if for any $x \in M$ and $P \subset T_xM$, the (p, n)-intermediate scalar curvature $s_{p,n}(x, P)$ is positive.

The intermediate scalar curvature, $s_{p,n}$, is a generalisation of scalar curvature in that it is a collection of curvatures interpolating between scalar and sectional curvature for $0 \leq p \leq n-2$. We provide three examples of $s_{p,n}$ for particular values of p to highlight this.

Example 2.1.9. *Letting $p = 0$, we have that P is a 0-plane. Thus $P^\perp = T_xM$, and the (p, n)-intermediate scalar curvature is*

$$s_{0,n}(\{0\}) = \sum_{i,j=1}^{n} K_x(\partial_i, \partial_j), \tag{2.11}$$

*In which it is referred to as p-curvature.

which is the scalar curvature of M at x.

Example 2.1.10. *Letting $p = n-2$, $P^\perp$ is a 2-dimensional subspace with basis $\{v, w\}$, and*

$$s_{n-2,n}(P) = K_x(v, w) + K_x(w, v) = 2K_x(P^\perp),$$

which is precisely twice the sectional curvature.

Example 2.1.11. *Letting $p = 1$ we have that $P = \langle \partial_k \rangle$, and therefore*

$$s_{1,n}(P) = \sum_{\substack{i,j=1 \\ i,j\neq k}}^{n} K_x(\partial_i, \partial_j) = \sum_{i,j=1}^{n} K_x(\partial_i, \partial_j) - \sum_{i=1}^{n} K_x(\partial_i, \partial_k) - \sum_{j=1}^{n} K_x(\partial_k, \partial_j) \tag{2.12}$$

$$= scal(x) - 2Ric_x(\partial_k), \tag{2.13}$$

which coincides with the Einstein curvature†*.*

Example 2.1.12. *If the sectional curvature is some constant c, then for all $0 \leq p \leq n-2$, we have that*

$$s_{p,n}(P) = \sum_{i,j=p+1}^{n} c = \sum_{\substack{i=p+1 \\ i\neq j}}^{n} \sum_{j=p+1}^{n} c$$

$$= \sum_{\substack{i=p+1 \\ i\neq j}}^{n} (n-p)c = (n-p-1)(n-p)c.$$

Importantly, we have the following relationship between intermediate scalar curvatures.

Proposition 2.1.13 ([22])**.** *If a Riemannian manifold (M, g) has positive (p, n)-intermediate scalar curvature, then it has positive (q, n)-intermediate scalar curvature for all non-negative $q \leq p$.*

Remark 2.1.14. *This notion of (p, n)-intermediate scalar curvature has been generalised to a quantity known as (p, q)-curvature, defined by M. Labbi in [24].*

According to Gray in [11] it is known that the scalar curvature can be related to the volume growth of geodesic balls in a Riemannian manifold M. In particular, at any point $x \in M$,

$$\frac{\mathrm{Vol}(B_M(x, \epsilon))}{\mathrm{Vol}(B_{\mathbb{E}^n}(\epsilon)} = 1 - \frac{s(x)}{6(n+2)}\epsilon^2 + O(r^4),$$

†See [23] for a detailed discussion on the definition and properties of this quantity.

where $\text{Vol}(B_M(x,\epsilon))$ denotes the volume of a ball of radius ϵ, centered at x, and $\text{Vol}(B_{\mathbb{E}^n}(\epsilon)$ denotes the volume of a ball of radius ϵ in n-dimensional Euclidean space. Furthermore, $s(x)$ denotes the scalar curvature, and the remaining $O(r^4)$ term includes higher order curvature terms. For sufficiently small $\epsilon > 0$, if the scalar curvature is positive, then the coefficient of ϵ^2 is negative, meaning that geodesic balls in manifolds of positive scalar curvature have less volume than their Euclidean counterparts. This can be interpreted as the volume-compression effect of positive scalar curvature, with such positively curved space contracting in comparison to Euclidean space.

Since the (p,n)-intermediate scalar curvature at $x \in M$ is the scalar curvature of the Riemannian submanifold $\exp_x(V)$, we can obtain the following similar volume growth expression for geodesic spheres, due to Labbi in [22].

$$\frac{\text{Vol}(S^{n-p-1}(x,\epsilon)))}{\text{Vol}(S^{n-p-1}_{\mathbb{R}^n}(\epsilon))} = 1 - \frac{s_{p,n}(x,P)}{6(n-p)}\epsilon^2 + \cdots ,$$

which, with some rearrangement, allows us to express the (p,n)-intermediate scalar curvature in terms of the volume-ratio:

$$s_{p,n}(x,P) = \lim_{\epsilon \to 0} \frac{6(n-p)}{\epsilon^2} \left(1 - \frac{\text{Vol}(S^{n-p-1}(x,\epsilon)))}{\text{Vol}(S^{n-p-1}_{\mathbb{R}^n}(\epsilon))} \right),$$

where $S^{n-p-1} = \{\exp_m(x); x \in P^\perp, ||x|| = \epsilon\}$, and $\epsilon > 0$ sufficiently small. From this expression we can see directly that the (p,n)-intermediate scalar curvature of a Riemannian manifold M is positive at $x \in M$ if and only if

$$\text{Vol}(S^{n-p-1}(x,\epsilon)) < \text{Vol}(S^{n-p-1}_{\mathbb{R}^n}(\epsilon)),$$

and negative if and only if

$$\text{Vol}(S^{n-p-1}(\epsilon)) > \text{Vol}(S^{n-p-1}_{\mathbb{R}^n}(\epsilon)).$$

As we mentioned in the introduction, as well as Labbi's intermediate scalar curvature, other notions of intermediate curvature are studied. One important example is the *k-Ricci curvature* defined by Wolfson in [53]. This is a smooth function on an n-dimensional Riemannian manifold (M,g), denoted Ric_k with $k \in \{1,\cdots,n\}$, defined at each point $x \in M$ as the sum of the k-smallest eigenvalues of the Ricci tensor at that point. Thus, it interpolates between the Ricci curvature (when $k=1$) and the scalar curvature (when $k=n$). As we said, there is no simple direct correspondence between the two curvature notions. However, the task of trying to extend techniques which apply in the positive scalar curvature context, but not the positive Ricci one, up this ladder of curvatures, applies here too.

2.2 Surgery and cobordism

Let X, Y be compact n-dimensional smooth manifolds. We say that X and Y are *cobordant* if there exists some $n+1$-dimensional manifold Z such that the boundary of Z is the disjoint union of X and Y, where Z is referred to as a *cobordism* between X and Y. Cobordism is an equivalence relation on compact manifolds of dimension n. This notion of equivalence is significantly coarser than homeomorphism or diffeomorphism. For example, consider removing a ball from the interior of a solid torus $S^1 \times D^2$ in $\mathbb{R}^3$. This results in a 3-dimensional manifold whose boundary is a disjoint union of the sphere S^2 and the torus T^2; see Figure 2.1. As a non-example in the context of surfaces, the sphere S^2 and the real-projective space $\mathbb{R}P^2$ are not cobordant as their Stiefel-Whitney numbers (which are a cobordism invariant) differ; see Chapter 4 of [34].

Definition 2.2.1. *An* $(n+1)$*-dimensional cobordism* $\{W; W_0, W_1\}$ *is an* $(n+1)$*-dimensional compact manifold,* W*, with boundary* $\partial W = W_0 \sqcup W_1$*, the disjoint union of closed* n*-dimensional manifolds* W_0 *and* W_1*.*

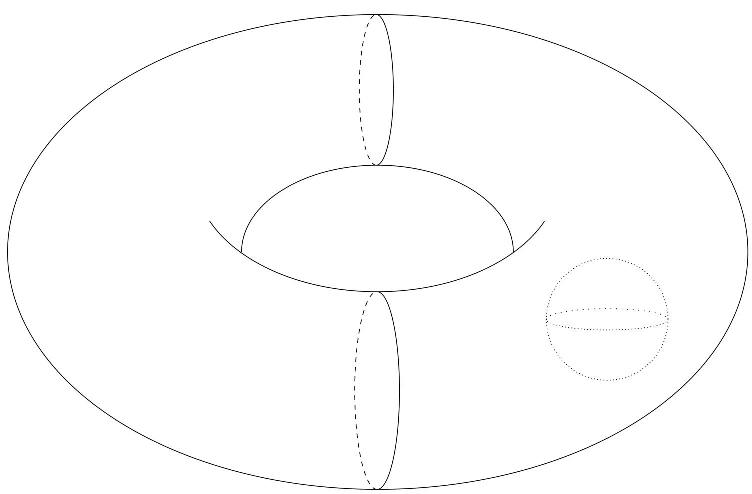


Figure 2.1: A cobordism between T^2 and S^2.

Related to cobordism is a technique for topologically modifying manifolds called *surgery*‡. Let M be a smooth oriented manifold of dimension $n = k + \ell + 1$ and consider a smooth, orientation preserving embedding $\phi : S^k \times D^{\ell+1} \hookrightarrow M$. A *surgery with respect to* ϕ is the

‡In the past, this notion was sometimes referred to as *χ-equivalence* ([30]), and *spherical-modification* ([47]).

procedure of forming a new manifold by removing the interior of $\phi(S^k \times D^{\ell+1})$ from M and attaching $D^{k+1} \times S^\ell$ along the common boundary $S^k \times S^\ell$, using $\phi|_{S^k \times S^\ell}$ as a "gluing map" (followed by some elementary smoothing at the attachment). In particular, this is an example of a dimension k-surgery or alternatively a surgery of codimension $\ell + 1$. This produces the new manifold:

$$M_\phi = (M - \text{int}\phi(S^k \times D^{\ell+1})) \cup_{\phi|_{S^k \times S^\ell}} (D^{k+1} \times S^\ell).$$

The manifold M_ϕ, and any manifold diffeomorphic to it, are said to be *surgery equivalent* to M. The notion of surgery equivalence plays a central role in classifying manifolds and studying their topological and geometric properties by systematically altering their structure through controlled modifications. We direct the reader to the paper [30] for discourse on surgery as a means to simplifying the homotopy groups of a manifold, and to the book [46] for discourse on surgery as a method by which one can study the homotopy types of manifolds.

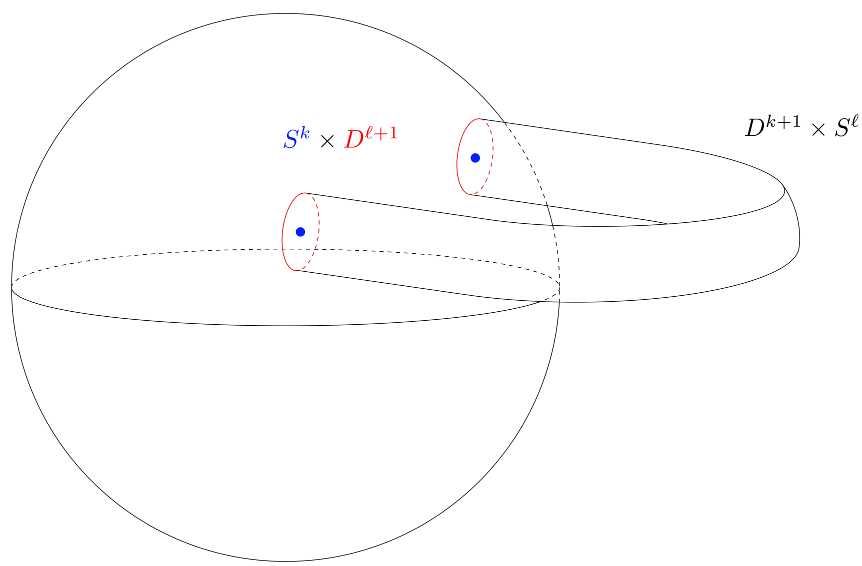


Figure 2.2: A surgery on S^n.

The following notion, called the trace of a surgery helps establish the link with cobordism.

Definition 2.2.2. *Let M^n be a smooth manifold of dimension $n = k + \ell + 1$ and $\phi : S^k \times D^{\ell+1} \hookrightarrow M$ a smooth, orientation preserving embedding. The trace of the surgery on ϕ, denoted $\bar{M}_\phi$ is the $(n+1)$-dimensional manifold*

$$\bar{M}_\phi := (M \times I) \cup_{\phi \times \{1\}} (D^{k+1} \times D^{\ell+1}),$$

where the map $\phi \times \{1\} : S^k \times D^{\ell+1} \hookrightarrow M \times \{1\}$ attaches the boundary component $(\partial D^{k+1}) \times D^{\ell+1}$ of the "handle" $D^{k+1} \times D^{\ell+1}$ to the cylinder $M \times I$ (followed by elementary smoothing). The *trace of* ϕ is a cobordism, $\{\bar{M}_\phi; M, M_\phi\}$, between M and M_ϕ, known as an *elementary cobordism.* A theorem due to Milnor and Wallace, known as the *Handle Presentation Theorem* (Chapter 7, [20]), states that any cobordism can be realised as a union of elementary cobordisms. This is achieved using Morse Theory, a subject we will discuss shortly.

At this stage, we should reintroduce some geometry. The technique of surgery offers a powerful method for constructing new manifolds with specific geometric properties, including those with positive scalar curvature under appropriate circumstances. In particular, it was utilised in a construction for *positive scalar curvature* metrics by Gromov-Lawson and, independently, Schoen-Yau by way of the following result regarding the stability of positive scalar curvature under surgery:

Theorem 2.2.3 ([12], [43])**.** *Let M be a compact manifold and M' a manifold obtained from M by surgeries in codimension at least* 3*. Then if M carries a Riemannian metric of positive scalar curvature, so does M'.*

This theorem asserts that if a manifold M possesses a Riemannian metric with positive scalar curvature, performing a surgery on M in codimension at least 3 results in a new manifold M' which also admits a Riemannian metric with positive scalar curvature, therefore showing that the condition of positive scalar curvature is "stable" under surgeries in codimension ≥ 3. Importantly, in proving this result, the authors provide an explicit construction, i.e. given a psc-metric g on M, the authors construct a psc-metric g' on the manifold M' obtained from the codimension ≥ 3 surgery.

Remark 2.2.4. *The codimension condition in this theorem is essential: consider the surgery in codimension* 2 *on S^2 which produces the torus T^2. The sphere admits a metric of positive scalar curvature, while T^2, as a consequence of the Gauss-Bonnet Theorem, does not.*

This result was later strengthened in [9] and [48] to show that a positive scalar curvature metric can be extended over the trace of a surgery to obtain a metric of positive scalar curvature which is a product metric near the boundary.

Theorem 2.2.5 ([9],[48])**.** *Let M be a Riemannian manifold equipped with a metric g of positive scalar curvature, and W the trace of a surgery on M in codimension at least* 3*. Then we can extend the metric g on M to a metric $\bar{g}$ on W which has positive scalar curvature and is a product near the boundary.*

The surgery theorem due to Gromov-Lawson, Schoen-Yau was generalised to the case of positive (p, n)-intermediate scalar curvature metrics by Labbi in [22], in which he shows that (p, n)-intermediate scalar curvature experiences similar surgery stability, albeit with an altered codimension condition which is a function of the index p.

Theorem 2.2.6 ([22])**.** *Let M be a compact n-dimensional manifold and M' a manifold obtained from M by surgeries in codimension at least $3 + p$, where $0 \leq p \leq n - 2$. Then if M admits a Riemannian metric of positive (p, n)-intermediate scalar curvature, so does M'.*

Regarding other *intermediate* curvatures, alongside Labbi's theorem in 2.2.6, we also have a result due to Wolfson in [53] which proves an analogous stability result for k-positive Ricci curvature. A more general result subsuming those of Labbi and Wolfson was later proved by Hoelzel in Theorem A of [17].

The geometric trace construction, Theorem 2.2.5, was extended to the (p, n)-intermediate scalar curvature case by M. Walsh, M. Burkemper and C. Searle in Theorem A of [5], which allows for the extension of $(s_{p,n} > 0)$-metrics over the trace of a codimension $\geq 3+p$ surgery to $(s_{p,n} > 0)$-metrics with a product structure near the boundary.

Theorem 2.2.7 (Theorem 6.3, [5])**.** *Let (M, g) be a smooth n-dimensional manifold, $\phi : S^k \times D^{\ell+1} \hookrightarrow M$ a smooth embedding, and $\{\bar{M}_\phi; M, M_\phi\}$ the trace of the surgery ϕ. Suppose furthermore that $n = k + \ell + 1$ and $n - k \geq 3 + p$. Then, if $g \in \mathcal{R}^{s_{p,n}>0}(M)$, there exist metrics g_ϕ on M_ϕ and $\bar{g}_\phi$ on $\bar{M}_\phi$ such that*

(i) $g_\phi \in \mathcal{R}^{s_{p,n}>0}(M_\phi)$, $\bar{g}_\phi \in \mathcal{R}^{s_{p,n+1}>0}(\bar{M}_\phi)$, and

(ii) near the boundary manifolds M and M_ϕ, $\bar{g}_\phi = g + dt^2$ and $\bar{g}_\phi = g_\phi + dt^2$ respectively.

There is no full analogue of the Gromov-Lawson surgery theorem for metrics of positive Ricci curvature, due to the fact that the positive Ricci curvature condition is more rigid and less stable under surgery than the positive scalar curvature condition. A restricted analogue of the Gromov-Lawson surgery theorem for positive Ricci curvature has been proven by Sha and Yang in [44], and strengthened by Wraith in [54]. The Ricci positive surgery theorem was furthered strengthened by Reiser in [38]. See Section 3 [55] for discourse on the circumstances under which surgery preserves Ricci positivity.

2.3 Isotopy and concordance

Let M be an n-dimensional smooth compact manifold and let $\mathcal{R}(M)$ denote the space of all Riemannian metrics on M. There is a natural topology, the smooth topology, on the space

$\mathcal{R}(M)$, induced by a family of seminorms and giving it the structure of a Fréchet manifold. Thus, locally $\mathcal{R}(M)$ looks like an infinite dimensional Banach space; see Chapter 1 of [56] for details. While the topology of $\mathcal{R}(M)$ is not so interesting (it is convex), there are various subspaces of metrics satisfying some or other curvature constraint which are topologically non-trivial. The most well-studied of these is the space of metrics of positive scalar curvature on M, $\mathcal{R}^{s>0}(M)$ and a good deal of progress has been made over the last twenty years in better understanding the topology of this space. We know from results of Botvinnik, Ebert and Randal-Williams in [3] for example, that in the case when M is a closed spin manifold of dimension at least five and admitting metrics of positive scalar curvature, this space has infinitely many non-trivial higher homotopy groups

In this work, we will mostly be interested in notions that relate to path-connectivity in this space and so it is worth recalling one of the earliest results in this regard due to Carr. This concerns the case when M is the standard sphere S^n.

Theorem 2.3.1 (Theorem 4, [6])**.** *The space of positive scalar curvature metrics on S^{4k-1}, for $k \geq 2$, has infinitely many path components.*

Very roughly, Carr constructs a certain infinite family of closed spin manifolds of dimension $4k$ (for each $k \geq 2$) with non-vanishing $\hat{A}$-genus, thus not admitting metrics of positive scalar curvature. He then shows that, if one removes a disc from one of these manifolds one can, using Gromov-Lawson surgery style methods, produce metrics of positive scalar curvature on the resulting manifolds with boundary (though not on the entire manifold). The boundary in each case is a $(4k-1)$-dimensional standard sphere and the metric Carr constructs has a product structure near the boundary. It follows that the boundary metric itself has positive scalar curvature and, moreover, cannot be in the same path component as the standard round metric. This is because such a metric could be shown to extend over the removed disc producing a psc-metric on a manifold which admits no such metrics, a contradiction. Similar arguments are used to show that all of the boundary sphere metrics produced in Carr's (countable) infinite family lie in distinct path components of the space $\mathcal{R}^{s>0}(S^{4k-1})$. In principle, such index obstruction methods are at the heart of all of these results on non-triviality in the topology of these spaces of psc-metrics.

Of course, the space of positive scalar curvature metrics, $\mathcal{R}^{s>0}(M)$, forms the largest of the collection of spaces of positive (p,n)-intermediate scalar curvature metrics on M, where $p \in \{0, 1, \cdots, n-2\}$, defined more generally by

$$R^{s_{p,n}>0}(M) = \{g \in \mathcal{R} \ : \ s_{p,n}(g) > 0\}.$$

Carr's result above generalises to these spaces in the following way.

Theorem 2.3.2 (Theorem 7.1, [5]). *Let $n \geq 2$ and M be a smooth, closed, spin manifold of dimension $4n-1$ which admits an $(s_{p,4n-1} > 0)$ for $p \in \{0, 1, 2, \cdots, 2n-3\}$. Then $\mathcal{R}^{s_{p,4n-1}>0}(M)$ has infinitely many path-components.*

For further general discussion on this topic we direct the reader to the articles [50] and [51], and to [7]. We introduce two notions which determine important equivalence relations on the space of positive (p, n)-intermediate scalar curvature metrics on M. These notions were implicitly used in our rough explanation of Carr's theorem that $\mathcal{R}^{s>0}(S^{4k-1})$ has infinitely many path components and are vital in proving both this theorem and its generalisation above, Theorem 2.3.2.

Definition 2.3.3. *Let g_0, g_1 be two $(s_{p,n} > 0)$-Riemannian metrics on an n-dimensional manifold M. We say that g_0 and g_1 are $(s_{p,n} > 0)$-isotopic if there exists a path $\gamma : t \mapsto g_t$, $t \in [0, 1]$ in $\mathcal{R}^{s_{p,n}>0}(M)$ such that $\gamma(0) = g_0$ and $\gamma(1) = g_1$. The path γ is referred to as a $(s_{p,n} > 0)$-isotopy.*

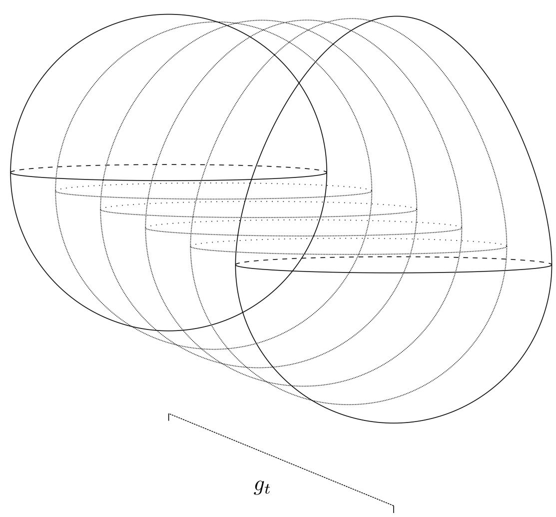


Figure 2.3: Visualising an isotopy, g_t, between round, g_0 and "egg" shaped, g_1, metrics on the sphere.

Definition 2.3.4. *Let g_0, g_1 be two $(s_{p,n} > 0)$-Riemannian metrics on an n-dimensional manifold M. The metrics g_0 and g_1 are said to be $(s_{p,n} > 0)$-concordant if, for some $\epsilon > 0$, there is a $(s_{p,n+1} > 0)$-metric $\bar{g}$ on the cylinder $M \times I$, such that*

$$\bar{g}\big|_{M\times[0,\epsilon]} = g_0 + dt^2 \text{ and}$$

$$\bar{g}\big|_{M\times[1-\epsilon,1]} = g_1 + dt^2,$$

where $\bar{g}$ is referred to as a $s_{p,n} > 0$-concordance.

Remark 2.3.5. *The reader should note that, though called an $s_{p,n} > 0$ concordance, as it is between metrics on an n-dimensional manifold, the concordance itself is a metric on an $n+1$-dimensional cylinder and one with $s_{p,n+1} > 0$ curvature. This coincides precisely with the definition of concordance for metrics of positive scalar curvature (the case when $p = 0$).*

An isotopy can be visualised (at least in low dimensions) as a sort of animation of the metric over time, as suggested in Figure 2.3 where the round sphere changes into an egg shape. A concordance is somewhat difficult to visualise, and we provide a simple schematic in Figure 2.4. A natural question one may ask then given two such equivalence relations: *to what extent, if any, do they determine each other?* Let g_t be an $(s_{p,n} > 0)$-isotopy between two metrics g_0, $g_1 \in \mathcal{R}^{s_{p,n}>0}(M)$. The following lemma is a generalisation of a well-known fact about positive scalar curvature metrics proved originally by Gromov and Lawson in [12].

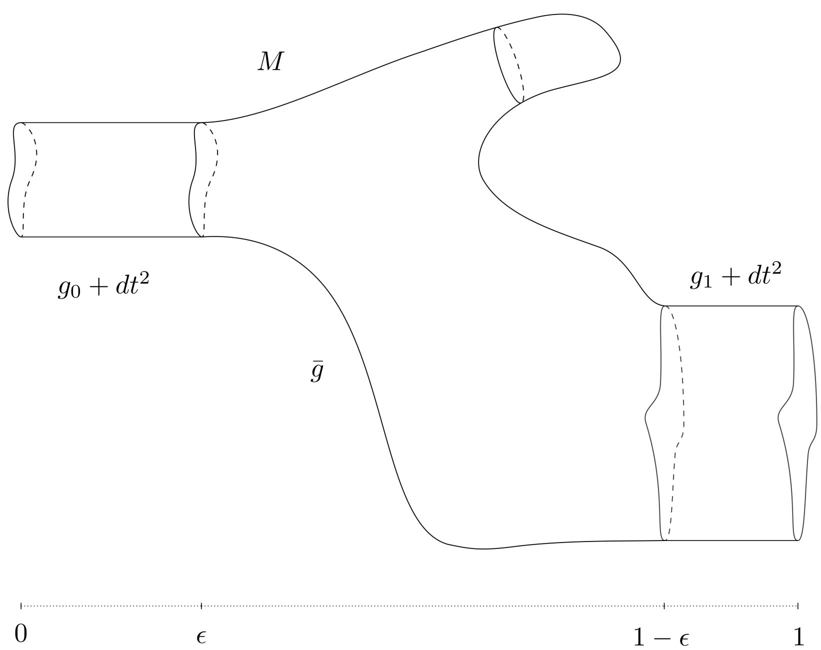


Figure 2.4: A schematic of a concordance between metrics g_0 and g_1.

Lemma 2.3.6 (Lemma 3.2, [5])**.** *Let M be a compact n-dimensional manifold and g_r with $r \in [0, 1]$ a smooth path of $(s_{p,n} > 0)$-metrics on M. Then there exists a positive constant $C \leq 1$ so that for every smooth function $f : \mathbb{R} \to [0, 1]$ with $|f'|, |f''| \leq C$ the metric $\bar{g} = g_{f(t)} + dt^2$ on $M \times \mathbb{R}$ has positive $(p, n+1)$-intermediate scalar curvature, $0 \leq p \leq n-2$.*

It follows immediately that $(s_{p,n} > 0)$-isotopic metrics are $(s_{p,n} > 0)$-concordant in $\mathcal{R}^{s_{p,n}>0}(M)$.

Corollary 2.3.7 (Proposition 3.3, [5]). *Let M be a smooth compact manifold of dimension n. Then, for any $p \in \{0, \cdots, n-2\}$, metrics which are $(s_{p,n} > 0)$-isotopic on M are $(s_{p,n} > 0)$-concordant.*

As we mentioned in the introduction, the converse problem of whether concordance implies isotopy (for any p) is much more difficult. Consider a pair of $(s_{p,n} > 0)$ concordant metrics, g_0 and g_1, on a manifold M. For simplicity, consider the case when M is the standard sphere S^n. How could one tell if these metrics are isotopic? We may be lucky in that the concordance may be very simple. For example, suppose the metrics g_0 and g_1 were connected by an isotopy, $g_t, t \in I$, in the space of $s_{p,n} > 0$ metrics on S^n giving rise by way of Corollary 2.3.7 to an $(s_{p,n} > 0)$-concordance of g_0 and g_1

$$\bar{g} = g_t + dt^2,$$

on $S^n \times I$. Recall, this means that the metric $\bar{g}$ actually has $s_{p,n+1} > 0$ curvature. (Strictly speaking the isotopy g_t may have to be rescaled, as in Lemma 2.3.6, but this is beside the point.) Importantly, such a concordance is "slicewise" in that the restriction of the metric, $\bar{g}$, to each slice $S^n \times \{t\}$ is an $s_{p,n} > 0$ metric and so an isotopy between g_0 and g_1 can easily be recovered. However, even such a nice concordance can be easily made into one which is in a sense "unrecognisable" and from which restoring or producing an isotopy between g_0 and g_1 seems extremely difficult. Consider the following modification.

Keeping the concordance $\bar{g}$ above, suppose we let m denote an arbitrary $(s_{p,n+1} > 0)$-metric on the sphere S^{n+1}. Using the method of Gromov-Lawson as enhanced by Labbi in Theorem 2.2.6, we can obtain the connected sum metric, taken on the interior of $S^n \times I$,

$$\bar{h} = \bar{g} \# m \in \mathcal{R}^{s_{p,n+1}>0}(S^n \times I),$$

itself a $(p, n+1)$-curvature-positive metric. Since taking the connected sum only alters the concordance $\bar{g}$ away from the boundary, the connected sum metric $\bar{h}$ is also an $(s_{p,n} > 0)$-concordance between g_0 and g_1.

The $(s_{p,n+1} > 0)$-metric m may be chosen arbitrarily, and so may be (especially if $n-p$ is large) an extremely complicated metric. Moreover, this connected sum adjustment to $\bar{g}$ may be carried out as many times as one likes with an arbitrary number of distinct arbitrarily chosen $(s_{p,n+1} > 0)$ metrics $m_1, m_2, \cdots, m_N$ on S^{n+1} for any positive integer N. Thus, the resulting concordance (still between the original metrics g_0 and g_1) can be made very complicated indeed. It is very difficult to see how such a concordance as

$$\bar{h} = \bar{g} \# m_1 \# m_2 \cdots \# m_N,$$

could be resolved into an isotopy between g_0 and g_1, or how any isotopy could be produced.

2.4 Morse theory

In this section we review some elementary notions from Morse theory, which we will make considerable use of later on. Morse Theory explains the relationship between the topology of a smooth manifold and the critical points of certain smooth real-valued functions defined on it, called Morse functions. This theory was developed by Marston Morse in the early 20th century, and restated in modern language by Milnor in his text [31]. Morse functions, have critical points which are all of the "simplest type" and can be used to decompose a manifold into a collection of elementary pieces called handlebodies[§]. As we will discuss, a Morse function can be used to decompose a cobordism of manifolds into a union of elementary cobordisms (each the trace of a single surgery). Via the various surgery based metric constructions we have just discussed, Morse functions can be used to produce interesting examples of Riemannian metrics, satisfying some or other curvature condition.

2.4.1 Morse functions

Let $n = k + \ell + 1$ and consider the space of smooth functions, denoted $\mathcal{F}(W)$, from the smooth cobordism $\{W^{n+1}; W_0, W_1\}$, to the unit inveral I such that

(i) $f^{-1}(0) = W_0$, $f^{-1}(1) = W_1$, and

(ii) All critical points of f are away from ∂W.

Recall that a point $w \in W$ is a *critical point* if $\mathrm{d}f(w) = 0$.

Definition 2.4.1. *Let $f \in \mathcal{F}(W)$. A critical point w of f is said to be non-degenerate if $det(Hf(w)) \neq 0$, where Hf denotes the Hessian of f.*

Definition 2.4.2. *The Morse index λ of a non-degenerate critical point w of a Morse function f is the number of negative eigenvalues of the Hessian of f at w.*

Definition 2.4.3. *A Morse function is a function $f \in \mathcal{F}(W)$, the critical points of which are all non-degenerate.*

An important lemma due to Marston Morse[¶] gives that at any of its critical points, a Morse function is locally equivalent to a quadratic polynomial. In particular, if w is a

[§]See Chapter 1.5 of [29].

[¶]A statement of this lemma can be found in Lemma 2.2 of [31] and further details can be found in Section 1.3 of [1]. For an exposition of Morse's ideas in his own words, we direct the reader to [35].

critical point of f, we can find a local coordinate system $(x_1, \cdots, x_{n+1})$ near w in which w is identified with the origin and f is a quadratical polynomial

$$f = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_{n+1}^2$$

where $c = f(w)$. The value λ is equal to the *Morse index* of w. Obviously this value is independent of the choice of coordinates $(x_1, \cdots, x_{n+1})$. We refer to the coordinate chart defined in the Morse lemma as a *Morse coordinate chart*, a schematic description (involving level sets) of which can be found in Figure 2.5. If, for some $\epsilon > 0$, $f^{-1}[c-\epsilon, c+\epsilon]$ contains a

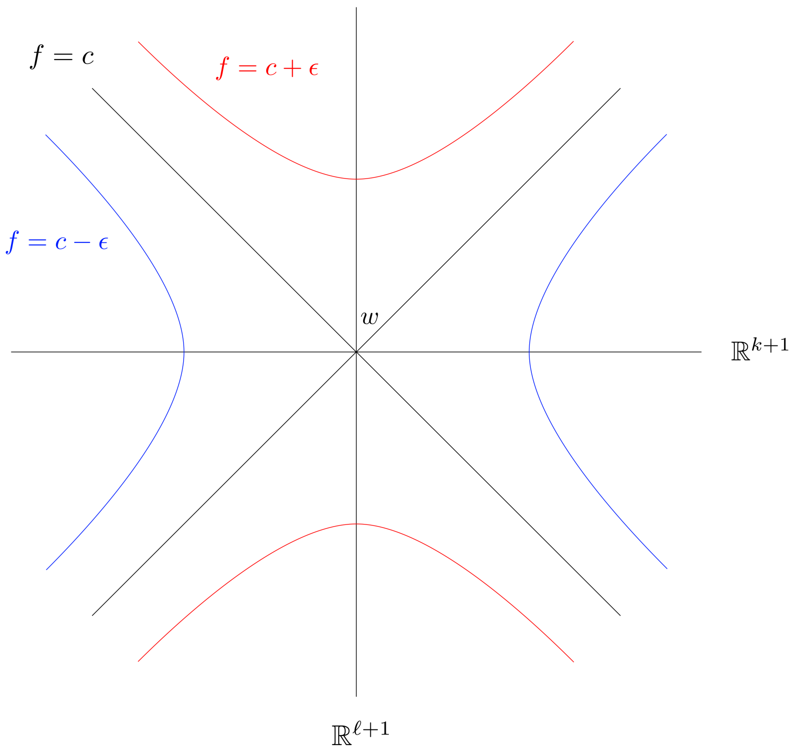


Figure 2.5: A Morse coordinate chart around a critical point w.

single critical point w, the this is said to be an *elementary cobordism*, between the level sets $f^{-1}(c-\epsilon)$ and $f^{-1}(c+\epsilon)$. Setting $\lambda = k+1$, where $k + \ell + 1 = n$, the level sets below $f = c$ are diffeomorphic to $S^k \times D^{\ell+1}$, and those above it are diffeomorphic to $D^{k+1} \times S^\ell$. Thus, $f^{-1}[c-\epsilon, c+\epsilon]$ is the trace of a k-surgery (codimension ℓ-surgery) on $f^{-1}(c-\epsilon)$, since f has a single critical point w of Morse index $\lambda = k+1$.

As a consequence of the Morse lemma, the critical points of a Morse function are necessarily isolated. Therefore, a Morse function $f : W \to [0, 1]$ on a compact cobordism W has finitely many critical points, which we denote w_i with index $p_i + 1$, where $0 \leq i \leq k$. Letting

$c_i = f(w_i)$, we assume that $0 < c_0 \leq c_1 \leq \cdots \leq c_k < 1$.

Definition 2.4.4. *A Morse function f is said to be well-indexed if critical points on the same level set have the same index, and the indices are ordered such that $p_i \leq p_{i+1}$ for all i.*

For some $\epsilon > 0$ small, denote $C_i = f^{-1}[c_{i-1} + \epsilon, c_i + \epsilon]$, a cobordism between the level sets $f^{-1}(c_{i-1} + \epsilon)$ and $f^{-1}(c_i + \epsilon)$ and the trace of a p_i-surgery on $f^{-1}(c_{i-1} + \epsilon)$. In particular $C_0 = f^{-1}([0, c_0 + \epsilon])$ and $C_k = f^{-1}([c_{k-1} + \epsilon, 1])$. If, for all i, $c_{i-1} < c_i$, then each C_i is an elementary cobordism containing the critical point w_i and W decomposes into the union

$$W = C_0 \cup C_1 \cup \cdots \cup C_k.$$

If the level sets of f have more than one critical point, the cobordisms in this composition will not necessarily be elementary. However, it is possible to adjust f through an arbitrarily small perturbation to a function such that the inequalities on the critical values are strict.

By specifying a Riemannian metric m on M, we can define the gradient vector field for the Morse function f, denoted $\text{grad}_m f$. We generalise this notion by defining *gradient-like vector fields* on M with respect to f and m.

Definition 2.4.5. *A gradient-like vector field with respect to f and m is a vector field V on M such that*

(i) $df_x(V_x) > 0 \; \forall x \in M$, and

(ii) around each critical point of f there is a neighbourhood on which $V_x = \text{grad}_m f(x)$.

As discussed earlier the surgery based metric constructions by Gromov-Lawson and others have important codimension conditions on the surgeries they permit. In the language of Morse theory these conditions translate to conditions on the Morse indices of the critical points of the Morse function. In our case, the codimension condition required by Labbi's Theorem 2.2.6 in the context of surgery on an $s_{p,n} > 0$-Riemannian manifold motivates the following definition.

Definition 2.4.6. *Let $\{W^{n+1}; W_0, W_1\}$ be a smooth, compact cobordism. A p-admissible Morse function is a triple (f, m, V) where $f : M \to I$ is a Morse function with critical points of index less than or equal to $n - 2 - p$, m is the background metric on W and V is a gradient-like vector field with respect to f and the Riemannian metric m.*

This generalises the notion of admissibility used in [48] which dealt only with the $p = 0$ (positive scalar curvature) case. The index condition in this case follows from the fact that $\lambda = k + 1$ and Labbi's Theorem, Theorem 2.2.6, requires that $\ell + 1 \geq 3 + p$.

2.4.2 Trajectory discs and spheres of a Morse function

We will now provide a brief review of so-called trajectory discs and spheres associated to a Morse critical point. Roughly speaking, these are embedded discs and spheres associated to each critical point of a Morse function and arising from the gradient flow of f (with respect to some arbitrarily chosen background Riemannian metric). To keep the discussion simple, we begin by considering the case when $f : W \to I$ is a Morse function on the cobordism $\{W; X_0, X_1\}$ with exactly one critical point, w, in the interior of W, of Morse index $\lambda = k+1$. As usual, we assume that $k+\ell+1 = n$. Most of this generalises easily to the case of multiple critical points. However, there are some interesting issues concerning the ways certain of these trajectory discs (associated to distinct critical points) may intersect. For full details, the reader should consult Milnor's classic texts [31] and [33].

We will assume that the cobordism W is equipped with an arbitrary "background" Riemannian metric for the purpose of defining the gradient of f. We impose no conditions on the metric and everything which follows in this section goes through regardless of the choice.

The critical point w has associated "trajectory discs", $D^{k+1}(w,-)$ and $D^{\ell+1}(w,+)$ embedded into W, where $\lambda = k+1$ is the Morse index of w, which correspond respectively to the negative and positive eigenspaces of the Hessian of f at w. These are formed as the unions of integral curves arising from the gradient flow of f and can be thought of as "flowing into" and "out of" w. They can be most easily seen (locally) as the respective images under the Morse coordinate map, of the subspaces $\mathbb{R}^{k+1}$ and $\mathbb{R}^{\ell+1}$ shown in Figure 2.5.

The trajectory disc $D^{k+1}(w,-)$ lies entirely below the critical level $f^{-1}(c)$ (where $c = f(w)$) and $D^{\ell+1}(w,+)$ lies entirely above. Importantly, the intersections of these trajectory discs with level sets below or above the critical level are transversal and result in embedded "trajectory" spheres of dimension k and ℓ respectively. More precisely, for any $t \in [0, c)$, we denote by $S^k(w,-)_t$, the embedded sphere which is the intersection of $f^{-1}(t)$ with $D^{k+1}(w,-)$ while for any $t \in (c, 1]$, $S^\ell(w,+)_t = f^{-1}(t) \cap D^{\ell+1}(w,+)$. It is common to denote the "beginning" and "end" spheres as

$$S^k(w,-) = S^k(w,-)_0 \subset X_0 \text{ and } S^\ell(w,+) = S^\ell(w,+)_1 \subset X_1.$$

As t approaches c from the left, $S^k(w,-)_t$ flows toward $f^{-1}(c)$, collapsing to the point w at $t = c$. The trajectory sphere $S^\ell(w,+)_t$ is best thought of as "emerging" out of w as t moves past c from left to right. Alternatively, we can rewind the flow and see the sphere $S^\ell(w,+)_t$ collapsing into w as t goes to c from the right. This is all schematically illustrated in Figure 2.6. Essentially, the embedded sphere, $S^k(w,-) \subset X_0$, is the sphere which is subject to (removed by) the surgery on X_0 while the outgoing trajectory sphere, $S^\ell(w,+) \subset X_1$, is the sphere attached (as part of a handle) to produce X_1.

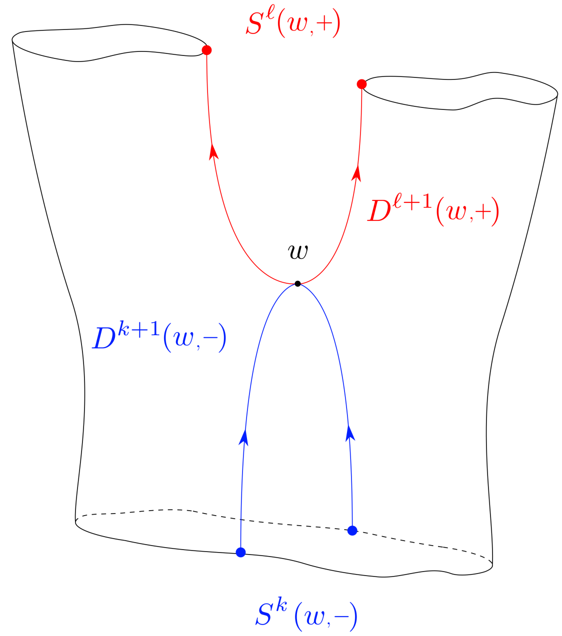


Figure 2.6: The critical point w and the outgoing trajectory sphere $S^\ell(w,+)$ and incoming trajectory sphere $S^k(w,-)$.

2.4.3 The weak and strong cancellation theorems

Having developed the necessary language of Morse theory, we proceed to discuss the *weak* and *strong cancellation theorems* from chapters 5 and 6 of [33], which provide criteria for eliminating certain "cancelling pairs" of critical points and therefore simplifying a Morse function. These cancelling critical points have consecutive Morse indices $\lambda, \lambda+1$ and roughly, correspond to a certain pair of surgeries, one undoing the effects of the other. More than this however, the underlying union of the corresponding traces turns out to be a cylinder.

The cancellation procedure can be understood as an "editing" of gradient-like vector fields, seen in Figure 2.8. Let C be an elementary cobordism of index λ and C' an elementary cobordism of index $\lambda+1$. We begin, by way of motivation, with the following question, discussed at length in [33]: *when is the composition CC' a cylinder?* Consider the very basic schematic described in Figure 2.7. In general this question is answered in Theorem 2.4.7 below.

Theorem 2.4.7 (The Weak Cancellation Theorem, Theorem 5.4 of [33])**.** *Let $\{W^{n+1}; W_0, W_1\}$ be a smooth compact cobordism, $n = k + \ell + 1$ and $f : W \to I$ be a Morse function on W which satisfies the following conditions.*

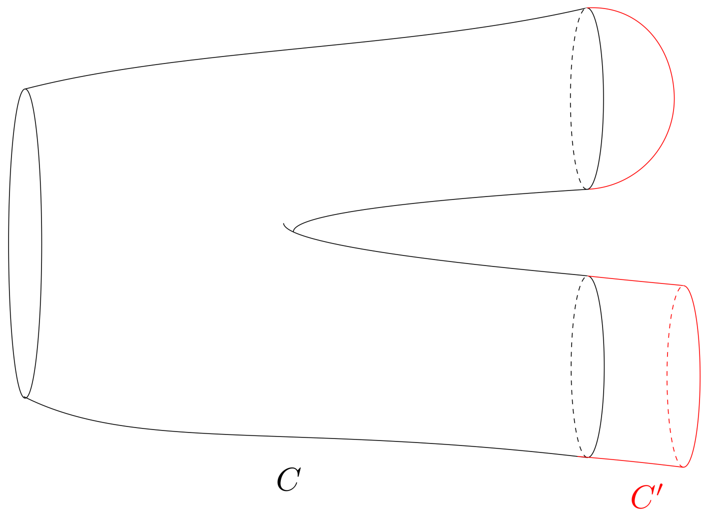


Figure 2.7: The concatenated cobordism CC'.

(i) The Morse function f has exactly 2 critical points w and w', and $0 < f(w) < f(w') < 1$;

(ii) the Morse index of $w = k+1$ and the Morse index of $w' = k+2$;

(iii) for each $t \in (f(w), f(w'))$, the intersection of the trajectory spheres $S^{\ell}(w,+)_t$ and $S^{k+1}(w',-)_t$, in the level set $f^{-1}(t)$, is transversal and consists of a single point.

Then the critical points w and w' of f cancel, meaning there is a diffeomorphism $W \cong W_0 \times I$.

The third condition above means that there is a single trajectory arc T from w to w' obtained as the image of a smooth injective path

$$[f(w), f(w')] \to W, t \mapsto \alpha(t) := S^{\ell}(w,+)_t \cap S^{k+1}(w',-)_t \in f(t), \tag{2.14}$$

connecting w to w'. Let U_T denote an arbitrarily small neighbourhood of T with a gradient-like vector field V. The vector field V can be altered on this neighbourhood giving a nowhere zero gradient-like vector field V' which agrees with V outside of U_T. This alteration provides a corresponding Morse function f' with gradient-like vector field V' which has no critical points and agrees with f outside of U_T. A diagram visualising this alteration can be found in Figure 2.8. The full proof of this theorem can be found in [33].

This theorem finds a strenghtening in the case where W, W_0 and W_1 are simply connected. First, let us review a prerequisite notion from [33] known as the *intersection number*. Let X_0 and X_1 be smooth submanifolds of dimensions d_0 and d_1 respectively in a smooth manifold Y of dimension $d_0 + d_1$ that intersect along points $\{x_i\}$ transversally. Suppose that X_0 is oriented and the normal bundle of X_1 in Y is oriented. At x_i we choose a positively-oriented

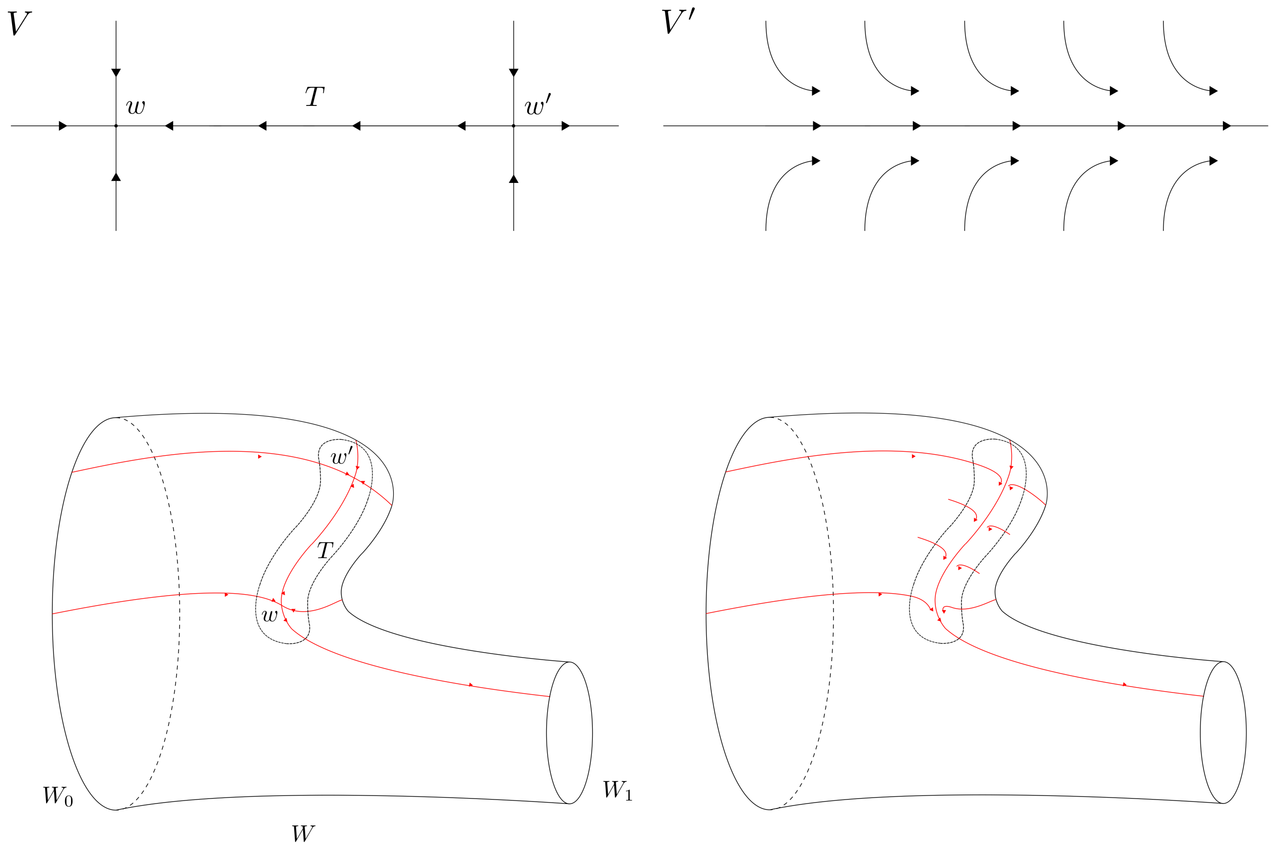


Figure 2.8: A diagram of the alteration on V.

d_0-frame $\{\xi_1, \cdots, \xi_{d_0}\}$ of linearly independent vectors spanning $T_{x_i}X_0$. Since the intersection at x_i is transverse, these vectors represent a basis for the fibre at x_i of the normal bundle of X_1.

Definition 2.4.8. *The intersection number of X_0 and X_1 at x_i is ± 1 if the basis $\{\xi_1, \cdots, \xi_{d_0}\}$ is positively or negatively oriented respectively. The intersection number of X_0 and X_1, denoted $X_0 \cdot X_1$, is the sum of the intersection numbers at each x_i.*

Theorem 2.4.9 (The Strong Cancellation Theorem, Theorem 6.4 of [33])**.** *Let $\{W^{n+1}; W_0, W_1\}$ be a smooth compact cobordism, $n = k + \ell + 1$ and $f : W \to I$ a Morse function on W. Suppose f has exactly two critical points w, w' with respective indices $k+1, k+2$ and satisfying $0 < f(w) < f(w') < 1$. In addition, suppose W, W_0 and W_1 are simply connected, and $1 \leq k \leq n-4$. If, for any $t \in (f(w), f(w'))$, $S^{k+1}(w', -)_t \cdot S^{\ell}(w, +)_t = \pm 1$, then W is diffeomorphic to $W_0 \times I$.*

Remark 2.4.10. *The proof of Theorem 2.4.9 involves showing that f can be altered near*

$f^{-1}(t)$ so that the trajectory spheres in $f^{-1}(t)$ intersect in a single point, transversely; and the conclusions of Theorem 2.4.7 apply.

2.4.4 The space of p-admissible Morse functions

Recall we will be interested in Morse functions whose critical points satisfy certain index conditions. Letting W^{n+1} denote a smooth compact cobordism $\{W; W_0, W_1\}$, we let $\mathcal{F}(W)$ denote the space of smooth functions $W \to I = [0,1]$ under the standard C^∞ topology; see [15] for a detailed description. As usual, we will require such functions to only have critical points on the interior of W. Contained in this is the subspace, denoted $\mathcal{M}(W)$, consisting of Morse functions. It is a well known fact, see [Theorem 2.7, [33]] for example, that $\mathcal{M}(W)$ is an open, dense subspace of $\mathcal{F}(W)$.

As before, let V denote a gradient-like vector field with respect to some background metric m and some function $f \in \mathcal{F}(W)$. This forms a triple (f, m, V). We consider the space of such triples denoted $\tilde{\mathcal{F}}(W)$. This contains, as a subspace, the space of such triples in which f is a Morse function, denoted $\tilde{\mathcal{M}}(W)$. Recalling that a p-admissible Morse function is one for which every critical point has index some λ satisfying $\lambda \leq n - 2 - p$, we denote the subspace of triples containing such functions, which we call p-admissible Morse triples, as $\tilde{\mathcal{M}}_{p-\mathrm{adm}}(W)$.

The spaces $\tilde{\mathcal{M}}(W)$ and $\tilde{\mathcal{M}}_{p-\mathrm{adm}}(W)$ are not path connected. In fact, two triples in the space $\tilde{\mathcal{M}}(W)$ can lie in the same path component only if they have the same number of index-λ critical points for each possible Morse index λ; see chapter four of [48] for discussion of these matters. Thus, the subspace $\tilde{\mathcal{M}}_{p-\mathrm{adm}}(W)$ is actually a union of path components of the space $\tilde{\mathcal{M}}(W)$.

We say then that a Morse triple is *well-indexed* if its Morse function is *well-indexed* (recall Definition 2.4.4). It is a well-known fact, which we state below, that every cobordism admits well-indexed Morse functions.

Theorem 2.4.11 (Theorem 4.8, [33]). *Any cobordism $W^{n+1} = \{W; W_0, W_1\}$ may be equipped with a Morse function $f : W \to I$ so that W may be expressed as a union of cobordisms*

$$W = C_0 \cup C_1 \cup \cdots \cup C_n$$

where each on each C_k, the Morse function f restricts to a Morse function with one critical level and having all critical points of index k.

In fact, given an arbitrary Morse function, f, on W, a continuous adjustment, through Morse functions, can be made to obtain such a well-indexed one of the type described above.

Lemma 2.4.12 (Theorem 4.2, [48]). *Every Morse triple $f \in \tilde{\mathcal{M}}(W)$ lies in the same path component as a well-indexed one $\bar{f}$.*

As Morse triples in the same path component have the same number of critical points of the same index we obtain the following corollary.

Corollary 2.4.13. *For any $p \in \{0, 1, \cdots, n-2\}$, any p-admissible Morse triple on W can be connected by a path through p-admissible Morse triples to a well-indexed one.*

Chapter 3

Warped Product Metrics

In this chapter we compute expressions for the (p, n)-intermediate scalar curvature of certain *warped product metrics*. Roughly, these are Riemannian metrics defined on product manifolds, where one factor is "warped" by a smooth scalar function along the direction of another. Warped product metrics and their "multiply-warped" generalisation are examples of *Riemannian submersions*, which we will discuss shortly. There are well known curvature formulae due to Gray and O'Neill for dealing with Riemannian submersion metrics* and we will make extensive use of them in our calculations.

We begin with the most basic case. Given a product manifold $M = B \times F$, and a smooth function $\beta : B \to (0, \infty)$, we consider the metric

$$g = g_B + \beta^2 g_F,$$

where g_B and g_F are metrics on B and F respectively. Such a metric is called a *warped product metric* and the function β is referred to as a *warping* function. For example, consider the case where the fibre F is the standard sphere, S^{n-1}, and the base B is the interval $(0, \pi)$. We can realise the round metric ds_n^2 on the sphere S^n as a warped product metric:

$$ds_n^2 = dt^2 + \sin^2 t ds_{n-1}^2.$$

on $(0, \pi) \times S^{n-1}$. This can be seen equivalently as the metric induced on $(0, \pi) \times S^{n-1}$ by the embedding

$$\begin{aligned} (0, \pi) \times S^{n-1} &\to \mathbb{R} \times \mathbb{R}^n \\ (t, \theta) &\mapsto (\cos t, \theta \sin t). \end{aligned}$$

*See [36] as well as Chapter 9 of [2] for details.

Furthermore, the round metric of radius ϵ can be expressed as $dt^2 + \epsilon^2 \sin^2\left(\frac{t}{\epsilon}\right) ds_{n-1}^2$ on $(0, \epsilon\pi) \times S^{n-1}$.

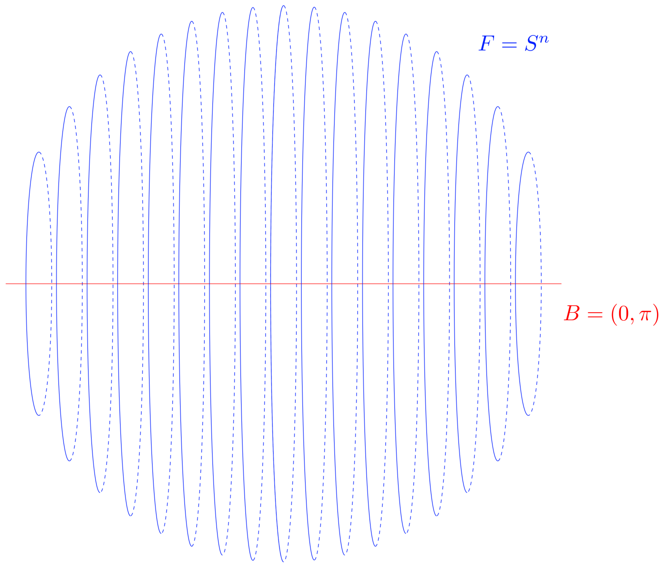


Figure 3.1: ds_n^2 realised as a singly-warped product.

Strictly speaking such metrics have been defined on the cylinder $S^{n-1} \times I$, however the behaviour of the warping function in this case causes the cylinder to close up near the end points in such a way as to guarantee the metric uniquely extends to the round metric on S^n. Thus, it is reasonable and convenient to describe $dt^2 + \sin^2 t ds_{n-1}^2$ as a metric on S^n. We will be particularly interested in warping functions which cause the cylinder to "close up" as a sphere in this way and so we consider the following lemma due to Petersen.

Lemma 3.0.1 (Chapter 1, Section 3.4 of [37])**.** *Let $f : (0, b) \to (0, \infty)$ be a smooth function with $f(0) = 0 = f(b)$. Then the metric $g = dt^2 + f^2(t) ds_{n-1}^2$ is a smooth metric on S^n if and only if*

$$f^{even}(0) = 0, \ f_t(0) = 1, \ f^{even}(b) = 0, \ \textit{and} \ f_t(b) = -1.$$

The notion of a warped product easily generalises on products of multiple manifolds. For example, we can equip a product manifold $B \times F_1 \times F_2$ with a metric of the form:

$$g = g_B + \alpha^2 g_{F_1} + \beta^2 g_{F_2},$$

where g_B, g_{F_1} and g_{F_2} are metrics on B, F_1 and F_2 respectively, and $\alpha, \beta : B \to (0, \infty)$ are smooth functions. Such a metric is called *double warped product metric*. Triply and N-tuply warped product metrics can be analogously defined although we need only double warped products here. The round metric ds_n^2 can also be realised as a doubly-warped product metric; the embedding (or collection of embeddings)

$$\begin{aligned} E_{k,\ell} : \left(0, \frac{\pi}{2}\right) \times S^k \times S^\ell &\to \mathbb{R}^{k+1} \times \mathbb{R}^{\ell+1} \\ (t, \phi, \theta) &\mapsto (\phi \cos t, \theta \sin t), \end{aligned} \tag{3.1}$$

induces the round unit radius sphere metric as a doubly-warped product metric

$$ds_n^2 = dt^2 + \cos^2 t ds_k^2 + \sin^2 t ds_\ell^2, \tag{3.2}$$

on $\left(0, \frac{\pi}{2}\right) \times S^k \times S^\ell$, where the fibre-sphere dimensions are such that $n = k + \ell + 1$. We can generalise this to a large class of metrics on S^n by replacing cos and sin with appropriate warping functions. Consider the embedding

$$\begin{aligned} (0, b) \times S^k \times S^\ell &\hookrightarrow \mathbb{R}^{k+1} \times \mathbb{R}^{\ell+1} \\ (t, \phi, \theta) &\mapsto (\alpha(t)\phi, \beta(t)\theta), \end{aligned}$$

where $b > 0$, and $\alpha, \beta : (0, b) \to (0, \infty)$ are smooth warping functions such that $\alpha'^2 + \beta'^2 = 1$. From this embedding, we define a doubly-warped product metric g as

$$\begin{aligned} g &= \sum_{i=1}^{k+1} \left(d\left(\alpha(t)\phi_i\right)\right)^2 + \sum_{j=1}^{\ell+1} \left(d\left(\beta(t)\theta_j\right)\right)^2 \\ &= \sum_{i=1}^{k+1} \left(\alpha'(t)\phi_i dt + \alpha(t) d\phi_i\right)^2 + \sum_{j=1}^{\ell+1} \left(\beta'(t)\theta_j dt + \beta(t) d\theta_j\right)^2 \\ &= \sum_{i=1}^{k+1} \left(\alpha'(t)^2 \phi_i^2 dt^2 + \alpha(t)^2 d\phi_i^2 + \alpha'(t)\alpha(t)\phi_i d\phi_i dt\right) \\ &\quad + \sum_{j=1}^{\ell+1} \left(\beta'(t)^2 \theta_j^2 dt^2 + \beta(t)^2 d\theta_j^2 + \beta'(t)\beta(t)\theta_j d\theta_j dt\right). \end{aligned}$$

By restricting $(\phi_1, \cdots, \phi_{k+1})$ and $(\theta_1, \cdots, \theta_{\ell+1})$ to S^k respectively S^ℓ, we obtain the following identities:

$$\begin{aligned} &\sum_{i=1}^{k+1} \phi_i^2 = \sum_{j=1}^{\ell+1} \theta_j^2 = 1 \text{ and, therefore} \\ &\sum_{i=1}^{k+1} \phi_i d\phi_i = \sum_{j=1}^{\ell+1} \theta_j d\theta_j = 0. \end{aligned}$$

Invoking these, we obtain the following expression for g:

$$\begin{aligned} g &= \left(\alpha'(t)^2 + \beta'(t)^2\right) dt^2 + \alpha^2(t) ds_k^2 + \beta^2(t) ds_\ell^2 \\ &= dt^2 + \alpha^2(t) ds_k^2 + \beta^2(t) ds_\ell^2. \end{aligned}$$

In order to guarantee that the resulting metric (defined on $(0, b) \times S^k \times S^\ell$) extends uniquely to a metric on S^n, we must impose the following conditions on α and β, following Lemma 4.1 and 4.2 of [37].

Lemma 3.0.2 (Lemma 4.1 and 4.2, [37])**.** *Let $\alpha, \beta : (0, b) \to (0, \infty)$ be smooth functions with $\alpha(b) = 0$ and $\beta(0) = 0$. Then the metric $g = dt^2 + \alpha^2(t) ds_k^2 + \beta^2(t) ds_\ell^2$ on $(0, b) \times S^k \times S^\ell$ uniquely extends to a smooth metric on S^n if and only if*

$$\begin{aligned} &\alpha(0) > 0, \quad \alpha^{(odd)}(0) = 0, \quad \alpha_t(b) = -1, \quad \alpha^{(even)}(b) = 0 \\ &\beta(b) > 0, \quad \beta^{(odd)}(b) = 0, \quad \beta_t(0) = 1, \quad \beta^{(even)}(0) = 0. \end{aligned}$$

In Chapter 4 we will consider several important metrics on the sphere and disc which take the form of warped and doubly warped products. Before that, we begin this chapter with a brief review of the theory of Riemannian submersions. In particular we will review the curvature formulae of O'Neill which will allow us to compute the intermediate scalar curvature of the relevant metrics in Chapter 4. The curvature expressions developed in this chapter will be used in Chapter 4 to show path-connectedness of particular subspaces of the space of metrics of *positive* (p, n)-intermediate scalar curvature.

3.1 Riemannian submersions

The goal of this introduction is to summarise the relevant sections of [2], Chapter 9 and [36] on the theory of Riemannian submersions, with a view to introducing a crucial expression for the Riemann curvature of a certain type of doubly-warped product metric derived in [5]. This will be foundational to the (p, n)-intermediate scalar curvature calculations in this chapter. For further information on Riemannian submersions in the context of (p, n)-intermediate scalar curvature, we direct the reader to Chapter 2 of [21].

In order to define a Riemannian submersion we need a good deal of notation. Let (M, g) and $(B, \check{g})$ be two Riemannian manifolds, and $\pi : M \to B$ a smooth submersion, with its derivative map at any $x \in M$ denoted $\pi_* : T_x M \to T_{\pi(x)} B$. For each $x \in M$ and $b = \pi(x)$, we let $F_b = \pi^{-1}(b)$, $\mathscr{V}_x := \ker \pi_* \subset T_x M$, the kernel of the derivative map π_* at x, and let $\mathscr{H}_x$ denote the orthogonal complement of $\mathscr{V}_x$.

We say that π is a *Riemannian submersion* if the derivative $\pi_* : T_xM \to T_bB$ induces an isometry from the horizontal subspace at x, $\mathscr{H}_x$, to T_bB for each $x \in M$. A schematic of this setup can be seen in Figure 3.2.

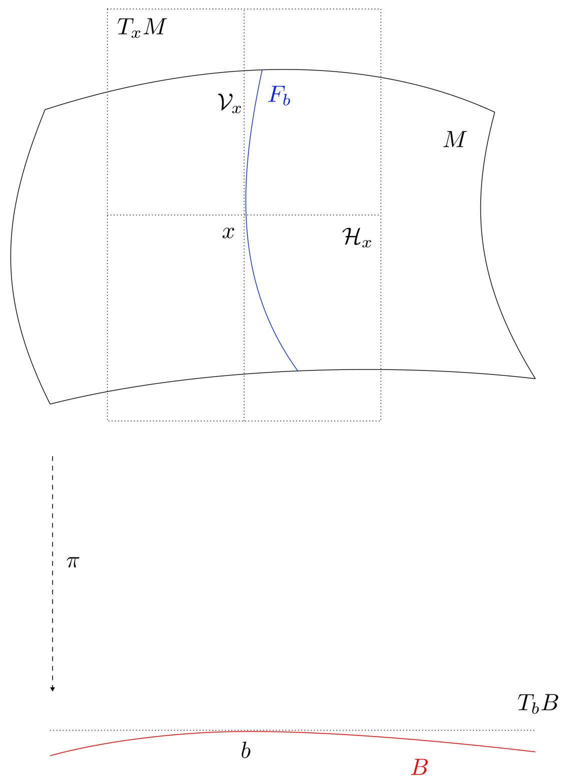


Figure 3.2: A schematic of a Riemannian submersion.

The warped and doubly warped product metrics we recently discussed can be viewed as Riemannian submersions. Recall the example of a product manifold, $M = B \times F$, (where B and F will often be referred to as the *base* and *fibre* respectively), equipped with a warped product metric

$$g = g_B + \beta^2 g_F,$$

where g_B is a metric on B, g_F a metric on F, and $\beta : B \to (0, \infty)$ is a smooth function. We denote the canonical projections of M onto its base and fibre as

$$\pi_B : M \to B, \ \pi_F : M \to F,$$

and it will be notationally convenient shortly to denote the image points $\check{x} = \pi_B(x)$ and $\hat{x} = \pi_F(x)$ respectively. The map $\pi_B : (M, g) \to (B, g_B)$ is a Riemannian submersion. (The map π_F also induces one if β is constantly 1, i.e. the metric is a standard Riemannian product.) Consider now the derivative maps of these projections.

$$(\pi_B)_* : T_x M \to T_{\check{x}} B, \ (\pi_F)_* : T_x M \to T_{\hat{x}} F.$$

Let $b = \pi_B(x) \in B$ for some $x \in M$, and define the *vertical* and *horizontal subspaces* at x as

$$\begin{aligned}&\mathscr{V}_x, \text{ the subspace tangent to } \pi_B^{-1}(b) \text{ in } T_x M, \text{and}\\ &\mathscr{H}_x, \text{ its orthogonal complement.}\end{aligned}$$

At each $x \in M$, the tangent space $T_x M$ splits as $T_x M = \mathscr{V}_x + \mathscr{H}_x$. This gives rise to a splitting of the tangent bundle TM into corresponding subbundles $\mathscr{V}$ and $\mathscr{H}$, the *vertical* and *horizontal distributions* respectively. Letting $u \in T_x M$, we denote the projections of this vector onto the vertical and horizontal distributions as

$$\begin{aligned}&u_F \in \mathscr{V}, \text{ and}\\ &u_B \in \mathscr{H},\end{aligned}$$

and the images of u under the tangent space maps as

$$\begin{aligned}&\check{u} = (\pi_B)_*(u), \text{ and}\\ &\hat{u} = (\pi_F)_*(u).\end{aligned}$$

In Section 4 of [5], the authors use the theory of Riemannian submersions, and in particular the Gray-O'Neill formulae ([36]) to derive an expression for the Riemann curvature of an arbitrary warped product metric. Making use of the notation above we state two important and relevant results from this paper:

Lemma 3.1.1 (Proposition 4.5, [5]). *Let $M = B^b \times F^n$ be equipped with the metric $g_B + \beta^2 g_F$. Let $\partial_1, \ldots, \partial_{b+n}$ denote local coordinate vector fields on M, with $\partial_1, \ldots, \partial_b$ tangent to B and $\partial_{b+1}, \ldots, \partial_{b+n}$ tangent to F. Let $P \subset T_x M$ be a 2-dimensional subspace for some $x \in M$. Let $v, w \in P$ be an arbitrary pair of linearly independent vectors, written in components as*

$v = \sum_{s=1}^{b+n} v_s \partial_s$ *and* $w = \sum_{s=1}^{b+n} w_s \partial_s$. *Then*

$$\begin{aligned} R(v,w,w,v) =& R_B(\check{v}_B, \check{w}_B, \check{w}_B, \check{v}_B) + \beta^2 R_F(\hat{v}_F, \hat{w}_F, \hat{w}_F, \hat{v}_F) \\ & - \sum_{i<j,\lambda} (v_i w_j - v_j w_i)^2 \beta^2 \beta_\lambda^2 g^B_{\lambda\lambda} \\ & - \sum_{i,\lambda,\mu} (w_i^2 v_\lambda v_\mu + v_i^2 w_\lambda w_\mu - 2 v_i w_i v_\lambda w_\mu) \left(\beta\beta_{\lambda\mu} + \sum_\nu \beta\beta_\nu (\partial_\lambda (g_B^{\mu\nu}) + g_B^{\nu\nu} \check{\Gamma}^\mu_{\lambda\nu}) g^B_{\mu\mu} \right). \end{aligned} \tag{3.3}$$

This lemma provides a general expression for the Riemann curvature tensor of a warped product. Furthermore, for a particular warped product metric the authors in [5] obtain the following sectional curvatures:

Lemma 3.1.2 (Lemma 4.6, [5])**.** *Let* $M = B \times F$, $B = (0,b_1) \times (0,b_2)$ *and* $F = S^n$, *where* $n \geq 2$. *This manifold* M *is equipped with the metric*

$$g = dr^2 + \omega(r,t)^2 dt^2 + \beta(r)^2 ds_n^2,$$

where $\beta : (0,b_1) \to (0,\infty)$ *and* $\omega : (0,b_1) \times (0,b_2) \to (0,\infty)$ *are smooth warping functions. The sectional curvatures of this manifold are given by*

$$K_{rt} = -\frac{\omega_{rr}}{\omega}, \; K_{ri} = -\frac{\beta_{rr}}{\beta}, \; K_{ti} = -\frac{\omega_r \beta_r}{\omega\beta}, \; K_{ij} = \frac{1-\beta^2}{\beta^2}.$$

3.2 Intermediate scalar curvature of a singly-warped product metric

In this section, we compute the (p,n)-intermediate scalar curvature of a *singly-warped product metric*. Specifically, we focus on a metric of the form

$$g = dt^2 + \alpha^2(t) ds_{n-1}^2,$$

on the product manifold $B \times F = (0,b) \times S^{n-1}$, where $\alpha : (0,b) \to (0,\infty)$ is a smooth warping function. The sectional curvatures of this metric, which can be found on page 121 of [37] and follow from Lemma 3.1.2, are given as

$$K_{ti} = -\frac{\alpha_{tt}}{\alpha}, \text{ and}$$
$$K_{ii'} = \frac{1-\alpha_t^2}{\alpha^2},$$

where $i, i' \in \{1, \cdots, n-1\}$. These sectional curvatures will be used in the computation of the (p,n)-intermediate scalar curvature of g, which is given in the following theorem.

Lemma 3.2.1. *Let $M = (0,b) \times S^{n-1}$, equipped with the metric*

$$g = dt^2 + \alpha^2(t) ds_{n-1}^2,$$

where $\alpha : (0,b) \to (0,\infty)$ is a smooth warping function. Let $x \in M$ and $P \subset T_xM$ such that dim $P = p \in \{0, \cdots, n-2\}$. Then the (p,n)-intermediate scalar curvature of M at x is given by

$$s_{p,n}(P) = (n-1-p)(n-2-p)\frac{1-\alpha_t^2}{\alpha^2} + A^2(1-\alpha_t^2) - B^2\alpha_{tt},$$

where $A = A(P,\alpha)$, $B = B(P,\alpha)$ are smooth, bounded, real-valued functions.

Proof. Let $\{\partial_t, \partial_1, \cdots, \partial_{n-1}\}$ denote local coordinate vector fields. Then at x we may assume that these form an orthonormal basis for T_xM. Let $P \subset T_xM$ be a p-plane for $p \in \{0, \cdots, n-2\}$. Then $P^\perp$ is an $(n-p)$-plane. Since the dimension of $P^\perp + T_xS^k$ can be at most the dimension of T_xM,

$$\begin{aligned}\dim(P^\perp \cap T_xS^k) &\geq \dim(P^\perp) + \dim(T_xS^k) - \dim(P^\perp + T_xS^k) \\ &\geq (n-p) + (n-1) - (n) \\ &= n-1-p.\end{aligned}$$

Therefore there are at least $n-1-p$ linearly independent vectors in $P^\perp$ which are tangent to S^{n-1}. Without loss of generality, we assume that $\{\partial_1, \cdots, \partial_{n-1-p}\}$ are in $P^\perp$ and tangent to S^{n-1}. As dim$P^\perp = n-p$, it can be seen that there could be a unit length vector v in $P^\perp$ not tangent to S^{n-1}. We therefore have two cases to consider, depending on the dimension of the subspace spanned by the projection of v onto T_xS^{n-1}.

Case 1. The projection of v into T_xS^{n-1} spans a 0-dimensional subspace. Thus, v has no fibre component and $v = \partial_t$. In particular, $P^\perp$ has orthogonal basis $\{\partial_t, \partial_1, \cdots, \partial_{n-1-p}\}$.

Therefore

$$s_{p,n}(P) = 2\sum_{i=1}^{n-1-p} K_{ti} + \sum_{\substack{i,i'=1\\ i\neq i'}}^{n-1-p} K_{ii'}$$

$$= (n-1-p)(n-2-p)\frac{1-\alpha_t^2}{\alpha^2} - 2(n-1-p)\frac{\alpha_{tt}}{\alpha}.$$

Case 2. The projection of v into T_xS^{n-1} spans a 1-dimensional subspace, orthogonal to $\{\partial_1, \cdots, \partial_{n-1-p}\}$ so without loss of generality we can assume it is spanned by ∂_{n-p}. Thus

$$v = v_t\partial_t + v_{n-p}\partial_{n-p}.$$

Using equation (3.3) from Lemma 3.1.1 for $1 \leq a \leq n-1-p$, we obtain the following expression for the Riemann curvature tensor

$$R(v,\partial_a,\partial_a,v) = v_t^2 R(\partial_t,\partial_a,\partial_a,\partial_t) + v_{n-p}^2 R(\partial_{n-p},\partial_a,\partial_a,\partial_{n-p}).$$

The curvature tensor subterms of which we compute as

$$R(\partial_t,\partial_a,\partial_a,\partial_t) = R_{tiit} = -\alpha\alpha_{tt}$$

$$R(\partial_{n-p},\partial_a,\partial_a,\partial_{n-p}) = R_{ii'i'i} = \alpha^2(1-\alpha_t^2).$$

Therefore, substituting these back into the original expression, we obtain

$$R(v,\partial_a,\partial_a,v) = -v_t^2\alpha\alpha_{tt} + v_{n-p}^2\alpha^2(1-\alpha_t^2).$$

Following the formula for the sectional curvature given in equation (2.8), we compute the following sectional curvature term

$$K(v,\partial_a) = \frac{1}{||v||^2||\partial_a||^2}\left(-v_t^2\alpha\alpha_{tt} + v_{n-p}^2\alpha^2(1-\alpha_t^2)\right)$$

$$= \frac{1}{||v||^2}\left(-v_t^2\frac{\alpha_{tt}}{\alpha} + v_{n-p}^2(1-\alpha_t^2)\right),$$

since $||\partial_a||^2 = \alpha^2||\partial_a||_F^2 = \alpha^2$. Now, averaging the sectional curvature over $P^\perp$, we obtain

$$\begin{aligned} s_{p,n}(P) &= \sum_{\substack{i\neq i'' \\ i,i'=1}}^{n-1-p} K_{ii'} + 2\sum_{a=1}^{n-1-p} K(v,\partial_a) \\ &= (n-1-p)(n-2-p)\frac{1-\alpha_t^2}{\alpha^2} + 2(n-1-p)\frac{1}{||v||^2}\left(-v_t^2\frac{\alpha_{tt}}{\alpha} + v_{n-p}^2(1-\alpha_t^2)\right). \end{aligned}$$

We express these two cases generally to obtain the formula for the (p,n)-intermediate scalar curvature as given in the theorem. □

3.3 Intermediate scalar curvature of a doubly-warped product metric

We now consider the manifold $(0,b)\times S^k \times S^\ell$ equipped with the doubly warped product metric

$$dt^2 + \alpha(t)^2 ds_k^2 + \beta(t)^2 ds_\ell^2,$$

where $\alpha,\ \beta : (0,b) \to (0,\infty)$ are smooth warping functions. Making use of calculations in the previous section for a warped product metric $g_B + \beta^2 g_F$ on $M = B\times F$, we take $B = (0,b)\times S^k$ and $F = S^\ell$. From Chapter 4, Section 4.2.4 of [37], the sectional curvatures of this metric are given by

$$K_{ti} = -\frac{\alpha_{tt}}{\alpha},\ K_{ii'} = \frac{1-\alpha_t^2}{\alpha^2},\ K_{tj} = -\frac{\beta_{tt}}{\beta},\ K_{ij} = -\frac{\alpha_t\beta_t}{\alpha\beta},\ K_{jj'} = \frac{1-\beta_t^2}{\beta^2}.$$

We use these sectional curvatures to compute the (p,n)-intermediate scalar curvature of this doubly-warped product metric.

Lemma 3.3.1. *Let $M = (0,b)\times S^k\times S^\ell$, equipped with the metric*

$$g = dt^2 + \alpha^2(t)ds_k^2 + \beta^2(t)ds_\ell^2,$$

where $\alpha,\beta : (0,b)\to(0,\infty)$ are smooth warping functions. Let $x\in M$ and $P\subset T_xM$ such that $\dim P = p \in \{0,\cdots,n-2\}$. Then the (p,n)-intermediate scalar curvature of M at x is given by

$$s_{p,n}(P) = 2(\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + A^2(1-\beta_t^2) + B^2(1-\alpha_t^2) - C^2\beta_{tt} - D^2\alpha_{tt} - E^2\alpha_t\beta_t,$$

where $A = A(P,\beta)$, $B = B(P,\alpha)$, $C = C(P,\beta)$, $D = D(P,\alpha)$, $E = E(P,\alpha,\beta)$ are real-valued functions.

Proof. Let $x = (t,\phi,\theta) \in (0,b) \times S^k \times S^\ell$. By way of notation, let $\{\partial_t, \partial_1, \cdots, \partial_k, \bar{\partial}_1, \cdots, \bar{\partial}_\ell\}$ be an orthonormal basis for T_xM, where ∂_t points in the t-direction, and $\{\partial_1, \cdots, \partial_k\}$ and $\{\bar{\partial}_1, \cdots, \bar{\partial}_\ell\}$ are tangent to S^k and S^ℓ respectively. Let $0 \leq p \leq k+\ell-1$ and let P be a p-plane in T_xM. Then $P^\perp$ is a $(1+k+\ell-p)$-plane. Since the dimension of $P^\perp + T_xS^\ell$ can be at most the dimension of T_xM,

$$\begin{aligned}\dim(P^\perp \cap T_xS^\ell) &\geq \dim(P^\perp) + \dim(T_xS^\ell) - \dim(P^\perp + T_xS^\ell)\\ &\geq (1+k+\ell-p) + \ell - (1+k+\ell)\\ &= \ell - p.\end{aligned}$$

We can conclude then that, since the dimension of this intersection is at least $\ell - p$, there are at least $\ell - p$ linearly independent vectors in $P^\perp$ which are tangent to S^ℓ. Let $\{\bar{\partial}_1, \cdots, \bar{\partial}_\ell\}$ denote the coordinate vector fields on S^ℓ, where $\{\bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$ are in $P^\perp$. By considering the dimension of $P^\perp$ we can see that we have at most $k+1$ unit length orthogonal vectors $\{v_0, v_1, \cdots, v_k\}$ in $P^\perp$ not tangent to S^ℓ.

Case 1. The projections of $\{v_0, v_1, \cdots, v_k\}$ into T_xS^ℓ span a 0-dimensional subspace. Thus $\{v_0, v_1, \cdots, v_k\}$ have no fibre component and $\text{span}\{v_0, v_1, \cdots, v_k\} = \text{span}\{\partial_t, \partial_1, \cdots, \partial_k\}$. In particular, $P^\perp$ has orthogonal basis $\{\partial_t, \partial_1, \cdots, \partial_k, \bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$. Therefore

$$\begin{aligned}s_p(P) &= 2\sum_{\substack{j,j'=1\\ j\neq j'}}^{\ell-p} K_{jj'} + 2\sum_{j=1}^{\ell-p} K_{tj} + 2\sum_{j=1}^{\ell-p}\sum_{i=1}^{k} K_{ij} + 2\sum_{i=1}^{k} K_{ti} + 2\sum_{\substack{i,i'=1\\ i\neq i'}}^{k} K_{ii'}\\ &= 2(\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + 2(\ell-p)\left(-\frac{\beta_{tt}}{\beta}\right) + 2k(\ell-p)\left(-\frac{\alpha_t\beta_t}{\alpha\beta}\right)\\ &\quad + 2k\left(-\frac{\alpha_{tt}}{\alpha}\right) + 2k(k-1)\frac{1-\alpha_t^2}{\alpha^2}.\end{aligned}$$

Case 2. The projections of $\{v_0, v_1, \cdots, v_k\}$ into T_xS^ℓ span a q-dimensional subspace, $1 \leq q \leq \min\{\ell, k+1\}$. This subspace is orthogonal to $\{\bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$ so without loss of generality we can assume it is spanned by $\{\bar{\partial}_{\ell-p+1}, \cdots, \bar{\partial}_{\ell-p+q}\}$. Thus

$$v_s = (v_s)_t \partial_t + \sum_{i=1}^{k} (v_s)_i \partial_i + \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j \bar{\partial}_j.$$

Using the curvature tensor expression for $1 \leq a \leq \ell - p$, we obtain

$$R(v_s, \bar{\partial}_a, \bar{\partial}_a, v_s) = R\left((v_s)_t \partial_t, \bar{\partial}_a, \bar{\partial}_a, (v_s)_t \partial_t\right) + R\left(\sum_{i=1}^{k} (v_s)_i \partial_i, \bar{\partial}_a, \bar{\partial}_a, \sum_{i=1}^{k} (v_s)_i \partial_i\right)$$

$$+ R\left(\sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j \bar{\partial}_j, \bar{\partial}_a, \bar{\partial}_a, \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j \bar{\partial}_j\right)$$

$$= (v_s)_t^2 R(\partial_t, \bar{\partial}_a, \bar{\partial}_a, \partial_t) + \sum_{i=1}^{k} (v_s)_i^2 R\left(\partial_i, \bar{\partial}_a, \bar{\partial}_a, \partial_i\right)$$

$$+ \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 R\left(\bar{\partial}_j, \bar{\partial}_a, \bar{\partial}_a, \bar{\partial}_j\right).$$

Given that this expression is a sum of Riemann curvature tensor subterms, it is necessary to isolate these subterms and compute them separately. This gives

$$\begin{aligned} R(\partial_t, \bar{\partial}_a, \bar{\partial}_a, \partial_t) &= -\beta \beta_{tt} \\ R\left(\partial_i, \bar{\partial}_a, \bar{\partial}_a, \partial_i\right) &= -\beta \beta_t \alpha \alpha_t \\ R\left(\bar{\partial}_j, \bar{\partial}_a, \bar{\partial}_a, \bar{\partial}_j\right) &= \beta^2 (1 - \beta_t^2). \end{aligned}$$

Therefore, substituting these subterms back into the original expression, we obtain the following expression for the Riemann curvature tensor:

$$R(v_s, \bar{\partial}_a, \bar{\partial}_a, v_s) = -(v_s)_t^2 \beta \beta_{tt} - \sum_{i=1}^{k} (v_s)_i^2 \beta \beta_t \alpha \alpha_t + \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 \beta^2 (1 - \beta_t^2),$$

which allows us to compute the following sectional curvature term:

$$K(v_s, \bar{\partial}_a) = \frac{1}{||v_s||^2||\bar{\partial}_a||^2}\left(-(v_s)_t^2\beta\beta_{tt} - \sum_{i=1}^{k}(v_s)_i^2\beta\beta_t\alpha\alpha_t + \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2\beta^2(1-\beta_t^2)\right)$$

$$= \frac{1}{||v_s||^2}\left(-(v_s)_t^2\frac{\beta_{tt}}{\beta} - \sum_{i=1}^{k}(v_s)_i^2\frac{\beta_t}{\beta}\alpha\alpha_t + \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2(1-\beta_t^2)\right),$$

since $||\bar{\partial}_a||^2 = \beta^2||\bar{\partial}_a||_F^2 = \beta^2$. Also

$$R(v_s, v_{s'}, v_{s'}, v_s) = \sum_{i<i'}\left((v_s)_i(v_{s'})_{i'} - (v_s)_{i'}(v_{s'})_i\right)^2\alpha^2(1-\alpha_t^2) - \sum_i((v_{s'})_i(v_s)_t - (v_s)_i(v_{s'})_t)^2\alpha\alpha_{tt}$$

$$+ \sum_{j<j'}((v_s)_j(v_{s'})_{j'} - (v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2) - \sum_j((v_{s'})_j(v_s)_t - (v_s)_j(v_{s'})_t)^2\beta\beta_{tt}$$

$$- \sum_{i,j}((v_{s'})_j(v_s)_i - (v_s)_j(v_{s'})_i)^2\beta\beta_t\alpha_t\alpha,$$

for $j, j' \in \{\ell-p+1, \cdots, \ell-p+q\}$. Using this expression for the Riemann curvature tensor, we compute the associated sectional curvature, obtaining

$$K(v_s, v_{s'}) = \frac{1}{||v_s||^2||v_{s'}||^2}\left(\sum_{i<i'}\left((v_s)_i(v_{s'})_{i'} - (v_s)_{i'}(v_{s'})_i\right)^2\alpha^2(1-\alpha_t^2)\right.$$

$$- \sum_i((v_{s'})_i(v_s)_t - (v_s)_i(v_{s'})_t)^2\alpha\alpha_{tt} + \sum_{j<j'}((v_s)_j(v_{s'})_{j'} - (v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2)$$

$$\left.- \sum_j((v_{s'})_j(v_s)_t - (v_s)_j(v_{s'})_t)^2\beta\beta_{tt} - \sum_{i,j}((v_{s'})_j(v_s)_i - (v_s)_j(v_{s'})_i)^2\beta\beta_t\alpha_t\alpha\right).$$

This concludes our calculation of the sectional curvatures of this metric. Averaging these sectional curvatures over $P^\perp$, we obtain the following expression for the (p, n)-intermediate scalar curvature:

$$
\begin{aligned}
s_{p,n}(P) &= 2\sum_{\substack{j<j'\\ j,j'=1}}^{\ell-p} K_{jj'} + 2\sum_s\sum_{a=1}^{\ell-p} K(v_s,\bar{\partial}_a) + 2\sum_{s,s'} K(v_s,v_{s'}) \\
&= 2(\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + 2(\ell-p)\sum_s \frac{1}{||v_s||^2}\Bigg(-(v_s)_t^2\frac{\beta_{tt}}{\beta} - \sum_{i=1}^{k}(v_s)_i^2\frac{\beta_t}{\beta}\alpha\alpha_t \\
&\quad + \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2(1-\beta_t^2)\Bigg) + \sum_{s,s'}\frac{2}{||v_s||^2||v_{s'}||^2}\Bigg(\sum_{i<i'}\left((v_s)_i(v_{s'})_{i'}-(v_s)_{i'}(v_{s'})_i\right)^2\alpha^2(1-\alpha_t^2) \\
&\quad - \sum_i((v_{s'})_i(v_s)_t-(v_s)_i(v_{s'})_t)^2\alpha\alpha_{tt} + \sum_{j<j'}((v_s)_j(v_{s'})_{j'}-(v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2) \\
&\quad - \sum_j((v_{s'})_j(v_s)_t-(v_s)_j(v_{s'})_t)^2\beta\beta_{tt} - \sum_{i,j}((v_{s'})_j(v_s)_i-(v_s)_j(v_{s'})_i)^2\beta\beta_t\alpha_t\alpha\Bigg).
\end{aligned}
$$

Summarising these cases, we arrive at the formula for the (p,n)-intermediate scalar curvature given in the theorem, where the coefficients of the functions are collected into the smooth functions A to E. □

Remark 3.3.2. *By letting $p = 0$, it can be seen that these formulae reduce to the expected scalar curvature expressions for a singly and doubly-warped product metric of spheres.*

Having computed expressions for the intermediate scalar curvature of a singly and doubly-warped product of spheres of dimension k and ℓ such that $n = k + \ell + 1$, we will, in the following chapter, apply these to construct isotopies of particular warped product metrics on S^n which have positive (p,n)-intermediate scalar curvature.

Chapter 4

Torpedo Metrics

In this chapter, we introduce *torpedo metrics*, which are a specific type of warped product metric of particular importance to the notion of surgery defined in Section 2.2. We begin by introducing particular spaces of warped product metrics in which torpedo metrics are contained, and demonstrating path-connectedness, to obtain results about the existence of isotopies between torpedo metrics in $\mathcal{R}^{s_{p,n}>0}(M)$. This chapter concludes with an introduction to *mixed* torpedo metrics, created by combining torpedo metrics on complementary subspheres, which will be of particular importance in the proof of the main theorem.

4.1 Singly-warped product spaces

Let $\mathcal{F}(0,b)$ denote the space of smooth functions $f : (0,b) \to (0,\infty)$ which satisfy the conditions of Lemma 3.0.1:

$$f^{\text{even}}(0) = 0,\ f_t(0) = 1,\ f^{\text{even}}(b) = 0,\ \text{and } f_t(b) = -1,$$

in addition to

$$\begin{aligned} &f_{tt} \leq 0, \\ &f_{ttt}(0) < 0, \\ &f_{ttt}(b) > 0,\ \text{and} \\ &f_{tt}(t) < 0 \text{ when } t \text{ is near but not at } 0 \text{ and } b, \end{aligned}$$

equipped with the C^∞-topology; the details of which are discussed in Chapter 2 of [15].

Proposition 4.1.1. *The space $\mathcal{F}(0,b)$ is convex.*

Proof. A sequence of elementary checks shows that the convex combination $(1-t)f+tg$ for $f, g \in \mathcal{F}(0,b)$, $t \in [0,1]$ satisfies the conditions which define $\mathcal{F}(0,b)$. □

Now we define the following space of metrics on S^n:

$$\mathcal{W}(0,b) = \{dt^2 + f^2(t)ds_{n-1}^2 : f \in \mathcal{F}(0,b)\}.$$

Lemma 4.1.2. *Let $n \geq 3$. The space $\mathcal{W}(0,b)$ is a path-connected subspace of $\mathcal{R}^{s_{p,n}>0}(S^n)$, for $p \leq n-3$.*

Proof. We begin by invoking Lemma 3.2.1, which states that the (p,n)-intermediate scalar curvature of an arbitrary metric $g \in \mathcal{W}(0,b)$:

$$s_{p,n}(P) = (n-1-p)(n-2-p)\frac{1-f_t^2}{f^2} + A^2(P,f)(1-f_t^2) - B^2(P,f)f_{tt},$$

where $P \subset T_xS^n$ is a p-dimensional subspace, and $A(P,f)$ and $B(P,f)$ are bounded real-valued functions. By definition $f_t(0) = 1$ and $f_t(b) = -1$, and f is concave downward. When $0 < t < b$, it follows that $|f_t(t)| < 1$ and $1 - f_t^2 > 0$. As $p \leq n-3$, each of the three summands in the above expression is positive when $0 < t < b$, and $s_{p,n} > 0$ in this case. At $t = 0$ and $t = b$, some elementary applications of l'Hopital's rule, as used in the proof of Poposition 1.6 of [48], show that $s_{p,n} > 0$. Therefore $\mathcal{W}(0,b) \subset \mathcal{R}^{s_{p,n}>0}(S^n)$. As $\mathcal{F}(0,b)$ is convex from Lemma 4.1.1, we deduce that $\mathcal{W}(0,b)$ is path-connected. □

We now define the space $\mathcal{W} = \bigcup_{b\in(0,\infty)} \mathcal{W}(0,b)$, and proceed to show that it is also a path-connected subspace.

Lemma 4.1.3. *Let $n \geq 3$. The space $\mathcal{W}$ is a path-connected subspace of $\mathcal{R}^{s_{p,n}>0}(S^n)$, for $p \leq n-3$.*

Proof. Let $g \in \mathcal{W}(0,b) \subset \mathcal{W}$. By Lemma 4.1.2, $\mathcal{W}(0,b)$ is path-connected, and therefore g is isotopic to the round metric of radius $\left(\frac{b}{\pi}\right)$ in $\mathcal{W}(0,b)$, given as

$$dt^2 + \left(\frac{b}{\pi}\right)^2 \sin^2\left(\frac{\pi t}{b}\right) ds_{n-1}^2.$$

All round metrics are isotopic via rescaling, thus we can rescale the prior round metric to one of radius $\left(\frac{b'}{\pi}\right)$ in $\mathcal{W}(0,b')$. Now having constructed an isotopy from $\mathcal{W}(0,b)$ to $\mathcal{W}(0,b')$, we can invoke path-connectedness of $\mathcal{W}(0,b)$ to isotopy an arbitrary $g \in \mathcal{W}(0,b)$ to an arbitrary $g' \in \mathcal{W}(0,b')$. Therefore $\mathcal{W}$ is path-connected. □

4.2 Single and double torpedo metrics

We proceed to define a particularly useful metric on the disc D^n, known as a *δ-torpedo metric*, an $O(n)$-symmetric metric which near the boundary of D^n is a product of the interval with the $(n-1)$-sphere of radius δ, and near the centre of D^n is the n-sphere of radius δ. Elementary calculations* show that these metrics have positive scalar curvature. Furthermore, Proposition 5.3 of [5] shows that torpedo metrics have positive (p, n)-intermediate scalar curvature. Such metrics are best thought of as warped product metrics; consider the following collection of warping functions:

Definition 4.2.1. *A δ-torpedo function, denoted f_δ, is a smooth real-valued function on $(0, \infty)$ which satisfies the following conditions:*

$$\begin{aligned} &f_\delta(t) = \delta \sin\left(\frac{t}{\delta}\right), \text{ for } t \text{ near } 0, \\ &f_\delta(t) = \delta, \text{ for } t \geq \delta\frac{\pi}{2}, \text{ and} \\ &\ddot{f}_\delta(t) \leq 0. \end{aligned}$$

As a consequence of Lemma 3.0.1, the metric $dr^2 + f_\delta^2(r)ds_{n-1}^2$ on $(0, \infty) \times S^{n-1}$ extends smoothly to a metric on $\mathbb{R}^n$, where r coincides with the radial parameter in $\mathbb{R}^n$. By restricting the interval from $(0, \infty)$ to $(0, b)$ for some $b > \delta\frac{\pi}{2}$, we obtain a metric on the disc D^n, defined formally below.

Definition 4.2.2. *We denote by $g_{tor}^n(\delta)$ the δ-torpedo metric, a metric on the disc D^n given by*

$$g_{tor}^n(\delta) = dt^2 + f_\delta^2(t)ds_{n-1}^2,$$

where f_δ is a torpedo function on $(0, b)$, $b > \delta\frac{\pi}{2}$.

From the torpedo metric on the disc D^n we can derive a metric on the sphere S^n, by considering the following element of $\mathcal{F}(0, b)$, defined as a "double" of torpedo functions.

Definition 4.2.3. *A δ-double torpedo function is a smooth function $\bar{f}_\delta : (0, b) \to (0, \infty)$ satisfying*

$$\begin{aligned} &\bar{f}_\delta(t) = f_\delta(t) : \ t \in \left(0, \frac{b}{2}\right), \text{ and} \\ &\bar{f}_\delta(t) = f_\delta(b - t) : \ t \in \left(\frac{b}{2}, b\right), \end{aligned}$$

*Section 1.3 [48].

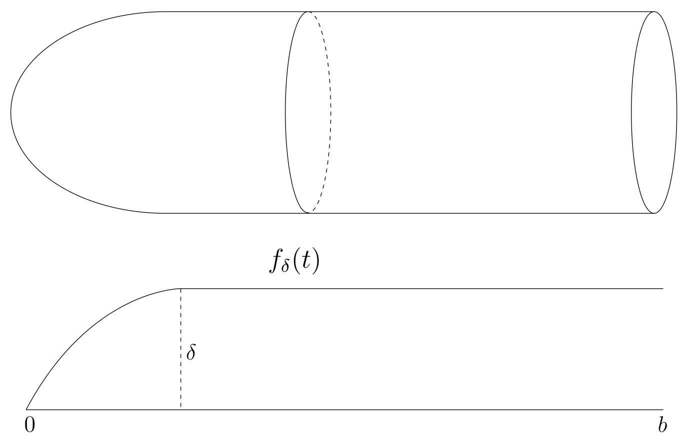


Figure 4.1: A schematic of a torpedo metric and function.

where $\frac{b}{2} > \delta\frac{\pi}{2}$.

Using $\bar{f}_\delta$ as a warping function allows us to construct a metric on the n-sphere, defined as follows.

Definition 4.2.4. *We denote by* $g^n_{Dtor}(\delta)$ *the* δ*-double torpedo metric*

$$g^n_{Dtor}(\delta) = dt^2 + \bar{f}^2_\delta(t)ds^2_{n-1},$$

where f_δ *is a torpedo function on* $(0, b)$, $b > \delta\frac{\pi}{2}$.

This metric has positive (p, n)-intermediate scalar curvature for $p \leq n-3$ as a consequence of Lemma 4.1.3, and therefore is an element of the space $\mathcal{W}$. Furthermore, as a consequence of the same result, this metric is isotopic to the round metric ds^2_n.

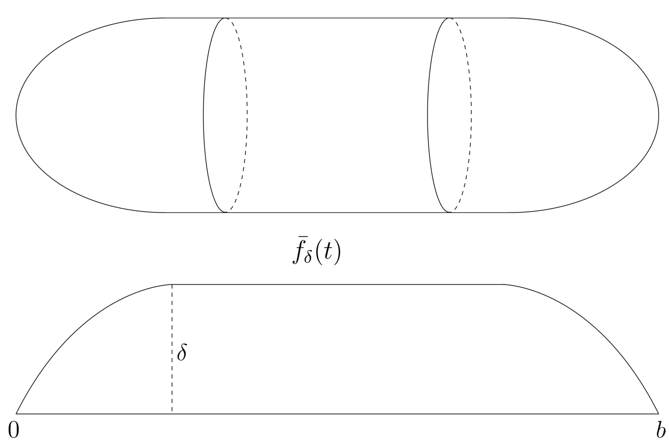


Figure 4.2: A schematic of a double torpedo function & metric.

Corollary 4.2.5. *The metric g^n_{Dtor} is isotopic to ds_n^2 in $\mathcal{R}^{s_{p,n}>0}(S^n)$, for $n \geq 3$ and $p \leq n-3$.*

Proof. This follows from Lemma 4.1.3. □

4.3 Doubly-warped product spaces

Let k, ℓ be non-negative integers such that $k + \ell + 1 = n$, $\alpha,\ \beta : (0, b) \to (0, \infty)$ smooth functions, and let $g = dt^2 + \alpha^2(t)ds_k^2 + \beta^2(t)ds_\ell^2$, $t \in (0, b)$ be the resulting doubly-warped product metric on the product manifold $(0, b) \times S^k \times S^\ell$, defined in Section 3.3. Let $\mathcal{F}_\alpha(0, b)$ denote the space of all smooth functions α which satisfy the conditions of Lemma 3.0.2, in addition to

$$\begin{aligned}
&\alpha_{tt} \leq 0,\\
&\alpha_{tt}(t) < 0 \text{ when } t \text{ is close to } b, \text{ and}\\
&\alpha_{ttt}(b) > 0.
\end{aligned}$$

Let $\mathcal{F}_\beta(0, b)$ denote the space of all smooth functions β which satisfy the conditions of Lemma 3.0.2:

$$\begin{aligned}
&\alpha(0) > 0, \quad \alpha^{(\text{odd})}(0) = 0, \quad \alpha_t(b) = -1, \quad \alpha^{(\text{even})}(b) = 0\\
&\beta(b) > 0, \quad \beta^{(\text{odd})}(b) = 0, \quad \beta_t(0) = 1, \quad \beta^{(\text{even})}(0) = 0,
\end{aligned}$$

in addition to

$$\begin{aligned}
&\beta_{tt} \leq 0,\\
&\beta_{tt}(t) < 0 \text{ when } t \text{ is close to } 0, \text{ and}\\
&\beta_{ttt}(0) < 0.
\end{aligned}$$

From Lemma 3.0.2, each pair $(\alpha, \beta) \in \mathcal{F}_\alpha(0, b) \times \mathcal{F}_\beta(0, b)$ gives a smooth doubly warped product metric on S^n. We define the space of such metrics as

$$\hat{\mathcal{W}}^{k,\ell}(0, b) = \{dt^2 + \alpha^2(t)ds_k^2 + \beta^2(t)ds_\ell^2 :\ (\alpha, \beta) \in \mathcal{F}_\alpha(0, b) \times \mathcal{F}_\beta(0, b)\}.$$

Remark 4.3.1. *These conditions give us functions α, β which, near b and 0, behave like cosine near $\frac{\pi}{2}$ and sine near 0 respectively. Moreover, this space contains the round metric ds_n^2 on S^n realised as the doubly warped product metric $dt^2 + \cos^2(t)ds_k^2 + \sin^2(t)ds_\ell^2$.*

Lemma 4.3.2. *Let $n \geq 3$ and let k, ℓ be non-negative integers such that $k + \ell + 1 = n$ and $\ell \geq 2$. Then, for $p \leq \ell - 2$, $\hat{\mathcal{W}}^{k,\ell}(0, b)$ is a path-connected subspace of $\mathcal{R}^{s_{p,n}>0}(S^n)$.*

Proof. In Lemma 3.3.1, we compute the expression for the (p, n)-intermediate scalar curvature of a doubly warped product metric:

$$s_{p,n}(P) = 2(\ell - p)(\ell - p - 1)\frac{1 - \beta_t^2}{\beta^2} + A^2(1 - \beta_t^2) + B^2(1 - \alpha_t^2) - C^2\beta_{tt} - D^2\alpha_{tt} - E^2\alpha_t\beta_t.$$

By definition, α and β are concave downward, that is to say $\alpha_{tt}, \beta_{tt} < 0$. We begin by establishing the following:

$$\begin{aligned} \alpha_t(0) = 0 \text{ and } \alpha_t(b) = -1 &\Rightarrow -1 < \alpha_t < 0 \text{ by downward concavity} \\ &\Rightarrow 0 < \alpha_t^2 < 1, \end{aligned}$$

and

$$\begin{aligned} \beta_t(0) = 1 \text{ and } \beta_t(b) = 0 &\Rightarrow 0 < \beta_t < 1 \text{ by downward concavity} \\ &\Rightarrow 0 < \beta_t^2 < 1. \end{aligned}$$

Therefore $1 - \alpha_t^2$, $1 - \beta_t^2 > 0$, and, invoking the precondition on p, we conclude that the first term of $s_{p,n}(P)$ is non-negative, and the second and third terms are positive. By the prior statements, $\alpha_{tt} < 0$, $\beta_{tt} < 0$, $\alpha_t\beta_t \leq 0$, and we conclude that the fourth and fifth terms in this sum are positive, and the sixth term is non-negative. Recall that α, β vanish at b, 0 respectively; in these limits positivity follows from an elementary application of l'Hopital's rule. Thus $\hat{\mathcal{W}}^{k,\ell}(0, b) \subset \mathcal{R}^{s_{p,n}>0}(S^n)$. Path-connectivity follows from the fact that $\mathcal{F}_\alpha(0, b) \times \mathcal{F}_\beta(0, b)$ is convex: let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{F}_\alpha(0, b) \times \mathcal{F}_\beta(0, b)$. Since this set is convex, $(1 - c)(\alpha_1, \beta_1) + c(\alpha_2, \beta_2) \in \mathcal{F}_\alpha(0, b) \times \mathcal{F}_\beta(0, b)$. Thus two arbitrary metrics in $\mathcal{W}^{k,\ell}(0, b)$

$$\begin{aligned} g_1 &= dt^2 + \alpha_1^2(t)ds_k^2 + \beta_1(t)^2 ds_\ell^2 \\ g_2 &= dt^2 + \alpha_2^2(t)ds_k^2 + \beta_2(t)^2 ds_\ell^2, \end{aligned}$$

are connected by a path

$$dt^2 + ((1 - c)\alpha_1(t) + c\alpha_2(t))^2\, ds_k^2 + ((1 - c)\beta_1(t) + c\beta_2(t))^2\, ds_\ell^2.$$

□

By way of notation, let

$$\mathcal{F}_\alpha \times \mathcal{F}_\beta = \bigcup_{b\in(0,\infty)} \mathcal{F}_\alpha(0,b) \times \mathcal{F}_\beta(0,b),$$
$$\hat{\mathcal{W}}^{k,\ell} = \bigcup_{b\in(0,\infty)} \hat{\mathcal{W}}^{k,\ell}(0,b), \text{ and}$$
$$\hat{\mathcal{W}} = \bigcup_{k+\ell+1=n} \hat{\mathcal{W}}^{k,\ell}.$$

Then we have the following result:

Lemma 4.3.3. *$\hat{\mathcal{W}}$ is a path-connected subspace of $\mathcal{R}^{s_{p,n}>0}(S^n)$, for $p \leq \ell - 2$.*

Proof. First, we will show that $\hat{\mathcal{W}}^{k,\ell}$ is path-connected. Let $g \in \hat{\mathcal{W}}^{k,\ell}(0,b)$ and $g' \in \hat{\mathcal{W}}^{k,\ell}(0,b')$. Since each of these spaces is path-connected, we can isotopy g and g', by a linear homotopy of warping functions, to the round metrics

$$dt^2 + \left(\frac{b}{\pi}\right)^2 \cos^2\left(\frac{\pi t}{b}\right) ds_k^2 + \left(\frac{b}{\pi}\right)^2 \sin^2\left(\frac{\pi t}{b}\right) ds_\ell^2, \text{ and}$$

$$dt^2 + \left(\frac{b'}{\pi}\right)^2 \cos^2\left(\frac{\pi t}{b'}\right) ds_k^2 + \left(\frac{b'}{\pi}\right)^2 \sin^2\left(\frac{\pi t}{b'}\right) ds_\ell^2,$$

respectively. These round metrics are isotopic by rescaling, therefore g, $g' \in \hat{\mathcal{W}}^{k,\ell}$ are isotopic. □

4.4 Mixed torpedo metrics

We introduce the concept of a mixed torpedo metric on S^n, a particular type of doubly-warped product metric where the warping functions are torpedo functions. We will show this metric is isotopic to the standard round metric in the space of positive (p,n)-intermediate scalar curvature metrics on S^n.

Definition 4.4.1. *Let $f_\epsilon, f_\delta : (0,b) \to (0,\infty)$ be torpedo functions where $b > max\{\epsilon\pi, \delta\pi\}$. A mixed torpedo metric, denoted $g^{k,\ell}_{Mtor}$, is the smooth metric on S^n determined by the following expression.*

$$g^{k,\ell}_{Mtor} = dt^2 + f_\epsilon^2(b-t)ds_k^2 + f_\delta^2(t)ds_\ell^2.$$

This metric is smooth and has positive (p,n)-intermediate scalar curvature as a consequence of Lemma 4.3.2, for $p \leq \ell - 2$. It follows from Lemma 3.0.2 that $g^{k,\ell}_{\mathrm{Mtor}}$ is a metric on S^n. To better understand this metric, consider decomposing the sphere S^n as a union of sphere-disc products as follows: recall the embedding of the sphere E from equation (3.1), which gives the round unit radius sphere metric as a doubly-warped product metric seen in equation (3.2). By restricting this embedding as

$$E_{k,\ell}\left(\left(0,\frac{\pi}{4}\right)\times S^k\times S^\ell\right)\cong S^k\times D^{\ell+1},$$
$$E_{k,\ell}\left(\left(\frac{\pi}{4},\frac{\pi}{2}\right)\times S^k\times S^\ell\right)\cong D^{k+1}\times S^\ell,$$

which, via the embedding E, we identify with the following standard decomposition of the sphere:

$$\begin{aligned} S^n &= \partial(D^{n+1}) \\ &\cong \partial(D^{k+1}\times D^{\ell+1}) \\ &\cong (S^k\times D^{\ell+1})\cup_{S^k\times S^\ell}(D^{k+1}\times S^\ell). \end{aligned}$$

For a schematic of such a decomposition of S^n equipped with the round metric ds_n^2, see Figure 4.3.

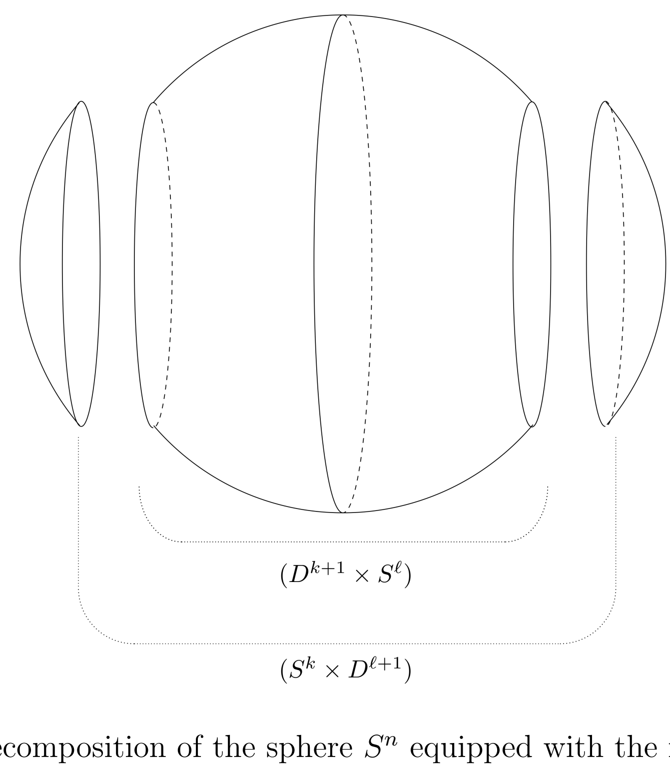


Figure 4.3: A decomposition of the sphere S^n equipped with the round metric ds_n^2.

Equipping $S^k\times D^{\ell+1}$ with the metric $\epsilon^2 ds_k^2+g_{\text{tor}}^{\ell+1}(\delta)$, and $D^{k+1}\times S^\ell$ with the metric $g_{\text{tor}}^{k+1}(\epsilon)+\delta^2 ds_\ell^2$, these metrics glue together smoothly along $S^k\times S^\ell$, forming the *mixed* *torpedo metric* $g^{k,\ell}_{\mathrm{Mtor}}$ on S^n as defined in Definition 4.4.1; see Figure 4.4.

Lemma 4.4.2. *Let $n \geq 3$, and k, ℓ be non-negative integers such that $n = k + \ell + 1$, $\ell \geq 2$, and $p \leq \ell - 2$. Let $f_\epsilon, f_\delta : (0, b) \to (0, \infty)$ be torpedo functions where $b > max\{\epsilon\pi, \delta\pi\}$. Then the mixed torpedo metric $g^{k,\ell}_{Mtor} = dt^2 + f^2_\epsilon(b-t)ds^2_k + f^2_\delta(t)ds^2_\ell$ is isotopic to the round metric ds^2_n in $\mathcal{R}^{s_{p,n}>0}(S^n)$.*

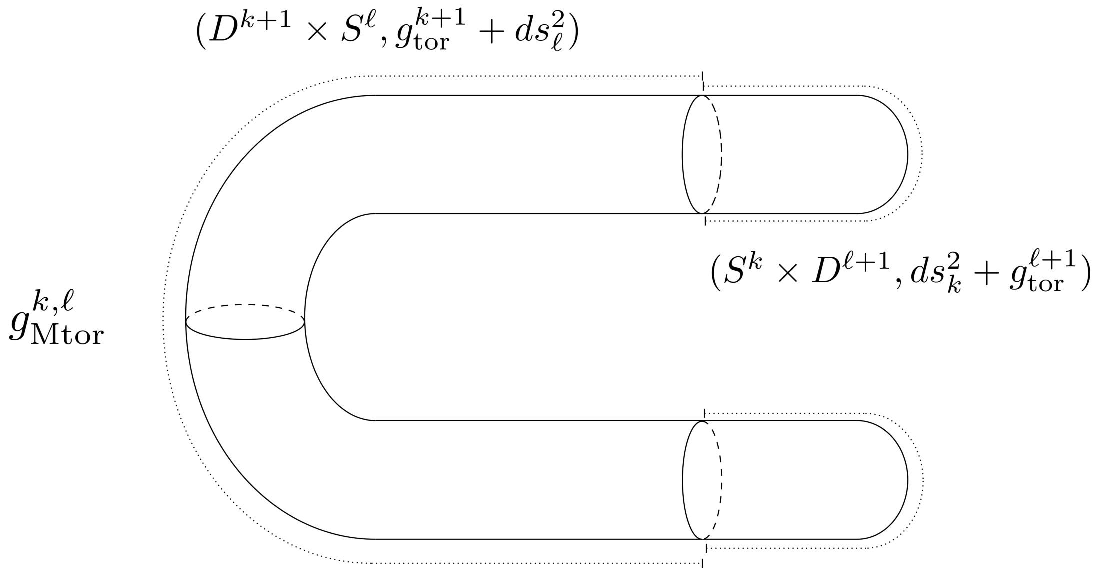
Proof. We begin by noting that the torpedo functions f_ϵ and f_δ lie in the function spaces $\mathcal{F}_\alpha(0, b)$ and $\mathcal{F}_\beta(0, b)$ respectively. Therefore the mixed torpedo metric is an element of $\hat{\mathcal{W}}^{k,\ell}(0, b)$. The standard round metric $ds^2_n = dt^2 + \cos^2(t)ds^2_k + \sin^2(t)ds^2_\ell$ is an element of $\hat{\mathcal{W}}^{k,\ell}(0, b)$, and therefore isotopy follows as a consequence of path-connectedness established in Lemma 4.3.3. Explicitly, such an isotopy takes the form

$$g_r = dt^2 + h^2_{1,r}(t)ds^2_k + h^2_{2,r}(t)ds^2_\ell, \ r \in [0, 1],$$

where $h_{1,r}(t)$, $h_{2,r}(t) : [0, 1] \times (0, b) \to \mathbb{R}^2$ are homotopies of curves in $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ respectively such that

$$\begin{aligned} h_{1,r}(t) &= (1 - r)f_\epsilon(b - t) + r\cos(t), \text{ and} \\ h_{2,r}(t) &= (1 - r)f_\delta(t) + r\sin(t). \end{aligned}$$

□



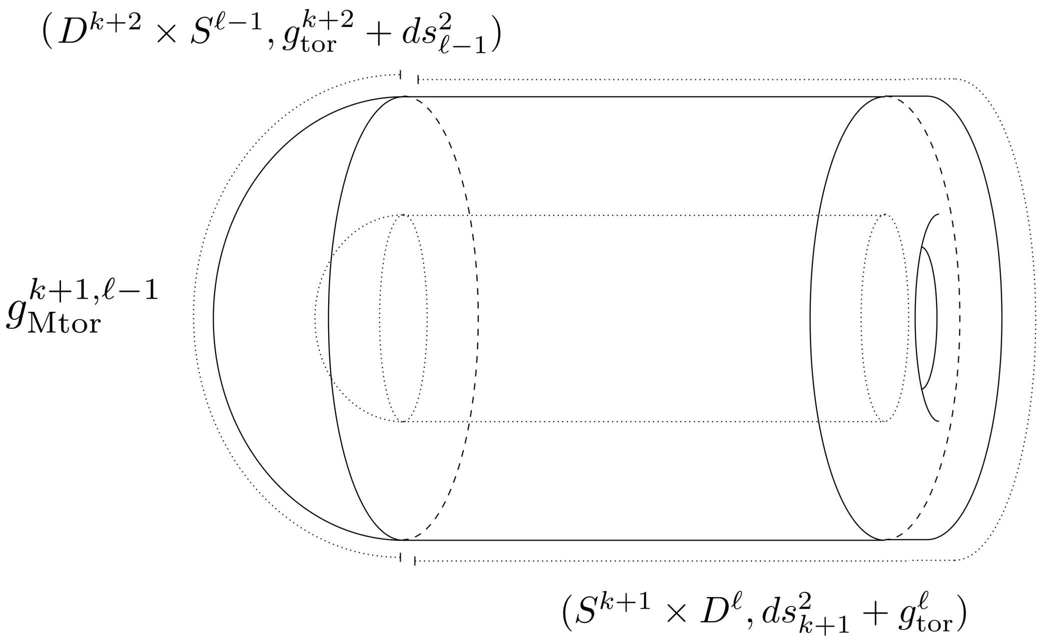


Figure 4.4: Schematics of the mixed torpedo metrics $g_{\text{Mtor}}^{k,\ell}$ and $g_{\text{Mtor}}^{k+1,\ell-1}$.

Chapter 5

A Relative Isotopy of Mixed Torpedo Metrics with Boundary

In the study of manifolds with metrics of positive (p, n)-intermediate scalar curvature, it is natural to consider the existence of such metrics on manifolds with boundary that behave well near the boundary. In this spirit, let M^n be a smooth, compact manifold with boundary ∂M, a closed $(n-1)$-dimensional manifold. Consider the space of metrics of positive (p, n)-intermediate scalar curvature on M which exhibit a product structure near the boundary. The consideration of this space will allow us to introduce a notion of *relative isotopy* * in the context of positive (p, n)-intermediate scalar curvature. The majority of this chapter will be devoted to the construction of a particular relative isotopy and the proof of its curvature-positivity. This relative isotopy will be a crucial tool in the proof of the main theorem of this thesis.

5.1 Relative isotopy

Let M^n be a smooth, closed manifold with boundary. We denote by $\mathcal{R}(M, \partial M) \subset \mathcal{R}(M)$ the space of Riemannian metrics on M which take on a product structure in proximity of the boundary ∂M. Specifying a *collar* embedding $c : [0, \epsilon) \times \partial M \hookrightarrow M$ we formally define this space as

$$\mathcal{R}(M, \partial M) = \{\bar{g} \in \mathcal{R}(M) : \ c^*\bar{g} = g + dt^2 \text{ for some } g \in \mathcal{R}(\partial M)\}.$$

We will consider the subspace $\mathcal{R}^{s_{p,n}>0}(M, \partial M)$ of such metrics which have positive (p, n)-intermediate scalar curvature. We consider a stronger notion of isotopy in this space, referred

*This concept is described in Section 2.1 of [49] in the context of positive scalar curvature.

to as *relative isotopy* and described in [49]. First, fix $g \in \mathcal{R}(\partial M)$ and consider the subspace of $\mathcal{R}(M, \partial M)$ consisting of metrics which restrict to g on the boundary, which we denote $\mathcal{R}(M, (\partial M, g))$. Furthermore, we denote by $\mathcal{R}^{s_{p,n}>0}(M, (\partial M, g))$ the space of such metrics with positive (p, n)-intermediate scalar curvature.

Definition 5.1.1. *Let $h, h' \in \mathcal{R}^{s_{p,n}>0}(M, (\partial M, g))$. These metrics are said to be isotopic relative to g if they lie in the same path component of this space of metrics.*

Consider the sphere S^{n+1} equipped with the mixed torpedo metric $g^{k,\ell+1}_{\mathrm{Mtor}}$, which, on the equator S^n restricts to the metric $g^{k,\ell}_{\mathrm{Mtor}}$. Let $g^{k,\ell+1}_{\mathrm{Mtor}}(+)$ denote the metric on the upper hemisphere, $S^{n+1}_+ \cong D^{n+1}$.

Notably, this metric does not have a product structure near the boundary. Ideally, we wish to construct a metric which is, roughly, obtained by gluing along the boundary the hemisphere $(S^{n+1}_+, g^{k,\ell+1}_{\mathrm{Mtor}}(+))$ to a cylinder of the boundary $(S^n \times I, g^{k,\ell}_{\mathrm{Mtor}} + dr^2)$; see Figure 5.2. While this almost works, the resulting "metric" on the slice $S^n \times \{1\}$ (which is identified with $\partial S^{n+1}_+ = S^n$) is not C^2-smooth.

Our goal in this section is to construct appropriate analogues of the mixed torpedo metrics (previously defined on the sphere) for a disc which, crucially, satisfy a product structure near the boundary. In particular, these *mixed torpedo metrics with boundary* will be defined on the disc D^{n+1} to satisfy the following conditions. On an annular region near the boundary of D^{n+1}, diffeomorphic to $S^n \times I$ such a metric is a product $(S^n \times I, g^{k,\ell}_{\mathrm{Mtor}} + dr^2)$. It then transitions to the metric $g^{k,\ell+1}_{\mathrm{Mtor}}(+)$ on a smaller disc in the centre of the orginal D^n. Ultimately we will want this metric to satisfy certain $s_{p,n} > 0$ curvature properties, however we will defer this for now. This metric (or rather collection of metrics as k, ℓ and n vary) will be denoted $\bar{g}^{k,\ell+1}_{\mathrm{Mtor}}$ and we will provide an explicit geometric description in the next section.

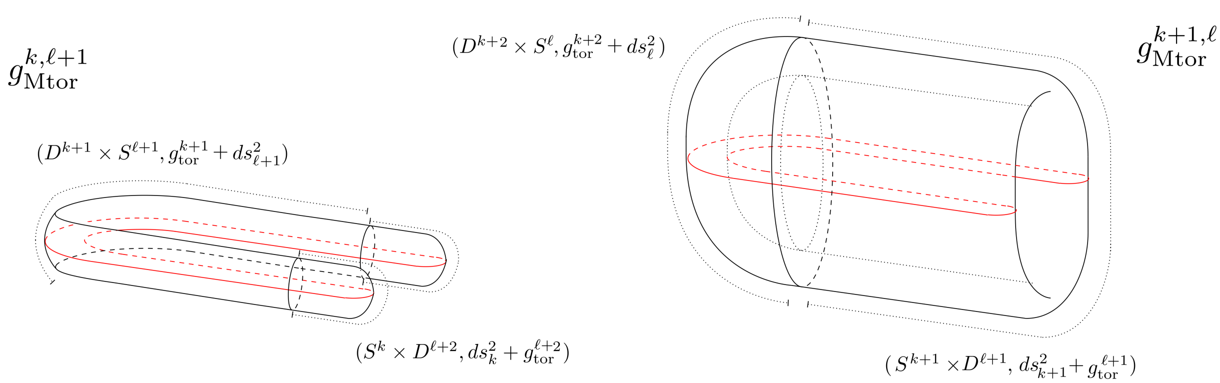


Figure 5.1: Schematics of $g^{k,\ell+1}_{\mathrm{Mtor}}$ and $g^{k+1,\ell}_{\mathrm{Mtor}}$.

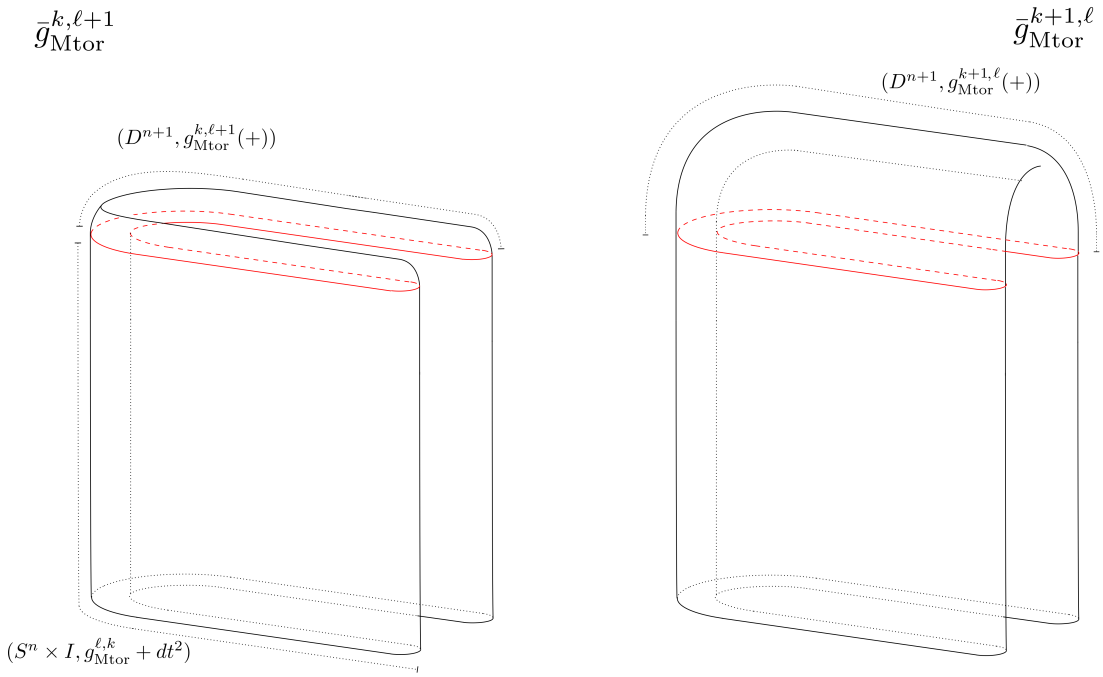


Figure 5.2: Rough sketch of mixed torpedo metrics with boundary, $\bar{g}^{k,\ell+1}_{\mathrm{Mtor}}$ and $\bar{g}^{k+1,\ell}_{\mathrm{Mtor}}$.

5.2 An analysis of the metric $\bar{g}^{k,\ell+1}_{\mathbf{Mtor}}$ on D^{n+1}

In order to construct the metric $\bar{g}^{k,\ell+1}_{\mathrm{Mtor}}$, as suggested in Figure 5.3, on D^{n+1}, we begin by decomposing the disc into three pieces.

A. an annular closed neighbourhood of the boundary diffeomorphic to $S^n \times I$,

B. a region diffeomorphic to $D^{k+1} \times D^{\ell+1}$,

C. a region diffeomorphic to $S^k \times D^{\ell+2}_+$

Recall that $D^{\ell+2}_+$ is the closed upper half disc in $D^{\ell+2}$. In the decomposition of S^{n+1} as $(0,b) \times S^k \times S^{\ell+1}$, choose an equator $S^\ell \subset S^{\ell+1}$, and consider the warped product metric restricted to $(0,b) \times S^k \times S^\ell$. This gives a unit round $S^n \subset S^{n+1}$, which is necessarily an equator. Thus we can split S^{n+1} into $S^{n+1}_\pm$ along this equator. The resulting pieces correspond to the splitting of $S^{\ell+1}$ within the product into $S^{\ell+1}_\pm$ along the chosen equator S^{ell}. Regions B and C then correspond to the products over $\left(0, \frac{b}{2}\right)$, $\left(\frac{b}{2}, b\right)$ respectively restricted to S^{n+1}_+:

$$\begin{aligned} S^{n+1}_+ &\cong S^k \times D^{\ell+2}_+ \cup_{S^k \times S^{\ell+1}} D^{k+1} \times S^{\ell+1}_+ \\ &\cong S^k \times D^{\ell+2}_+ \cup_{S^k \times S^{\ell+1}} D^{k+1} \times D^{\ell+1}. \end{aligned}$$

Note that, though not explicitly shown in Figure 5.3, this same embedding, restricted on the boundary sphere S^n extends in the obvious slice-wise way to the following decomposition of the annular boundary region, A:

$$A \cong S^n \times I \cong (S^k \times D^{\ell+1} \cup_{S^k \times S^\ell} D^{k+1} \times S^\ell) \times I.$$

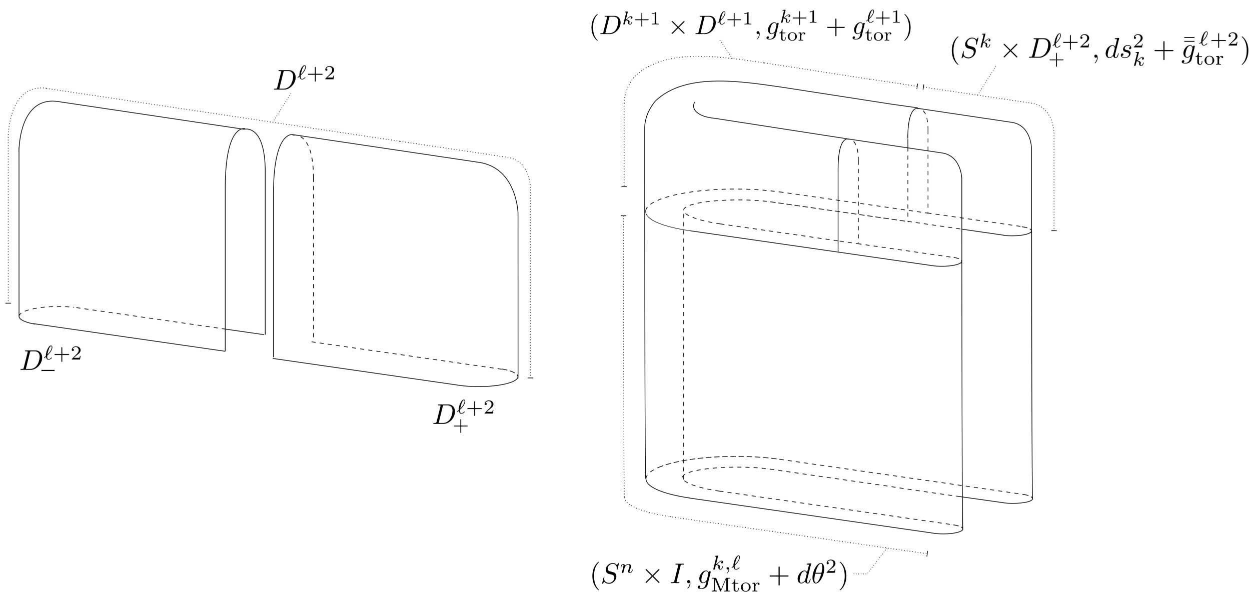


Figure 5.3: Schematics of a decomposition of $D^{\ell+2}$ into two manifolds with corners, and a decomposition of $\bar{g}_{\mathrm{Mtor}}^{k,\ell+2}$.

To define $\bar{g}_{\mathrm{Mtor}}^{k,\ell+1}$, we now equip these regions with metrics which will smoothly glue together. These are already suggested in Figure 5.3, although at least one factor metric, $\bar{\bar{g}}_{\mathrm{tor}}^{\ell+2}$ on the half-disc $D_+^{\ell+2}$, has yet to be defined. We will come back to this shortly. We begin with the annular region, A, diffeomorphic to $S^n \times I$. The metric here will be the standard product of mixed torpedo metrics on S^n, $g_{\mathrm{Mtor}}^{k,\ell} + dr^2$ where r is the coordinate on the interval I. Note that this restricts on the sub-regions of A specified in the decomposition above as

$$(S^k \times D^{\ell+1} \times I, ds_k^2 + g_{\mathrm{tor}}^{\ell+1} + dr^2),$$

and

$$(D^{k+1} \times S^\ell \times I, g_{\mathrm{tor}}^{k+1} + ds_\ell^2 + dr^2).$$

Next we equip Region B, diffeomorphic to $D^{k+1} \times D^{\ell+1}$ with the product of torpedo metrics $g_{\mathrm{tor}}^{k+1} + g_{\mathrm{tor}}^{\ell+1}$. Note that this region (a manifold with corners) intersects Region A along an embedded $D^{k+1} \times S^\ell \subset S^n \times \{1\}$. Letting r denote the radial coordinate for the disc factor $D^{\ell+1}$ and so writing the torpedo metric $g_{\text{tor}}^{\ell+1} = dr^2 + \beta(r)^2 ds_\ell^2$ where $\beta \equiv 1$ near the boundary of $D^{\ell+1}$ (where the metric is cylindrical) it is clear that Region B and Region A equipped with their respective metrics smoothly attach. It remains to extend this metric on Region C.

Region C is diffeomorphic to the product of discs $S^k \times D_+^{\ell+2}$, where $D_+^{\ell+2}$ denotes the closed upper "half" of $D^{\ell+2}$ seen in Figure 5.3. Its boundary intersection with Region A is an embedded $S^k \times D^{\ell+1}$. Its boundary intersection with Region B is also an embedded $S^k \times D^{\ell+1}$ although these appear orthogonal to each other in Figure 5.3. Near these boundaries, the metrics on A and B respectively take the forms $ds_k^2 + g_{\text{tor}}^{\ell+1} + dr^2$ and $ds_k^2 + g_{\text{tor}}^{\ell+1} + dt^2$. Thus, near the boundaries the coordinate r should be thought, according to the layout in Figure 5.3 to run vertically, while the t coordinate runs horizontally.

Thus, to ensure smoothness, the metric we define on Region C will be defined to extend these product forms in a neighbourhood of the boundary of C. The S^k-factor is common across this region, and on the neighbouring parts of A and C the metrics defined there restrict on it as the standard metric ds_k^2. In order to extend smoothly, we do the same on Region C and simply equip the S^k factor with ds_k^2. Ultimately, we will form a product metric on C and it remains to specify the metric on the half-disc factor, $D_+^{\ell+2}$. This metric will be denoted $\bar{\bar{g}}_{\text{tor}}^{\ell+2}$ and is schematically illustrated in Figure 5.4.

Metrics of this sort were described in Section 5.1 of [5] as being formed by carefully gluing a torpedo-metric hemisphere $g_{\text{tor}+}^{\ell+2}$ to the cylinder of torpedo metrics $g_{\text{tor}}^{\ell+1} + dt^2$. For our purposes, we provide an alternative description; see the left hand side of Figure 5.4.

We begin by identifying the half disc $D_+^{\ell+2}$ with the half-hemisphere (quarter sphere) $S_{++}^{\ell+2} = \{x = (x_1, \cdots, x_{\ell+3}) \in \mathbb{S}^{\ell+2} : x_{\ell+2}, x_{\ell+3} \geq 0)\}$. This is done by restricting the projection map $S_{++}^{\ell+2} \to D_+^{\ell+2}$ which simply drops the last coordinate.

Recall once again, the family of sphere embeddings described in equation (3.1). In the case when $n = \ell + 2$, the embedding $E_{\ell,2}$ decomposes $S^{\ell+2}$ as

$$S^{\ell+2} \cong S^\ell \times D^2 \cup D^{\ell+1} \times S^1.$$

Restricting to the quarter sphere $S_{++}^{\ell+2}$, we obtain

$$D_+^{\ell+2} = S_{++}^{\ell+2} \cong S^\ell \times D_{++}^2 \cup D^{\ell+1} \times (S_{++}^1 \cong I).$$

Here, D_{++}^2 is simply the closed quarter disc of the unit disc. Consider again the description of S_+^{n+1} on page 60. Now choose a second equator $S^\ell \subset S^{\ell+1}$, perpendicular to the original choice. This gives an alternative S_+^{n+1}. Writing down the intersection of these two hemispheres leads to a decomposition of the quarter-sphere S_{++}^{n+1}. We obtain the above

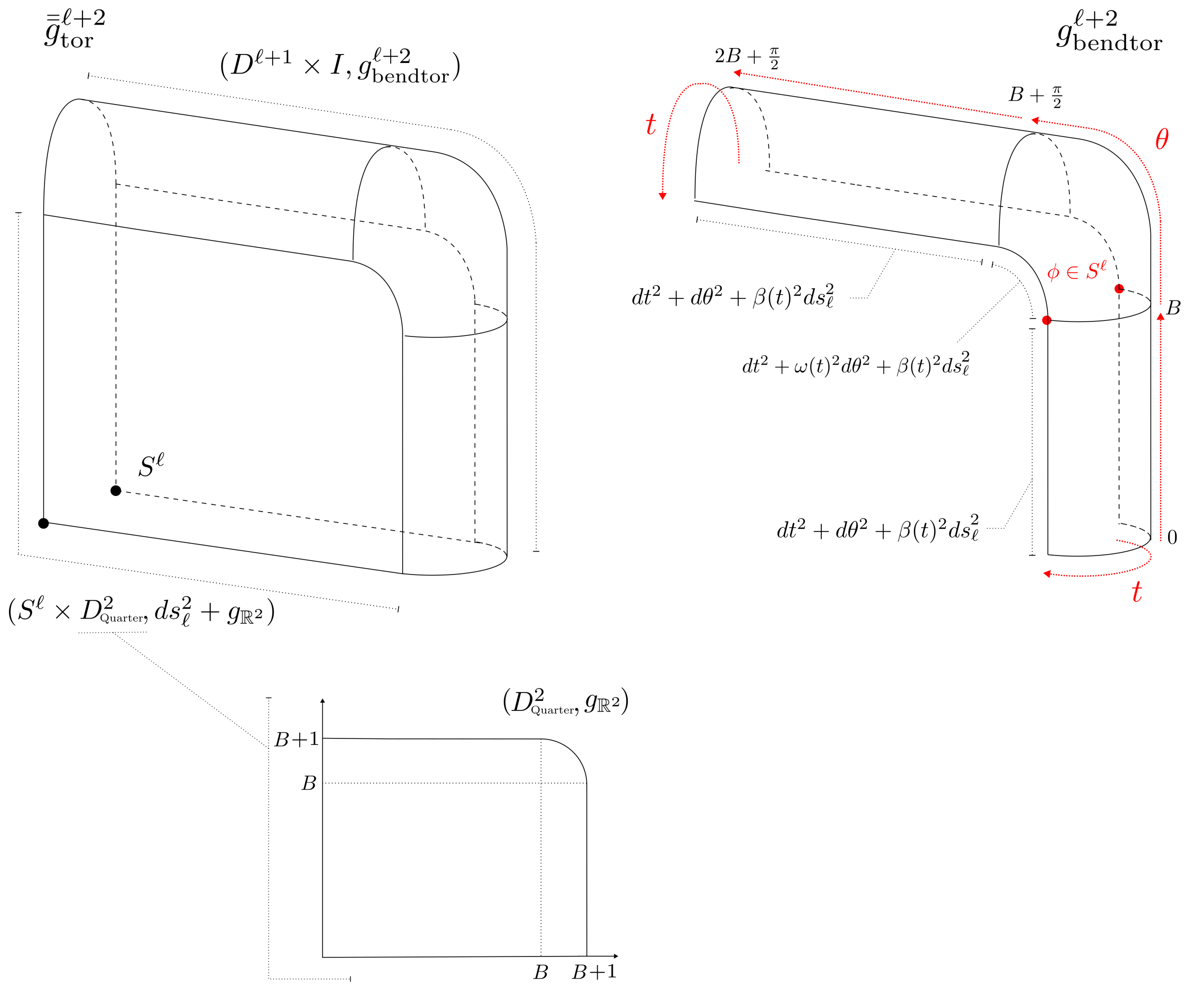


Figure 5.4: Schematics of the metrics $\bar{\bar{g}}^{\ell+2}_{\text{tor}}$, $g_{\mathbb{R}^2}$ and $g^{\ell+2}_{\text{bendtor}}$.

decomposition of $S^{\ell+2}_{++}$ from this model by replacing $n+1$ with $\ell+2$ on the left-hand side, and k by ℓ, ℓ by 0 on the right-hand side.

Remark 5.2.1. *It would be more precise to say $S^1_{++} = [0, \frac{\pi}{2}]$ than $S^1_{++} \cong I$. However, as we will have need to stretch and rescale this interval, it makes sense to only emphasise its topology.*

We will need a slightly adjusted version of D^2_{++}, one where the circular part of the boundaries is straightened near the corners. We will denote this region, D^2_{Quarter}, obtained from the square $I \times I$ smoothly rounding off the corner at $(1,1)$ as shown in lower image in Figure 5.4.

At this stage we have decomposed Region C into two subregions. The one diffeomorphic to $S^k \times D^{\ell+1} \times I$ we call Region 1 and the other, diffeomorphic to $S^k \times S^\ell + D^2_{\text{Quarter}}$, we refer to as Region 2; see Figure 5.8 for details.

On Region 2, diffeomorphic to $S^k \times S^\ell \times D^2_{\text{Quarter}}$, we define $\bar{\bar{g}}^{\ell+2}_{\text{tor}}$ to be the product $ds_k^2 + ds_\ell^2 + g_{\mathbb{R}^2}$ where $g_{\mathbb{R}^2}$ is the standard Euclidean metric on $\mathbb{R}^2$ restricted to this region of the plane. We will now extend this metric over Region 1, diffeomorphic to $S^k \times D^{\ell+1} \times I$. On the sphere factor, we have already decided the metric will the standard one ds_k^2. Thus, we focus exclusively on the cylinder $D^{\ell+1} \times I$. Essentially, we want to construct a "product" of $D^{\ell+1}$-torpedo metrics which bends around a corner, as suggested by Figure 5.4. This metric will be called $g^{\ell+2}_{\text{bendtor}}$ and we construct it below.

Let $b \in (0,1)$, $B > 0$, and let $\Omega : (0,b) \times [0, 2B + \frac{\pi}{2}] \to [0,1]$ be a smooth function, defined as

$$\Omega(t,\theta) = 1 - \mu(\theta) + \mu(\theta)\omega(t),$$

where $\mu : [0, 2B + \frac{\pi}{2}] \to [0,1]$ is a family of smooth bump functions of the type shown in Figure 5.5, and $\omega : (0,b) \to (0,1)$ is a smooth function such that

$$\omega''(t) \leq 0 \text{ for all } t,$$
$$\omega(t) = \cos(t) \text{ near } 0,$$
$$\omega(t) = 1 - t \text{ near } b.$$

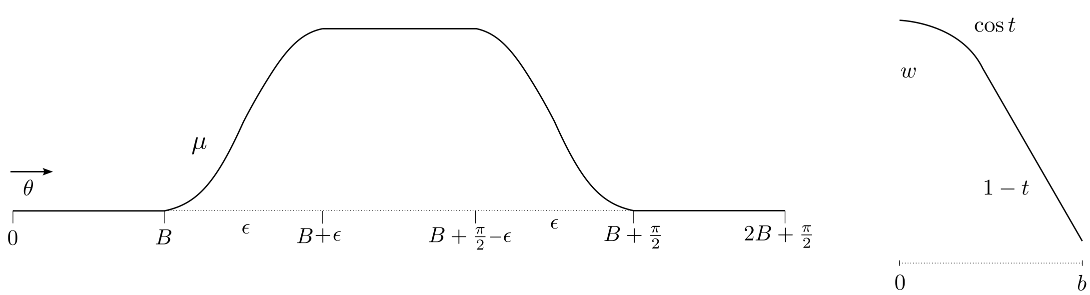


Figure 5.5: Schematics of the functions μ and ω.

The function Ω, seen in Figure 5.6, by means of the smooth family of cut-off functions μ, smoothly interpolates between the constant function 1 and the smooth function ω on the interval $[0, 2B + \frac{\pi}{2}]$ for some positive constant $B > 0$.

The metric $g^{\ell+2}_{\text{bendtor}}$ is therefore given as the warped product metric of the form

$$g^{\ell+2}_{\text{bendtor}} = dt^2 + \Omega(t,\theta)^2 d\theta^2 + \beta(t)^2 ds_\ell,$$

where $\beta : (0,b) \to (0,\infty)$ is a torpedo function such that $(\omega')^2 + (\beta')^2 = 1$. See the right hand side of Figure 5.4. Recall we referred to the region in C diffeomorphic to $S^k \times D^{\ell^1} \times I$

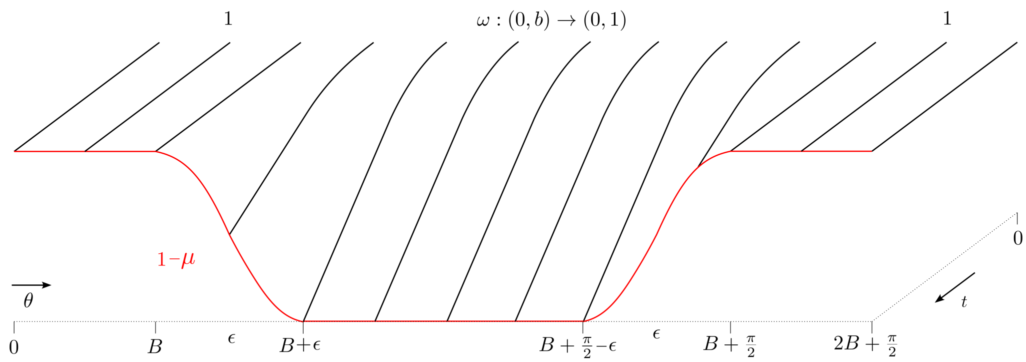


Figure 5.6: A schematic of the function $\Omega(t, \theta)$.

as Region 1. On it, we specify the metric

$$ds_k^2 + g_{\text{bendtor}}^{\ell+2} = ds_k^2 + dt^2 + \Omega(t,\theta)^2 d\theta^2 + \beta(t)^2 ds_\ell$$

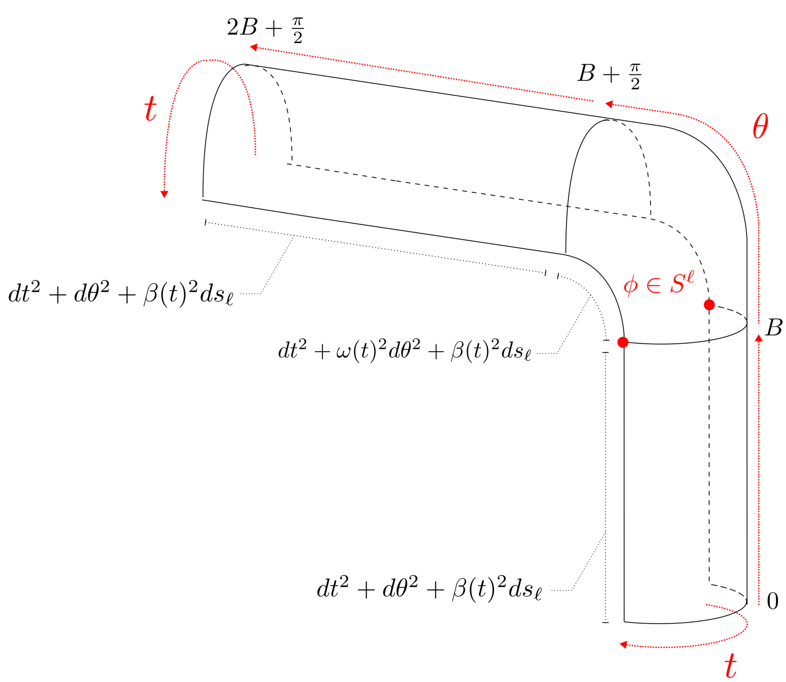


Figure 5.7: A schematic of the metric $g_{\text{bendtor}}^{\ell+2}$.

Recombining, regions 1 and 2 of C with the respective metrics $ds_k^2 + g_{\text{bendtor}}^{\ell+2}$ and $ds_k^2 + ds_\ell^2 + g_{\mathbb{R}^2}$ determines the desired metric $\bar{\bar{g}}_{\text{tor}}^{\ell+2}$ on D^{n+1}. To help the reader see how all of this fits together, we provide Figure 5.8. Here, we show the subdivision of Region C into regions 1 and 2, and provide an analogous subdivision of Region B into regions 3 and 4 indicating the metric restrictions here. Region A, the annular product $S^n \times I$ is relabelled as Region 5.

In the next theorem, we will establish conditions for which the metrics $\bar{\bar{g}}_{\text{tor}}^{\ell+2}$ have positive $s_{p,n}$-curvature and show certain isotopy relations among them. Importantly, these concern relative isotopies which fix metrics on the boundary.

Main Theorem A. *Let $n \geq 3$ and suppose k and ℓ satisfy $n = k + \ell + 1$ with $\ell \geq 2$. Then for all p satisfying $0 \leq p \leq \ell - 2$, the metrics $\bar{g}_{Mtor}^{k,\ell+1}$ and $\bar{g}_{Mtor}^{k+1,\ell}$ are $(s_{p,n+1} > 0)$ isotopic, relative to the metric $g_{Mtor}^{k,\ell}$ on S^n, in the space $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$.*

While this theorem is an existence result, its proof will be entirely constructive. We will explicitly describe an isotopy $G_\tau, \tau \in [0,1]$ which connects the metrics $\bar{g}_{\text{Mtor}}^{k,\ell+1}$ to $\bar{g}_{\text{Mtor}}^{k+1,\ell}$ showing that both the curvature constraint and the requirement that the boundary metric is fixed are satisfied throughout.

This theorem will be of particular importance to the proof of the main result of this thesis, and therefore the remainder of this chapter is devoted to its proof. Before stating it, it is worth emphasising that, once it is established that the metrics $\bar{g}_{\text{Mtor}}^{k,\ell+1}$ and $\bar{g}_{\text{Mtor}}^{k+1,\ell}$ satisfy the $s_{p,n} > 0$, showing these metrics are isotopic in the space $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$ is quite straightforward. In this case, the only constraint (aside from the curvature one) is to maintain a product structure near the boundary. However, the boundary metric itself may vary. Insisting that the boundary metric stay fixed adds a considerable challenge. In any case, we state the following proposition and corollaries concerning this easier case with the weaker boundary condition.

Proposition 5.2.2. *Let $n \geq 3$. The metric $\bar{g}_{Mtor}^{k,\ell+1}$ has positive $(p, n+1)$-intermediate scalar curvature, for $p \leq \ell - 2$ and $n = k + \ell + 1$. In particular, $\bar{g}_{Mtor}^{k,\ell+1} \in \mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$.*

The proof of this proposition is subsumed into that of Main Theorem A in the next section; it is the case when the isotopy constructed in that proof, G_τ, is at $\tau = 0$.

Corollary 5.2.3. *Let $n \geq 3$, $n = k + \ell + 1$ and $p \leq \ell - 2$. Then $\bar{g}_{Mtor}^{k,\ell+1}$ is isotopic to g_{tor}^{n+1} in $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$.*

Proof. The isotopy between these metrics is given explicitly as

$$dt^2 + h_1^2 ds_k^2 + \tilde{h}_2^2 ds_\ell^2|_{S_+^\ell},$$

where h_1 is a straight line homotopy between $\cos(t)$ and $\alpha(t)$, and $\tilde{h}_2 = 1 - \mu(r) + \mu(r)h_2$, with $h_2(t, r)$ a straight line homotopy between sin and β. Curvature-positivity of the metrics in this isotopy follows from the convexity of the function spaces in which the homotopies are contained, in a fashion analogous to Lemma 4.4.2. ∎

Corollary 5.2.4. *Let $n = k + \ell + 1$ and $p \leq \ell - 2$. $\bar{g}^{k,\ell+1}_{Mtor}$ and $\bar{g}^{k+1,\ell}_{Mtor}$ are isotopic in $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1}, S^n)$.*

Proof. This follows immediately from Corollary 5.2.3. $\square$

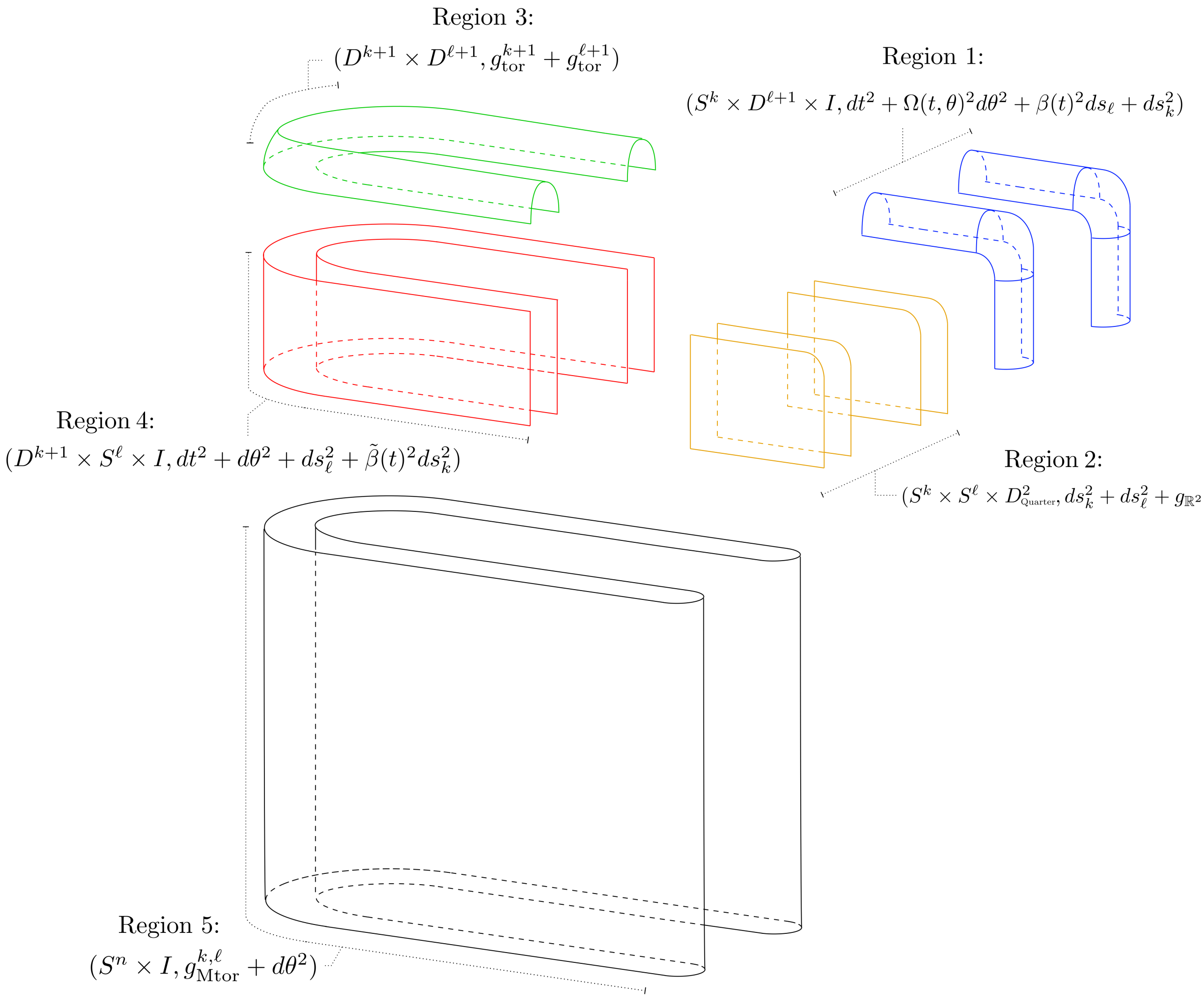


Figure 5.8: A decomposition of the metric $\bar{g}^{k,\ell}_{\text{Mtor}}$.

5.3 The path of metrics G_τ and its components

To prove Main Theorem A, we construct an explicit *relative* isotopy between the Riemannian metrics $\bar{g}^{k,\ell+1}_{\text{Mtor}}$ and $\bar{g}^{k+1,\ell}_{\text{Mtor}}$ in the space $\mathcal{R}^{s_{p,n+1}>0}(D^{n+1},(S^n,g^{k,\ell}_{\text{Mtor}}))$ (which will implicitly show both metrics are elements of this space), which at each stage in the isotopy restricts to $g^{k,\ell}_{\text{Mtor}}$ on the boundary S^n and retains positive $(p,n+1)$-intermediate scalar curvature throughout. This isotopy is denoted G_τ, where $\tau\in\left[0,\frac{\pi}{2}\right]$, such that $G_0=\bar{g}^{k,\ell+1}_{\text{Mtor}}$, $G_{\frac{\pi}{2}}=\bar{g}^{k+1,\ell}_{\text{Mtor}}$, and on the boundary $G_\tau|_{S^n}=g^{k,\ell}_{\text{Mtor}}$ for all τ.

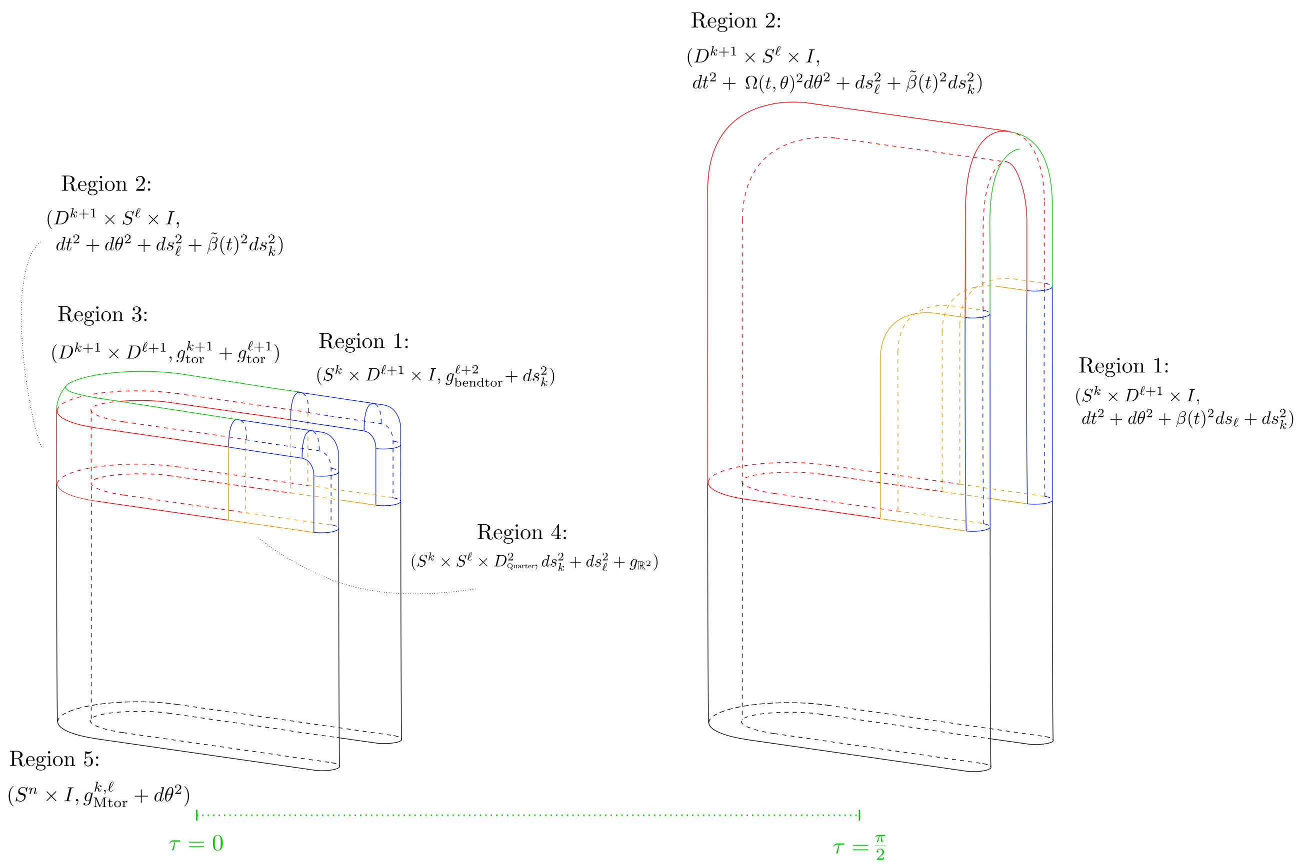


Figure 5.9: A detailed schematic of the metrics $\bar{g}^{k,\ell+1}_{\text{Mtor}}$ and $\bar{g}^{k+1,\ell}_{\text{Mtor}}$.

Utilising our decomposition of the metric $\bar{g}^{k,\ell+1}_{\text{Mtor}}$, it is useful to consider how this isotopy transforms the restricted metrics on each region. The metric $g^{\text{Region 1}}$ transforms from a "bent" state to an "unbent" state across the isotopy. Conversely, the metric $g^{\text{Region 2}}$ transforms from an "unbent" state to a "bent" state throughout the isotopy. The metric $g^{\text{Region 4}}$ transforms in such a fashion that the curvature of the metric on this region remains unchanged, and $g^{\text{Region 3}}$ and $g^{\text{Region 5}}$ remain fixed throughout the isotopy.

What can be said then immediately regarding the curvature of the metric at each stage in this isotopy? The only variation in curvature which can occur will occur on the metrics on Region 1 and Region 2. Therefore curvature positivity of this isotopy follows from the curvature positivity of its restrictions to Region 1 and Region 2, given that, as we will show, the remaining regions begin with positive intermediate scalar curvature.

The proof of Main Theorem A will therefore proceed as follows: we will construct the restrictions of G_τ to Region 1 and Region 2, which for simplicity we denote $g_\tau^{\text{Region 1}}$ and $g_\tau^{\text{Region 2}}$ and compute expressions for the intermediate scalar curvature. The proof therefore is a direct proof of positivity given these expressions.

The isotopy of metrics is induced by the one-parameter family of functions $\Omega_\tau(t,\theta)$, in the parameter $\tau \in \left[0, \frac{\pi}{2}\right]$. This one-parameter family continuously deforms $\Omega_0(t,\theta) = \Omega(t,\theta)$ into the constant function 1, thereby "unbending" the region. That is to say, at $\tau = 0$ we have the metric $g^{\text{Region 1}}$, and at $\tau = \frac{\pi}{2}$ we have the metric

$$dt^2 + d\theta^2 + \beta^2(t)ds_\ell^2 + ds_k^2 = d\theta^2 + g_{\text{tor}}^{\ell+1} + ds_k^2.$$

We therefore alter $\Omega(t,\theta)$ appropriately to obtain this one-parameter family of metrics. Let $b \in (0,1)$, $B > 0$, and let $\Omega_\tau : (0,b) \times [0, 2B + \frac{\pi}{2} - \tau] \to [0,1]$ be a smooth one-parameter family of functions, defined as

$$\Omega_\tau(t,\theta) = 1 - \mu_\tau(\theta) + \mu_\tau(\theta)\omega_\tau(t),$$

where

$$\omega_\tau(t) = \frac{2}{\pi}\tau + \left(1 - \frac{2}{\pi}\tau\right)\omega(t),$$

and

$$\mu_\tau(\theta) = \frac{2}{\pi}\tau + \left(1 - \frac{2}{\pi}\tau\right)\mu(\theta),$$

are straight-line homotopies between the constant function 1 and ω, μ respectively. The behaviour of $\Omega_\tau(t,\theta)$ as $\tau \to \frac{\pi}{2}$ is displayed in Figure 5.10. When $\tau = \frac{\pi}{2}$, the function $\Omega_\tau(t,\theta)$ becomes identically 1 for all θ, yielding a cylindrical product metric. This continuous deformation captures the "unbending" process, whose effect is visualised in Figure 5.11. We define the metric $g_\tau^{\text{Region 1}}$ in the obvious fashion

$$g_\tau^{\text{Region 1}} = dt^2 + \Omega_\tau(t,\theta)^2 d\theta^2 + \beta^2(t)ds_\ell^2 + ds_k^2.$$

In an analogous fashion, the isotopy of Region 2 is given as:

$$g_\tau^{\text{Region 2}} = dt^2 + \Omega_{\frac{\pi}{2}-\tau}(t,\theta)^2 d\theta^2 + ds_\ell^2 + \tilde{\beta}(t)^2 ds_k^2.$$

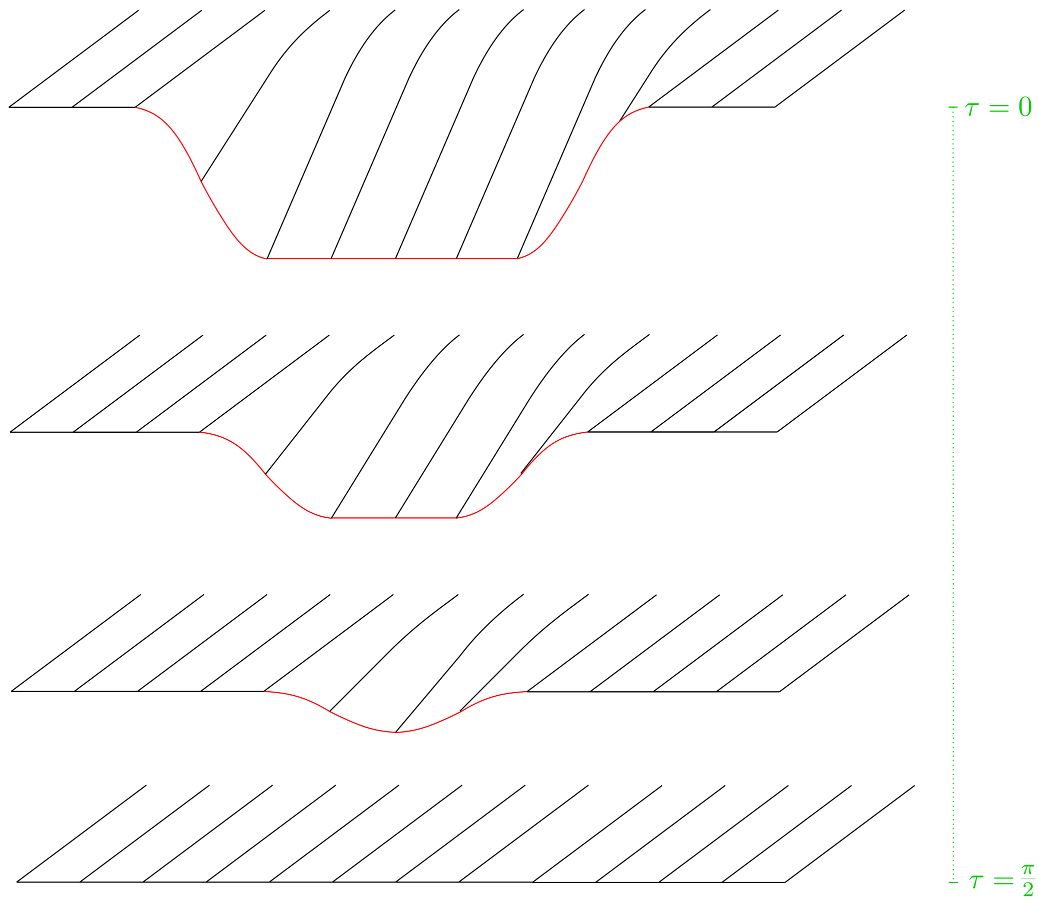


Figure 5.10: A schematic characterising the behaviour of $\Omega_\tau(t,\theta)$, as τ varies.

Opposite to the transformation on Region 1, $\Omega_{\frac{\pi}{2}-\tau}$ is the constant function 1 at $\tau = 0$ and a warping function with smooth ends at $\tau = \frac{\pi}{2}$; consider the suggested deformation in Figure 5.10 reversed. Therefore the metric $g_\tau^{\text{Region 2}}$ at $\tau = 0$ is the metric $g^{\text{Region 2}}$, and at $\tau = \frac{\pi}{2}$ is

$$dt^2 + \Omega(t,\theta)^2 d\theta^2 + ds_\ell^2 + \tilde{\beta}(t)^2 ds_k^2.$$

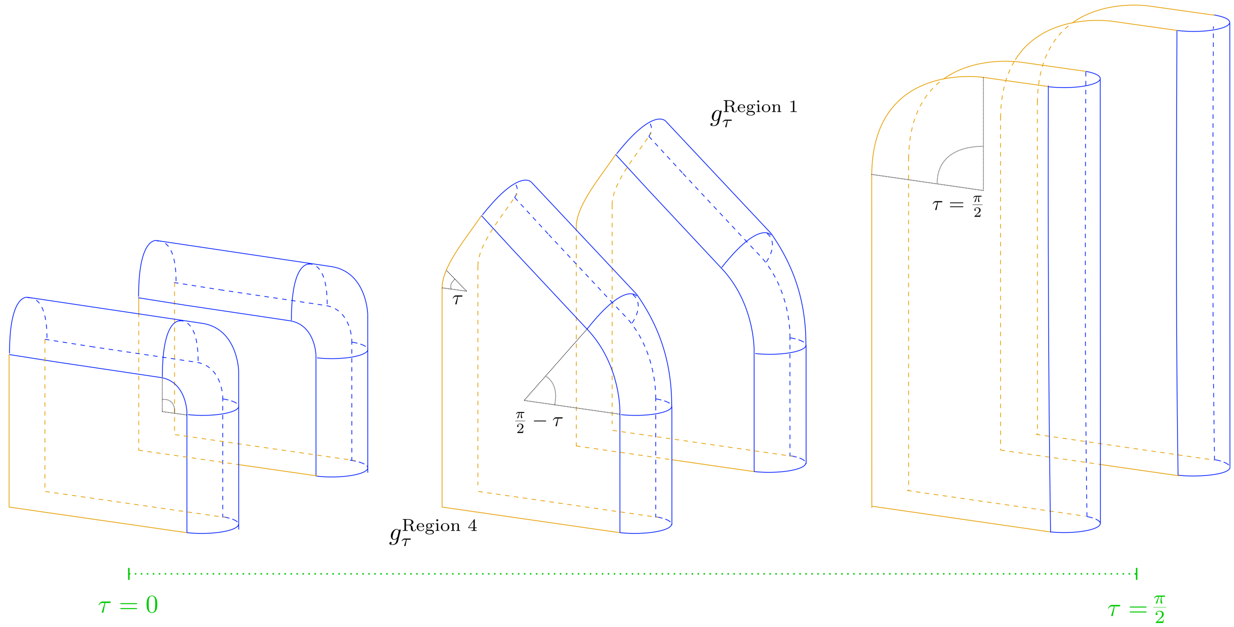


Figure 5.11: The intermediate and final stages of the bending.

5.4 The curvature of the components

We will now compute an expression for the $(p, n+1)$-intermediate scalar curvature of the metric $g_\tau^{\text{Region 1}}$ on the product manifold $M = S^k \times D^{\ell+1} \times I$, where $\tau \in \left[0, \frac{\pi}{2}\right]$. We begin by computing the relevant sectional curvatures and then take appropriate sums to obtain the intermediate scalar curvature. The curvature calculation techniques used in the following proof were developed in the proof of Proposition 4.7 from [5] and therefore we draw heavily from the work of M. Burkemper, C. Searle and M. Walsh in the statement of this lemma and its proof.

Lemma 5.4.1. *Let $M = B \times F$ with $B = S^k$ and $F = D^{\ell+1} \times I$, equipped with the metric $g_\tau^{\text{Region 1}} = dt^2 + \Omega_\tau(t,\theta)^2 d\theta^2 + \beta^2(t) ds_\ell^2 + ds_k^2$ where $\tau \in \left[0, \frac{\pi}{2}\right]$. If $x \in M$ and P is a p-plane in T_xM for $p \in \{0, \cdots, k+\ell\}$, then the $(p, k+\ell+2)$-intermediate scalar curvature of P has the form*

$$s_{p,k+\ell+2}(P) = (\ell - p)(\ell - p - 1)\frac{1-\beta_t^2}{\beta^2} + A^2(1-\beta_t^2) - B^2(\Omega_\tau)_{tt} - C^2\beta_{tt} - D^2\beta_t(\Omega_\tau)_t + E^2$$

for some real-valued functions $A = A(P,\beta)$, $B = B(P,\Omega_\tau)$, $C = C(P,\beta)$, $D = D(P,\beta,\Omega_\tau)$ and $E = E(P)$.

Proof. We regard the metric $g_\tau^{\text{Region 1}}$ as being composed of a base metric and a fibre metric.

In particular,

$$g_\tau^{\text{Region 1}} = g_B + g_F,$$

such that

$$\begin{aligned} g_B &= ds_k^2 \text{ and,} \\ g_F &= dt^2 + \Omega_\tau(t,\theta)^2 d\theta^2 + \beta^2(t) ds_\ell^2. \end{aligned}$$

Let $v, w \in T_x M$. Then the Riemann curvature tensor of M can be expressed as

$$R(v,w,w,v) = R_B(\check{v}_B, \check{w}_B, \check{w}_B, \check{v}_B) + R_F(\hat{v}_F, \hat{w}_F, \hat{w}_F, \hat{v}_F).$$

Let $\{\partial_1, \cdots, \partial_k, \partial_t, \partial_\theta, \bar{\partial}_1 \cdots, \bar{\partial}_\ell\}$ be an orthonormal basis for $T_x M$ and $i, i' \in \{1, \cdots, k\}$, $j, j' \in \{1, \cdots, \ell\}$. We compute the Riemann curvature tensor of the base:

$$\begin{aligned} R_B = R_{S^k} &= |\hat{v}_{S^k} \wedge \hat{w}_{S^k}|^2 K_{S^k} \\ &= |\hat{v}_{S^k} \wedge \hat{w}_{S^k}|^2 \\ &= \sum_{i<i'} (v_i w_{i'} - w_i v_{i'})^2. \end{aligned}$$

Invoking equation (4.10) of the proof of Lemma 4.6 in [5] we obtain the following expression for the Riemann curvature tensor of the fibre:

$$\begin{aligned} R_F = &- \Omega_\tau (\Omega_\tau)_{tt} (v_\theta w_t - v_t w_\theta)^2 + \sum_{j<j'} (v_j w_{j'} - v_{j'} w_j)^2 \beta^2 (1 - \beta_t^2) \\ &- \sum_j (v_j w_t - w_j v_t)^2 \beta \beta_{tt} - \sum_j (v_j w_\theta - w_j v_\theta)^2 \beta \beta_t \Omega_\tau (\Omega_\tau)_t. \end{aligned}$$

Combining these expressions for the curvature tensor of the base and fibre, R_B and R_F, we obtain the following overall expression for the curvature tensor:

$$\begin{aligned} R(v,w,w,v) = &\sum_{i<i'} (v_i w_{i'} - w_i v_{i'})^2 - \Omega_\tau (\Omega_\tau)_{tt} (v_\theta w_t - v_t w_\theta)^2 + \sum_{j<j'} (v_j w_{j'} - v_{j'} w_j)^2 \beta^2 (1 - \beta_t^2) \\ &- \sum_j (v_j w_t - w_j v_t)^2 \beta \beta_{tt} - \sum_j (v_j w_\theta - w_j v_\theta)^2 \beta \beta_t \Omega_\tau (\Omega_\tau)_t. \end{aligned}$$

We will use the Riemann curvature tensor to compute various sectional curvatures for use in the computation of the intermediate scalar curvature. The following sectional curvatures of $g_\tau^{\text{Region 1}}$ on M are given as

$$\begin{aligned} K_{t\theta} &= -\frac{(\Omega_\tau)_{tt}}{\Omega_\tau}, \\ K_{tj} &= -\frac{\beta_{tt}}{\beta}, \\ K_{\theta j} &= -\frac{(\Omega_\tau)_t \beta_t}{\Omega_\tau \beta}, \\ K_{jj'} &= \frac{1-\beta_t^2}{\beta^2}, \\ K_{ii'} &= 1, \text{ and} \\ K_{ti} &= K_{\theta i} = K_{ij} = 0. \end{aligned}$$

Now we will compute the $(p, k+\ell+2)$-intermediate scalar curvature of $(M, g_\tau^{\text{Region 1}})$. Let P be a p-plane in T_xM, where $0 \leq p \leq k+\ell$. Then $\dim P^\perp = (k+\ell+2-p)$. The dimension of $P^\perp + T_xF$ can be at most the dimension of T_xM, therefore

$$\begin{aligned} \dim(P^\perp \cap T_xF) &\geq \dim(P^\perp) + \dim(T_xF) - \dim(P^\perp + T_xF) \\ &\geq (k+\ell+2-p) + (\ell+2) - (k+\ell+2) \\ &= \ell + 2 - p, \end{aligned}$$

from which we conclude that there are at least $\ell+2-p$ linearly independent vectors in $P^\perp$ tangent to F, which we denote $\{\partial_t, \partial_\theta, \bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$. Completing a basis for $P^\perp$, there are at most k vectors not in T_xF, which we denote $\{v_1, v_2, \cdots, v_k\}$. We split our computation of the intermediate scalar curvature into two separate cases, determined by the dimension of the subspace spanned by the projections of $\{v_1, v_2, \cdots, v_k\}$ onto the tangent space of the fibre.

Case 1. The projections of $\{v_1, v_2, \cdots, v_k\}$ into T_xF span a 0-dimensional subspace, and therefore have no fibre component: $\text{span}\{v_1, \cdots, v_k\} = \text{span}\{\partial_1, \cdots, \partial_k\}$. Furthermore, $P^\perp$ has orthogonal basis $\{\partial_1 \cdots, \partial_k, \partial_t, \partial_\theta, \bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$. Therefore

$$s_{p,k+\ell+2}(P) = 2\sum_{j<j'}^{\ell-p} K_{jj'} + 2\sum_{j=1}^{\ell-p} K_{tj} + 2\sum_{j=1}^{\ell-p} K_{\theta j} + 2K_{t\theta} + 2\sum_{i\neq i'}^{k} K_{ii'}$$

$$= 2(\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + 2(\ell-p)\left(-\frac{\beta_{tt}}{\beta}\right)$$

$$+ 2(\ell-p)\left(-\frac{(\Omega_\tau)_t\beta_t}{\Omega_\tau\beta}\right) - 2\frac{(\Omega_\tau)_{tt}}{\Omega_\tau} + 2k(k-1).$$

Case 2: The projections of $\{v_1, v_2, \cdots, v_k\}$ into T_xF span a q-dimensional subspace, $1 \leq q \leq \min\{\ell+2, k\}$. This subspace is orthogonal to $\{\partial_t, \partial_\theta, \bar{\partial}_1, \cdots, \bar{\partial}_{\ell-p}\}$ so without loss of generality we can assume it is spanned by $\{\bar{\partial}_{\ell-p+1}, \cdots, \bar{\partial}_{\ell-p+q}\}$. Thus

$$v_s = \sum_{i=1}^{k}(v_s)_i\partial_i + \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j\bar{\partial}_j.$$

$$R(v_s, \partial_t, \partial_t, v_s) = \sum_{i=1}^{k}(v_s)_i^2 R(\partial_i, \partial_t, \partial_t, \partial_i) + \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2 R(\bar{\partial}_j, \partial_t, \partial_t, \bar{\partial}_j)$$

$$= \sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2\left(-\frac{\beta_{tt}}{\beta}\right).$$

$$K(v_s, \partial_t) = \frac{1}{||v_s||^2||\partial_t||^2}\sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2\left(-\frac{\beta_{tt}}{\beta}\right).$$

$$R(v_s, \partial_\theta, \partial_\theta, v_s) = \sum_{i=1}^{k} (v_s)_i^2 R(\partial_i, \partial_\theta, \partial_\theta, \partial_i) + \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 R(\bar{\partial}_j, \partial_\theta, \partial_\theta, \bar{\partial}_j)$$

$$= \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 \left(-\frac{(\Omega_\tau)_t \beta_t}{\Omega_\tau \beta} \right).$$

$$K(v_s, \partial_\theta) = \frac{1}{||v_s||^2 ||\partial_\theta||^2} \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 \left(-\frac{(\Omega_\tau)_t \beta_t}{\Omega_\tau \beta} \right).$$

Let $0 \leq j \leq \ell - p$, then

$$R(v_s, \bar{\partial}_j, \bar{\partial}_j, v_s) = \sum_{i=1}^{k} (v_s)_i^2 R(\partial_i, \bar{\partial}_j, \bar{\partial}_j, \partial_i) + \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 R(\bar{\partial}_j, \bar{\partial}_j, \bar{\partial}_j, \bar{\partial}_j)$$

$$= \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 \left(\frac{1 - \beta_t^2}{\beta^2} \right).$$

$$K(v_s, \bar{\partial}_j) = \frac{1}{||v_s||^2 ||\bar{\partial}_j||^2} \sum_{j=\ell-p+1}^{\ell-p+q} (v_s)_j^2 \left(\frac{1 - \beta_t^2}{\beta^2} \right).$$

Furthermore

$$
\begin{aligned}
R(v_s, v_{s'}, v_{s'}, v_s) = & -\Omega_\tau(\Omega_\tau)_{tt}((v_s)_\theta(v_{s'})_t - (v_s)_t(v_{s'})_\theta)^2 \\
& + \sum_{j<j'}((v_s)_j(v_{s'})_{j'} - (v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2) \\
& - \sum_j((v_s)_j(v_{s'})_t - (v_{s'})_j(v_s)_t)^2\beta\beta_{tt} \\
& - \sum_j((v_s)_j(v_{s'})_\theta - (v_{s'})_j(v_s)_\theta)^2\beta\beta_t\Omega_\tau(\Omega_\tau)_t \\
& + \sum_{i<i'}((v_s)_i(v_{s'})_{i'} - (v_{s'})_i(v_s)_{i'})^2,
\end{aligned}
$$

and

$$
\begin{aligned}
K(v_s, v_{s'}) = \frac{1}{||v_s||^2||v_{s'}||^2}\Bigg(& -\Omega_\tau(\Omega_\tau)_{tt}((v_s)_\theta(v_{s'})_t - (v_s)_t(v_{s'})_\theta)^2 \\
& + \sum_{j<j'}((v_s)_j(v_{s'})_{j'} - (v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2) - \sum_j((v_s)_j(v_{s'})_t - (v_{s'})_j(v_s)_t)^2\beta\beta_{tt} \\
& - \sum_j((v_s)_j(v_{s'})_\theta - (v_{s'})_j(v_s)_\theta)^2\beta\beta_t\Omega_\tau(\Omega_\tau)_t + \sum_{i<i'}((v_s)_i(v_{s'})_{i'} - (v_{s'})_i(v_s)_{i'})^2\Bigg).
\end{aligned}
$$

Therefore

$$s_{p,k+\ell+2}(P) = 2\sum_{\substack{j<j' \\ j,j'=1}}^{\ell-p} K_{jj'} + 2\sum_{s}\sum_{j=1}^{\ell-p} K(v_s,\bar{\partial}_j) + 2\sum_{s,s'} K(v_s,v_{s'}) + 2\sum_{j=1}^{\ell-p} K_{\theta j} + 2\sum_{j=1}^{\ell-p} K_{tj} + 2K_{t\theta}$$

$$= 2(\ell-p)(\ell-p-1)K_{jj'} + 2(\ell-p)\sum_{s} K(v_s,\bar{\partial}_j)$$

$$+ 2\sum_{s,s'} K(v_s,v_{s'}) + 2(\ell-p)K_{\theta j} + 2(\ell-p)K_{tj} + 2K_{t\theta}$$

$$= 2(\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + 2(\ell-p)\sum_{s}\frac{1}{||v_s||^2||\bar{\partial}_a||^2}\sum_{j=\ell-p+1}^{\ell-p+q}(v_s)_j^2\left(\frac{1-\beta_t^2}{\beta^2}\right)$$

$$+ 2\sum_{s,s'}\frac{1}{||v_s||^2||v_{s'}||^2}\Bigg(-\Omega_\tau(\Omega_\tau)_{tt}((v_s)_\theta(v_{s'})_t - (v_s)_t(v_{s'})_\theta)^2$$

$$+ \sum_{j<j'}((v_s)_j(v_{s'})_{j'} - (v_s)_{j'}(v_{s'})_j)^2\beta^2(1-\beta_t^2) - \sum_{j}((v_s)_j(v_{s'})_t - (v_{s'})_j(v_s)_t)^2\beta\beta_{tt}$$

$$- \sum_{j}((v_s)_j(v_{s'})_\theta - (v_{s'})_j(v_s)_\theta)^2\beta\beta_t\Omega_\tau(\Omega_\tau)_t + \sum_{i<i'}((v_s)_i(v_{s'})_{i'} - (v_{s'})_i(v_s)_{i'})^2\Bigg)$$

$$+ 2(\ell-p)\left(-\frac{(\Omega_\tau)_t\beta_t}{\Omega_\tau\beta}\right) + 2(\ell-p)\left(-\frac{\beta_{tt}}{\beta}\right) + 2\left(-\frac{(\Omega_\tau)_{tt}}{\Omega_\tau}\right).$$

Collecting these cases into a single expression in terms of the smooth, bounded, real-valued functions given in the statement of the theorem, we obtain the posited expression for the $(p, k+\ell+2)$-intermediate scalar curvature of $g_\tau^{\text{Region 1}}$. □

Following the computation of the curvature expression for the metric on Region 1, we arrive at an expression for the $(p, n+1)$-intermediate scalar curvature of the metric $g_\tau^{\text{Region 2}}$ on the product manifold $M = D^{k+1} \times I \times S^\ell$, where $\tau \in \left[0, \frac{\pi}{2}\right]$.

Corollary 5.4.2. *Let $M = B \times F$ with $B = D^{k+1} \times I$ and $F = S^\ell$, equipped with the metric $g_\tau^{\text{Region 2}} = dt^2 + \Omega_{\frac{\pi}{2}-\tau}(t,\theta)^2 d\theta^2 + ds_\ell^2 + \tilde{\beta}(t)^2 ds_k^2$ where $\tau \in \left[0, \frac{\pi}{2}\right]$. If $x \in M$ and P is a* *p-plane in T_xM for $p \in \{0, \cdots, k+\ell\}$, then the $(p, k+\ell+2)$-intermediate scalar curvature of P has the form*

$$s_{p,k+\ell+2}(P) = (\ell-p)(\ell-p-1) + A^2(\ell-p) + B^2(1-\tilde{\beta}_t^2) - C^2(\Omega_{\frac{\pi}{2}-\tau})_{tt} - D^2\tilde{\beta}_{tt} - E^2\tilde{\beta}_t(\Omega_{\frac{\pi}{2}-\tau})_t$$

for some real-valued functions $A = A(P, \tilde{\beta})$, $B = B(P, \Omega_{\frac{\pi}{2}-\tau})$, $C = C(P, \tilde{\beta})$, $D = D(P, \tilde{\beta}, \Omega_{\frac{\pi}{2}-\tau})$ and $E = E(P)$.

Proof. The metric $g_\tau^{\text{Region 2}}$ is essentially the same as $g_\tau^{\text{Region 1}}$ from Lemma 5.4.1. Therefore the same calculation applies, and this result follows as a corollary. □

5.5 Proof of Main Theorem A

Having obtained a curvature expression for $g_\tau^{\text{Region 1}}$ and $g_\tau^{\text{Region 2}}$ in Lemma 5.4.1 and Corollary 5.4.2 respectively, we are in a position to prove Main Theorem A, by showing that the metric G_τ, for all τ, has positive intermediate scalar curvature and fixes the boundary metric for all τ. The isotopy G_τ occurs across five regions, however as mentioned prior, only the metrics on Regions 1 and 2 experience changes in curvature.

Proof. As noted in the introduction to this chapter, the relative isotopy we have constructed deforms Region 1 in the opposite "direction" to Region 2. That is to say: as Region 1 "unbends", Region 2 "bends", and vice versa. We recall the restriction of G_τ to Region 1:

$$g_\tau^{\text{Region 1}} = ds_k^2 + dr^2 + \Omega_\tau(r,\theta)^2 d\theta^2 + \beta^2(r) ds_\ell^2,$$

and to Region 2:

$$g_\tau^{\text{Region 2}} = dt^2 + \Omega_{\frac{\pi}{2}-\tau}(t,\theta)^2 d\theta^2 + ds_\ell^2 + \tilde{\beta}(t)^2 ds_k^2.$$

We wish now to show that G_τ defines an $(s_{p,k+\ell+2} > 0)$-isotopy between $\bar{g}_{\text{Mtor}}^{k,\ell+1}$ and $\bar{g}_{\text{Mtor}}^{k+1,\ell}$, where curvature changes occur exclusively in Region 1 and Region 2. Therefore, it suffices to prove that, for $\tau \in \left[0, \frac{\pi}{2}\right]$, the metrics $g_\tau^{\text{Region 1}}$ and $g_\tau^{\text{Region 2}}$ have positive $(p, k+\ell+2)$-intermediate scalar curvature. Recall the curvature expression from Lemma 5.4.1:

$$s_{p,k+\ell+2}(P) = (\ell-p)(\ell-p-1)\frac{1-\beta_t^2}{\beta^2} + A^2(1-\beta_t^2) - B^2(\Omega_\tau)_{tt} - C^2\beta_{tt} - D^2\beta_t(\Omega_\tau)_t + E^2.$$

Since $p \leq \ell - 2$, $(\ell-p)(\ell-p-1) > 0$. Furthermore, $1-\beta_t^2 \geq 0$ from the definition of β. By applying L'Hopital's rule at $t = 0$, we conclude that the first term is non-negative. It follows immediately from the definition of β that the second term is positive. The remaining terms contain $-(\Omega_\tau)_{tt}$, $-\beta_{tt}$ and $-\beta_t(\Omega_\tau)_{tt}$. By definition ω_{tt} and β_{tt} are downward concave. When non-zero, ω_t and β_t have opposite signs, and therefore $-\omega_t\beta_t$ is positive. By definition of Ω_τ these properties which pertain to ω also apply to Ω_τ, therefore the remaining terms are non-negative. Therefore $g_\tau^{\text{Region 1}}$ has positive intermediate scalar curvature.

Recall the curvature expression from Corollary 5.4.2:

$$s_{p,k+\ell+2}(P) = (\ell-p)(\ell-p-1) + A^2(\ell-p) + B^2(1-\tilde{\beta}_t^2) - C^2(\Omega_{\frac{\pi}{2}-\tau})_{tt} - D^2\tilde{\beta}_{tt} - E^2\tilde{\beta}_t(\Omega_{\frac{\pi}{2}-\tau})_t$$

Given that the warping functions which define Region 2 are defined analogously to those of Region 1, analogous identities hold. Since $p \leq \ell - 2$, the first term is positive, while the remaining terms are non-negative, and it follows immediately that $g_\tau^{\text{Region 2}}$ has positive intermediate scalar curvature. It follows from Lemma 2.3.6 that the metric $g^{\text{Region 5}} = g_{\text{Mtor}}^{k,\ell} + d\theta^2$ has positive intermediate scalar curvature. Finally, positivity of the metrics on Region 3 and Region 4 follows from positivity of the metric $g^{\text{Region 1}}$. From the expressions of $g_\tau^{\text{Region 1}}$ and $g_\tau^{\text{Region 2}}$ it is clear that G_τ fixes the boundary; the only variation as τ moves between 0 and $\frac{\pi}{2}$ occurs in the θ-direction. Therefore these metrics are isotopic *relative* to the boundary metric $g_{\text{Mtor}}^{k,\ell}$ on S^n. □

We have therefore constructed a *relative* isotopy of metrics G_τ interpolating between $\bar{g}_{\text{Mtor}}^{k,\ell+1}$ and $\bar{g}_{\text{Mtor}}^{k+1,\ell}$, fixed on the boundary and possessing strictly positive $(p, n+1)$-intermediate scalar curvature at each stage. In the next chapter we will investigate the concept of a *Gromov-Lawson concordance*. The proof of the main theorem, which asserts that, under certain conditions such a concordance is in fact an isotopy, will invoke the relative isotopy constructed in this chapter.

Chapter 6

Gromov-Lawson Cobordism and Concordance

As discussed in earlier chapters, a fundamental technique in the construction and manipulation of Riemannian metrics with positive scalar curvature involves surgery and Morse-theoretic decompositions of manifolds. This method enables, under reasonable conditions, the extension of positive scalar curvature metrics over a certain cobordism to metrics of positive scalar curvature which satisfy nice boundary conditions. In particular, we recall Theorem 0.2 of [48].

Theorem 6.0.1 (Theorem 0.2, [48]). *Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism. Suppose g_0 is a metric of positive scalar curvature on X_0 and $f : W \to I$ is an admissible Morse function. Then there is a particular psc-metric $\bar{g} = g(g_0, f)$ on W, determined by the metric g_0 and the Morse function f, which extends g_0 and has a product structure near the boundary.*

The metric on W arisng from the inputs g_0 and f, denoted $\bar{g}(g_0, f)$ is called a *Gromov-Lawson cobordism.*

Remark 6.0.2. *Technically, the construction of $\bar{g} = \bar{g}(g_0, f)$, while primarily dependent on g_0 and f, involves a number of minor additional choices. These choices have very little effect on the resulting metric and varying them results only in very minor perturbations on $\bar{g}$. This does not affect the isotopy type, which is our main concern here.*

Recall from Definition 2.4.6 that the term *admissible* with respect to the Morse function means, in the positive scalar curvature context ($s_{p,n} > 0$ when $p = 0$), that f has critical points of index less than or equal to $n - 2$ only.

The relevant special case for us is when the cobordism W is a cylinder $M \times I$, for some closed smooth manifold M, g_0 is a metric of positive scalar curvature on M and $f : M \times I \to I$ is an admissible Morse function. Then the resulting Gromov-Lawson Cobordism metric $\bar{g} = \bar{g}(g_0, f)$ is actually a positive scalar curvature concordance between g_0 and $g_1 = \bar{g}|_{M \times \{1\}}$ on the manifold M. Such a concordance, arising in this way, is referred to as a *Gromov-Lawson concordance.*

These notions have been extended to the setting of (p, n)-intermediate scalar curvature by M. Burkemper, M. Walsh, and C. Searle. In [5] they generalise Theorem 6.0.1 as follows.

Theorem 6.0.3. *[Theorem 6.3, [5]] Let (M, g) be a smooth n-dimensional manifold, $\phi : S^k \times D^{\ell+1} \hookrightarrow M$ a smooth embedding, and $\{\bar{M}_\phi; M, M_\phi\}$ the trace of the surgery ϕ. Suppose furthermore that $n = k + \ell + 1$ and $n - k \geq 3 + p$. Then, if $g \in \mathcal{R}^{s_{p,n}>0}(M)$, there exist metrics g_ϕ on M_ϕ and $\bar{g}_\phi$ on $\bar{M}_\phi$ such that*

(i) $g_\phi \in \mathcal{R}^{s_{p,n}>0}(M_\phi)$, $\bar{g}_\phi \in \mathcal{R}^{s_{p,n+1}>0}(\bar{M}_\phi)$, and

(ii) near the boundary manifolds M and M_ϕ, $\bar{g}_\phi = g + dt^2$ and $\bar{g}_\phi = g_\phi + dt^2$ respectively.

This result shows that under appropriate surgery-codimension conditions, a $(s_{p,n} > 0)$-metric on a manifold can be extended over the trace of a surgery to a $(s_{p,n+1} > 0)$-metric that exhibits a product structure near the boundary. From this theorem, it follows that any p-admissible Morse function (defined in Definition 2.4.6 to mean that critical points of f have Morse index less than or equal to $n - 2 - p$) on a cobordism can be used to extend a $(s_{p,n} > 0)$-metric on a boundary manifold to a $(s_{p,n+1} > 0)$-metric on the cobordism.

Theorem 6.0.4. *Let $\{W^{n+1}; M_0, M_1\}$ be a smooth compact cobordism, $f : W \to I$ a p-admissible Morse function and $g_0 \in \mathcal{R}^{s_{p,n}>0}(M_0)$. Then there is a metric $\bar{g} = \bar{g}(g_0, f) \in \mathcal{R}^{s_{p,n+1}>0}(W)$ which extends g_0 and has a product structure near the boundary.*

Proof. On a compact manifold f will have a finite number of isolated critical points, each of which correspond to an elementary cobordism. After a minor perturbation of f if necessary, we can realise the compact cobordism W as a union of elementary cobordisms, each containing a lone critical point of f. It suffices then to invoke Theorem 2.2.7 sequentially over each of these elementary cobordisms, and therefore over all of W, to obtain the desired positive (p, n)-intermediate scalar curvature metric. $\square$

The metric $\bar{g} = \bar{g}(g_0, f)$, obtained in Theorem 6.0.4 is called a *Gromov-Lawson cobordism*, with respect to g_0 and f. If we restrict the metric $\bar{g}$ to a regular level set of the Morse function f, away from the next critical level above, then we obtain the metric given by applying the Gromov-Lawson method to a sequence of surgeries corresponding to the critical points below that regular level set. (Near the next critical level above, the metric is undergoes an isotopic standardisation preparing it for the next surgery.)

Remark 6.0.5. *Full details of the construction are provided in [48] and [5]. However it is worth emphasising that the metric $\bar{g}$ above is obtained as a union of elementary cobordism metrics of the type obtained in Theorem 6.0.3. Indeed, for any pair of consecutive critical points, w, w', where $0 < f(w) < f(w') < 1$ and for which $f^{-1}((f(w), f(w')))$ has no critical points, then for some small $\epsilon > 0$, the metric $\bar{g}$ restricts on $f^{-1}([f(w)+\epsilon, f(w')-\epsilon])$ as a product $\bar{g}|_{f^{-1}(f(w)+\epsilon)} + dt^2$, where $\bar{g}|_{f^{-1}(f(w)+\epsilon)}$ is the metric obtained by applying the Gromov-Lawson construction sequentially to all surgeries associated to all critical points below and including w, on the manifold (M_0, g_0).*

The following theorem is a special case of Theorem 0.5 of [48], essentially showing that the above constructed metric $\bar{g}(g_0, f) \in \mathcal{R}^{s_{p,n+1}>0}(W)$ varies continuously with a continuouos variation of the Morse function, f, in the space of p-admissible Morse functions on W.

Theorem 6.0.6. *Let $\{W^{n+1}; M_0, M_1\}$ be a smooth compact cobordism, $f, f' : W \to I$ a pair of p-admissible Morse function and $g_0 \in \mathcal{R}^{s_{p,n}>0}(M_0)$ which lie in the same path component of the space of p-admissible Morse functions on W. Then the metrics $\bar{g} = \bar{g}(g_0, f)$ and $\bar{g}' = \bar{g}(g_0, f') \in \mathcal{R}^{s_{p,n+1}>0}(W)$, obtained in Theorem 6.0.4 are $s_{p,n+1} > 0$ isotopic in the space $\mathcal{R}^{s_{p,n+1}>0}(W)$.*

Corollary 6.0.7. *Under the hypotheses of Theorem 6.0.6 above, the restriction metrics $\bar{g}|_{M_1}$ and $\bar{g}'|_{M_1}$ are $s_{p,n} > 0$ isotopic metrics on M_1.*

Consider the case where the cobordism W is the cylinder $M_0 \times I$. Equipping M_0 with an $(s_{p,n} > 0)$-metric g_0 and this cylinder with a p-admissible Morse function f, we can invoke Theorem 6.0.3 to extend g_0 to a Gromov-Lawson cobordism $\bar{g}(g_0, f)$ on $M_0 \times I$, which has positive (p, n)-intermediate scalar curvature, obtaining a *Gromov-Lawson concordance* in the more general context of $s_{p,n} > 0$ curvature; see Figure 6.1.

6.1 An elementary Gromov-Lawson concordance

In this section, we will perform a metric analysis of a certain type of Gromov-Lawson concordance, which will form a "building block" example in the proof of our main theorem. In particular, we will consider the following case. Let M^n be a smooth closed manifold of dimension n. Suppose f is a Morse function mapping the cylinder $M \times I$ to I which satisfies the hypotheses of Theorem 2.4.7, the weak cancellation theorem. In particular, this means

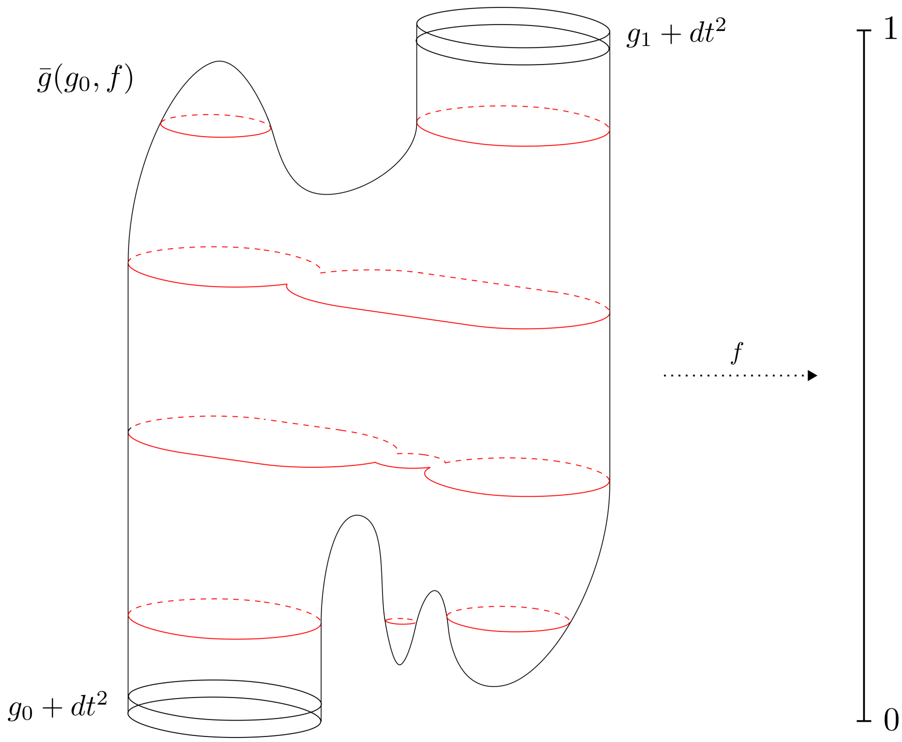


Figure 6.1: A schematic of a Gromov-Lawson concordance $\bar{g}(g_0, f)$.

(i) The Morse function f has exactly 2 critical points w and w' in the interior of $M \times I$, and $0 < f(w) < f(w') < 1$.

(ii) For some non-negative integers k and ℓ, satisfying $k + \ell + 1 = n$, the Morse index of $w = k + 1$ and the Morse index of $w' = k + 2$;

(iii) for each $t \in (f(w), f(w'))$, the intersection of the spheres $S^k_+(w)_t$ and $S^\ell_-(w')_t$, in the level set $f^{-1}(t)$, is transversal and consists of a single point.

Now suppose that g is a Riemannian metric on M satisfying $s_{p,n} > 0$ for some $p \in \{0, 1, \cdots, n-2\}$ and that the Morse function f is p-admissible. Thus the Morse indices $k+1$ and $k+2$ of the critical points must be less than or equal $n-2-p$. Simple arithmetic shows this requires $k \leq n - 4 - p$.

We are now in a position to apply Theorem 6.0.3 and obtain a Gromov-Lawson concordance $\bar{g} = \bar{g}(g, f)$ on the cylinder $M \times I$. In particular, $\bar{g}$ is an $s_{p,n+1} > 0$ metric on $M \times I$, which has a product structure near the boundary and restricts as $\bar{g}|_{M\times\{0\}} = g$, the $s_{p,n} > 0$ metric we began with.

We are especially interested in analysing the $s_{p,n} > 0$ metric, $\bar{g}|_{M\times\{1\}}$, also on M obtained by restriction of $\bar{g}$ to the top end of the cylinder $M \times I$. This metric is produced by applying the Gromov-Lawson surgery technique (as adapted by Labbi) over two surgeries (one for each critical point). The first surgery on (M, g) is a k-surgery associated with the critical point w. It produces a new manifold (topologically distinct from M) and a new $s_{p,n} > 0$ metric on this manifold. We denote this pair (M', g'). Essentially (M', g') is the Riemannian manifold obtained by restriction of $\bar{g}$ to a level set between the critical levels, i.e. $f^{-1}(t)$ where $t \in (f(w), f(w'))$.

The second surgery, a $(k+1)$-surgery associated with the second critical point w', necessarily restores M' to the original smooth topology of M but gives rise to a metric, which we denote $g'' := \bar{g}|_{M\times\{1\}}$, on M which looks very different from the original metric g. It is important at this stage to recall how the technique of surgery on metrics of $s_{p,n} > 0$ can produce metrics on a manifold which are not isotopic. For example, in Theorem 2.3.2, applying a certain sequence of surgeries to the standard sphere S^{4k-1} ($k \geq 2$), with its standard round metric, eventually restores the smooth topology of the sphere, but the Gromov-Lawson technique radically changes the round metric to a new metric, still satisfying $s_{p,n} > 0$ for $p \leq 2n - 3$, but in a distinct path component in the space of $s_{p,n} > 0$ metrics on S^{4k-1}.

After carefully analysing the metric g'' (to see how it differs from g) we will show that this type of "cancellation of surgeries" does not change the isotopy type of the underlying metrics and that g and g'' are in fact isotopic. Moreover, we will demonstrate this by constructing an isotopy.

We denote by ϕ the smooth embedding

$$\phi : S^k \times D^{\ell+1} \hookrightarrow M,$$

describing the surgery associated with the first critical point of f. Note that an explicit description of such an embedding, ϕ, can be obtained from the gradient flow of f, identifying M with $M \times \{0\}$. Following the work of Labbi in [22], we adjust the metric g in the image of the embedding (near the embedded sphere $\phi(S^k \times \{0\})$) to make it "standard" there, obtaining a new $s_{p,n} > 0$ metric on M which we call g_{std}. Recall this means that on the sub-region which is the image of the restricted embedding $\phi_{\frac{1}{2}} := \phi|_{S^k\times D^{\ell+1}(\frac{1}{2})}$, the metric g_{std} satisfies

$$\phi^*_{\frac{1}{2}} g_{\text{std}} = ds_k^2 + g_{\text{tor}}^{\ell+1}.$$

Importantly, from Lemma 6.2 of [5], the standardised metric g_{std}, which is formed by a controlled "pushing out" of small geodesic spheres in the disc fibres of the embedding ϕ (see lower image in Figure 6.2) is actually isotopic through $s_{p,n} > 0$ metrics to the original metric g.

The standardised metric is now "surgery-ready"; see Figure 6.3. We perform the k-

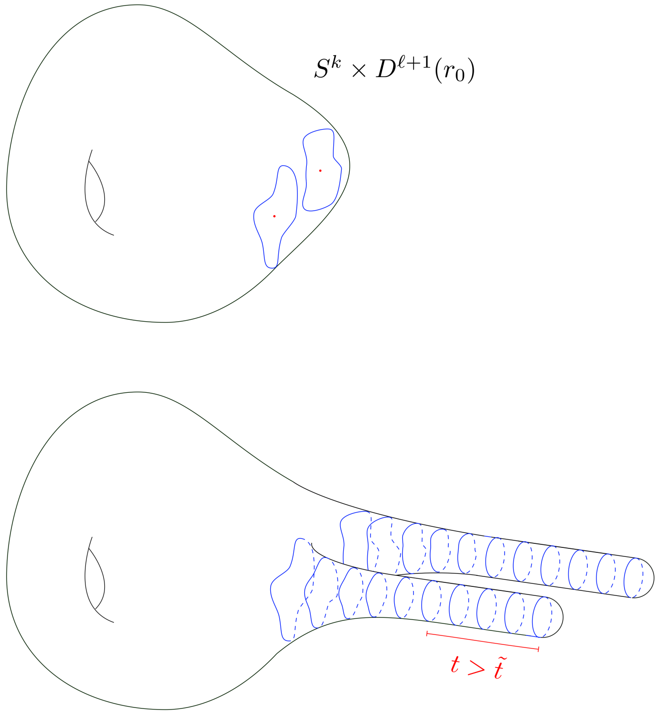


Figure 6.2: The creation of the standard region.

surgery, removing the interior of the image of $S^k \times D^{\ell+1}$ under $\phi_{\frac{1}{2}}$ from M and attaching the interior of $D^{\ell+1} \times S^k$ along the boundary spheres $S^k \times S^\ell$. The product we attach, $D^{\ell+1} \times S^k$, is equipped with the standard metric $g_{\text{tor}}^{k+1} + ds_\ell^2$, which glues, via $\phi_{\frac{1}{2}}$, smoothly onto the boundary of the metrically standard region of $M - \text{int}\phi_{\frac{1}{2}}\left(S^k \times D^{\ell+1}\right)$ to form an $s_{p,n} > 0$ metric g' on the new manifold M'. This is depicted in Figure 6.4.

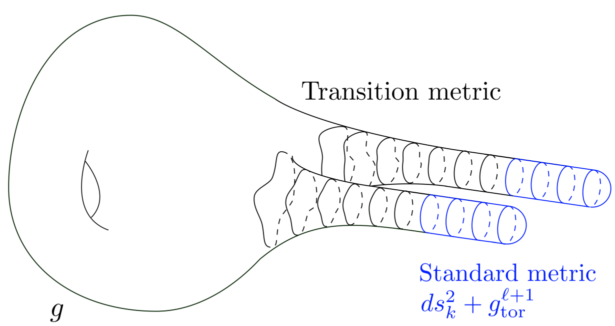


Figure 6.3: The *surgery-ready* metric obtained from g.

Both metrics g and g' can be thought of as comprising of three pieces, suggested in Figures 6.3 and 6.4. On the piece diffeomorphic to $\left(M - \text{int}\phi\left(S^k \times D^{\ell+1}\right)\right)$, $g_{\text{std}} = g' = g$. Attached to this is a piece diffeomorphic to $S^k \times S^\ell \times I$, on which $g' = g_{\text{std}}$ takes the form of what we refer to as the *transition metric*. This transition metric is described in the positive

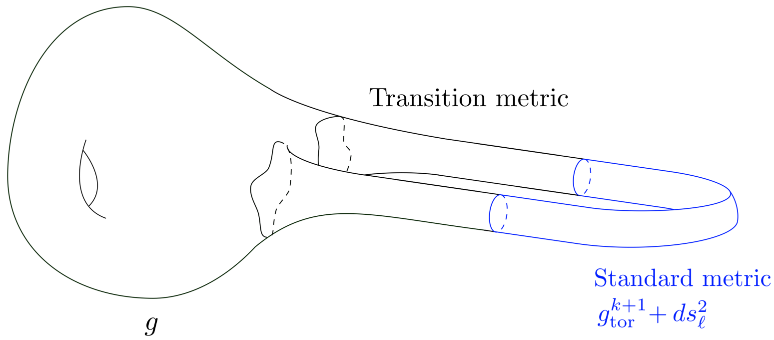


Figure 6.4: The metric g'.

scalar curvature ($p = 0$) case in the original papers of Gromov-Lawson [12] and Schoen-Yau [43] (with a more detailed discussion in [48]) and in the more general $s_{p,n} > 0$ case by Labbi in [22]. The third piece is where the metric takes a standard form. In the case of g_{std}, on the original manifold M, this is $(S^k \times D^{\ell+1}, ds_k^2 + g_{\text{tor}}^{k+1})$ while in the case of g', on the new manifold M' we have $(D^{k+1} \times S^\ell, g' = g_{\text{tor}}^{k+1} + ds_\ell^2)$.

The second critical point of f, w', determines a $(k+1)$-surgery on an embedding we call $\psi : S^{k+1} \times D^\ell \hookrightarrow M'$; see Figure 6.5. This surgery undoes the effects of the first one and restores the original smooth topology of M. Because of the hypotheses (of the weak cancellation theorem) on f, the embedded sphere S^k (on which we did the first surgery), is actually contractible in M. In particular, it bounds a $(k+1)-$dimensional disc in M. While there are many choices of disc, which can be embedded into M with boundary $\phi(S^k \times \{0\})$, there is a natural choice, which we will simply call $D^{k+1}(w, w')$. This disc can be determined by the gradient flow associated to f, recalling that $M' \cong f^{-1}(t)$ for some $t \in (f(w), f(w'))$. In terms of the gradient flow, the first surgery involves a collapsing of an embedded k-sphere, namely $\phi(S^k \times \{0\}) \cong S^k(w, -)_t \subset f^{-1}(t)$ for some $t \in [0, f(w))$. There is a specific disc with boundary $\phi(S^k \times \{0\})$ (obtainable by rewinding the flow) which, as a result of this boundary collapse at $t = f(w)$, emerges as the sphere $\psi(S^{k+1} \times \{0\}) \cong S^{k+1}(w, +)_t \subset f^{-1}(t)$ for some $t \in (f(w), f(w'))$. It is this disc, embedded in M and with boundary $\phi(S^k \times \{0\})$, we call $D^{k+1}(w, w')$.

By way of the gradient flow associated to f, there is a slicewise diffeomorphism

$$\Phi : M' \times (f(w), f(w')) \to f^{-1}((f(w), f(w'))),$$

which sends each slice $M' \times \{t\}$ to the level set $f^{-1}(t)$. Pulling back via this diffeomorphism is used in extending the surgery metric g', produced by the Gromov-Lawson procedure, as a product $g' + dt^2$ on $f^{-1}((f(w), f(w')))$ (at least, away from the boundary), as part of the construction of the Gromov-Lawson cobordism metric in Theorem 6.0.4; see details in [48] and [5]. (Near the boundary, the metric must be adjusted for handle attachment and to "get past" the critical level but we are not considering that yet.)

In particular, recalling that $M' = \big(M \setminus \mathrm{int}\phi\left(S^k \times D^{\ell+1}\right) \cup D^{k+1} \times S^{\ell}\big)$, consider the restriction of Φ to the interior of the handle $D^{k+1} \times S^{\ell}$ in M'. This provides a very useful set of coordinates in $f^{-1}(f(w), f(w')) \subset W$ when it comes to our later metric description. Before that however, we make an important technical observation. It will be useful, a little later on, to make a slight isotopic adjustment to the embedding ψ, so that its image locally coincides in M' more "neatly" with the handle $D^{k+1} \times S^{\ell}$ attached in forming M' during the first surgery. This is explained below.

6.1.1 Straightening out the second surgery embedding

Recall the trajectory arc T (defined in formula 2.14) is the image of a path connecting the critical points w and w' and obtained by intersecting the trajectory spheres as $\alpha(t) := S^{\ell}(w,+)_t \cap S^{k+1}(w',-)_t$ along $t \in [f(w), f(w')]$. This trajectory arc (or at least its interior) is contained in the image of the restricted diffeomorphism $\Phi|_{\mathrm{int}}(D^{k+1} \times S^{\ell} \times (f(w), f(w')))$. Because of the way Φ is constructed (from the gradient flow of f), we know that for each $t \in (f(w), f(w'))$,

$$\Phi(\{0\} \times S^{\ell}) \times \{t\} = S^{\ell}(w,+)_t,$$

the outgoing trajectory sphere from the critical point w. In particular, there is a path, $\beta(t) = \Phi^{-1}(\alpha(t))$ with image in the domain $D^{k+1} \times S^{\ell} \times (f(w), f(w'))$ and satisfying $\beta(t) \in \{0\} \times S^{\ell} \times \{t\}$. Consider now, for some fixed $t \in (f(w), f(w'))$, the inclusion into the handle part of M', $D^{k+1} \times \{\beta(t)\} \hookrightarrow D^{k+1} \times S^{\ell} \subset M'$. This inclusion intersects with $\psi(S^{k+1} \times \{0\})$ at the point $\beta(t)$.

Lemma 6.1.1. *[Lemma 4.7, [48]] The embedding $\psi : S^{k+1} \times D^{\ell} \hookrightarrow M' \cong f^{-1}(t)$ may be isotopically adjusted so that, on a neighbourhood of $\beta(t) \in D^{k+1} \times S^{\ell} \subset M'$, the image of the restriction $\psi|_{S^{k+1} \times \{0\}}$ coincides with the inclusion $D^{k+1} \times \{\beta(t)\} \hookrightarrow M'$.*

Proof. This is done in Lemma 4.7 of [48]. The key idea is as follows. Transversality means that, for any $t \in (f(w), f(w'))$, the trajectory sphere $S^{k+1}(w,+)_t$ is locally the graph of a function over an open subset of $\Phi(D^{k+1} \times \{\beta(t)\})$ containing $\alpha(t) = \Phi(\{0\} \times \{\beta(t)\})$. Now we can adjust the graph, as done in Corollary 1.7 of [20], by continuously pressing it down to its domain on some small ball $(k+1)$-dimensional ball around $\alpha(t) \in \Phi(D^{k+1} \times \{\beta(t)\})$ and transitioning back along some arbitrarily small annular region around that ball to the original $S^{k+1}(w,+)_t$. □

In effect, this means the trajectory sphere $S^{k+1}(w,+)_t \subset f^{-1}(t) \cong M'$ is locally "straightened out" to coincide with the image of the handle disc $\Phi(D^{k+1} \times \{\beta(t)\})$; see Figure 6.5. This will make certain metric adjustments much easier later on. We return to the embedding $\psi : S^{k+1} \times D^{\ell} \hookrightarrow M' \cong f^{-1}(t)$, where $t \in (f(w), f(w'))$. We denote the closure of the disc shaped subset of the S^{k+1}-factor, which is "straightened out" in Proposition 6.1.1, by D^{k+1}_{str} and the closure of its complement (also diffeomorphic to D^{k+1}), where the original embedding ψ still applies, by D^{k+1}_{ψ}.

As mentioned, the second surgery, ψ, undoes the topological changes induced previously and restores M. Via the Gromov-Lawson-Labbi technique, it leads to a new metric on M, which we call g'', with positive (p,n)-intermediate scalar curvature; see Figure 6.6.

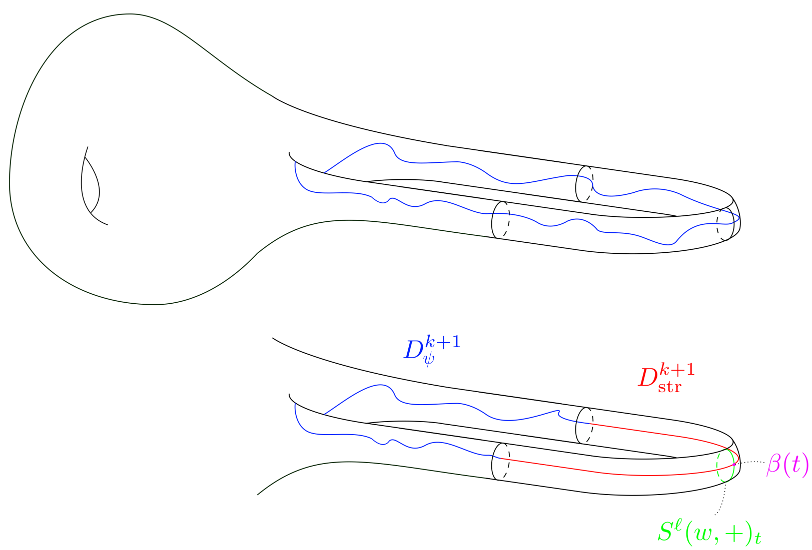


Figure 6.5: Straightening out the second surgery embedding.

6.2 An analysis of the metric g'' on M

The following description of g'' will follow closely the work done in [48], replicating it in the positive (p,n)-intermediate scalar curvature setting. Following this we will diverge from the methods of [48] by using Main Theorem A to construct an isotopy between g'' and g in the positive (p,n)-intermediate scalar curvature setting, a vital tool in the proof of our main theorem. As a consequence of Main Theorem A, the isotopy we construct is considerably simpler than that constructed in [48] in the positive scalar curvature setting.

As was done with g', we can obtain a more precise understanding of the metric g'' by characterising it as consisting of various pieces. This is done carefully in [48] but to aid the reader, we summarise it here referring to the diagram in Figure 6.7. The reader should

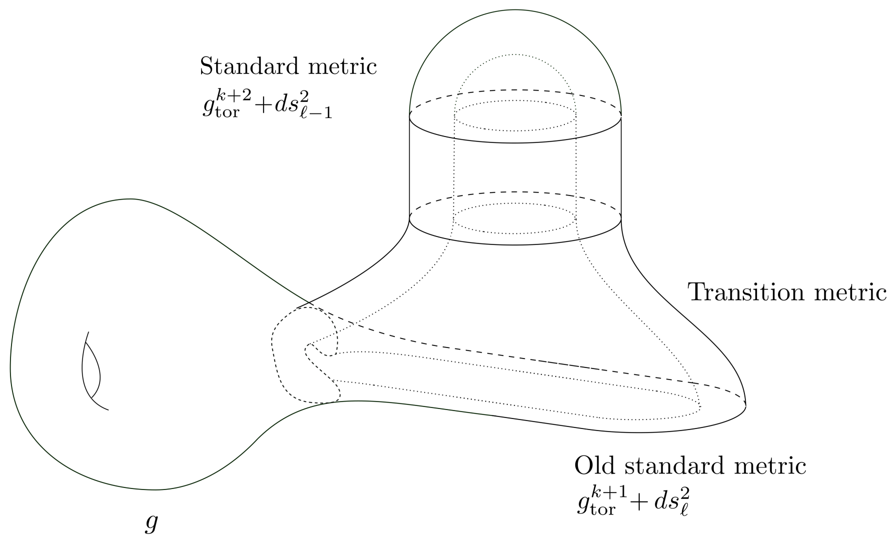


Figure 6.6: The metric g''.

note that the first surgery adjusted the metric on a region diffeomorphic to $S^k \times D^{\ell+1} \subset M$ while the second, having restored M left the original metric adjusted on a region of M diffeomorphic to the disc, D^n. This is a result of the way the images of the embeddings ϕ and ψ overlap on the original manifold M. While technically, ψ has image in M', part of its image is a tubular neighbourhood of the embedded disc, bounded by $\phi(S^k \times \{0\})$, which we called $D^{k+1}(w, w')$. This tubular neighbourhood in union with the image of ϕ is topologically a disc D^n in M. We will abbreviate this disc as D for now.

Importantly, it is only on this disc shaped region, D, that the original metric g on M undergoes any change. Off of the region, D, where the effects of both surgeries on the metric are felt (which is topologically just $M \setminus D$), $g'' = g$, the original metric.

The region D itself decomposes into four sub-regions:

$$D \cong C_1 \cup B_k \cup C_2 \cup B_{k+1},$$

specified as follows.

The sub-region C_1 is the cylinder (which we call a "connecting cylinder"), diffeomorphic to $S^{n-1} \times I$ in which the original metric g transitions via the Gromov-Lawson-Labbi construction, to a more standard form. Technically, it combines the standardising transition of both surgeries and may be very complicated (especially as we know little about the original metric g); see Figure 6.8. As we will see however, our later constructions do not make any adjustments to this region and so no further analysis is required.

The region B_k is topologically a cobordism between S^{n-1} which is attached to C_1, and the product $S^{k+1} \times S^{\ell}$. On it, the metric is well understood and the various pieces are labelled in the diagram, Figure 6.7. In particular, the subregion labelled "easy transition metric" is where the Gromov-Lawson construction has been applied to part of the embedded (second) surgery sphere via $\psi(S^{k+1} \times \{0\})$ in the region of M' where the metric g' is a standard handle product metric, $g_{\text{tor}}^{k+1} \times ds_{\ell}^2$. Making use of Lemma 6.1.1 to locally adjust the embedding ψ in this region, we may assume that the resulting surgery metric g'' takes a particularly nice and well-understood form here, one we discuss in the next chapter. (This is in contrast to the transition metrics obtained when the Gromov-Lawson construction is applied to an arbitrary $s_{p,n} > 0$-curvature metric, like the original metric g.). Near the end attached to C_1, the metric g'' on B_k, though well understood is not a product near the boundary. However, near the other boundary component diffeomorphic to $S^{k+1} \times S^{\ell}$, this metric takes a standard product form, $g_{\text{Dtor}}^{k+1} + ds_{\ell}^2 + ds^2$.

Finally, region C_2 is the second connecting cylinder, diffeomorphic to $S^{k+1} \times S^{\ell} \times I$ where the metric transitions easily to a fully standard form, allowing for the attachment of the well-known metric $g_{\text{tor}^{k+2}} + ds_{\ell-2}^2$ on the remaining region $B_{k+1} \cong D^{k+2} \times S^{\ell-1}$.

As will be seen in the next chapter, the metric g'' will be the starting point in an $(s_{p,n} > 0)$-isotopy which will move g'' continuously back to the original metric g, maintaining $s_{p,n} > 0$ at every stage. This is Lemma 7.1.1, which states that Gromov-Lawson concordance implies isotopy in $\mathcal{R}^{s_{p,n}>0}(M)$ in this basic case of two cancelling surgeries. This result is a building block result. The proof of Main Theorem B makes use of this fact in the general case of an admissible Morse function which, as we will see, can be reduced to a finite union of iterations of the two critical point case we have described.

Regarding the analysis of the metric g'' above, the pertinent fact to remember here is this. Within the region $B_k \cup C_2 \cup B_{k+1} \subset D$, the metric g'' is essentially a collection of standard pieces transitioning nicely into one another. It is only within the region C_1, the first connecting cylinder, that the metric g'' (transitioning form an arbitrary $s_{p,n} > 0$-curvature metric g) may be very complicated indeed. Thus, when it comes to constructing an isotopy between g and g'' as part of our main theorem, we will carefully avoid making any adjustments to the metric g'' on C_1 and work only on the region $B_k \cup C_2 \cup B_{k+1}$.

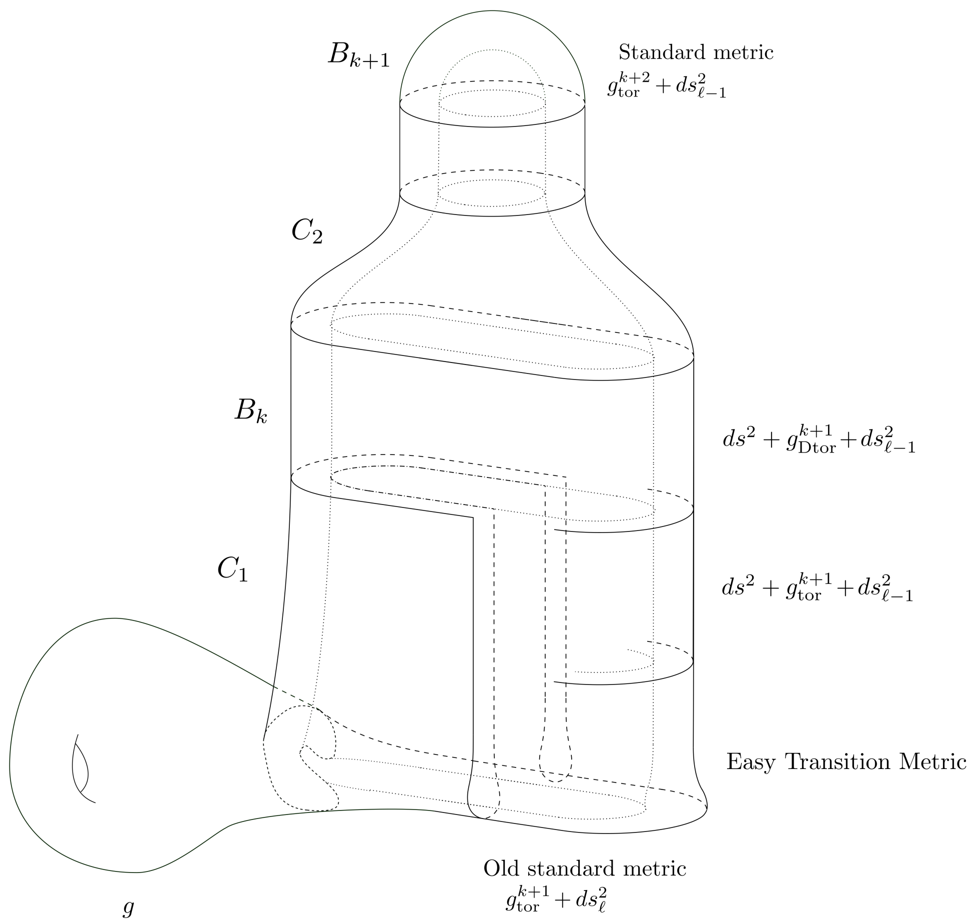


Figure 6.7: A detailed schematic of the metric g''.

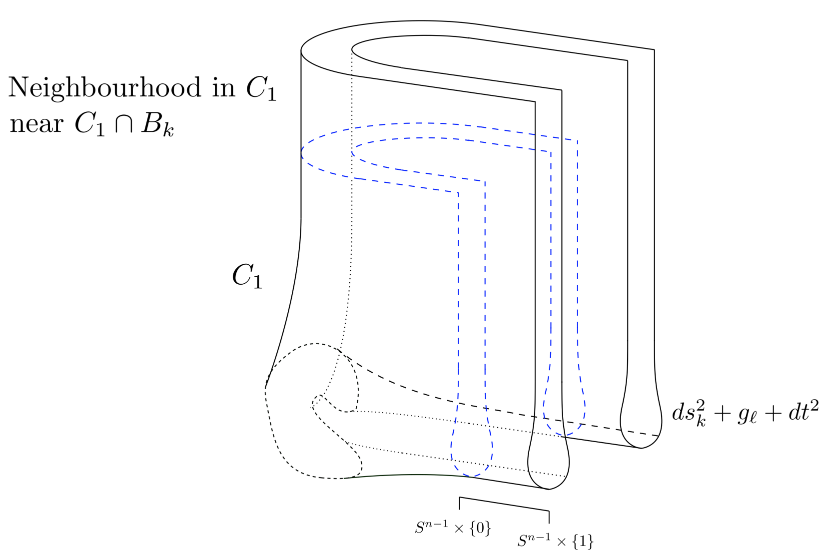


Figure 6.8: A schematic of the connecting cylinder.

Chapter 7

Gromov-Lawson Concordance and Isotopy for Positive Intermediate Scalar Curvature

The question of when Gromov-Lawson concordant metrics are isotopic in the space of (p, n)-intermediate scalar curvature metrics will be answered in the main theorem of this thesis, Main Theorem B, which is the crux of this chapter. First, we identify a relatively tractable instance of this question: when the Gromov-Lawson concordance is generated by an admissible Morse function with a pair of cancelling surgeries, as investigated in Section 6.1. This restricted case, dealt with in Lemma 7.1.1 below provides a foundational isotopy result which will be proven using the relative isotopy developed in Main Theorem A, and will itself be used in the proof of Main Theorem B. The first section of this chapter contains the proof of this restricted case, which is a highly geometric argument. The second section of this chapter contains the proof of the main theorem, relying strongly on the results introduced in Section 2.4, which are Morse-theoretic and concern the arrangement of cancelling pairs of critical points.

7.1 A pair of cancelling surgeries

We begin by returning to the situation discussed in the previous chapter. Thus, we have a closed smooth manifold M of dimension n, and a Morse function $f : M \times I \to I$ satisfying the conditions below.

(i) The Morse function f has exactly 2 critical points w and w' in the interior of $M \times I$, and $0 < f(w) < f(w') < 1$.

(ii) For some non-negative integers k and ℓ, satisfying $k + \ell + 1 = n$, the Morse index of $w = k + 1$ and the Morse index of $w' = k + 2$;

(iii) for each $t \in (f(w), f(w'))$, the intersection of the spheres $S^k_+(w)_t$ and $S^\ell_-(w')_t$, in the level set $f^{-1}(t)$, is transversal and consists of a single point.

We are now in a position to state our "building block" Lemma, which proves the main theorem in the special case of an admissible Morse function with a single cancelling pair of critical points.

Lemma 7.1.1. *Let M^n be a closed smooth manifold of dimension n, equipped with a Riemannian metric g satisfying $s_{p,n} > 0$ for some $p \in \{0, 1, \cdots, n-2\}$ and let f be a p-admissible Morse function $f : M \times I \to I$ satisfying the three conditions listed above. Let $\bar{g} = \bar{g}(g, f)$, denote the Gromov-Lawson concordance obtained by application of Theorem 6.0.4, an $s_{p,n} > 0$ concordance of $s_{p,n} > 0$ metrics g and $g'' = \bar{g}|_{M \times \{1\}}$. Then the metric g'' is actually $s_{p,n} > 0$ isotopic to the original metric g, in the space of $s_{p,n} > 0$ curvature metrics on M.*

Remark 7.1.2. *Recall that the p-admissibility condition above means that $k \leq n - 4 - p$, where the critical points of f have index $k + 1$ and index $k + 2$. Since $k + \ell + 1 = n$, this is equivalent to saying that $p + 3 \leq \ell$. Note that this seems to differ from the traditional condition of Labbi and Gromov-Lawson, namely that the codimension should satisfy $\ell \geq 2 + p$ (or codimension $\ell + 1 \geq 3 + p$, codimension ≥ 3 in the positive scalar curvature ($p = 0$) case). This is only because we are dealing with two surgeries of differing codimension and so we must adapt our hypotheses to cope with the worst case scenario, when the Morse index is $k + 2$. Thus, in the positive scalar curvature situation, the first surgery would need to be in codimension at least four.*

Note: Before proceeding with the proof, it is important to acknowledge that in this and all remaining proofs of the thesis, we will implicitly make use of Theorem 6.0.6, which implies that we can safely make continuous adjustments to the Morse function f without changing the $s_{p,n} > 0$ isotopy type of the resulting Gromov-Lawson cobordism (or concordance) metric, $\bar{g}(g_0, f)$.

Proof. We will draw upon the description of the metric g'' given in the previous chapter. We will construct over several stages an $s_{p,n} > 0$ isotopy from g'' to g. This will initially follow the method given by Walsh in [48] but will differ in drawing upon Main Theorem A to greatly simplify the construction.

We begin by making an adjustment to the metric g'' in the region B_k, near where it connects to the first connecting cylinder C_1. We will focus here on a cylindrical neighbourhood of $C_1 \cap B_k$ parameterised by a cylinder $S^{n-1} \times I$ as shown in Fig. 6.8, where t denotes the coordinate along the interval I. We will assume at this stage that we have employed Lemma 6.1.1 to adjust the second surgery embedding so that the embedded surgery $(k+1)$-sphere, determined by the embedding ψ, coincides nicely with the disc factor D^{k+1} on part of the attached handle. Recall, we referred to this neatly overlapping disc region as D^{k+1}_{str}. This means that the metric g' (on M', before the second surgery), restricts on D^{k+1}_{str} as the standard torpedo metric g^{k+1}_{tor}.

As shown in Figure 6.8, the intersection $C_1 \cap B_k$ naturally decomposes as $C_1 \cap B_k \cong S^{n-1} \cong S^k \times D^\ell \cup D^{k+1} \times S^{\ell-1}$ and the metric g'' restricts here as a product metric of the form

$$g''|_{C_1 \cap B_k} = ds_k^2 + g_\ell + dt^2 \cap g^{k+1}_{\text{tor}} + ds_\ell^2,$$

where g_ℓ is the restriction to the disc factor D^ℓ. The metric g_ℓ is obtained from the Gromov-Lawson construction by continuously pushing out a "torpedo" on a neighbourhood of a round ℓ-sphere metric and so is easily seen to be isotopic to a standard torpedo metric on D^ℓ.

Within B_k, but near the $S^k \times D^\ell$ part of the boundary component $B_k \cap C_1$, the metric g'' is a cylinder metric, $g'' = ds_k^2 + g_\ell + dt^2$; see the right of Figure 6.8. As a result of the Gromov-Lawson construction, the metric that g_ℓ (which is derived from "pushing out" a torpedo from a standard round metric) on the disc D^{k+1} is an $O(k+1)$-symmetric metric takes the form

$$g_\ell = dr^2 + F(r)^2 ds_k^2$$

, for some function F of the type shown in Figure 7.1. Thus, we can induce an $(s_{p,n} > 0)$-isotopy from the metric g_ℓ to the torpedo metric g^ℓ_{tor} by means of a linear homotopy from the function F to an appropriate torpedo function η, visualised in Figure 7.1 and given explicitly as

$$h_t(r) = t\eta(r) + (1-t)F(r).$$

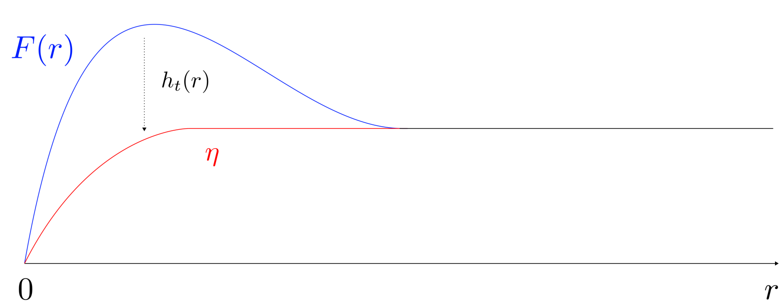


Figure 7.1: The linear homotopy h_t from F to the torpedo function η.

Let $g_t = dr^2 + h_t(r)^2 ds_k^2$ on D^{k+1}. Since $h_t(r)$ satisfies the same conditions (described in

Proposition 5.2 of [5]) as the torpedo function η, the argument given in Proposition 5.3* of [5] can be applied to g_t, and we conclude that $s_{p,k+1}(g_t) > 0$ for all $t \in [0,1]$. Hence g_t gives a $(s_{p,n} > 0)$-isotopy from g_ℓ to g^ℓ_{tor}.

By Lemma 2.3.6, we obtain an $s_{p,n} > 0$ concordance on $S^{n-1} \times I$ which agrees with the original metric $ds_k^2 + g_\ell + dt^2$ near $S^{n-1} \times \{0\}$ and changes it to

$$ds_k^2 + g^\ell_{\text{tor}} + dt^2,$$

near $S^{n-1} \times \{1\}$, as depicted in Figure 7.2. It can be seen that g'' can be isotopied, adjusting only near the $S^k \times D^\ell$ part of $C_1 \cap B_k$, to obtain a new (but only slightly modified) metric which is now, on B_k, a restriction of $g^{k+1}_{\text{Dtor}} + g^\ell_{\text{tor}}$ as depicted in Figure 7.3.

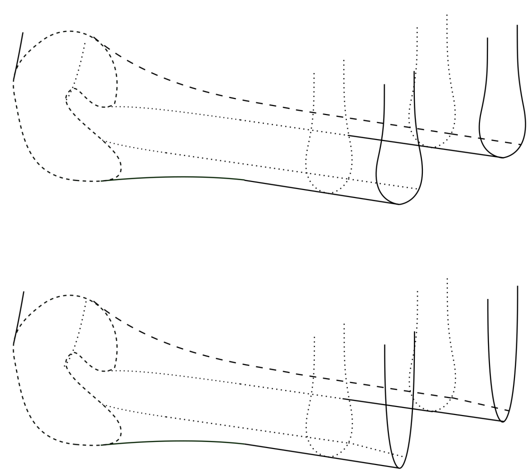


Figure 7.2: The result of the isotopy between g_ℓ and a torpedo metric.

Having performed the first adjustment on the metric g'', wish now to adjust the metric on the new standard component $g^{k+2}_{\text{tor}} + ds^2_{\ell-1}$. Note that, given the definition of the torpedo metric, we can express the metric $g^{k+2}_{\text{tor}} + ds^2_{\ell-1}$ as

$$ds^2 + \eta^2(s) ds^2_{k+1} + ds_{\ell-1}.$$

The second component of the metric above is round metric on S^{k+1} scaled by a torpedo function. Recall that the $(k+1)$-dimensional round metric is $(s_{p,n} > 0)$-isotopic to the $(k+1)$-dimensional double torpedo metric according to Lemma 4.2.5. By applying this isotopy to ds^2_{k+1}, we isotopy the new standard metric to a metric of the form

$$ds^2 + \eta(s)^2 g^{k+1}_{\text{Dtor}} + ds_{\ell-1}.$$

*In which it is shown that torpedo metrics have positive intermediate scalar curvature.

We denote the metric obtained from these two adjustments g_1, the various components of which are detailed in Figure 7.3. As with g'', we regard g_1 as being composed of a *standard* region and a *transition* region which is the connecting cylinder discussed previously. As we proceed, the adjustments we make to g_1 will occur only in the standard region and will fix the connecting cylinder.

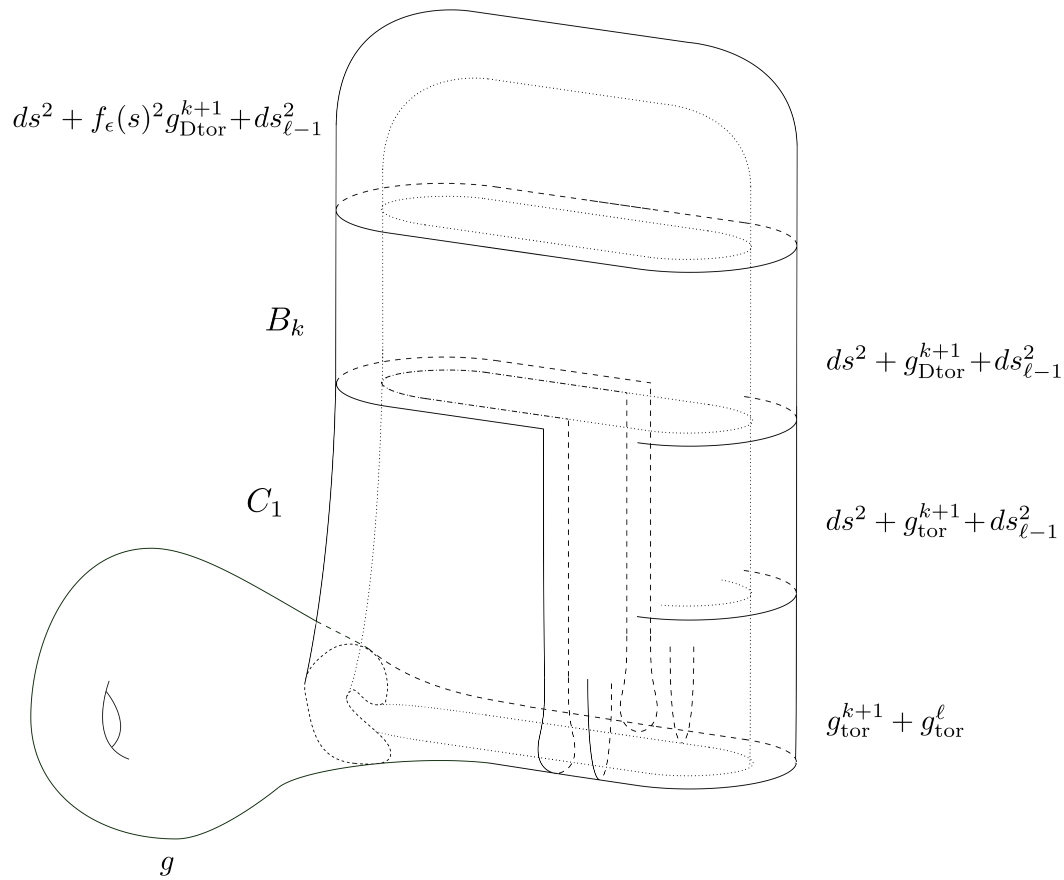


Figure 7.3: A schematic of the metric g'' following adjustments.

Recall the relative isotopy G_τ constructed in the proof of Main Theorem A, Chapter 5. This is an $(s_{p,n+1} > 0)$-isotopy and deforms $G_0 = \bar{g}_{\text{Mtor}}^{k,\ell+1}$ to $G_1 = \bar{g}_{\text{Mtor}}^{k+1,\ell}$ while fixing the boundary metric $g_{\text{Mtor}}^{k,\ell}$. The right-hand side of the standard region of g_1, detailed in Figure 7.4 in red, is the metric $\bar{g}_{\text{Mtor}}^{k+1,\ell-1}$ on D^n. Invoking Main Theorem A we conclude that this metric is $(s_{p,n+1} > 0)$-isotopic to the metric $\bar{g}_{\text{Mtor}}^{k,\ell}$, relative to their mutual boundary $g_{\text{Mtor}}^{k,\ell-1}$. This initial application of the isotopy G_τ results in the metric g_2. The top of the standard region of this metric contains the metric $\bar{g}_{\text{Mtor}}^{k+1,\ell-1}$, detailed in blue in Figure 7.4 and which, by a second application of Main Theorem A is isotopic to $\bar{g}_{\text{Mtor}}^{k,\ell}$ relative to $g_{\text{Mtor}}^{k,\ell-1}$. We denote the result of concatenating this pair of isotopies by g_3.

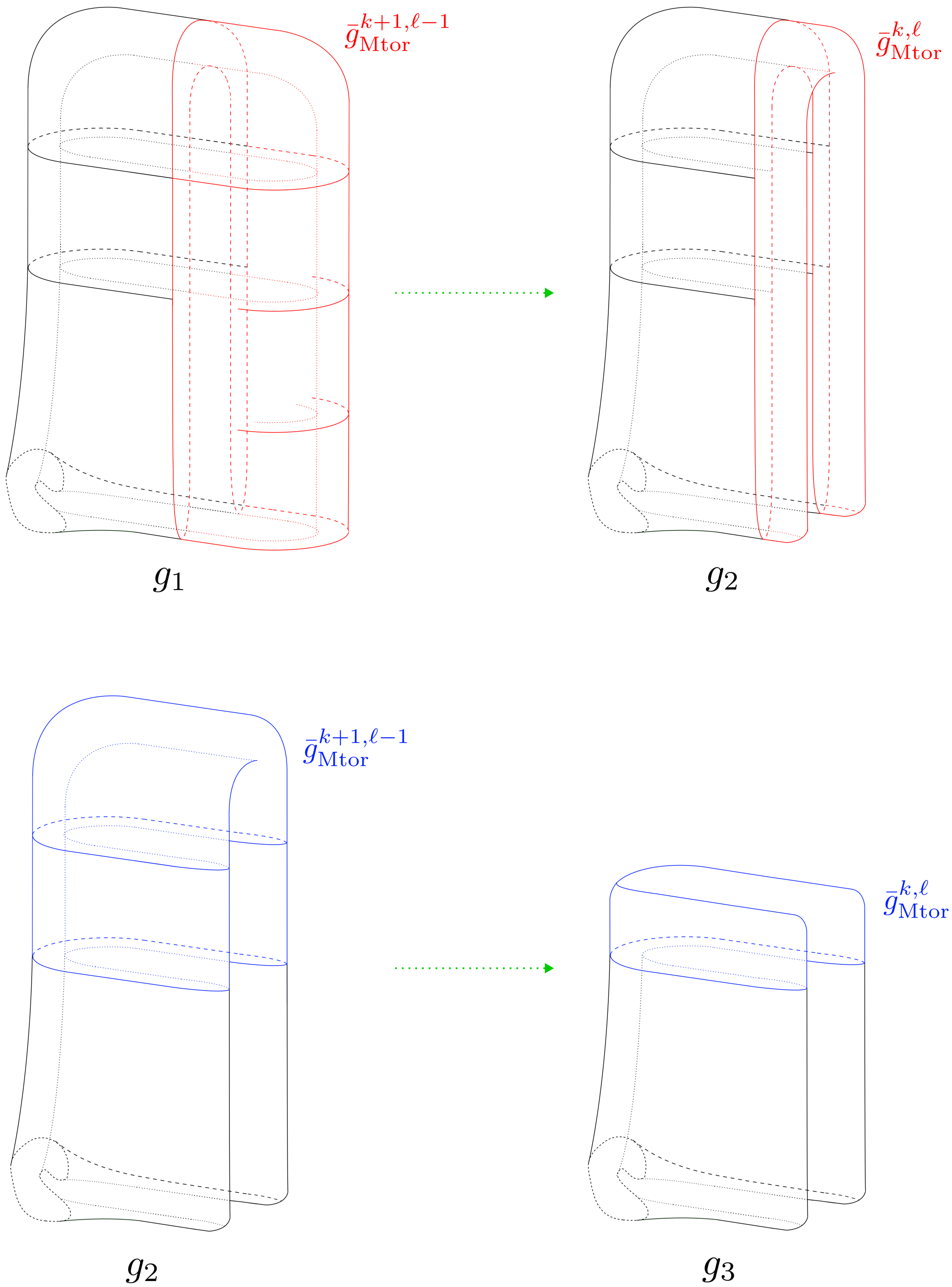


Figure 7.4: A double application of the relative isotopy G_τ.

At this stage we have built an isotopy, through $s_{p,n} > 0$ metrics which moves g'' to g_3 described above. We now complete the isotopy between g and g'' by building an $s_{p,n} > 0$ isotopy between g and g_3.

To construct an isotopy from g to g_3, we first reconstruct the $s_{p,n} > 0$ isotopy from g to g_{std} making use of Lemma 6.2 of [5] (which in turn utilises Theorem 3.1 of [19]) and the (first) surgery information arising from the embedding ϕ. The difference here is that we do not go through with the actual surgery on ϕ which gave us the metric g'. However, all of the non-trivial information (concerning the standardisation of the metric near the embedded surgery sphere) is preserved.

We now consider the disc $D^{k+1}(w, w')$ specified earlier which is bounded by the embedded surgery sphere S^k. Recall that the gradient flow of f moves this disc into the embedded sphere $\psi(S^{k+1})$ corresponding to the second surgery. Following the procedure in [48], in particular applying another standardising isotopy near $D^{k+1}(w, w')$ (preserving the non-trivial metric information contained in C_1 concerning both surgeries but without doing any surgery), this metric can be further isotopied to one that agrees with the metric g'' on all regions excluding the standard region. We denote this adjusted g_{std} by $\tilde{g}_{\text{std}}$.

This adjustment is characterised in the left-hand side of Figure 7.5. Extending this isotopy to the the old standard region, we obtain the metric characterised in the middle right of Figure 7.5, denoted $\tilde{\tilde{g}}_{\text{std}}$. As before, we use a linear homotopy of warping functions to induce an isotopy from $\tilde{\tilde{g}}_{\text{std}}$ to g_3, essentially the reverse homotopy as that described in Figure 7.1 and Figure 7.2. This gives an $(s_{p,n} > 0)$-isotopy from g to g_3 which, when concatenated with the $(s_{p,n} > 0)$-isotopy from g'' to g_3, gives an $(s_{p,n} > 0)$-isotopy from g to g'', proving the claim. □

7.2 Proof of Main Theorem B

Having dealt with the specific case of a pair of cancelling surgeries, we proceed to prove Main Theorem B. First, we establish some preliminaries. Let $\{W^{n+1}; M_0, M_1\}$ be a smooth compact cobordism such that M_0 and M_1 are closed, n-dimensional manifolds, equipped with a p-admissible Morse function[†] which, for simplicity, we denote f. By Theorem 2.4.12, we can assume that f is well-indexed. Recall from Section 2.4 that f can be used to decompose the compact cobordism W^{n+1} into a union of cobordisms

$$C_0 \cup C_1 \cup \cdots \cup C_{n+1},$$

[†]Recall that this object is a triple (f, m, V), defined in Section 2.4.

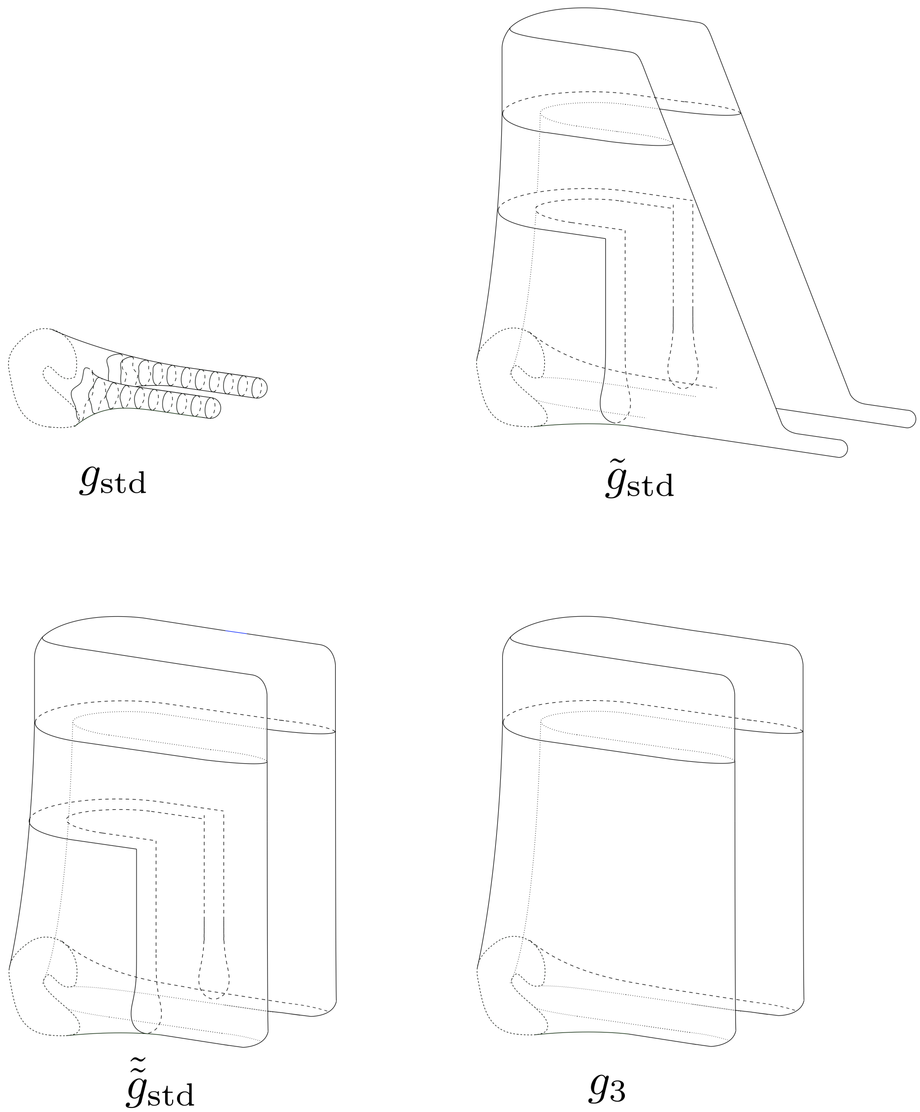


Figure 7.5: The isotopy from g_{std} to g_3.

where each C_i is such that all critical points of f in it have index i. We begin with a restatement of Theorem 6.1 of [48], adjusted for positive (p, n)-intermediate scalar curvature case.

Lemma 7.2.1. *Let M^n be a closed smooth n-dimensional manifold where $n \geq 5$, equipped with a metric $g_0 \in \mathcal{R}^{s_{p,n}>0}(M)$. Suppose $\bar{g} = \bar{g}(g_0, f)$ is a Gromov-Lawson concordance on $M \times I$, where f is a p-admissible Morse function with no critical points of index* 0 *or* 1*. Then the metrics g_0 and $g_1 = \bar{g}_{M\times\{1\}}$ are $(s_{p,n} > 0)$-isotopic.*

Proof. Assuming that the Morse function f is p-admissible and well-indexed, we may decompose the cylinder $M \times I$ into into a union of cobordisms

$$M \times I = C_2 \cup \cdots C_{n-p-2},$$

so that the restriction $f : C_k \to I$ has only critical points of index k.

At this point, the proof follows exactly that of Theorem 6.1 in [48], which is the $p = 0$ case. The p-admissible Morse function f can be isotopied through admissible Morse functions to one which decomposes into a modified version of the decomposition above:

$$M \times I = C_{2,3} \cup C_{3,4} \cup \cdots \cup C_{n-3-p,n-2-p}.$$

Here, $C_{k,k+1}$ is a cobordism consisting of all pairs of cancelling critical points of respective indices k and $k + 1$. The proof in Theorem 6.1 of [48], using standard Morse theoretic arguments, of the type used by Milnor in [33], only requires that the indices of the Morse function be greater than zero or one. (There is an issue in performing this isotopy with such critical points, which requires the added hypothesis of simple connectivity, and is taken care of in the more general theorem below.) The fact that the Morse function is p-admissible, as opposed to just $(p = 0)$-admissible makes no difference.

Importantly, the boundary of $C_{k,k+1}$ is diffeomorphic to a disjoint union $M \sqcup M$. Moreover, the trajectory arcs connecting each pair are all mutually disjoint and f can be further adjusted on each $C_{k,k+1}$ to make $C_{k,k+1}$ a disjoint union of cobordisms, each with a single pair of $k, k + 1$-indexed canceling critical points.

Thus, $M \times I$ can be decomposed into a finite union of cobordisms, where on each, f satisfies the conditions of Lemma 7.1.1. Repeated application of Lemma 7.1.1 proves Lemma 7.2.1. □

We proceed to the statement and proof of Main Theorem B, which, following the format of [48], involves extending Lemma 7.2.1 to critical points of index 0 and 1. In order to do this, we make use of an additional hypothesis that M is simply connected.

Main Theorem B. *Let M^n be a closed, simply-connected manifold of dimension $n \geq 5$. Let g_0 denote an $(s_{p,n} > 0)$-metric on M, $f : M \times I \to I$, a p-admissible Morse function and $\bar{g} = \bar{g}(g_0, f)$, the corresponding Gromov-Lawson concordance metric on $M \times I$. Then the metrics g_0 and $g_1 = \bar{g}|_{M\times\{1\}}$ are $(s_{p,n} > 0)$-isotopic.*

Proof. This follows exactly the method used to prove Theorem 0.8 in [48]. Assume again that f is well-indexed and p-admissible, and recall that this implies that all critical points of f have index less than or equal to $n - p - 2$. We use our Morse function f to decompose $M \times I$ into a union of cobordisms

$$C_0 \cup C_1 \cup \cdots \cup C_{n-p-2}.$$

Suppose first that f has critical points of index 0. Then f can be perturbed such that for some sufficiently small $\epsilon > 0$, the cobordism $f^{-1}[0, \epsilon]$ contains all critical points of index 0 and an equal number of critical points of index 1. It has been shown that in Theorem 8.1 of [33], that provided M is simply connected and has dimension at least 5, these critical points can be arranged into (index 0) - (index 1) pairs which satisfy the conditions of the *Weak Cancellation Theorem*. Therefore invoking Lemma 7.1.1, we obtain an isotopy between g_0 and $g_\epsilon = \bar{g}|_{f^{-1}(\epsilon)}$, the latter of which, if there are no more index 1 critical points, is isotopic to g_1 by virtue of Lemma 7.2.1.

Suppose then that there are more index 1 critical points which remain following this procedure. The solution to this problem amounts to replacing these critical points with an equal number of index 3 critical points and applying Lemma 7.2.1, synopsised as follows (full details can be found in [48] and it's cited texts, in particular [33]). We begin by adding in (index 2)-(index 3) cancelling pairs of critical points which have trajectory spheres which intersect at a point and that satisfy the conditions of the *Weak Cancellation Theorem*, Theorem 2.4.7. Importantly, these may be added such that the outgoing trajectory spheres of the index 1 critical points intersect transversally with the incoming trajectory spheres of the new index 2 critical points, forming de facto (index 1)-(index 2) pairs, which can be cancelled using Lemma 7.1.1. The proof is then completed with an application of Lemma 7.2.1.

□

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Symbols

$\bar{g}(g_0, f)$	Gromov-Lawson concordance
$\bar{M}_\phi$	trace of surgery
$\mathcal{C}$	elementary cobordism
$\mathcal{R}(M)$	space of Riemannian metrics on M
$\mathcal{R}^+(M)$	subspace with $s > 0$
$\mathcal{R}^{s_{p,n}>0}(M)$	subspace with $s_{p,n} > 0$
Γ_{ij}^k	Christoffel symbols
Gr_p	p-Grassmann bundle
Ric	Ricci curvature tensor
ds_n^2	round metric
f	Morse function
g	Riemannian metric on M
g_{ij}	matrix of metric coefficients
g^{ij}	inverse of the metric matrix
g_{tor}	torpedo metric
g_{Dtor}	double torpedo metric
g_{Mtor}	mixed torpedo metric
$\bar{g}_{\mathrm{Mtor}}$	mixed torpedo metric with boundary
H_x, V_x	horizontal and vertical subspaces of T_xM
$\mathscr{H}, \mathscr{V}$	horizontal and vertical distributions
K	sectional curvature
λ	Morse index
M_ϕ	result of surgery
∇	Levi-Civita connection
∂_i	coordinate vector fields
ϕ	surgery embedding
R	Riemann curvature tensor
s	scalar curvature
$s_{p,n}$	(p, n)-intermediate scalar curvature
u_F, u_B	projections of $u \in T_xM$ onto fiber/base directions