STRONGLY REAL 2-BLOCKS AND THE FROBENIUS-SCHUR INDICATOR

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Let G be a finite group. In this paper we investigate the permutation module of G acting by conjugation on its involutions, over a field of characteristic 2. This develops the main theme of [10] and [8]. In the former paper G. R. Robinson considered the projective components of this module. In the latter paper the author showed that each such component is irreducible and self-dual and belongs to a 2-blocks of defect zero. Here we investigate which 2-blocks have a composition factor in the involution module. There are two apparently different ways of characterising such blocks. One method is local and uses the *defect classes* of the block. This gives rise to the definition of a *strongly real* 2-block. The other method is global and uses the Frobenius-Schur indicators of the irreducible characters in the block. Our main result is Theorem 2. The proof of this theorem requires Corollaries 4, 15, 18 and 20.

J. A. Green proved a number of results about p-blocks, using the observation that the group algebra of G is a module for the group $G \times G$. Here we shall exploit the additional fact that the group algebra is a module for the wreath product of G with a cyclic group of order 2. This was also an essential tool in [8].

Throughout this paper k will be an algebraically closed field of characteristic 2. There are various reasons why we limit ourselves to characteristic 2. Our wreath product group is an extension of $G \times G$ by a group of order 2. It is thus fairly uninteresting, from the point of view of blocks over a field of characteristic not equal to 2. In addition, the prime 2 is useful for studying the contragradient operator and real blocks, as pairing arguments of various kinds can be employed.

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Recall also the following classical result. The Frobenius-Schur indicator $\nu(\chi)$ of a generalized character χ of G is the integer $\frac{1}{|G|}\sum_{g\in G}\chi(g^2)$. If χ is absolutely irreducible then $\nu(\chi)=0$ or $1,-1$, depending on whether χ is not real-valued, or χ is real-valued and a χ-module affords a symmetric, respectively anti-symmetric non-degenerate G-invariant bilinear form. Then Frobenius and Schur proved that $|\Omega|=\sum\nu(\chi)\chi(1_G)$.

The reader may be interested to know that in odd characteristic, the geometric type (quadratic or sympletic) of an irreducible self dual module is determined by the Frobenius-Schur indicator of a real valued character which contains the Brauer character of the module with odd multiplicity. It is an open problem as to whether there is an analogous Frobenius-Schur indicator in characteristic 2. See [11] for details.

A *component* of a module is a direct summand of the module that is indecomposable. Following Green, a 2-block of G is a component of kG, considered as a $G\times G$-module in the usual way. For the rest of the paper we use B to denote a 2-block of G.

A *defect class* of B is a conjugacy class of G whose sum appears with non-zero multiplicity in the block idempotent 1_B, and on which the central character ω_B of B does not vanish. Defect classes are known to exist and to consist of elements of odd order.

The irreducible complex characters, Brauer characters and indecomposable modules of G are partitioned among its 2-blocks. We use $\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$ and $\mathrm{Pic}(B)$ to denote, respectively, the set of irreducible characters, the Brauer characters and the principal indecomposable characters of G that belong to B. We use ψ to indicate the irreducible Brauer character associated to $\Psi\in\mathrm{Pic}(B)$. If M is a G-module, $M{\downarrow_H}$ denotes the restriction of M to $H\leq G$ and $M{\uparrow^K}$ denotes the induction of M to $K\geq G$. Identical notation applies to the restriction and induction of characters. See [9] for any additional unexplained notation.

The contragradient map ^{o} is defined by $(\sum\alpha_g g)^o=\sum\alpha_g g^{-1}$. It is a k-algebra involutary anti-automorphism of kG. A block B is said to be *real* if $B^o=B$. A conjugacy class C of G is said to be *real* if it coincides with the class C^o of the inverses of its elements. It is one of the main results of [5] that each 2-block has at least one defect class that is real.

A real conjugacy class of G is said to be *strongly real* if it is the trivial class or if its elements are inverted by involutions. This leads to the following key definition:

Definition **1.** *A* strongly real 2*-block is a real* 2*-block that has a strongly real defect class.*

It turns out that if B is strongly real then each of its real defect classes is strongly real. This was proved by Gow in [4]. Notice that the principal 2-block is strongly real; the identity class is a strongly real defect class.

We use 1_G both for the identity element of G and its trivial character. The involutions in G, together with 1_G, form a G-set $\Omega := \{g \in G \mid g^2 = 1_G\}$ under conjugation. We denote the kG-module with permutation basis Ω by $k\Omega$. Our main result in this paper is:

Theorem 2. *Let B be a 2-block of G. Then the following are equivalent:*

(i) *$k\Omega$ has a B-composition factor;*

(ii) $\sum_{\chi\in \mathrm{Irr}(B)} \nu(\chi)\chi(1_G) > 0$*;*

(iii) *B is strongly real.*

Let g be an element of G. There is a unique decomposition of $< g >$ into a direct product of a cyclic 2-group E and a cyclic $2'$-group O. So $g = g_2 g_{2'} = g_{2'} g_2$, for some $g_2 \in E$ and $g_{2'} \in O$. We call g_2 the 2-part, and $g_{2'}$ the $2'$-part, of g. Both are uniquely determined by g.

In our first lemma we compute the multiplicity of an irreducible kG-module as a composition factor of $k\Omega$.

Lemma 3. *Let $\overline{P}$ be an irreducible kG-module, let P be the projective cover of $\overline{P}$, and let $\Phi \in \mathrm{Pic}(G)$ be principal indecomposable character of P. Then $\overline{P}$ occurs with multiplicity $\nu(\Phi)$ as a composition factor of $k\Omega$. In particular, $\nu(\Phi) \geq 0$.*

Proof. The number of solutions in G to the equation $x^2 = g$, for fixed $g \in G$, is given by $\sum_{\chi\in \mathrm{Irr}(G)} \nu(\chi)\chi(g)$. Let $g \in G$ be 2-regular and let $x \in G$ satisfy $x^2 = g$. As x_2 and $x_{2'}$ commute, we have $g = x_2^2 x_{2'}^2 = x_{2'}^2 x_2^2$. So $x_2^2 = 1_G$ and $x_{2'}^2 = g_{2'}$. It follows that $x_2 \in \Omega(C_G(g))$, while $x_{2'} = g_{2'}^{1/2}$ is uniquely determined. Conversely, given any involution $t \in C_G(t)$, then $tg_{2'}^{1/2}$ is a solution to $x^2 = g$ in G. We conclude that $\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(g) = |\Omega \cap C_G(g)|$ coincides with the Brauer character of $k\Omega$. The lemma follows once we note that the virtual Brauer character of the restriction of the generalized character $\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi$ to 2-regular elements is given by $\sum_{\Psi\in\mathrm{Pic}(G)} \nu(\Psi)\psi$. □

Our Corollary shows that (i) $\Longleftrightarrow$ (ii) in Theorem 2.

Corollary 4. *The dimension of the sum of all submodules of $k\Omega$ that belong to B is given by* $\sum\limits_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(1_G)$. *In particular* $\sum\limits_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(1_G) \geq 0$.

Proof. Both statements follow from Lemma 3 and the fact that

$$\sum_{\chi\in\mathrm{Irr}(B)} \chi(1_G)\chi = \sum_{\Psi\in\mathrm{Pic}(B)} \psi(1_G)\Psi.$$

□

Let $\overline{P}, P$ and Φ be as in Lemma 3. Suppose that $t \in \Omega$. The Frobenius-Nakayama reciprocity formula [9, 3.1.27] shows that $\overline{P}$ occurs with multiplicity $< \Phi_{C_G(t)}, 1_{C_G(t)} >$ as a composition factor of $k_{C_G(t)}{\uparrow}^G$. Then, using the previous lemma, we get (c.f. [10, Lemma 1])

$$\nu(\Phi) \;=< \Phi, 1_G > + \sum_t < \Phi_{C_G(t)}, 1_{C_G(t)} >,$$

where t ranges over a set of representatives for the conjugacy classes of involutions in G.

We now proceed to the main construction needed for the proof of Theorem 2.

The wreath product group $G \wr \Sigma$ is the semi-direct product of $G \times G$ with the symmetric group Σ on two symbols. Here the conjugation action of the involution $\sigma \in \Sigma$ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. We shall use the notations

$$\underline{g} := (g, g) \in G \wr \Sigma, \quad \text{for each } g \in G\text{, and}$$

$$\underline{X} := \{\underline{x} \mid x \in X\} \subseteq G \wr \Sigma, \quad \text{for each } X \subseteq G.$$

We highlight the following crucial fact:

Lemma 5. *The centralizer of σ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$.*

Let R be a commutative ring. Then the group algebra RG is a right $RG \wr \Sigma$-module. For, RG is a $RG \times G$-module via $x \cdot (g_1, g_2) = g_1^{-1} x g_2$, for each $x \in RG$ and $g_1, g_2 \in G$. The action of σ on RG is given by the contragradient involution $x \cdot \sigma = x^o$. In more detail we have:

Lemma 6. *The $RG \wr \Sigma$-module RG is isomorphic to the permutation module $(R_{\underline{G} \times \Sigma})\uparrow^{G \wr \Sigma}$.*

Proof. The elements of G form an $RG \wr \Sigma$-invariant basis of RG. Moreover if $g_1, g_2 \in G$, then $g_2 = g_1 \cdot (g_1, g_2)$. So G is a transitive $G \wr \Sigma$-set. The stabilizer of $1_G \in RG$ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$. The lemma follows from these facts. $\square$

Suppose that E is a block algebra of RG. Set $E^* := E + E^o$. Then E^* is an $RG \wr \Sigma$-submodule of RG. If $E \neq E^o$, we have $E^* = E\uparrow^{G \wr \Sigma}$. If $E = E^o$, it is still useful to distinguish between the $RG \times G$-module E and the $RG \wr \Sigma$-module E^*, even though the underlying R-modules are the same.

Lemma 7. *Let $E_1, \ldots, E_r$ be the real blocks and $E_{r+1}, E_{r+1}^o, \ldots, E_{r+s}, E_{r+s}^o$ be the nonreal blocks of RG. Then there is a unique indecomposable decomposition of RG as $RG \wr \Sigma$-module:*

$$RG = E_1^* \oplus \ldots \oplus E_{r+s}^*.$$

Proof. This follows from the indecomposable decomposition of RG into a direct sum of its blocks algebras, as $R(G \times G)$-module. $\square$

As a particular case, consider when $R = \mathbb{C}$. Let $\chi \in \mathrm{Irr}(G)$ and let M be a $\mathbb{C}G$-module that affords χ. We use $E(\chi)$ to denote the corresponding Wedderburn component $\mathrm{End}_{\mathbb{C}}(M)$ of $\mathbb{C}G$. Clearly $E(\chi)$ has $G \times G$-character $\overline{\chi} \otimes \chi : (g_1, g_2) \to \chi(g_1^{-1})\chi(g_2)$, for $g_1, g_2 \in G$. Suppose now that $\chi = \overline{\chi}$ is real valued. Then $\overline{\chi} \otimes \chi = \chi \otimes \chi$ has two (irreducible) extensions to $G \wr \Sigma$. These will be denoted by χ^{+1} and χ^{-1}. Here if $\varepsilon \in \{\pm 1\}$ then $\chi^{\varepsilon}(g_1, g_2)\sigma = \varepsilon \chi(g_1 g_2)$, for all $g_1, g_2 \in G$. When $\overline{\chi} \neq \chi$, the next lemma shows why it is useful to denote the induced $G \wr \Sigma$-character $(\overline{\chi} \otimes \chi)\uparrow^{G \wr \Sigma}$ by χ^0.

Lemma 8. *Let χ be an irreducible character of G and let $E(\chi)^*$ be the corresponding $G \wr \Sigma$-component of $\mathbb{C}G$. Then $E(\chi)^*$ has character $\chi^{\nu(\chi)}$.*

Proof. This is obvious when $\nu(\chi) = 0$. So we may assume that $\chi = \overline{\chi}$. Then

$$< \chi^{\pm 1}\downarrow_{\underline{G}\times\Sigma}, 1_{\underline{G}\times\Sigma} >= \frac{1}{2|G|}\sum_{g\in G}\big(\chi(g^{-1})\chi(g) \pm \chi(g^2)\big) = (1 \pm \nu(\chi))/2.$$

The result now follows from Lemma 6 and Frobenius reciprocity. $\square$

Recall the following result [7, Theorem 1] of Green. A modern proof is [9, 5.10.8].

Lemma 9. *Let D be a defect group of B. Then B has vertex $\underline{D}$, as indecomposable $k(G \times G)$-module.*

We use this to make a preliminary observation about the vertices of the component B^* of the $G \wr \Sigma$-module kG. This will be refined in Proposition 14.

Lemma 10. *Let D be a defect group of B. If B is not real then $\underline{D}$ is a vertex of B^*; if B is real then there exists $e \in N_G(D)$, with $e^2 \in D$, such that $\underline{D} < \underline{e}\sigma >$ is a vertex of B^*.*

Proof. Suppose first that B is not real. So $B^* = B\uparrow^{G\wr\Sigma}$. It then follows from Lemma 9 that B^* has vertex $\underline{D}$.

Suppose then that B is real. Lemma 6 shows that B^* is $\underline{G} \times \Sigma$-projective. We choose a vertex V of B^* so that $V \leq \underline{G} \times \Sigma$. Now B^* is a quasi-permutation module, $G \times G$ is a normal subgroup of $G \wr \Sigma$, and $B^*\downarrow_{G\times G} = B$ is indecomposable. A variant of Lemma 9.7 of [2] then implies that $V \cap (G \times G) = V \cap \underline{G}$ is a vertex of B. Using Lemma 9, we may choose D so that $V \cap \underline{G} = \underline{D}$. As $G \wr \Sigma/G \times G$ is a 2-group, and as $B^*\downarrow_{G\times G}$ is indecomposable, Green's indecomposability theorem [6, Theorem 8], implies that $V \not\leq (G \times G)$. The last statement of the lemma follows from this. $\square$

It is easy to compute the stabilizer subgroup of an element g of the $G \wr \Sigma$-set G in the group $\underline{G} \times \Sigma$. This subgroup will be denoted $C_{\underline{G}\times\Sigma}(g)$. We hope that the reader will not confuse this group with $C_{\underline{G}\times\Sigma}(\underline{g}) = \underline{C_G(g)} \times \Sigma$.

Lemma 11. *Let $g \in G$. If g is not G-conjugate to g^{-1}, then $C_{\underline{G}\times\Sigma}(g) = \underline{C_G(g)}$. If $g^t = g^{-1}$, for $t \in G$, then $C_{\underline{G}\times\Sigma}(g) = \underline{C_G(g)} < \underline{t}\sigma >$.*

Proof. These statements follow from the fact that $g \cdot \sigma = g^{-1}$. $\square$

For $H \leq G$ and M a kG-module, let M^H denote the sum of all the trivial H-submodules of $M{\downarrow_H}$. The relative trace map $\mathrm{Tr}_H^G : M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) = \sum m \cdot g$, for $m \in M^H$. Here g ranges over any set of representatives for the right cosets of H in G. We write $\mathrm{Tr}_H^G(M)$, instead of $\mathrm{Tr}_H^G(M^H)$, for the image of the trace map on M^H.

The reader is warned that B^* is generally not a $\underline{G} \times \Sigma$-algebra, in the sense of Green [7]. In particular, there is no Mackey-type decomposition of a product of the form $\mathrm{Tr}_X^{\underline{G}\times\Sigma}(a)\mathrm{Tr}_Y^{\underline{G}\times\Sigma}(b)$, for $X, Y \leq \underline{G} \times \Sigma$. However, we do have the following useful result.

Lemma 12. *Let A be a k-algebra and a kG-module such that each element of G acts on A as a k-algebra automorphism or as a k-algebra anti-automorphism. Suppose also that A^G is contained in the centre $Z(A)$ of A. Then A^G a subalgebra of $Z(A)$. Also $\mathrm{Tr}_H^G(A)$ is an ideal of A^G, for each $H \leq G$.*

Proof. Write the G-action on A in exponential form. It is obvious that $\mathrm{Tr}_H^G(A)$ is a k-subspace of A. Let $a \in A$, $z \in A^G$ and $g \in G$. Suppose that g acts as a k-algebra anti-automorphism. Then $a^g z = (z^{g^{-1}} a)^g = (za)^g = (az)^g$. Similarly $a^g z = (az)^g$, if g acts as a k-algebra automorphism. It follows that the map $a \to az$ is a kG-endomorphism of A. In particular, if $a \in A^H$, then $\mathrm{Tr}_H^G(a)z = \mathrm{Tr}_H^G(az)$. Taking $H = G$, we get that A^G is a subalgebra of $Z(A)$. More generally, we can conclude that $\mathrm{Tr}_H^G(A)$ is an ideal of A^G. $\square$

We will apply this Lemma to the algebra B^* and the group $\underline{G} \times \Sigma$. Denote by $Z^*(kG)$ the σ-fixed point subalgebra of $Z(kG)$. It has k-basis $\{(C \cup C^o)^+\}$, where C ranges over the conjugacy classes of G. Note that $Z(kG) = kG^{\underline{G}}$ and $Z^*(kG) = kG^{\underline{G}\times\Sigma}$.

Corollary 13. *Let P be a 2-subgroup of G and let $q \in N_G(P)$ with $q^2 \in P$. Then*

(i) $\mathrm{Tr}_{\underline{P}}^{\underline{G}\times\Sigma}(kG)$ *is an ideal of $Z^*(kG)$ with k-basis $\{(X \cup X^o)^+\}$. Here X ranges over the set of non-real conjugacy classes of G such that P contains a Sylow 2-subgroup of $C_G(x)$, for some $x \in X$.*

(ii) $\mathrm{Tr}_{\underline{P}<\underline{q}\sigma>}^{\underline{G}\times\Sigma}(kG)$ *is an ideal of* $Z^*(kG)$ *with* k*-basis* $\{(X\cup X^o)^+\}\cup\{Y^+\}$. *Here* X *has the same meaning as in* (i), *while* Y *ranges over the the set of real conjugacy classes of* G *such that* P *contains a Sylow* 2*-subgroup of* $C_G(y)$, *and* $y^{pq}=y^{-1}$, *for some* $y\in Y$ *and* $p\in P$.

Proof. Lemma 12 implies that both $\mathrm{Tr}_{\underline{P}}^{\underline{G}\times\Sigma}(kG)$ and $\mathrm{Tr}_{\underline{P}<\underline{q}\sigma>}^{\underline{G}\times\Sigma}(kG)$ are ideals of $Z^*(kG)$.

In general, suppose that G is a finite group, H is a subgroup of G, and M is a permutation kG-module. Then it is well know that the k-space $\mathrm{Tr}_H^G(M)$ has basis of the form $\{O^+\}$. Here O ranges over the G-orbits on the permutation basis such that H contains a Sylow 2-subgroup of the stabilizer subgroup of some element of O in G. The Corollary follows by applying this, and Lemma 11, to the group $\underline{G}\times\Sigma$, its subgroups $\underline{P}$ and $\underline{P}<\underline{q}\sigma>$ and the module kG. □

We can now identify the vertices of B^*.

Proposition 14. *Suppose that* B *is real. Then* B *has a real defect class. Let* $c\in G$ *belong to a real defect class of* B, *let* D *be a Sylow* 2*-subgroup of* $C_G(c)$ *and let* $D<e>$ *be a Sylow* 2*-subgroup of* $C_G^*(c)$. *Then* $1_B\in\mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, *for* $E\leq\underline{G}\times\Sigma$ *if and only if* $\underline{D}<\underline{e}\sigma>\ \leq_{\underline{G}} E$. *Also* $\underline{D}<\underline{e}\sigma>$ *is a vertex of* B^*.

Proof. To show that B has a real defect class, we repeat the original argument of Gow, from [4, Lemma 1.2], for the convenience of the reader. Write $1_B=\sum\lambda_K K^+$, where K runs over the conjugacy classes of G and $\lambda_K\in k$, for each class K. Then

$$1_k=\omega_B(1_B)=\sum\lambda_K\,\omega_B(K^+).$$

Now $\lambda_K=\lambda_{K^o}$ and $\omega_B(K^+)=\omega_B(K^{o+})$, as B is real. It follows that the contribution of a nonreal class K and its inverse class K^o to the above sum is $2\,\lambda_K\,\omega_B(K^+)=0_k$. So there must exist a real class C such that $\lambda_C\,\omega_B(C^+)\neq 0_k$. Each such C is a real defect class of B.

Now fix a real defect class C of B. So $\lambda_C\neq 0_k$, using the notation of the previous paragraph. Suppose that $1_B\in\mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, where $E\leq\underline{G}\times\Sigma$. Then Corollary 13 implies that there exists $c \in C$ and D a Sylow 2-subgroup of $C_G(c)$ and $D < e >$ a Sylow 2-subgroup of $C^*_G(c)$, such that $\underline{D}{<}\underline{e}\sigma{>} \leq E$.

Mackey's Theorem implies that $(B^*)^{\underline{G}\times\Sigma} \subseteq \sum \mathrm{Tr}^{\underline{G}\times\Sigma}_{P\cap(\underline{G}\times\Sigma)}(kG)$. Here P ranges over the vertices of B^*. Corollary 13 implies that each subspace $\mathrm{Tr}^{\underline{G}\times\Sigma}_{P\cap(\underline{G}\times\Sigma)}(kG)$ is an ideal of $Z^*(kG)$. But 1_B is a primitive idempotent in $Z^*(kG)$. So by Rosenberg's Lemma [9, 5.1.1], there exists a vertex V of B^* such that $1_B \in \mathrm{Tr}^{\underline{G}\times\Sigma}_{V\cap\underline{G}\times\Sigma}(kG)$.

The last two paragraphs imply that $\underline{D} < \underline{e}\sigma > \leq_G V$. But $|\underline{D} < \underline{e}\sigma >| = |V|$, as a consequence of Lemma 10. It follows that $1_B \in \mathrm{Tr}^{\underline{G}\times\Sigma}_{\underline{D}<\underline{e}\sigma>}(kG)$, and also that $\underline{D} < \underline{e}\sigma >$ is a vertex of B^*. □

Our Corollary shows that (ii) $\Longrightarrow$ (iii) in Theorem 2.

Corollary 15. *Suppose that there exists $g \in G$ such that $\sum\limits_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(g) > 0$. Then B is real. Let $\underline{D} < \underline{e}\sigma >$ be a vertex of B^*. Then there exists $d \in D$ such that g_2 is G-conjugate to $(de)^2$. In particular, if g can be chosen to be 2-regular, then B is strongly real.*

Proof. When B is the principal 2-block of G, the result is true. So assume otherwise. The hypothesis implies that B is real, as it forces $\nu(\chi) \neq 0$, for some $\chi \in \mathrm{Irr}(B)$.

Let (R, F, k) be a 2-modular system for G. Suppose that $\hat{B}$ is the block algebra of RG such that $B = \hat{B}/J(R)\hat{B}$. Then Lemma 8 shows that the $G \wr \Sigma$-character of $\hat{B}^*$ is $\chi_B := \sum_{\chi\in\mathrm{Irr}(B)} \chi^{\nu(\chi)}$. Now $\hat{B}^*$ and B^* have the same vertices, as both are trivial source modules [9, 4.8.9]. So $\underline{D} < \underline{e}\sigma >$ is a vertex of B^*. As $\chi_B((1_G, g)\sigma) = \sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(g)$, the hypothesis is that $\chi_B((1_G, g)\sigma) \neq 0$. It then follows from a theorem of Green [9, 4.7.4] that the 2-part of $(1_G, g)\sigma$ is contained in a vertex of $\hat{B}^*$. But $((1_G, g)\sigma)_2 = (1_G, g_2)(g_{2'}^{-1/2}, g_{2'}^{1/2})\sigma$ and $\left((g_{2'}^{-1/2}, g_{2'}^{1/2})\sigma\right)^{(1_G, g_{2'})} = \sigma$. In particular $((1_G, g)\sigma)_2$ is $G \wr \Sigma$-conjugate to $(1_G, g_2)\sigma$. So there exists $g_1, g_2 \in G$ and $d \in D$ such that $(1_G, g_2)\sigma = (\underline{de}\sigma)^{(g_1, g_2)} = (g_1^{-1}deg_2, g_2^{-1}deg_1)\sigma$. This gives $g_2^{-1} = g_1^{-1}de$, and hence also $g_2 = [(de)^2]^{g_1}$.

Suppose that $g_2 = 1_G$. Then $(de)^2 = 1_G$. So de is an involution that belongs to $D < e > \backslash D$. Then, using Proposition 14, we see that each real defect class of B is strongly real, whence B is strongly real. □

Let K be a field and let τ be a field automorphism of of K. Suppose that γ is a K-representation of G. Then we may form the representation γ^τ of G by applying τ to the matrix entries in $\gamma(g)$, for each $g \in G$. If M is the KG-module corresponding to γ, we let M^τ denote the KG-module corresponding to γ^τ. This construction also applies if τ is an automorphism of a subfield K_0 of K, and γ is realisable over K_0.

We use this to define the Frobenius twist of a module or character. The Frobenius automorphism Fr of k is given by $\lambda \to \lambda^2$, for $\lambda \in k$. Every $\mathbb{C}$-representation of G can be realized over $\mathbb{Q}(\zeta)$, where ζ is a primitive $|G|^{th}$ root of unity. There is Galois automorphism Fr of $\mathbb{Q}(\zeta)$ given by $\zeta \to \zeta_2\zeta_{2'}^2$, for $\zeta \in \mathbb{Q}(\zeta)$. In either case, if M is a G-module, with Brauer or ordinary character χ, then the Frobenius twist module M^{Fr} has character $\chi^{\mathrm{Fr}} : g \to \chi(g_{2'}^2 g_2)$, for each g in the domain of definition of χ.

Note that $\{\chi^{\mathrm{Fr}} \mid \chi \in \mathrm{Irr}(B)\}$ is the set of irreducible characters in a 2-block B^{Fr} of G. If M is an indecomposable kG-module, then M belongs to B if and only if M^{Fr} belongs to B^{Fr}.

We identify G and $\underline{G}$ and let $\underline{B}$ be the 2-block of $\underline{G}$ corresponding to B. There is a unique 2-block of $\underline{G} \times \Sigma$ that covers $\underline{B}$. We denote this block by $\underline{B} \times \Sigma$. Clearly $(\underline{B} \times \Sigma)^{\mathrm{Fr}} = \underline{B}^{\mathrm{Fr}} \times \Sigma$.

Lemma 16. *The Brauer induced block $(\underline{B}^{\mathrm{Fr}} \times \Sigma)^{G\wr\Sigma}$ is defined. It is the unique 2-block of $G \wr \Sigma$ that covers the block $B \otimes B$ of $G \times G$.*

Proof. Let D be a defect group of B. Then $\underline{D} \times \Sigma$ is a defect group of $\underline{B}^{\mathrm{Fr}} \times \Sigma$. Since $C_{G\wr\Sigma}(\underline{D} \times \Sigma) = \underline{C_G(D)} \times \Sigma$ is contained in $\underline{G} \times \Sigma$, the induced block $(\underline{B}^{\mathrm{Fr}} \times \Sigma)^{G\wr\Sigma}$ is defined [9, 5.3.6].

Let $B^{\otimes 2}$ be the unique 2-block of $G \wr \Sigma$ that covers the 2-block $B \otimes B$ of $G \times G$. So χ^{+1} belongs to $B^{\otimes 2}$, whenever $\chi \in \mathrm{Irr}(B)$.

Now $C_{G\wr\Sigma}(\sigma) = \underline{G} \times \Sigma$. Each Brauer character of $\underline{G} \times \Sigma$ can be identified with a Brauer character of $\underline{G}$. Using Brauer's second main theorem [9, 5.4.2], we have

$$(1) \qquad \chi^{+1}(\underline{g}\sigma) = \sum_{\theta \in \mathrm{IBr}(\underline{G})} d^\sigma_{\chi,\theta}\theta(g), \qquad \text{for all } g \in G \text{ of odd order,}$$

where the $d^\sigma_{\chi,\theta}$ are algebraic integers with the property that $d^\sigma_{\chi,\theta} = 0$, unless θ belongs to a 2-block B_1 of G such that $(\underline{B_1} \times \Sigma)^{G \wr \Sigma} = B^{\otimes 2}$. On the other hand, the definition gives

$$(2) \qquad \chi^{+1}(\underline{g}\sigma) \;=\; \chi(g^2) = \chi^{\mathrm{Fr}}(g) \;=\; \sum_{\theta \in \mathrm{IBr}(\underline{G})} d_{\chi^{\mathrm{Fr}},\theta}\theta(g), \qquad \text{for all } g \in G \text{ of odd order.}$$

But the irreducible Brauer characters of $\underline{G}$ are linearly independent on the 2-regular classes of $\underline{G}$. So (1) and (2) imply that $d^\sigma_{\chi,\theta} = d_{\chi^{\mathrm{Fr}},\theta}$, for all $\theta \in \mathrm{IBr}(\underline{G})$. As $d_{\chi^{\mathrm{Fr}},\theta} \neq 0$, for some $\theta \in \mathrm{IBr}(B^{\mathrm{Fr}})$, we conclude that $(\underline{B}^{\mathrm{Fr}} \times \Sigma)^{G \wr \Sigma} = B^{\otimes 2}$. □

The following lemma is a key step in the proof of Theorem 2.

Lemma 17. *Restriction* $\downarrow^{\underline{G}\times\Sigma}_{\underline{G}}$ *establishes a multiplicity preserving bijection between the components of* $B^*\!\downarrow_{\underline{G}\times\Sigma}$ *that have a vertex containing* Σ *and the components of* $k\Omega$ *that belong to* $\underline{B}^{\mathrm{Fr}}$.

Proof. Let M be a component of $B^*\!\downarrow_{\underline{G}\times\Sigma}$ that has a vertex V containing Σ. Then Σ is contained in the kernel of M, as M is a trivial source module. Thus M coincides with the inflation of the indecomposable $\underline{G}$-module $M\!\downarrow_{\underline{G}}$ to $\underline{G} \times \Sigma$. In addition, $M\!\downarrow_{\underline{G}}$ has a vertex V/Σ.

The orbits of $\underline{G} \times \Sigma$ on the $G \wr \Sigma$-set G are $\{C \cup C^o \mid C \text{ a conjugacy class of } G\}$. Lemma 6 and Mackey's theorem imply that

$$kG\!\downarrow_{\underline{G}\times\Sigma} \;=\; \sum_{C,C^o} k(C \cup C^o).$$

Also $k(C \cup C^o) \;=\; k_{C_{\underline{G}\times\Sigma}(c)}\!\uparrow^{\underline{G}\times\Sigma}$, for each $c \in C \cup C^o$. Now Lemma 11 implies that $\Sigma \leq C_{\underline{G}\times\Sigma}(c)$ if and only if $c \in \Omega$. Then by the Krull-Schmidt theorem M is a component of $k_{C_{\underline{G}\times\Sigma}(t)}\!\uparrow^{\underline{G}\times\Sigma}$, for some $t \in \Omega$. But $(k_{C_{\underline{G}\times\Sigma}(t)}\!\uparrow^{\underline{G}\times\Sigma})\!\downarrow_{\underline{G}} = k_{\underline{C_G(t)}}\!\uparrow^{\underline{G}}$. We conclude that $M\!\downarrow_{\underline{G}}$ is a component of $k\Omega$.

Let B_1 be the 2-block of G such that $\underline{B_1}^{\mathrm{Fr}} \times \Sigma$ contains M. As $\Sigma \leq V$, Lemma 5 forces $C_{G\wr\Sigma}(V) \leq \underline{G} \times \Sigma$. So by a Theorem of Nagao-Green [9, 5.3.12], the induced block $(\underline{B_1}^{\mathrm{Fr}} \times \Sigma)^{G\wr\Sigma}$ contains B^*. But $(\underline{B_1}^{\mathrm{Fr}} \times \Sigma)^{G\wr\Sigma} \;=\; B_1^{\otimes 2}$, by Lemma 16. So B^* belongs to $B_1^{\otimes 2}$. This forces the $G \times G$-module B to belong to $B_1 \otimes B_1$. It follows easily that $B = B_1$.

We can reverse the above argument to show that if N is a B^{Fr}-component of $k\Omega$, then the inflation of N to $\underline{G}\times\Sigma$ is a component of $B^*{\downarrow_{\underline{G}\times\Sigma}}$ that has a vertex containing Σ. $\square$

Corollaries 4 and 15 give the implication (i) $\Longrightarrow$ (iii) of Theorem 2. Our next result gives a direct proof of this and also produces some information on the vertices of the components of $k\Omega$.

Corollary 18. *Suppose that some component of $k\Omega$ belongs to B. Then B is strongly real. More precisely, let $c \in G$ belongs to a real defect class of B, let D be a Sylow 2-subgroup of $C_G(c)$, and let E be a Sylow 2-subgroup of $C_G^*(c)$ that contains D. Suppose that N is a component of $k\Omega$ that belongs to B. Then there exists $t \in \Omega\cap(E\backslash D)$ such that N has a vertex $V \leq C_D(t)$.*

Proof. Clearly N^{Fr} is also a component of $k\Omega$. Also $\underline{V}$ is a vertex of N^{Fr}, as $\underline{G}$-module. Let M be the inflation of N^{Fr} to $\underline{G}\times\Sigma$. Then Lemma 17 implies that M is a component of $B^*{\downarrow_{\underline{G}\times\Sigma}}$. Now M has vertex $\underline{V}\times\Sigma$. As M is a component of $B^*{\downarrow_{\underline{G}\times\Sigma}}$, it follows that some vertex of B^* contains $\underline{V}\times\Sigma$.

We established in Proposition 14 that $\underline{D} < \underline{e}\sigma >$ is a vertex of B^*. Then by the previous paragraph there exists $(g_1, g_2) \in G\times G$ such that $(\underline{V}\times\Sigma)^{(g_1,g_2)} \leq \underline{D} < \underline{e}\sigma >$. In particular $\sigma^{(g_1,g_2)} = (g_1^{-1}g_2, (g_1^{-1}g_2)^{-1})\sigma$ belongs to $\underline{D} < \underline{e}\sigma >$. Choose $d \in D$ such that $de = g_1^{-1}g_2$. Then $t := de$ belongs to Ω. So $E = D < t >$ splits over D. In particular B is strongly real. Also $\underline{V}^{(g_1,g_2)} \leq C_{\underline{D}}(\underline{t}\sigma)$. So $V \leq_G C_D(t)$. $\square$

An important special case is that of a real 2-block of defect zero. The next result was proved by R. Gow (in unpublished work), unifying results in [4] and [10]. Gow's proof used a pairing argument on L. L. Scott's 'orbital characters' defined with respect to the involution module. Our proof uses Alperin-Scott modules.

Theorem 19. *Suppose that B is a real 2-block of defect 0. Let χ be the unique irreducible character in B. Then $\nu(\chi) = +1$. Let C be a real defect class of B and let $t \in \Omega$. Then $< \chi{\downarrow_{C_G(t)}}, 1_{C_G(t)} >= 1$ or 0, depending on whether or not t inverts an element of C.*

Proof. Note that χ is the unique principal indecomposable character in B. The fact that $\nu(\chi) = +1$ appears in [10] and independently in several other places. Here it is a simple consequence of Lemma 3. It also follows from that lemma that the G-character of $\mathbb{C}\Omega$ contains χ with multiplicity 1.

Let $c \in C$ and let $t \in \Omega$ be such that $c^t = c^{-1}$. The proof will be completed by showing that $< \chi\!\downarrow_{C_G(t)}, 1_{C_G(t)} > \neq 0$.

Now $Z(B^*) = (B^*)^{\underline{G}\times\Sigma}$ is spanned by 1_B. Also B^* has a trivial source. So $B^*\!\downarrow_{\underline{G}\times\Sigma}$ has a unique Scott component, and this component has socle spanned by 1_B. Proposition 14 implies that $1_B \in \mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, for $E \leq \underline{G} \times \Sigma$ if and only if $<\underline{t}\sigma> \leq_{\underline{G}} E$. It follows that the Scott component of $B^*\!\downarrow_{\underline{G}\times\Sigma}$ has vertex $< \underline{t}\sigma >$.

An easy calculation shows that $\underline{C_G(t)} \times \Sigma$ coincides with the normalizer of $< \underline{t}\sigma >$ in both $C_G(t) \wr \Sigma$ and $\underline{G} \times \Sigma$. The Green correspondence preserves Scott modules. Then, using a result of D. Burry [9, 4.4.7], and the previous paragraph, the Scott module $S(C_G(t) \wr \Sigma; < \underline{t}\sigma >)$ has multiplicity 1 as a component of $B^*\!\downarrow_{C_G(t)\wr\Sigma}$. The restriction of $S(C_G(t) \wr \Sigma; < \underline{t}\sigma >)$ to $C_G(t) \times C_G(t)$ has a projective Scott component. We deduce that $B\!\downarrow_{C_G(t)\times C_G(t)}$ has a Scott component.

Let $\hat{B}$ be a lift of B to a $G \times G$-module over a field of characteristic 0. Then $\hat{B}$ has character $\overline{\chi} \otimes \chi$. Thus $< (\overline{\chi} \otimes \chi)\!\downarrow_{C_G(t)\times C_G(t)}, 1_{C_G(t)\times C_G(t)} >$ is the number of Scott components of $B\!\downarrow_{C_G(t)\times C_G(t)}$. But this inner product is

$$\sum_{c_1,c_2\in C_G(t)} \chi(c_1^{-1})\chi(c_2) = < \chi\!\downarrow_{C_G(t)}, 1_{C_G(t)} >^2 .$$

It then follows from the previous paragraph that $< \chi\!\downarrow_{C_G(t)}, 1_{C_G(t)} > \neq 0$. □

We can now prove that (iii) $\Longrightarrow$ (ii) in Theorem 2. This completes the proof of that theorem.

Corollary 20. *Suppose that B is strongly real. Then $\nu(B) > 0$.*

Proof. Let $c \in G$ belong to a real defect class C of B. Fix a Sylow 2-subgroup D of $C_G(c)$ and a a Sylow 2-subgroup E of $C_G^*(c)$ that contains D. Write $E = D < t >$, where $t \in \Omega$.

We need a version of Brauer's first main theorem. Let N be the normalizer of D in G and let b be the Brauer correspondent of B with respect to (G, N, D). Now b is a real 2-block of N, as $(b^o)^G = (b^G)^o = B$, and b is the unique block of kN such that $b^G = B$. If $1_B = \sum_K \lambda_K K^+$, then $1_b = \sum_K \lambda_K (K \cap C_G(D))^+$. Here K ranges over the classes of G. Also $\omega_B(K^+) = \omega_b\left((K \cap C_G(D))^+\right)$, for each conjugacy class K of G. It follows that $C_1 := C \cap C_G(D)$ is a real defect class of b.

Set $\overline{N} := N/D$ and let μ be the natural k-algebra projection $kN \to k\overline{N}$. Let $\overline{C}$ be the conjugacy class of $\overline{N}$ that contains $\overline{c} = Dc$. Then $\mu(C_1^+) = \overline{C}^+$, by [9, 5.8.9]. It follows that $\mu(1_b) \neq 0$. Write $\mu(1_b) = \sum_{i=1}^s 1_{\beta_i}$, where $\beta_1, \ldots, \beta_s$ are distinct blocks of $\overline{N}$. As $1_{b^o} = 1_b$, it follows that there is a permutation τ of $\{1, \ldots, s\}$ such that $\beta_i^o = \beta_{i\tau}$, for $i = 1, \ldots, s$. If $i \neq i\tau$, an easy argument shows that $1_{\beta_i} + 1_{\beta_{i\tau}}$ is supported on the non-real classes of $\overline{N}$. But $\overline{C}$ is a real class of $\overline{N}$ whose sum appears with non-zero multiplicity in $\mu(1_b)$. We deduce that there exists i such that β_i is a real 2-block of $\overline{N}$ and $\overline{C}^+$ appears with non-zero multiplicity in 1_{β_i}. Set $\beta := \beta_i$.

Now β has a trivial defect group, by [9, 5.8.7(ii)], and $C_{\overline{N}}(\overline{c})$ is odd, by [9, 5.8.9(ii)]. It follows that $\overline{C}$ is a real defect class of β. Let χ be the unique irreducible character in β. Now $\overline{t}$ is an involution in $\overline{N}$ that inverts an element of $\overline{C}$. So by Theorem 19 we have $< \chi_{C_{\overline{N}}(\overline{t})}, 1_{C_{\overline{N}}(\overline{t})} >= 1$. The preimage of $C_{\overline{N}}(\overline{t})$ in N is $C_N(Dt) := \{n \in N \mid t^n \in Dt\}$. Inflating χ to N, we get $< \chi_{C_N(Dt)}, 1_{C_N(Dt)} >= 1$. But $C_N(t) \leq C_N(Dt)$. So $< \chi_{C_N(t)}, 1_{C_N(t)} >\neq 0$. Let M be the unique irreducible β-module. Then we have just shown that M is a b-composition factor of $k_{C_N(t)}\uparrow^N$. We deduce from Lemma 17 that $b^*\downarrow_{\underline{N}\times\Sigma}$ has a component with a vertex that contains Σ.

J. L. Alperin proved in [1] that b is a component of $B\downarrow_{N\times N}$. So b^* is a component of $B^*\downarrow_{N\times N}$. This and the previous paragraph show that $B^*\downarrow_{\underline{N}\times\Sigma}$ has a component with a vertex that contains Σ. Applying Lemma 17, we deduce that $k\Omega$ has a B-composition factor. We conclude from Corollary 4 that $\nu(B) > 0$. $\square$

We conclude our paper with a small application of Theorem 2. R. Gow proved the following result in [3, 5.6]:

Proposition 21. *Let B be a real 2-block, let $c \in G$ belong to a real defect class of B, let D be a Sylow 2-subgroup of $C_G(c)$ and let $D < e >$ be a Sylow 2-subgroup of $C_G^*(c)$. Then B contains a real-valued irreducible character of height 0 and Frobenius-Schur indicator -1 if and only if $D < e > /D'$ does not split over D/D'.*

It is known that each real 2-block has a real-valued irreducible character of Frobenius-Schur indicator $+1$. So Theorem 2 and Proposition 21 combine to give:

Corollary 22. *Let B, D and $D < e >$ be as in Proposition 21. Suppose that $D < e > /D'$ splits over D/D' but $D < e >$ does not split over D. Then B contains a real-valued irreducible character of height greater that 0 and Frobenius-Schur indicator -1.*

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