BLOCKS OF CENTRALISER ALGEBRAS AND DEGENERATE AFFINE HECKE ALGEBRAS

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Abstract. Let k be a field and let $l < m$ be positive integers. We compute the blocks of the centralizer of the symmetric group Σ_l in the group algebra $k\Sigma_m$, for $m - l \leq 3$. We also determine the possible decomposition matrices of these blocks. Our proof uses formal characters of degenerate affine Hecke algebras and abacus combinatorics.

1. Introduction

Throughout this paper, we fix the following notation:

(i) p is a prime number,
(ii) (R, F, k) is a p-modular system (as in [7]) with k algebraically closed,
(iii) m is a positive integer and Σ_m is the symmetric group of degree m,
(iv) n is a non-negative integer with $0 \leq n < m$ and $l = m - n$,
(v) Σ'_n is the stabilizer of $\{1, \ldots, l\}$ and Σ_l is identified with the stabilizer subgroup of $\{l+1, \ldots, m\}$ in Σ_m (so $\Sigma'_n \cong \Sigma_n$ and $\Sigma_l \times \Sigma'_n \leq \Sigma_m$),
(vi) $R\Sigma_m^{\Sigma_l} = \{a \in R\Sigma_m \mid ab = ba \text{ for all } b \in R\Sigma_l\}$.

Our long-term goal is to understand the representation theory of the centralizer algebra $R\Sigma_m^{\Sigma_l}$. In particular, we would like to find the simple modules and the blocks of $k\Sigma_m^{\Sigma_l}$. We would also like to find the decomposition matrices for the algebra $R\Sigma_m^{\Sigma_l}$. For example, if $n = 0$ and $l = m$, $R\Sigma_m^{\Sigma_m}$ is the centre $Z(R\Sigma_m)$ of the group algebra, and the blocks are determined by the Murnagham-Nakayama conjecture [8]. As the centre is commutative, each of its blocks contains a unique simple module. In [6] we completed a similar analysis for the case $n = 1$.

The problem appears to be substantially more difficult when $n > 1$. In particular we do not yet have a good description of the centre $Z(k\Sigma_m^{\Sigma_l})$ of $k\Sigma_m^{\Sigma_l}$. In this paper we develop an approach to these problems that depends on understanding the formal characters of the degenerate affine Hecke algebra $\mathcal{H}_n^k$. We solve all three problems completely, for $n = 2, 3$. It is likely that our methods would work for $n = 4$, subject to an unpleasant case-by-case analysis. But it will become apparent that the combinatorics renders this approach impractical, for $n > 4$, at least without the introduction of some new ideas.

1.1. **Background on Degenerate Affine Hecke Algebras and Representations of the Symmetric Group.** If $\mathcal{O}$ is any commutative ring and n is a positive integer, the *degenerate affine Hecke algebra of degree n over $\mathcal{O}$*, denoted by $\mathcal{H}_n^{\mathcal{O}}$, is

Date: July 5, 2010.
Key words and phrases. Symmetric Group, Centralizer Algebra, Degenerate Affine Hecke Algebra, Abacus, Block.

the unital, associative $\mathcal{O}$ algebra with generators $x_1, \dots, x_n, s_1, \dots s_{n-1}$ subject to the following relations:

(1) $$x_i x_j = x_j x_i,$$

(2) $$(s_i s_j)^{m_{ij}} = 1, \text{ where } m_{ii} = 1,\ m_{ii+1} = 3 \text{ and } m_{ij} = 2, \text{ for } |i-j| > 1,$$

(3) $$s_i x_j = x_j s_i, \quad \text{if } j \neq i, i+1,$$

(4) $$s_i x_i = x_{i+1} s_i - 1 \quad \text{or equivalently} \quad s_i x_{i+1} = x_i s_i + 1.$$

The subalgebra of $\mathcal{H}_n^{\mathcal{O}}$ generated by all x_i is isomorphic to the polynomial algebra $\mathcal{O}[x_1, \dots x_n]$. The relations in (2) are the familiar Coxeter relations defining the symmetric group Σ_n; it follows that the subalgebra generated by all s_i is isomorphic to the group algebra $\mathcal{O}\Sigma_n$. As an $\mathcal{O}$-module, $\mathcal{H}_n^{\mathcal{O}}$ is just $O[x_1, \dots, x_n] \otimes_{\mathcal{O}} \mathcal{O}\Sigma_n$. Relations (3) and (4) show how to multiply an element of the polynomial subalgebra by an element of the group ring subalgebra. For more information about degenerate affine Hecke algebras, see Kleshchev's book [9].

The relations (3) and (4) are motivated at least in part by the fact they are satisfied when x_i is replaced by the ith Jucys-Murphy element L_i. (Recall that L_i is the sum of all transpositions in Σ_i that are not in Σ_{i-1}, with $L_1 = 0$.) It follows that there is a surjective algebra homomorphism $\mathcal{H}_n^{\mathcal{O}} \to \mathcal{O}\Sigma_n$ given by $s_i \mapsto (i, i+1)$ and $x_i \mapsto L_i$. This homomorphism and variations on it are central this paper.

It is a remarkable fact that the composition factors of an $\mathcal{H}_n^k$-module M that is finite dimensional over k are determined by the composition factors of the restriction of M to the polynomial subalgebra. This allows us to define formal characters of $\mathcal{H}_n^k$-modules, as follows. Because k is algebraically closed, every simple $k[x_1, \dots, x_n]$-module is one-dimensional. There is a bijection between the set of isomorphism types of simple $k[x_1, \dots, x_n]$-modules and k^n, in which a simple module V corresponds to $(a_1, \dots, a_n) \in k^n$ if the variable x_i acts as the scalar a_i on the one-dimensional k-space V. The *formal character* of a $k[x_1, \dots, x_n]$-module is defined to be the formal $\mathbb{Z}$-linear combination of elements of k^n corresponding to its composition factors. The *formal character* of an $\mathcal{H}_n^k$-module M that is finite dimensional over k is defined to be the formal character of the restriction $M{\downarrow}_{k[x_1,\dots,x_n]}$. By Theorem 5.3.1 in [9], the composition factors of M are determined by its formal character. Because of the surjective map $\mathcal{H}_n^k \to k\Sigma_n$, each simple $k\Sigma_n$-module can be inflated to a simple $\mathcal{H}_n^k$-module, and so every simple $k\Sigma_n$-module has a formal character. The same thing is true if k is replaced by $\mathbb{C}$.

When studying blocks, it is useful to know that the centre of $\mathcal{H}_n^{\mathcal{O}}$ consists of all symmetric polynomials in the variables $x_1, \dots, x_n$. This was first proved by Bernstein, see [2]. Murphy [10] (and independently Jucys) proved that the restriction of the map $\mathcal{H}_n^{\mathcal{O}} \to \mathcal{O}\Sigma_n$ to the centres is a surjective homomorphism $\mathrm{Z}(\mathcal{H}_n^{\mathcal{O}}) \to \mathrm{Z}(\mathcal{O}\Sigma_n)$.

The *central character* of $\mathcal{H}_n^k$ associated to an indecomposable $\mathcal{H}_n^k$-module M is the map $Z(\mathcal{H}_n^k) \to k$ given by $f \to f(a_1, \dots, a_n)$, where $(a_1, \dots, a_n)$ occurs in the support of the formal character of M. This does not depend on the order of the entries in $(a_1, \dots, a_n)$, as all such $(a_1, \dots, a_n)$ lie in a single Σ_n-orbit. Sometimes we say that $(a_1, \dots, a_n)$ is a central character of M.

The formal characters of the simple $\mathbb{C}\Sigma_n$-modules are easy to compute. Let S^λ be the Specht module (defined over $\mathbb{C}$) corresponding to the partition λ of n. Consider the Young diagram corresponding to λ. The *content* of the node in the c-th column and r-th row of the diagram is $c - r$. The Young diagram can be built up in one or more ways by starting with the node in the upper left hand corner, then adding nodes one by one subject to the condition that after each node is added the result is a Young diagram. Each one of these ways to build the Young diagram is recorded in an n-tuple $(a_1, \ldots, a_n)$, where a_i is the content of i-th node added. There is one such n-tuple for each standard λ-tableau. The formal character of S^λ is the formal sum of these elements of $\mathbb{C}^n$. For example, the partition $[3,1]$ has 3-standard tableau t_1, t_2, t_3 and the formal character of $S^{[3,1]}$ is $(0,-1,1,2) + (0,1,-1,2) + (0,1,2,-1)$ because:

$$\begin{array}{cccc}
\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \\
\text{contents} & t_1 & t_2 & t_3
\end{array}$$

Note that the formal characters of simple $\mathbb{C}\Sigma_n$-modules are linear combinations of elements of $\mathbb{Z}^n$.

If the formal characters of the simple $\mathcal{H}_n^k$-modules are known, it is an easy mechanical task to compute the decomposition matrix for Σ_n. Because of the surjective map $\mathcal{H}_n^k \to k\Sigma_n$, each simple $k\Sigma_n$-module D^μ corresponds to a simple $\mathcal{H}_n^k$-module; let ψ^μ be the formal character of this module. To find the row of the decomposition matrix labelled by the partition λ, one starts with the formal character of S^λ and reduces each entry of each n-tuple modulo p to obtain a $\mathbb{Z}$-linear combination of elements of k^n; this combination is equal to $\sum_\mu d_{\lambda\mu}\psi^\mu$, where $d_{\lambda\mu}$ is the entry on the λ row and μ column of the decomposition matrix. Finding the integers $d_{\lambda\mu}$ from the formal characters is just a matter of solving a system of linear equations.

1.2. **The Degenerate Hecke algebra $\mathcal{H}_n^k$ and the centralizer algebra $k\Sigma_m^{\Sigma_l}$.** In this paper, we develop a very similar strategy for investigating the blocks and decomposition matrices of $R\Sigma_m^{\Sigma_l}$.

First, we need a parametrization of the simple $F\Sigma_m^{\Sigma_l}$-modules. Because $F\Sigma_m$ and $F\Sigma_l$ are split semisimple algebras, it follows from 1.0.1 in [9] that each irreducible $F\Sigma_m^{\Sigma_l}$-module has the form $S^{\beta\backslash\alpha} := \mathrm{Hom}_{F\Sigma_l}(S^\alpha, S^\beta)$, where α is a partition of l and β is a partition of m such that the Specht module S^α is a constituent of the restriction $(S^\beta){\downarrow_{F\Sigma_l}}$. Thus, using the classical branching rule, the irreducible $F\Sigma_m^{\Sigma_l}$-modules are parametrized by pairs (α, β), where α is a partition of l whose Young diagram has been obtained from that for the partition β of m by removing $m - l$ nodes.

Next, we fix a block idempotent E of $R\Sigma_l$, and let $\overline{E}$ be the corresponding block idempotent of $k\Sigma_l$. Recall that $n = m - l$. In place of the surjective map $\mathcal{H}_n^k \to k\Sigma_n$ above, we use the following:

$$\begin{array}{rcll}
\phi : \mathcal{H}_n^k & \to & k\Sigma_m^{\Sigma_l}\overline{E} & \\
s_i & \mapsto & (l+i, l+i+1)\overline{E}, & i = 1, \ldots, n-1 \\
x_i & \mapsto & L_{l+i}\overline{E}, & i = 1, \ldots, n
\end{array}$$

By Proposition 2.1.1 in [9], the algebra $k\Sigma_m^{\Sigma_l}$ is generated by $\mathrm{Z}(k\Sigma_l)$, the Murphy elements $L_{l+1}, \ldots, L_m$, and the subalgebra $k\Sigma_n'$. (This remains true when k is replaced by any commutative ring.) Since $\mathrm{Z}(k\Sigma_l\overline{E})$ is a commutative local ring, it follows that the map $\mathcal{H}_m^k \to k\Sigma_m^{\Sigma_l}\overline{E}/\mathrm{JZ}(k\Sigma_l\overline{E})$ is surjective, where $\mathrm{JZ}(k\Sigma_l\overline{E})$ is the ideal of $k\Sigma_m^{\Sigma_l}\overline{E}$ generated by the Jacobson radical of $\mathrm{Z}(k\Sigma_l\overline{E})$. Since $\mathrm{JZ}(k\Sigma_l\overline{E})$ is contained in the Jacobson radical of $k\Sigma_m^{\Sigma_l}\overline{E}$, the algebras $k\Sigma_m^{\Sigma_l}\overline{E}$ and $k\Sigma_m^{\Sigma_l}\overline{E}/\mathrm{JZ}(k\Sigma_l\overline{E})$ have the same simple modules. A theorem of Dade (12.9 in [7]) tells us that $k\Sigma_m^{\Sigma_l}\overline{E}$ and $k\Sigma_m^{\Sigma_l}\overline{E}/\mathrm{JZ}(k\Sigma_l\overline{E})$ also have the same blocks.

We can use the displayed map ϕ to associate to each simple $k\Sigma_m^{\Sigma_l}\overline{E}$-module the formal character of a simple $\mathcal{H}_n^k$-module. Provided we know the formal characters of the simple $\mathcal{H}_n^k$-modules, we can then find the decomposition matrix for $R\Sigma_m^{\Sigma_l}E$. The row of the decomposition matrix corresponding to the irreducible $F\Sigma_m^{\Sigma_l}E$-module $S^{\beta\backslash\alpha}$ is computed as follows. The Young diagram for β is obtained from the Young diagram for α in one or more ways by adding a series of nodes. The element $(a_1, a_2, \ldots, a_n)$ of $\mathbb{Z}^n$ is associated to adding a node of content a_1, then a node of content a_2, etc. There is one n-tuple for each standard skew $[\beta\backslash\alpha]$-tableau. Take the formal sum of all these, then reduce each entry in each n-tuple modulo p to obtain a formal $\mathbb{Z}$-linear combination of elements of k^n. The result can be expressed in a unique way as a $\mathbb{Z}$-linear combination of formal characters associated to simple $k\Sigma_m^{\Sigma_l}\overline{E}$-modules. The resulting coefficients are the entries in the row of the decomposition matrix.

The precise decomposition matrix of $R\Sigma_m^{\Sigma_l}E$ depends on m, l, and E. However, knowing only $n = m - l$ already tell us quite a lot. We use n to write down a short "menu" from which the rows of the decomposition matrix must be chosen. Whether each possible row appears, and how many times it appears, are then determined by m and E, and the combinatorics of James' abacus (see Section 3 below).

Our main result says that if $n \leq 3$, then every block idempotent of $R\Sigma_m^{\Sigma_l}$ has the from E_1E_2, where E_1 is a block idempotent of $R\Sigma_m$ and E_2 is a block idempotent of $R\Sigma_l$. We obtain this information from our knowledge of the decomposition matrices. Using Proposition 6 and a theorem of Bessenrodt [3], we show that enough of the rows on the "menu" actually occur to link any two irreducible $F\Sigma_m^{\Sigma_l}E_1E_2$-modules.

It is natural to conjecture that the centre of $R\Sigma_m^{\Sigma_l}$ is generated by $\mathrm{Z}(R\Sigma_m)$ and $\mathrm{Z}(R\Sigma_l)$. In an earlier paper [6], we proved that this is the case when $n = 1$. Computer calculations performed by Danz, some of them making use of a recent theorem of Alperin [1], have shown that this the case for all l and m with $m \leq 8$. (Alperin's theorem provides us with a way to compute, for any group G and subgroup H, the order of the finite abelian group $\mathrm{Z}(\mathbb{Z}G^H)/\langle \mathrm{Z}(\mathbb{Z}G), \mathrm{Z}(\mathbb{Z}H)\rangle$, where as usual $\mathbb{Z}$ denotes the integers; this group has order equal to the product of the elementary divisors of a certain matrix called the reduced class-coset table for the groups ΔH and $H \times G$.) It is not true that for any group G and subgroup H, the centre of RG^H is generated by $\mathrm{Z}(RG)$ and $\mathrm{Z}(RH)$; for counterexamples, see [5].

There is a surjective algebra homomorphism $\mathcal{H}_n^{\mathrm{Z}(R\Sigma_l)} \to R\Sigma_m^{\Sigma_l}$ that acts as the identity on $\mathrm{Z}(R\Sigma_l)$, sends x_i to L_{l+i}, and sends s_i to the transposition $(l{+}i, l{+}i{+}1)$. The conjecture that the centre of $R\Sigma_m^{\Sigma_l}$ is generated by $\mathrm{Z}(R\Sigma_m)$ and $\mathrm{Z}(R\Sigma_l)$ is equivalent to the conjecture that this homomorphism restricts to a surjective map $\mathrm{Z}(\mathcal{H}_n^{\mathrm{Z}(R\Sigma_l)}) \to \mathrm{Z}(R\Sigma_m^{\Sigma_l})$. It is interesting to compare this to a theorem of Brundan [4]. Brundan proves that for any commutative ring $\mathcal{O}$ and any monic polynomial $f(x_1) \in \mathcal{O}[x_1]$, the map $\mathcal{H}_n^{\mathcal{O}} \to \mathcal{H}_n^{\mathcal{O}}/\langle f\rangle$ restricts to a surjective map between the centres.

2. Formal characters of simple $\mathcal{H}_n^k$-modules for $n \leq 3$

The main purpose of this section is to enumerate the formal characters of all simple $\mathcal{H}_n^k$-modules when $n \leq 3$. However, we start with some very general remarks about modules over rings. Suppose that A and B are rings and $\phi : A \to B$ is a ring homomorphism. Then each B-module can be inflated to an A-module along ϕ.

Lemma 1. *Let V be a B-module and let U be a submodule of V.*

(i) *U is an A-submodule of V.*
(ii) *Assume that ϕ is surjective. Then each A-submodule of V is a B-submodule of V.*
(iii) *Assume that $B = \phi(A) + J(B)$. If V is simple as an B-module, then it is simple as an A-module.*

Proof. Only the third statement requires any comment. Assume that V is simple as an B-module. Let $\phi' : A \to B/J(B)$ be ϕ followed by the natural map. Then ϕ' is surjective, by the hypothesis. Since V is simple as a B-module, it is well-defined and simple as a $B/J(B)$-module. Then the second statement shows that V is simple as an A-module. □

Corollary 2. *Assume that $B = \phi(A) + J(B)$. If*

$$0 \subseteq U_1 \subseteq U_2 \subseteq \ldots \subseteq U_r = V$$

is a composition series for V as a B-module, then it is a composition series for V as an A-module.

Proof. The proof is by induction on r. By the previous Lemma, U_1 is a simple A-submodule of the A-module V. Then V/U_1 has a shorter composition length as a B-module. □

The map $\phi : \mathcal{H}_n^k \to k\Sigma_m^{\Sigma_l}\overline{E}$ from the introduction satisfies condition (iii) from this Lemma. This gives us the following proposition.

Proposition 3. *Let M be an R-free $R\Sigma_m^{\Sigma_l}E$-module such that $F \otimes_R M$ is an irreducible $F\Sigma_m^{\Sigma_l}$-module. Let $\overline{M} = M/\mathrm{J}(R)M$. Then the composition factors of $\overline{M}$ as an $\mathcal{H}_n^k$-module are the inflations along ϕ of the composition factors of $\overline{M}$ as $k\Sigma_m^{\Sigma_l}$-module.*

Proof. We may assume that $EM = M$. The result now follows from an application of Corollary 2 with $A = \mathcal{H}_n^k$, $B = k\Sigma_m^{\Sigma_l}\overline{E}$, and $V = \overline{M}$. □

Now we turn to formal characters of simple modules over a degenerate affine Hecke algebra. The rest of this section only requires k to be algebraically closed; it does not have to be of finite characteristic. Much of this analysis (and more) already appears in A. S. Kleshchev's book [9], or chapter 4 of M. Vazirani's Ph.D. thesis [11].

The relations among the generators of $\mathcal{H}_n^k$ ensure that each of its 1-dimensional modules has formal character $(a, a+1, a+2\ldots)$ or $(\ldots, a+2, a+1, a)$, for some $a \in k$. Suppose that $\underline{a} \in k^n$ is such that $a_i - a_j \neq \pm 1$, for all $i \neq j$. Then by Theorem 6.1.4 of [9], there is an simple $\mathcal{H}_n^k$-module that has dimension $n!$ and formal character $\sum_{\sigma \in \Sigma_n} \sigma(\underline{a})$. For example, the 1-dimensional $\mathcal{H}_2^k$-modules have formal character $(a, a-1)$ or $(a-1, a)$, while each 2-dimensional $\mathcal{H}_2^k$-modules has formal character $(a, b) + (b, a)$, for some $a, b \in k$ with $b \neq a \pm 1$. These account for all simple $\mathcal{H}_2^k$-modules.

Lemma 4. *Let $a, b, c \in k$ with $a \neq b \pm 1$, $a \neq c \pm 1$ and $b \neq c \pm 1$. Apart from the families of 1-dimensional modules $M_1 = (a-2, a-1, a)$ and $M_2 = (a, a-1, a-2)$, there are two families of simple $\mathcal{H}_3^k$-modules which have central character $(a-2, a-1, a)$, and these have formal characters*

$$M_3 := (a-1, a, a-2) + (a-1, a-2, a) \quad \text{and}$$
$$M_4 := (a-2, a, a-1) + (a, a-2, a-1).$$

There are two families of simple $\mathcal{H}_3^k$-modules which have central character $(a-1, a, b)$, and these have formal characters

$$M_5 := (a-1, a, b) + (a-1, b, a) + (b, a-1, a) \quad \text{and}$$
$$M_6 := (b, a, a-1) + (a, b, a-1) + (a, a-1, b).$$

In addition, $\mathcal{H}_3^k$ has a single family of simple 6-dimensional modules M_7 with formal character is $\sum_{\sigma \in \Sigma_3} \sigma(a, b, c)$.

Proof. We adopt the notation of [9]. By the *shuffle lemma*, the induced module $(a-2, a-1) \otimes (a)\uparrow^{\mathcal{H}_3^k}$ has formal character $(a-2, a-1, a) + (a-2, a, a-1) + (a, a-2, a-1)$. In particular M_1 is its unique 1-dimensional composition factor. It follows from this that there is a 2-dimensional composition factor M_3 whose formal character is $(a-2, a, a-1) + (a, a-2, a-1)$. Repeating the argument with $(a-1, a-2) \otimes (a)\uparrow^{\mathcal{H}_3^k}$ we get a 2-dimensional composition factor M_4 whose formal character is $(a-1, a-2, a) + (a-1, a, a-2)$. Also $(a-2) \otimes (a-1) \otimes (a)\uparrow^{\mathcal{H}_3^k} = M_1 + M_2 + M_3 + M_4$, in the Grothendieck group of $\mathcal{H}_3^k$.

Now consider those modules with central character $(a-1, a, b)$. The module $M_5 := (a-1, a) \otimes (b)\uparrow^{\mathcal{H}_3^k}$ has formal character $(a-1, a, b) + (a-1, b, a) + (b, a-1, a)$. This is simple, as it has no 1-dimensional composition factors. Similarly $M_6 := (a, a-1) \otimes (b)\uparrow^{\mathcal{H}_3^k}$ is an simple module with formal character $(a, a-1, b) + (a, b, a-1) + (b, a, a-1)$. Also $(a-1) \otimes (a) \otimes (b)\uparrow^{\mathcal{H}_3^k} = M_5 + M_6$, in the Grothendieck group of $\mathcal{H}_3^k$.

Finally, we have already discussed the existence of the 6-dimensional simple $\mathcal{H}_3^k$-module M_7, before this lemma.

It follows from Lemma 6.1.1 in [9] that these are all the simple $\mathcal{H}_3^k$-modules. □

3. Some results on the abacus

We introduce an idea of G. James [8]. Let e be a positive integer. Then we can represent a partition α as an arrangement of beads on an abacus with e-runners. It is convenient to assume that the total number of beads is a fixed multiple of e. The runners are labelled from left to right by the residues $0, 1, \ldots, e-1$ mod e. If i is a positive integer, with residue r mod e, then we may refer to the r-th runner as the i-th runner. The positions on runner i are labelled, from top to bottom, by the integers $i, i+e, i+2e, \ldots$.

Moving a bead due left (right) into an empty position on an adjacent runner corresponds to removing (adding) a single node from (to) α. Moving a bead one position up (down) into an empty space on the same runner corresponds to removing (adding) an e-rim hook from (to) α. More generally, moving a bead from position i to an empty position j corresponds to adding a rim hook of length $j-i$ to (if $i < j$) or removing a rim hook of length $i-j$ from (if $i > j$) from α. We refer to $i \to j$ as an $|i-j|$-rim hook at i (or on runner i).

If $i \to j$ is a rim-hook of α, the *leg-length* of H is the number of positions strictly between i and j that are occupied by beads. For integers $l \leq h$ we use $a_h^l(\alpha)$ and $r_h^l(\alpha)$ to denote the number of addable or removable h-rim hooks of leg-length l that can be added or removed from α, respectively. We will use the following elegant result, due to C. Bessenrodt [3]:

Lemma 5. *Given a partition α and a positive integer h, then for $l = 0, \ldots, h-1$ we have*

$$a_h^l(\alpha) = 1 + r_h^l(\alpha).$$

Now the *e-core* of α is obtained by removing all e-rim hooks from the abacus of α. The number of bead moves required to do this is called the *e-weight* of α. The collection of all partitions of a fixed integer that have a given e-core is called an *e-block*.

Suppose that α is a partition of n and that $i \neq j$ are integers such that α has a removable node at i and an addable node at $j-1$. Moving the bead at i into position $i-1$ and the bead at position $j-1$ into position j produces another partition of n. We denote this partition by $\alpha(i,j)$. Then $\alpha(i,j)$ is said to be a *one-box-shift* of α. Clearly α and $\alpha(i,j)$ belong to the same e-block if and only if $i \equiv j \bmod e$. The rest of this section will be concerned with proving:

Proposition 6. *Let α and β be partitions that belong to an e-block B. Suppose that α has a removable node at position i. Then either β has a removable node on runner i or $\alpha(i,j)$ exists for some $j \equiv i \bmod e$.*

For *integers $i > j$* we set

$$q_{ij} := \#\{f \in \mathbb{N} \mid j < ef \leq i\}. \tag{5}$$

Thus q_{ij} counts the number of rows of the abacus that contain i or that are directly above i, and that are also below but not containing j. Note that if k is an integer with $i > k > j$ then $q_{ij} = q_{ik} + q_{kj}$.

Suppose that B is an e-block of partitions. Then for each residue $i \bmod e$, the number x_i of beads on runner i of the e-abacus representing $\alpha \in B$ does not depend on α. We use $w_i = w_i(\alpha)$ to denote the number of bead moves required to move all beads as far up as possible on runner i of α. In particular $w_0 + \ldots + w_{p-1}$ is the weight of B. For a positive integer h, we set $a_{ih} = a_{ih}(\alpha)$ as the number of addable h rim-hooks with initial position on runner i and $r_{ih} = r_{ih}(\alpha)$ as the number of removable h rim-hooks with initial position on runner i. These notations can be extended to the case that i is an integer, by considering the residue of $i \bmod e$. For integers $i > j$, we define

$$d_{ij} := q_{ij} + x_j - x_i. \tag{6}$$

This depends only on the e-block B of α.

Lemma 7. *Let $i > j$ and e be positive integers. Set $h := i - j$. Then for each partition α we have $a_{jh}(\alpha) = d_{ij} + r_{ih}(\alpha)$.*

Proof. Represent α on an e-abacus, with enough beads so that there are no empty positions in the top $q_{i,j}$-rows. For each position a on runner j, there is a corresponding position $a + h$ on runner i. All but the topmost q_{ij} positions on runner i correspond to positions on runner j. Each removable h-rim hook on runner i represents an empty position on runner j and a bead on the corresponding position on runner i. Each addable h rim-hook on runner j represents a bead on runner j and a corresponding empty position on runner i. Otherwise corresponding positions are either both empty or both occupied by a bead. Counting also the top q_{ij} beads on runner i, we see that $x_i - x_j = q_{ij} + r_{ih} - a_{jh}$. The lemma follows from this. □

Corollary 8. *Let B be an e-block of partitions and let $i > j$ be integers. Set $h = i - j$. Then either all partitions in B have a removable h-rim hook on runner i, or for each partition in B, the number of addable h-rim hooks on runner j is greater than or equal to the number of removable h-rim hook on runner i.*

Proof of Proposition 6. If $d_{i,i-1} < 0$ then Lemma 7 implies that β has a removable node on runner i. So we may assume that $d_{i,i-1} \geq 0$. Now $r_{i,1}(\alpha) > 0$, by hypothesis. So Lemma 7 implies that $a_{i-1,1}(\alpha) > 0$. This means that we may choose $j \equiv i \bmod e$ such that α has an addable node at position $j-1$. It follows from this that the partition $\alpha(i,j)$ exists and belongs to B. □

4. The p-blocks of $R\Sigma_{l+2}^{\Sigma_l}$

In this section $l > 0$ and $n = 2$. So $G = \Sigma_{l+2}$ and $H = \Sigma_l$. By the work in Section 2, for any field k there are three possible families of simple kG^H-modules:

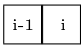
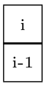
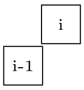
name	formal character
M_1	$(a, a-1)$
M_2	$(a-1, a)$
M_3	$(a, b) + (b, a), \quad b \neq a \pm 1$

Now let (F, R, k) be a p-modular system. Let E_1 be a block (i.e. central primitive) idempotent in RG and let E_2 be a block idempotent in RH such that $E_1E_2 \neq 0$. Then there exist a partition $\beta \in E_1$ and a partition $\alpha \in E_2$, such that $[\alpha]$ can be obtained from $[\beta]$ by the successive removal of two nodes. Recall that the formal character of $S^{\beta\backslash\alpha}$ is the sum of all 3-tuples of the residues of the nodes occupied by $n-1$ and n in all standard skew $[\beta\backslash\alpha]$-tableau and the modular character of $S^{\beta\backslash\alpha}$ is the reduction modulo p of its formal character. This coincides with the formal character of the p-modular reduction of $S^{\beta\backslash\alpha}$.

Theorem 9. *Suppose that $p \neq 2$. Then $RG^H E_1E_2$ is a block of RG^H.*

Proof. If $kG^H\overline{E_1}\,\overline{E_2}$ has only one irreducible module then the result is trivial. If it has a simple module of type M_3, then it cannot have simple modules of type M_1 or M_2. So we may assume that it has simple modules of both types M_1 and M_2. Moreover, cutting kG^H by the idempotent $\overline{E_1}\,\overline{E_2}$ fixes the central character of these modules as $(i, i-1)$, for some $i \in \mathbb{F}_p \subseteq k$.

We have enumerated the irreducible FG^H-modules as being of type M_1, M_2 or M_3. We label the p-modular reduction of M_u by N_u, for $u = 1, 2, 3$. As $p \neq 2$, a module of type M_1 or M_2 has p-modular reduction with central character $(i, i-1)$ if and only if $\overline{a} = i$. Similarly, a module of type M_3 has p-modular reduction with central character $(i, i-1)$ if and only if $\{\overline{a}, \overline{b}\} = \{i, i-1\}$. The work above shows that each row of the decomposition matrix of $RG^H E_1 E_2$ is a row of the matrix:

$[\beta\backslash\alpha]$	name	M_1	M_2
	N_1	1	0
	N_2	0	1
	N_3	1	1

If $FG^H E_1 E_2$ were not connected, it would contain at least one module of type N_1 and another of type N_2. Suppose that this situation occurs. The proof is completed by showing that $FG^H E_1 E_2$ contains at least one module of type N_3.

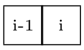
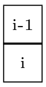
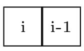
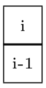
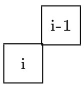
Suppose that $d_{i,i-1} < 0$. Let $\beta \in E_1$ be of type N_2. Then Lemma 7 implies that β has a removable 1-rim-hook on runner i. In particular, β is also of type N_3.

Suppose then that $d_{i,i-1} \geq 0$. Let β be of type N_1. So $r_{i,1}(\beta) > 0$. Lemma 7 implies that $a_{i-1,1}(\beta) > 0$. So there are integers $u, v \equiv i \bmod p$ such that β has beads at positions u and $v-1$ and no beads at positions $u-1, u-2$ and $v+1$. Move the bead at position u into position v and move the bead at position $v-1$ into position $u-1$. The resulting partition belongs to E_1 and has removable 1-rim hooks on runners i and $i-1$. It follows that in either case $FG^H E_1 E_2$ contains at least one module of type N_3. □

Theorem 10. *Suppose that $p = 2$. Then $RG^H E_1 E_2$ is a block of RG^H.*

Proof. As in the previous theorem, we may assume that $kG^H \overline{E_1}\,\overline{E_2}$ has two one-dimensional simple modules, each with central character $(0, 1)$.

We have enumerated the irreducible FG^H-modules as being of type M_1, M_2 or M_3. A module of type M_1, M_2 always has 2-modular reduction with central character $(i, i+1)$. A module of type M_3 has 2-modular reduction with central character $(i, i+1)$ if and only if $\overline{a} \neq \overline{b}$. We label the possible 2-modular reduction of M_1 by N_1, N_3, that of M_2 by N_2, N_4 and that of M_3 by N_5. The work above shows that each row of the decomposition matrix of $RG^H E_1 E_2$ is a row of the matrix:

$[\beta\backslash\alpha]$	name	M_1	M_2
	N_1	1	0
	N_2	1	0
	N_3	0	1
	N_4	0	1
	N_5	1	1

If $FG^H E_1 E_2$ were not connected, it would contain at least one module of type N_1 or N_2, another of type N_3 or N_4. Suppose that this situation occurs. Then, just as in the proof of Theorem 9, $FG^H E_1 E_2$ contains at least one module of type N_5. □

5. The p-blocks of $R\Sigma_{l+3}^{\Sigma_l}$ for $p \neq 3$

In this section $l > 0$, $n = 3$ and $p \neq 3$. So $G = \Sigma_{l+3}$, and $H = \Sigma_l$. As before (F, R, k) is a p-modular system. Let E_1 be a p-block idempotent of RG and let E_2 be a p-block idempotent of RH such that $E_1 E_2 \neq 0$. Then there exist a partition $\beta \in E_1$ and a partition $\alpha \in E_2$, such that $[\alpha]$ can be obtained from $[\beta]$ by the successive removal of three nodes. As in the previous section, we will be concerned with the module $S^{[\beta \backslash \alpha]}$, its formal character, and the formal character of its p-modular reduction.

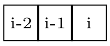
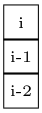
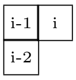
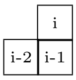
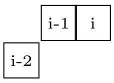
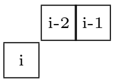
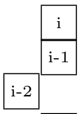
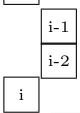
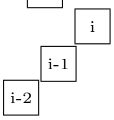
Suppose that $kG^H \overline{E_1}\,\overline{E_2}$ has an simple module with central character (i, j, k), where $i, j, k \in \mathbb{F}_p$ and $j, k \neq i \pm 1$, $k \neq j \pm 1$. Then this is the unique simple $kG^H \overline{E_1}\,\overline{E_2}$-modules. But any algebra with a single class of simple modules is indecomposable. So $kG^H \overline{E_1}\,\overline{E_2}$ is a block algebra in this case.

Suppose now that $kG^H \overline{E_1}\,\overline{E_2}$ has an simple module with central character $(i, i-1, j)$, where $i, j \in \mathbb{F}_p$ and $j \neq i-2, i-1, i, i+1$. Then we may assume that $kG^H \overline{E_1}\,\overline{E_2}$ has simple modules of type M_5 and M_6, with $\overline{a} = i$ and $\overline{b} = j$, in the notation of Lemma 4. The modules have formal characters $(i-1, i, j) + (i-1, j, i) + (j, i-1, i)$ and $(i, i-1, j) + (i, j, i-1) + (j, i, i-1)$, respectively. Using similar arguments to those used in Theorems 9 and 10, we see that $kG^H \overline{E_1}\,\overline{E_2}$ is a block algebra in this case also.

From now on we assume that $kG^H \overline{E_1}\,\overline{E_2}$ has an simple character with central character $(i, i-1, i-2)$, for some $i \in \mathbb{F}_p$. Then there are four simple $\mathcal{H}_3^k$-modules that have central character $(i-2, i-1, i)$. These are the modules M_1, M_2, M_3 or M_4, with $a = i$, in the notation of Lemma 4.

Now consider the irreducible $\mathcal{H}_3^F$-modules of Lemma 4 whose parameters a, b, c are all integers. We give necessary and sufficient conditions on the parameters to ensure that the p-modular reduction has central character $(i-2, i-1, i)$. For modules of type M_1, M_2, M_3 or M_4 the condition is that $a \equiv i \bmod p$. We label these possibilities as N_1, N_2, N_3 or N_4, respectively. For modules of type M_5 or M_6 we require $\overline{a} = i$ and $\overline{b} = i-2$, or $\overline{a} = i-1$ and $\overline{b} = i$. We label these possibilities as $N_{5,1}, N_{6,1}, N_{5,0}$ or $N_{6,0}$, respectively. For modules of type M_7 the condition is that $\{\overline{a}, \overline{b}, \overline{c}\} = \{i-2, i-1, i\}$. We label any one of these modules by N_7.

We construct the matrix below as follows. The rows are indexed by the possible irreducible $FG^H E_1 E_2$-modules. The columns are indexed by the simple $\mathcal{H}_3^k$-modules which have central character $(i-2, i-1, i)$. Each entry gives the composition multiplicity of the column module in the p-modular reduction of the row module.

$[\beta\backslash\alpha]$	name	M_1	M_2	M_3	M_4
	N_1	1	0	0	0
	N_2	0	1	0	0
	N_3	0	0	1	0
	N_4	0	0	0	1
	$N_{5,1}$	1	0	1	0
	$N_{5,0}$	1	0	0	1
	$N_{6,1}$	0	1	0	1
	$N_{6,0}$	0	1	1	0
	N_7	1	1	1	1

The following result describes a symmetry of the decomposition matrix of RG^H:

Lemma 11. *The permutation $i \to -i$ on formal characters induces an involutary map on the irreducible FG^H-modules and on the simple kG^H-modules. For FG^H it is induced by transposition of partitions. The map commutes with p-modular reduction and acts on the type of modules as follows:*

$$\begin{array}{lll} M_1 \leftrightarrow M_2, & M_3 \leftrightarrow M_3, & M_4 \leftrightarrow M_4, \\ N_1 \leftrightarrow N_2, & N_3 \leftrightarrow N_3, & N_4 \leftrightarrow N_4, \\ N_{5,i} \leftrightarrow N_{6,1-i}, & N_7 \leftrightarrow N_7. & \end{array}$$

Theorem 12. *Suppose that $p \neq 3$. Then $RG^H E_1 E_2$ is a block of RG^H.*

Proof. Suppose first that $p \neq 2$. Consider the following table:

FG^H-modules	kG^H-module	critical residue	Additional FG^H-modules
N_1, N_2	$-$	i	$N_{6,0}$ or $N_{5,0}$
N_1, N_3	$-$	$i-2$	$N_{5,1}$
N_1, N_4	$-$	i	$N_{5,0}$
$N_1, N_{5,1}$	M_1	$-$	$-$
$N_1, N_{5,0}$	M_1	$-$	$-$
$N_1, N_{6,1}$	$-$	i	N_7 or $N_{5,0}$
$N_1, N_{6,0}$	$-$	$i-2$	$N_{5,1}$ or N_7
N_3, N_4	$-$	$i-1$	N_7
$N_3, N_{5,1}$	M_3	$-$	$-$
$N_3, N_{5,0}$	$-$	$i-1$	N_7
$N_4, N_{5,1}$	$-$	$i-1$	N_7
$N_4, N_{5,0}$	M_4	$-$	$-$
$N_{5,1}, N_{5,0}$	M_1	$-$	$-$
$N_{5,1}, N_{6,1}$	$-$	$i-1$	N_7
$N_{5,1}, N_{6,0}$	M_3	$-$	$-$
$N_{5,0}, N_{6,1}$	M_4	$-$	$-$

Its row are labelled by pairs of possible irreducible $FG^H E_1 E_2$-modules. We omit any pair of modules if the pair of transpose modules (as given in Lemma 11) has already been listed. If the pair has a common p-modular constituent, this is listed in the second column. Otherwise, we apply Proposition 6 with the runner given in the third paragraph. In this case the existence of the two modules implies the existence of one of the simple modules given in the final column. We do not consider pairs involving N_7, as the p reduction of this module contains all simple $kG^H \overline{E_1}\,\overline{E_2}$-modules as composition factors.

For example, the first row considers the case that $FG^H E_1 E_2$ contains irreducible modules of type N_1 and N_2. These modules have no common p-modular composition factor. For $i = 1, 2$, let β_i be a partition in E_1 such that $S^{[\beta_i \setminus \alpha_i]}$ is of type N_i, for some $\alpha_i \in E_2$. In the first row we consider runner i. If β_2 has a removable node on runner i, then there exists $\alpha_3 \in E_2$ such that $S^{[\beta_2 \setminus \alpha_3]}$ is of type $N_{6,0}$. Otherwise, Proposition 6 tells us that there exists $j \equiv i \bmod p$ such that $\beta_1(i, j)$ belongs to E_1. But there exists $\alpha_4 \in E_2$ such that $S^{[\beta_1 \setminus \alpha_4]}$ is of type $N_{5,0}$. This accounts for the possibilities listed in the last column of the first row.

Suppose that $FG^H E_1 E_2$ has modules of type X_1, X_2 whose 3-modular decompositions do not have a common composition factor. The table can be used to show that there exists an irreducible $FG^H E_1 E_2$-module Y such that:

- $\overline{X_i}$ and $\overline{Y}$ share an simple composition factor, for $i = 1, 2$, or
- $\dim(Y) > \dim(X_i)$, for $i = 1, 2$.

This is enough to show that the p-decomposition matrix of $RG^H E_1 E_2$ is connected. It follows that $RG^H E_1 E_2$ is a p-block of FG^H.

When $p = 2$, we must modify our arguments. For $\mathcal{H}_3^K$ has only three families of simple modules, namely those of type M_1, M_3, M_4 (as each module of type M_2 is already of type M_1). We get the same list of potential FG^H modules $N_1, \ldots, N_7$. However now more of these modules have common p-modular composition factors. The proof is then completed using a table similar to that given above for $p > 3$. $\square$

Example 13. *The partition $\beta = [5, 1^3]$ is the unique partition in a 5-block E_1 of Σ_8, as it has 5-weight 0. We may strip nodes of content $2, 3, 4$ from β in 4 different ways, giving partitions that belong to the principal 5-block E_2 of Σ_5. The corresponding block $R\Sigma_8^{\Sigma_5} E_1 E_2$ contains modules of type N_1, N_2, N_5 and N_7.*

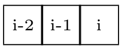
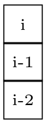
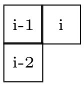
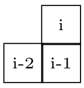
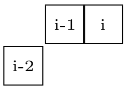
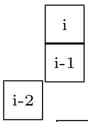
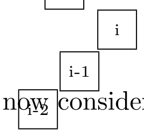
Example 14. *The principal 5-block E_1 of Σ_5 contains 5 partitions. We may strip nodes of residue $2, 3$ and 4 mod 5 from 4 of these, to give the partition [2] in the principal 5-block E_2 of Σ_2. The corresponding block $R\Sigma_5^{\Sigma_2} E_1 E_2$ contains modules of type N_1, N_2, N_6 and N_8.*

6. The p-blocks of $R\Sigma_{l+3}^{\Sigma_l}$ for $p = 3$

In this section $p = 3$ and (R, F, k) is a 3-modular system. We continue to assume that $G = \Sigma_{l+3}$, $H = \Sigma_l$ and $n = 3$. There are six simple $\mathcal{H}_3^k$-modules that have central character $(1, 2, 0)$. These are modules of type M_1 or M_2, with a an integer, in the notation of Lemma 4. For $i \in \mathbb{F}_3$, we use $M_{1,i}$ and $M_{2,i}$ for the modules whose characters are $(i-2, i-1, i)$ and $(i, i-1, i-2)$, respectively.

Now consider the irreducible $\mathcal{H}_3^F$-modules of Lemma 4 whose parameters a, b, c are all integers. All of these have 3-modular reduction with central character $(i-2, i-1, i)$, assuming that a, b, c are distinct mod 3. We label a module of type M_j by $N_{j,i}$, where $i \equiv a$ mod 3.

We construct a matrix as follows. The columns are indexed by the simple $\mathcal{H}_3^k$-modules which have central character $(0, 1, 2)$. Each row, apart from the last, represents one of three possible irreducible $FG^H E_1 E_2$-modules, depending on the value of $i \in \mathbb{F}_3$. Each entry gives the composition multiplicity of the column module in the p-modular reduction of the row module. So the matrix has 19 rows.

$[\beta\backslash\alpha]$	name	$M_{1,i}$	$M_{1,i-1}$	$M_{1,i-2}$	$M_{2,i}$	$M_{2,i-1}$	$M_{2,i-2}$
	$N_{1,i}$	1	0	0	0	0	0
	$N_{2,i}$	0	0	0	1	0	0
	$N_{3,i}$	0	0	1	0	1	0
	$N_{4,i}$	0	1	0	0	0	1
	$N_{5,i}$	1	0	1	0	1	0
	$N_{6,i}$	0	1	0	1	0	1
	N_7	1	1	1	1	1	1

We now consider the effect of transposition of partitions:

Lemma 15. *The permutation $i \to -i$ on formal characters induces an involutary map on the irreducible FG^H-modules and on the simple kG^H-modules. For FG^H it is induced by transposition of partitions. The map commutes with p-modular reduction and acts on the type of modules as follows:*

$$M_{1,i} \leftrightarrow M_{2,2-i},$$
$$N_{1,i} \leftrightarrow N_{2,2-i}, \quad N_{3,i} \leftrightarrow N_{3,2-i}, \quad N_{4,i} \leftrightarrow N_{4,2-i},$$
$$N_{5,i} \leftrightarrow N_{6,1-i}, \quad N_7 \leftrightarrow N_7.$$

Theorem 16. *Suppose that $p = 3$. Then $RG^H E_1 E_2$ is a block of RG^H.*

Proof. Consider the following table:

FG^H-modules	kG^H-module	critical runner	Additional FG^H-modules
$N_{1,i}, N_{1,i-1}$	–	i	$N_{5,i-1}$
$N_{1,i}, N_{2,i}$	–	$i-2$	$N_{5,i}$ or $N_{6,i}$
$N_{1,i}, N_{2,i-1}$	–	–	$N_{3,j}$ or $N_{4,j}$
$N_{1,i}, N_{2,i-2}$	–	–	$N_{3,j}$ or $N_{4,j}$
$N_{1,i}, N_{3,i}$	–	$i-2$	$N_{5,i}$
$N_{1,i}, N_{3,i-1}$	$M_{1,i}$	–	–
$N_{1,i}, N_{3,i-2}$	–	$i-2$	$N_{5,i}$ or $N_{6,i}$
$N_{1,i}, N_{4,i}$	–	i	$N_{5,i-1}$
$N_{1,i}, N_{4,i-1}$	–	$i-2$	$N_{5,i}$ or N_7
$N_{1,i}, N_{4,i-2}$	$M_{1,i}$	–	–
$N_{1,i}, N_{5,i}$	$M_{1,i}$	–	–
$N_{1,i}, N_{5,i-1}$	$M_{1,i}$	–	–
$N_{1,i}, N_{5,i-2}$	–	$i-2$	$N_{5,i}$ or N_7
$N_{1,i}, N_{6,i}$	–	i	$N_{5,i-1}$ or N_7
$N_{1,i}, N_{6,i-1}$	–	$i-2$	$N_{5,i}$ or N_7
$N_{1,i}, N_{6,i-2}$	$M_{1,i}$	–	–
$N_{3,i}, N_{3,i-1}$	–	$i-2$	$N_{5,i}$ or N_7
$N_{3,i}, N_{4,i}$	–	$i-1$	N_7
$N_{3,i}, N_{4,i-1}$	$M_{1,i-2}$	–	–
$N_{3,i}, N_{4,i-2}$	$M_{2,i-1}$	–	–
$N_{3,i}, N_{5,i}$	$M_{1,i-2}$ and $M_{2,i-1}$	–	–
$N_{3,i}, N_{5,i-1}$	–	$i-1$	N_7
$N_{3,i}, N_{5,i-2}$	$M_{1,i-2}$	–	–
$N_{4,i}, N_{4,i-1}$	–	$i-2$	$N_{6,i}$ or N_7
$N_{4,i}, N_{5,i}$	–	i	$N_{5,i-1}$ or N_7
$N_{4,i}, N_{5,i-1}$	$M_{1,i-1}$ and $M_{2,i-2}$	–	–
$N_{4,i}, N_{5,i-2}$	$M_{1,i-1}$	–	–
$N_{5,i}, N_{5,i-1}$	$M_{1,i}$	–	–
$N_{5,i}, N_{6,i}$	–	i	N_7
$N_{5,i}, N_{6,i-1}$	$M_{1,i-2}$ and $M_{2,i-1}$	–	–
$N_{5,i}, N_{6,i-2}$	$M_{1,i}$ and $M_{2,i-1}$	–	–

Its row are labelled by pairs of possible irreducible $FG^H E_1 E_2$-modules. We omit any pair of modules if the pair of transpose modules (as given in Lemma 15) has already been listed. If the pair has a common 3-modular constituent, this is listed in the second column. Consider the rows labelled by $N_{1,i}, N_{2,i-1}$ and $N_{1,i}, N_{2,i-2}$. The existence of any one of these modules implies that E_1 has positive 3-weight. It then follows from Lemma 5 that $FG^H E_1 E_2$ has one of the modules labelled $N_{3,j}$ or $N_{4,j}$, for some j. Otherwise, we can apply Proposition 6 with the runner given in the third paragraph. In this case the existence of the two modules implies the existence of one of the simple modules given in the final column. We do not consider pairs involving N_7, as the p reduction of this module contains all simple $kG^H E_1 E_2$-modules as composition factors.

Suppose that $FG^H E_1 E_2$ has modules of type X_1, X_2 whose 3-modular decompositions do not have a common composition factor. The table can be used to show that there exists an irreducible $FG^H E_1 E_2$-module Y such that:

- $\overline{X_i}$ and $\overline{Y}$ share an simple composition factor, for $i = 1, 2$, or
- $\dim(Y) > \dim(X_i)$, for $i = 1, 2$.

This is enough to show that the 3-decomposition matrix of $RG^H E_1 E_2$ is connected. It follows that $RG^H E_1 E_2$ is a block of RG^H. □

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