The Case of Equality in the Dobrushin–Deutsch–Zenger Bound

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Dedicated to Professor Shmuel Friedland on his 65–th birthday

Abstract

Suppose that $A = (a_{i,j})$ is $n \times n$ real matrix with constant row sums μ. Then the *Dobrushin–Deutsch–Zenger* (DDZ) *bound* on the eigenvalues of A other than μ is given by $\mathcal{Z}(A) = \dfrac{1}{2} \max_{1 \le s,t \le n} \sum_{r=1}^{n} |a_{s,r} - a_{t,r}|$. When A a transition matrix of a finite homogeneous Markov chain so that $\mu = 1$, $\mathcal{Z}(A)$ is called the *coefficient of ergodicity of the chain* as it bounds the asymptotic rate of convergence, namely, $\max\{|\lambda| \mid \lambda \in \sigma(A) \setminus \{1\}\}$, of the iteration $x_i^T = x_{i-1}^T A$, to the stationary distribution vector of the chain.

In this paper we study the structure of real matrices for which the DDZ bound is sharp. We apply our results to the study of the class of

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graphs for which the transition matrix arising from a random walk on the graph attains the bound. We also characterize the eigenvalues λ of A for which $|\lambda| = \mathcal{Z}(A)$ for some stochastic matrix A.

Key words: Stochastic Matrices, Coefficient of Ergodicity, Graphs, Random Walks, Eigenvalues of Stochastic Matrices

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1 Introduction and Preliminaries

Let $A = (a_{i,j})$ be an $n \times n$ real matrix with constant row sums, that is, there exists a number $\mu \in \mathbb{R}$, such that

$$\sum_{j=1}^{n} a_{i,j} \;=\; \mu, \quad \text{for all } i = 1, \ldots, n.$$

It is easily seen that μ is an eigenvalue of A corresponding to the n–vector of all ones, $\mathbf{1}$. Then an upper bound on the largest (in modulus) eigenvalue of A other than μ is given by

$$|\lambda| \;\leq\; \mathcal{Z}(A) \tag{1.1}$$

where

$$Z(A) \;:=\; \frac{1}{2} \max_{1 \leq s,t \leq n} \sum_{r=1}^{n} |a_{s,r} - a_{t,r}|.$$

The bound is due to Eckart Deutsch and Zenger [7]. In Seneta [19, p.62–63] a self–contained proof is given for this bound. We shall return to elements of this proof later.

Now let $A \in \mathbb{R}^{n,n}$ be a transition matrix for an ergodic homogeneous Markov chain on n states. Then A is an $n \times n$ nonnegative, row-stochastic, and irreducible matrix so that, by the Perron–Frobenius theory, the spectral radius of A, which is an eigenvalue of A, is 1. In this case the quantity

$$\gamma(A) \;=\; \max_{\lambda \in \sigma(A) \setminus \{1\}} |\lambda|, \tag{1.2}$$

when it is smaller than 1, determines the *asymptotic rate of convergence* of the iteration process $z_i^T = z_{i-1}^T A$ to the stationary distribution vector of the chain. In this context of transition matrices, Dobrushin [10] has shown that

$$\gamma(A) \;\leq\; \mathcal{Z}(A) \tag{1.3}$$

and called $\mathcal{Z}(A)$ the *coefficient of ergodicity* of the chain. In view of the aforementioned history, we shall call $\mathcal{Z}(A)$ the Dobrushin–Deutsch–Zenger bound or the DDZ bound for short.

The main purpose of this paper is to study the properties and structure of nonnegative, stochastic, and irreducible matrices A for which equality holds in (1.3) and to apply these results to random walks for which the equality holds for the underlying transition matrix. We commence our investigation, however, in Section 2 by assuming only that $A \in \mathbb{R}^{n,n}$ is a matrix whose row sums are a constant which we shall take to be 1. Observe that there is no loss of generality in that assumption, since we can always add a suitable rank one matrix $\mathbf{1}y^T$ to A to put it in that form. Throughout the paper we shall call an eigenvalue λ of A *subdominant* if $|\lambda| = \gamma(A)$ and usually denote this fact by writing λ as $\lambda_{\rm sub}$.

One application in which which there is equality in the DDZ bound is, in fact, in the Google matrix. Suppose that the web has n pages and that for each $i = 1, \ldots, n$, page i has $d_i > 0$ outgoing links. (The assumption that each page has at least one outgoing link does not affect the validity of the conclusion below.) We now construct a stochastic matrix $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ as follows. If $d_i \geq 1$, then each link from page i to page j, we set $a_{i,j} = 1/d_i$. If there is no link from page i to page j, then we set $a_{i,j} = 0$. Assume now that the web contains a union of $k \geq 2$ disjoint strongly connected components so that A has the form:

$$A \;=\; \begin{pmatrix} A_{1,1} & 0 & \cdots & \cdots & 0 \\ 0 & A_{2,2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & A_{k,k} \\ A_{k+1,1} & A_{k+1,2} & \cdots & \cdots & A_{k+1,k+1} \end{pmatrix},$$

with $A_{\ell,\ell} \in \mathbb{R}^{m_\ell,m_\ell}$, for $\ell = 1, \ldots, k+1$, with $m_1 + \ldots m_{k+1} = n$, and with each $A_{\ell,\ell}$ being a stochastic matrix, $\ell = 1, \ldots, k$. Let $\alpha \in (0,1)$. Then the Google matrix is given by

$$G_\alpha \;=\; (1-\alpha)A + \alpha e v^T,$$

where v is a positive vector with $\|v\|_1 = 1$, that is, v is a probability vector. Then as shown in Ipsen and Kirkland [11, Corollary 7.2],

$$\gamma(G_\alpha) \;=\; \mathcal{Z}(G_\alpha) \;=\; 1-\alpha.$$

The DDZ eigenvalue bound in (1.1) has been applied in contexts other than transition matrices of Markov chains. As an example, let $\mathcal{G}$ be an unweighted undirected graph on n vertices $v_1, \ldots, v_n$ whose degrees are $d_1, \ldots, d_n$, respectively. Let M be the $(0,1)$ adjacency matrix of $\mathcal{G}$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then $L = D - M$ is the *Laplacian matrix* associated with $\mathcal{G}$. It is easy to see that L has zero row sums and hence the DDZ bound is applicable to the eigenvalues of L. We note in passing that L is also a positive semidefinite M–matrix. We comment that there is much interest in the literature in the eigenvalues of L and hence in finding good bounds on them. For example, the second smallest eigenvalue of L is known as the *algebraic connectivity* of $\mathcal{G}$. In the situation we describe here, we clearly have that $\rho(L)$, the *spectral radius* of L, is bounded above by $\mathcal{Z}(L)$ and several recent papers have investigated the structure of graphs $\mathcal{G}$ for which $\mathcal{Z}(L) = \rho(L)$, see, for example, Rojo, Soto, and Rojo [17] and Das [5, 6].

As mentioned above, we shall also seek to use the equality case in the DDZ bound to determine the structure of certain graphs, but in a different sense than in the papers [17] and [5, 6]. Let $\mathcal{G}$ be an undirected unweighted connected graph on n vertices and let the matrices D and M be as above. It is easy to see that the matrix $A(\mathcal{G}) = D^{-1}M \in \mathbb{R}^{n,n}$, which is nonnegative and irreducible, is the *transition matrix* for a *random walk* on $\mathcal{G}$. It is also straightforward to see that A is diagonally similar to the symmetric matrix $D^{-\frac{1}{2}}MD^{-\frac{1}{2}}$ so that, in particular, all the eigenvalues of A are real. As an aside, we note that the so-called *normalized Laplacian matrix* for $\mathcal{G}$ (see [4]) is given by $\mathcal{L} = I - D^{-\frac{1}{2}}MD^{-\frac{1}{2}}$, so that eigenvalue bounds for A will generate corresponding eigenvalue bounds for $\mathcal{L}$.

In Section 2 we develop some preliminary results, while in Section 3 we characterize the complex numbers that can be attained as an eigenvalue of a stochastic matrix yielding equality in (1.1). In Section 4 we study random walks on various families of graphs for which $\gamma(D^{-1}M) = \mathcal{Z}(D^{-1}M)$. Generally speaking the transition matrices for these random walks exhibit a certain nonzero–zero block structure.

We close this introductory section by giving two contrasting examples. The first is a graph $\mathcal{G}$ whose Laplacian matrix yields equality in the DDZ eigenvalue bound, but whose random walk transition matrix yields strict inequality in the DDZ bound. The second example is a graph for which equality holds for the

DDZ eigenvalue bound for the transition matrix of the corresponding random walk, but not for the corresponding Laplacian matrix. For the first example take:

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

in which case $D = \mathrm{diag}(3, 4, 4, 4, 4, 5)$. Then for the Laplacian $L = D - M$, we find that $\sigma(L) = \{0, 3, 4, 5, 6, 6\}$ and $\mathcal{Z}(L) = 6$, while for the associated transition matrix $D^{-1}M$ of the random walk we find that: $\sigma(D^{-1}M) = \{1, 0.1059, 0, -0.2500, -0.2673, -0.5886\}$ and $\mathcal{Z}(A) = .75$ so that $\mathcal{Z}(A) > \gamma(A)$. For the second example take the 14×14 adjacency matrix:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$\sigma(D^{-1}M) = \{1,\ 0.5774,\ 0.5774,\ 0.5774,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -0.5774,\ -0.5774,\ -0.5774,\ -1\}$$

so that $\gamma(A) = |-1| = 1 = \mathcal{Z}(A)$. A computation now shows that for $L = D - A$, $\rho(L) = 7$, while $\mathcal{Z}(L) = 8$.

2 The DDZ Bound

Seneta's proof of the DDZ bound (1.1) rests on the following bound on the inner product of two vectors, one of which is orthogonal to the ones vector.

Lemma 2.1 (Paz [16, Chp. IIa], Seneta [19, p.63]) *Let $z = (z_1, \ldots, z_n)$ be an arbitrary row vector of complex numbers. Then for any real vector $\delta \neq 0$ with $\delta^T \mathbf{1} = 0$,*

$$|z^T\delta| \;\leq\; \frac{1}{2} \max_{1\leq i,j\leq n} |z_i - z_j| \|\delta\|_1. \tag{2.4}$$

To facilitate the study in this paper of the equality case in (1.1) we need the characterization of the case of equality in (2.4). The following theorem comes from [13].

Theorem 2.2 (Kirkland, Neumann, and Shader [13, Theorem 2.1]) *Let $\delta \in \mathbb{R}^n$ be a vector such that $\delta^T\mathbf{1} = 0$ and let $z \in \mathbb{C}^n$. Then equality holds in (2.4), viz.*

$$|z^T\delta| \;=\; \frac{1}{2} \max_{1\leq i,j\leq n} |z_i - z_j| \|\delta\|_1$$

if and only if z and δ can be reordered simultaneously such that

$$\delta \;=\; \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \\ -\delta_{m+1} \\ \vdots \\ -\delta_{m+k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and } z \;=\; \begin{bmatrix} a \\ \vdots \\ a \\ b \\ \vdots \\ b \\ c_1 \\ \vdots \\ c_{n-k-m} \end{bmatrix}, \tag{2.5}$$

and where

$$\max_{1\leq i,j\leq n} |z_i - z_j| \;=\; |a - b| \;\text{ and }\; \delta_i > 0,\; i = 1, \ldots, k+m. \tag{2.6}$$

Throughout the remainder of this section $A = (a_{i,j})$ will always be an $n \times n$ real matrix with row sums 1 and subdominant eigenvalue $\lambda_{\text{sub}}(A)$, in which case we can write that:

$$\mathcal{Z}(A) \;=\; \max_{1\leq i,j\leq n} \left\{ \frac{1}{2} \left\|(e_i^T - e_j^T)A\right\|_1 \right\} \;\geq\; |\lambda_{\text{sub}}(A)|. \tag{2.7}$$

We comment that in the case that A is also a nonnegative matrix, it readily follows from the stochasticity of A and the definition of $\mathcal{Z}(A)$ that

$$\mathcal{Z}(A) \ \le \ \frac{1}{2}(2\|A\|_1) \ \le \ 1. \tag{2.8}$$

In our first lemma on the equality case of the DDZ bound on A we describe some of the quantitative structure of the entries of A.

Lemma 2.3 *Suppose that equality holds in* (2.7) *and let z be an eigenvector of A corresponding to $\lambda_{\rm sub}$. Then for any pair of indices $1 \le i, j \le n$ such that*

$$|z_i - z_j| \ = \ \max_{1\le p,q\le n}\{|z_p - z_q| \ |\},$$

we have that

$$\frac{1}{2}\left\|(e_i^T - e_j^T)A\right\|_1 \ = \ \mathcal{Z}(A).$$

Further, there are entries a and b of z with $|a-b| = |z_i - z_j|$ such that for any k, we have that

i) $z_k = a$ *whenever* $a_{i,k} - a_{j,k} > 0$,

and

ii) $z_k = b$ *whenever* $a_{i,k} - a_{j,k} < 0$.

In particular, if for some index k we have $z_k \ne a, b$, then $a_{i,k} = a_{j,k}$.

Proof: For any vector $v \in \mathbb{C}^n$, let $f(v) = \max_{1\le p,q,\le n}\{|v_p - v_q|\}$. We then have that

$$|\lambda_{\rm sub}(A)|f(z) \ = \ |\lambda_{\rm sub}(A)||z_i - z_j| \ = \ \left|(e_i^T - e_j^T)Az\right|$$

$$\le \ \tfrac{1}{2}\left\|(e_i^T - e_j^T)A\right\|_1 f(z) \ \le \ \mathcal{Z}(A)f(z) = |\lambda_{\rm sub}(A)|f(z).$$

Consequently, it must be the case that $\frac{1}{2}\left\|(e_i^T - e_j^T)A\right|_1 = \mathcal{Z}(A)$. Furthermore, we must also have that

$$\left\|(e_i^T - e_j^T)Az\right\|_1 \ = \ \frac{1}{2}\left\|(e_i^T - e_j^T)A\right\|_1 f(z),$$

and appealing to Theorem 2.2, we find that conclusions (i) and (ii) follow. $\square$

As an example consider the matrix

$$A = \begin{bmatrix} 0.1683 & 0.2683 & 0.2183 & 0.1850 & 0.1600 \\ 0.2683 & 0.1683 & 0.2183 & 0.1850 & 0.1600 \\ 0.2183 & 0.2183 & 0.2183 & 0.1850 & 0.1600 \\ 0.1850 & 0.1850 & 0.1850 & 0.2850 & 0.1600 \\ 0.1600 & 0.1600 & 0.1600 & 0.1600 & 0.3600 \end{bmatrix}.$$

Here $\mathcal{Z}(A) = .2$ and the spectrum of A is given by $\sigma(A) = \{1, .2, .1, 0, -.1\}$. Thus the (only) subdominant eigenvalue of A is $\lambda_{\text{sub}} = .2 = |\lambda_{\text{sub}}(A)| = \mathcal{Z}(A)$ and the conditions of Lemma 2.7 are applicable. The corresponding eigenvector to λ_{sub} is given by

$$z = \begin{bmatrix} 0.2236 \\ 0.2236 \\ 0.2236 \\ 0.2236 \\ -0.8944 \end{bmatrix},$$

in which case we see that $a = 0.2336$, $b = -0.8944$, and we observe that the indices i and j for which

$$|z_i - z_j| = \max_{1 \le p,q \le n} \{|z_p - z_q|\},$$

are given by $i = 1, 2, 3$, and 4, and $j = 5$, respectively. Taking, for example, the difference of rows 2 and 5 of A, we get that it is given by the vector

$$[0.1083, 0.0083, 0.0583, 0.0250, -0.2000].$$

Notice that for $k = 1, \ldots, 4$, $a_{2,k} - a_{5,k} > 0$ and we expect that $z_k = 0.2336$ which we see is true, while for $k = 5$, $a_{2,5} - a_{5,5} < 0$ and, as we expect from the lemma, $z_5 = -0.8944$.

Based on Lemma 2.3 we can prove a further inequality on the entries of A when equality holds in (2.7).

Theorem 2.4 *Suppose that equality holds in* (2.7)*. Let z be a λ_{sub} eigenvector, and suppose that i and j are indices such that $|z_i - z_j| = \max_{1 \le p,q \le n}\{|z_p - z_q|\}$. Then*

$$(a_{i,i} - a_{j,i})(a_{i,j} - a_{j,j}) \le 0.$$

Proof: Suppose to the contrary that $(a_{i,i} - a_{j,i})(a_{i,j} - a_{j,j}) > 0$. Without loss of generality, we may assume that $i = 1, j = 2,\ a_{1,1} > a_{2,1}$, and $a_{1,2} > a_{2,2}$ (otherwise we can simultaneously permute the rows and columns of A so that it has the desired form). We may also assume that the remaining rows and columns have been ordered so that $a_{1,p} > a_{2,p},$, for $p = 3, \ldots, m,\ a_{1,p} < a_{2,p}$, for $p = m+1, \ldots, m+q$, and $a_{1,p} = a_{2,p}$, for $p = m+q+1, \ldots, n$. It follows from Lemma 2.3 that we have

$$e_1^T A \ = \ \left[\begin{array}{c|c|c} u_1^T & v_1^T & w^T \end{array}\right],\ e_2^T A \left[\begin{array}{c|c|c} u_2^T & v_2^T & w^T \end{array}\right],$$

and

$$z \ = \ \begin{bmatrix} a\mathbf{1} \\ b\mathbf{1} \\ c \end{bmatrix},$$

where the partitions are conformal and where we have $u_1 > u_2,\ v_1 < v_2$, and $|a - b| = \max_{1 \le p,q \le n}\{|z_p - z_q|\}$.

From the eigenequation $Az = \lambda_{\rm sub} z$ we have $\lambda_{\rm sub} a = a u_1^T \mathbf{1} + b v_1^T \mathbf{1} + w^T c$ and $\lambda_{\rm sub} a = a u_2^T \mathbf{1} + b v_2^T \mathbf{1} + w^T c$, so that $a(u_1 - u_2)^T \mathbf{1} + b(v_1 - v_2)^T \mathbf{1} = 0$. Now $1 - w^T \mathbf{1} = u_1^T \mathbf{1} + v_1^T \mathbf{1} = u_2^T \mathbf{1} + v_2^T \mathbf{1}$, so that $(u_1 - u_2)^T \mathbf{1} = (v_2^T - v_1^T)\mathbf{1}$. We conclude that $(a - b)(u_1 - u_2)^T \mathbf{1} = 0$, and hence that $a = b$, a contradiction. □

A refinement of the results in Theorem 2.4 is given in the following lemma:

Lemma 2.5 *Suppose that equality holds in* (2.7). *Let z be a $\lambda_{\rm sub}$ eigenvector, and suppose that i and j are indices such that $i < j$ and $|z_i - z_j| = \max_{1 \le p,q \le n}\{|z_p - z_q|\}$. Suppose further that $(a_{i,i} - a_{j,i})(a_{i,j} - a_{j,j}) < 0$, and define the following sets of indices:*

$$\Sigma_1 \ = \ \{k | a_{i,k} > a_{j,k}\},$$

$$\Sigma_2 \ = \ \{k | a_{i,k} < a_{j,k}\},$$

and

$$\Sigma_3 \ = \ \{k | a_{i,k} = a_{j,k}\}.$$

Set

$$\sigma_{i,1} \;:=\; \sum_{k\in\Sigma_1} a_{i,k},$$

$$\sigma_{i,2} \;:=\; \sum_{k\in\Sigma_2} a_{i,k},$$

$$\sigma_{j,1} \;:=\; \sum_{k\in\Sigma_1} a_{j,k},$$

and

$$\sigma_{j,1} \;:=\; \sum_{k\in\Sigma_2} a_{j,k}.$$

Then

a) *if* $a_{i,i} > a_{j,i}$ *and* $a_{i,j} < a_{j,j}$, *then* $\lambda_{\rm sub} = \sigma_{i,1} - \sigma_{j,1}$.

and

b) *if* $a_{i,i} < a_{j,i}$ *and* $a_{i,j} > a_{j,j}$, *then* $\lambda_{\rm sub} = \sigma_{j,1} - \sigma_{i,1}$.

Proof: a) Without loss of generality, we assume that $i = 1$ and $j = 2$. Further, we may simultaneously reorder indices $3, \ldots, n$ so that

$$e_1^T A \;=\; \left[\begin{array}{cc|c|c|c} a_{1,1} & a_{1,2} & u_1^T & v_1^T & w^T \end{array}\right],\; e_2^T A \;=\; \left[\begin{array}{cc|c|c|c} a_{2,1} & a_{2,2} & u_2^T & v_2^T & w^T \end{array}\right],$$

and

$$z \;=\; \left[\begin{array}{c} a \\ b \\ a\mathbf{1} \\ b\mathbf{1} \\ c \end{array}\right],$$

where the partitions are conformal and where $u_1 > u_2,\; v_1 < v_2$, and $|a - b| = \max_{1\le p,q\le n}\{|z_p - z_q|\}$. From the eigenequation $Az = \lambda_{\rm sub} z$ it follows that

$$\lambda_{\rm sub} a \;=\; a\sigma_{1,1} + b\sigma_{1,2} + w^T c$$

and

$$\lambda_{\rm sub} b \;=\; a\sigma_{2,1} + b\sigma_{2,2} + w^T c.$$

Subtracting the two equations we find that

$$\lambda_{\rm sub}(a - b) \;=\; a(\sigma_{1,1} - \sigma_{2,1}) + b(\sigma_{1,2} - \sigma_{2,2}).$$

Now since

$$1 - w^T\mathbf{1} \;=\; \sigma_{1,1} + \sigma_{1,2} \;=\; \sigma_{2,1} + \sigma_{2,2},$$

we find that

$$\sigma_{1,2} - \sigma_{2,2} \;=\; -(\sigma_{1,1} - \sigma_{2,1}).$$

Hence

$$\lambda_{\text{sub}}(a-b) \;=\; (\sigma_{1,1}-\sigma_{2,1})(a-b),$$

and conclusion (a) follows. The proof of (b) is analogous. □

The equality case in (2.7) allows us to prove results about the Jordan block structure corresponding to the subdominant eigenvalues of A. We begin with the following lemma:

Lemma 2.6 *Suppose that equality holds in* (2.7) *for the matrix A. Then for any $k \in I\!N$, equality also holds in* (2.7) *for the matrix A^k, with $\lambda_{\text{sub}}(A^k) = \mathcal{Z}(A^k) = (\mathcal{Z}(A))^k$.*

Proof: From the proof of Proposition 1.4 on p.70 of Paz [16] (see also Seneta [19, Lemma 4.3]) it follows readily that $\mathcal{Z}(A^k) \le (\mathcal{Z}(A))^k$, for each $k \in I\!N$. But then for any such k, we have $(\mathcal{Z}(A))^k = |\lambda_{\text{sub}}(A)|^k = |\lambda_{\text{sub}}(A^k)| \le \mathcal{Z}(A^k) \le (\mathcal{Z}(A))^k$. Whence $\mathcal{Z}(A^k) = |\lambda_{\text{sub}}(A^k)|$, for each $k \in I\!N$. □

We can now prove:

Theorem 2.7 *Suppose that equality holds in* (2.7) *for the matrix A. Then for any eigenvalue $\lambda \neq 1$ such that $|\lambda| = \mathcal{Z}(A)$, the geometric and algebraic multiplicities of λ coincide.*

Proof: If $\lambda = 0$, the result follows readily from the fact that in that case, A must have rank 1. So, henceforth we take λ to be nonzero.

Suppose to the contrary that the geometric muliplicity of λ is less than the algebraic multiplicity of λ. Then there are vectors x^T and y^T such that $x^TA = \lambda x^T$, $y^TA = \lambda y^T + x^T$, and $||y^T||_1 = 1$. Observe that necessarily $y^T\mathbf{1} = 0 = x^T\mathbf{1}$. A straightforward proof by induction shows that

$$y^TA^k \;=\; \lambda^k y^T + k\lambda^{k-1}x^T \;=\; \lambda^k\left(y^T + \frac{k}{\lambda}x^T\right).$$

Note that

$$\left\|\lambda^k\left(y^T + \frac{k}{\lambda}x^T\right)\right\|_1 \;\ge\; |\lambda|^k\left(\frac{k}{|\lambda|}\lambda\left\|x^T\right\|_1 - \left\|y^T\right\|_1\right) \;=\; (\mathcal{Z}(A))^k\left(\frac{k}{|\lambda|}\left\|x^T\right\|_1 - 1\right).$$

In particular, we find that for all sufficiently large $k \in I\!N$, $\left\|y^TA^k\right\|_1 > (\mathcal{Z}(A))^k = \mathcal{Z}\left(A^k\right)$. This last contradicts Lemma 2.6. We thus conclude that the geometric and algebraic multiplicities of λ must be equal. □

We comment that the converse of Theorem 2.7 does not hold as the following example shows. Let

$$A = \begin{bmatrix} 0.1183 & 0.3183 & 0.2183 & 0.1850 & 0.1600 \\ 0.3183 & 0.1183 & 0.2183 & 0.1850 & 0.1600 \\ 0.2183 & 0.2183 & 0.2183 & 0.1850 & 0.1600 \\ 0.1850 & 0.1850 & 0.1850 & 0.2850 & 0.1600 \\ 0.1600 & 0.1600 & 0.1600 & 0.1600 & 0.3600 \end{bmatrix}.$$

Then the spectrum of A is given by $\sigma(A) = \{1, .2, .1, 0, -.2\}$, so that the geometric and algebraic multiplicities of both subdominant eigenvalues, $\pm.2$ are 1, yet $\mathcal{Z}(A) = 0.2417 > .2 = |\lambda_{\text{sub}}(A)|$.

Until now we have considered the equality case in the DDZ bound for any real matrix. Let us now assume that A is an $n \times n$ nonnegative and irreducible matrix whose row sum is a constant 1. In this case A is row–stochastic and can be regarded as a transition matrix of a finite homogeneous ergodic Markov chain on n states. For such a Markov chain, Meyer [14] has shown that virtually any important parameter of the chain can be read from the *group generalized inverse*[1] $Q^{\#}$ of the singular and irreducible M–matrix[2] $Q = I - A$. Cleary $Q\mathbf{1} = 0$ and it is known that $Q^{\#}\mathbf{1} = Q\mathbf{1}$. It is further known that $\sigma(Q^{\#}) = \{0\} \cup \{\frac{1}{1-\lambda} | \lambda \in \sigma(A) \setminus \{1\}$. Thus, on applying the DDZ eigenvalue bound we can write that:

$$\frac{1}{|1-\lambda_{\text{sub}}|} \leq \frac{1}{\min_{\lambda \in \sigma(A)\setminus\{1\}} |1-\lambda|} \leq \mathcal{Z}(Q^{\#}) \leq \frac{1}{1-\mathcal{Z}(A)}, \tag{2.9}$$

where the rightmost inequality is due to Seneta, see [19].

Suppose now that for A as above, the equality case in the DDZ bound (1.1) holds. In this case we can write that

$$\frac{1}{|1-\lambda_{\text{sub}}(A)|} \leq \frac{1}{\min_{\lambda \in \sigma(A)\setminus\{1\}} |1-\lambda|} \leq \mathcal{Z}(Q^{\#}) \leq \frac{1}{1-|\lambda_{\text{sub}}(A)|}, \tag{2.10}$$

for any $\lambda_{\text{sub}}(A) \in \sigma(A)$. It is now straight forward to prove the following result:

[1]For comprehensive accounts on group generalized inverses of matrices, including when they exist, see Ben–Israel and Greville [1] and Campbell and Meyer [3].

[2]For a comprehensive account on the Perron–Frobenius theory for nonnegative matrices and on M–matrices se see Berman and Plemmons [2].

Theorem 2.8 *Suppose that A is an $n \times n$ nonnegative, stochastic, and irreducible matrix for which equality holds in (1.3). If $\gamma(A) < 1$ and A has an eigenvalue $\lambda_{\mathrm{sub}}(A) \in \mathbb{R}^+$, then for $Q = I - A$, we have*

$$\mathcal{Z}(Q^{\#}) \;=\; \frac{1}{1 - |\lambda_{\mathrm{sub}}(A)|}.$$

Let is give two examples. First take:

$$A \;=\; \begin{bmatrix} 0.12 & 0.52 & 0.12 & 0.12 & 0.12 \\ 0.12 & 0.12 & 0.52 & 0.12 & 0.12 \\ 0.12 & 0.12 & 0.12 & 0.52 & 0.12 \\ 0.12 & 0.12 & 0.12 & 0.12 & 0.52 \\ 0.52 & 0.12 & 0.12 & 0.12 & 0.12 \end{bmatrix}. \tag{2.11}$$

Then

$$\sigma(A) \;=\; \{1.0000,\ 0.1236{+}0.3804i,\ 0.1236{-}0.3804i,\ -0.3236{+}0.2351i,\ -0.3236{-}0.2351i].$$

Here $\mathcal{Z}(A) = .4 = |\lambda_{\mathrm{sub}}(A)|$, but we see that A could not possibly fulfill the conditions of Theorem 2.8. Indeed we find that for $Q = I - A$,

$$Q^{\#} \;=\; \begin{bmatrix} 0.6770 & 0.07081 & -0.1717 & -0.2687 & -0.3075 \\ -0.3075 & 0.6770 & 0.07081 & -0.1717 & -0.2687 \\ -0.2687 & -0.3075 & 0.6770 & 0.07081 & -0.1717 \\ -0.1717 & -0.2687 & -0.3075 & 0.6770 & 0.07081 \\ 0.07081 & -0.1717 & -0.2687 & -0.3075 & 0.6770 \end{bmatrix}$$

for which

$$\frac{1}{\max_{\lambda \in \sigma(A)\setminus\{1\}} |1-\lambda|} \;=\; 1.0467 \;<\; \mathcal{Z}(Q^{\#}) \;=\; 1.3240 \;<\; 1.6667 \;=\; \frac{1}{1-.4} \;=\; \frac{1}{1-\mathcal{Z}(A)}.$$

As a second example consider

$$A \;=\; \begin{bmatrix} 0.8750 & 0.06250 & 0.0 & 0.06250 & 0.0 \\ 0.5000 & 0.0 & 0.5000 & 0.0 & 0.0 \\ 0.5000 & 0.5000 & 0.0 & 0.0 & 0.0 \\ 0.5000 & 0.0 & 0.0 & 0.0 & 0.5000 \\ 0.5000 & 0.0 & 0.0 & 0.5000 & 0.0 \end{bmatrix}.$$

Here

$$\sigma(A) \ = \ [1,\ 0.375, .5\ \ -0.5,\ -.5]$$

so that $|\lambda_{\rm sub}(A)| = .5$. Furthermore we find that $\mathcal{Z}(A) = .5$ and hence $\mathcal{Z}(A) = |\lambda_{\rm sub}(A)|$ and so for this A the conditions of Theorem 2.8 are fulfilled. On computing the group inverse of $Q = I - A$ we obtain that:

$$\mathcal{Z}(Q^{\#}) \ = \ \begin{bmatrix} 0.3200 & -0.08444 & -0.07556 & -0.08444 & -0.07556 \\ -1.280 & 1.116 & 0.5244 & -0.2178 & -0.1422 \\ -1.280 & 0.4489 & 1.191 & -0.2178 & -0.1422 \\ -1.280 & -0.2178 & -0.1422 & 1.116 & 0.5244 \\ -1.280 & -0.2178 & -0.1422 & 0.4489 & 1.191 \end{bmatrix}$$

and that $\mathcal{Z}(Q^{\#}) = 2 = 1/(1-1/2) = 1/(1-\mathcal{Z}(A))$.

3 The complex eigenvalues yielding equality in the DDZ inequality for stochastic matrices

Much is known about the eigenvalues of stochastic matrices A. For example, Dmitriev and Dynkin [8, 9], and Karpelevich [12] determined the region within the unit circle in which the eigenvalues of an $n \times n$ stochastic matrix must lie (see Minc [15]) for a more accessible acount of the result of Dmitriev and Dynkin). Romanovsky [18] (see also Varga [21, Corollary, p.39]) showed that if A is an $n \times n$ cyclic matrix of index $k \geq 2$, and so A is, in particular imprimitive, then its characteristic polynomial is given by

$$\phi(t) \ = \ \lambda^m \left[t^k - \rho^k(A)\right] \left[t^k - \delta_2 \rho(A)^k\right] \cdots [t^k - \delta_r \rho^k(A)],$$

where $|\delta_i| < 1$, for $1 < i \leq r$, if $r > 1$.

Observe for example, that Romanovsky's theorem does not tell us about the nature of the eigenvalues other than 1 when A is irreducible, but not k–cyclic, that is, when A is primitive. This is illustrated in the example given in (2.11), where the four eigenvalues other than 1 of A "continue" to be the four non–real roots of the equation $t^5 = .4$.

In this section we show that if A is a stochastic matrix for which the equality case in the DDZ bound holds, then the subdominant eignvalues of A satisfy equations of the form $t^k = \alpha$, where $k \le n$, with $\alpha \in \mathbb{R}$, $|\alpha| \le 1$, and with further restrictions on k when $\alpha < 0$.

We begin with the following construction. Suppose that we have n distinct complex numbers $z_1, \ldots, z_n$, and let $\rho = \max_{1 \le i,j \le n}\{|z_i - z_j|\}$. The corresponding *diameter graph* for the vector $z = \begin{bmatrix} z_1 & z_2 & \ldots & z_n \end{bmatrix}^T$ is the graph $\Gamma(z)$ on vertices $1, \ldots, n$ with $i \sim j$ in $\Gamma(z)$ if and only if $|z_i - z_j| = \rho$.

We begin with a useful lemma.

Lemma 3.1 *Let $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Suppose that there is an $n \times n$ stochastic matrix A having eigenvalue $\lambda \ne 1$ for which $\mathcal{Z}(A) = |\lambda|$. Then there is a stochastic matrix M of order at most n and an eigenvector z such that $Mz = \lambda z$, $\mathcal{Z}(M) = |\lambda|$, z has distinct entries, and the diameter graph of z has no isolated vertices.*

Proof: Let x be an eigenvector of A corresponding to λ. Suppose that x does not have distinct entries; for concreteness we take $x_1 = x_2$ without loss of generality. Write A and x as

$$A = \left[\begin{array}{cc|c} a_{11} & a_{12} & r_1^T \\ a_{21} & a_{22} & r_2^T \\ \hline c_1 & c_2 & \overline{A} \end{array}\right] \quad \textbf{and} \quad x = \left[\begin{array}{c} x_1 \\ x_2 \\ \hline \overline{x} \end{array}\right].$$

Next, consider the matrix $\hat{B}$ of order $n-1$ and the vector y given as follows:

$$\hat{B} = \left[\begin{array}{c|c} a_{11} + a_{12} & r_1^T \\ \hline c_1 + c_2 & \overline{A} \end{array}\right] \quad \textbf{and} \quad y = \left[\begin{array}{c} x_1 \\ \hline \overline{x} \end{array}\right].$$

Evidently $\hat{B}y = \lambda y$, and it is readily verified that $\mathcal{Z}(\hat{B}) \le \mathcal{Z}(A)$. Further, since $\mathcal{Z}(A) = |\lambda| \le \mathcal{Z}(\hat{B}) \le \mathcal{Z}(A)$, we see that in fact $|\lambda| = \mathcal{Z}(\hat{B})$. Now, applying an induction step on the order of the matrix, it follows that we can find a matrix B and vector u such that $Bu = \lambda u, \mathcal{Z}(B) = |\lambda|$ and u has distinct entries. If it happens that the diameter graph of u has no isolated vertices, then we are done.

So, suppose that the diameter graph of u has some isolated vertices. Without loss of generality, we have $B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$, and $u = \left[\begin{array}{c} u_1 \\ \hline u_2 \end{array}\right]$, where the subvector u_2 corresponds to all of the isolated vertices in the diameter graph of u. It follows from Lemma 2.3 that B_{12} is rank 1 and of the form $\mathbf{1}w^T$ for some nonnegative vector w^T. Note also that $w^T\mathbf{1} < 1$, otherwise we have $B_{11} = 0$, from which it follows that u_1 is multiple of $\mathbf{1}$, a contradiction. From the eigenequation, we have $B_{11}u_1 + (w^T u_2)\mathbf{1} = \lambda u_1$. Next, consider the matrix $M = B_{11} + \frac{w^T\mathbf{1}}{1-w^T\mathbf{1}}\mathbf{1}e_1^T B_{11}$. Note that M is stochastic, and that $\mathcal{Z}(M) = \mathcal{Z}(B_{11}) = \mathcal{Z}(A)$. A straightforward computation reveals that the vector $z = u_1 + \frac{\lambda w^T\mathbf{1}e_1^T u_1}{(\lambda-1)(1-w^T\mathbf{1})}\mathbf{1}$ is an eigenvector for M with corresponding eigenvalue λ. Further, note that z has distinct entries, and that its diameter graph has no isolated vertices. □

The following result will be applied to the diameter graph of a suitable eigenvector in Theorem 3.3 below.

Lemma 3.2 *Suppose that $\mathcal{G}$ is a graph on n vertices with no isolated vertices and maximum degree at least two. Let A be an $n \times n$ real matrix such that $\mathcal{Z}(A) = 1, A$ has constant row sums, and all rows of A are distinct. Suppose that for each pair of indices $i, j = 1, \ldots, n$ such that $i \sim j$ in $\mathcal{G}$, we have $(e_i - e_j)^T A = (e_k - e_l)^T$, for some $k \sim l$ in $\mathcal{G}$. Then A can be written as $A = \mathbf{1}y^T \pm S$, where S is a $(0, 1, -1)$ matrix with the properties thatS has a single zero row, and for some index i and every nonzero row of S is of the form $(e_i - e_j)^T$ for some suitable j.*

Proof: Suppose without loss of generality that vertex 1 of $\mathcal{G}$ has maximum degree, with 1 adjacent to vertices $2, 3, \ldots, k$. Let $S = A - \mathbf{1}e_1^T A$, which has an all zero first row. Then there are indices a, b, c, and d, with $a \sim b, c \sim d$ in $\mathcal{G}$, such that $(e_2 - e_1)^T S = (e_a - e_b)^T$ and $(e_3 - e_1)^T S = (e_c - e_d)^T$ and hence $e_2^T S = (e_a - e_b)^T, e_3^T S = (e_c - e_d)^T$. Since $\mathcal{Z}(S) = \mathcal{Z}(A) = 1$, we find that necessarily either $a = c$ or $b = d$, otherwise $\mathcal{Z}(S) > 1$. Without loss of generality, we suppose that $a = b$ (in the case that $b = d$, we consider $-S$ instead of S). Note that since S has distinct rows, necessarily $b \neq d$. If $k \geq 4$, we find as above that for each $j = 4, \ldots, k$, there are indices p_j and q_j such that $e_j^T S = (e_{p_j} - e_{q_j})^T$. Furthermore, for each such j, if $p_j \neq a$, then necessarily $q_j = b$ and $q_j = d$, a contradiction. We conclude that $p_j = a$ for each $j = 4, \ldots, k$.

Suppose now that $p \sim q$ is an edge of $\mathcal{G}$ that is not incident with vertex

1. Let $e_p^T S = x^T$ so that for some indices i and j with $i \sim j$ in $\mathcal{G}$, we have $e_q^T S = x^T + e_i^T - e_j^T$. For concreteness, we will henceforth take rows $2, \ldots, k$ of S to be $e_1^T - e_2^T, \ldots, e_1^T - e_k^T$, respectively, and we will take $p = k+$ and $q = k+2$, all without loss of generality. Thus the first $k+1$ rows of S have the following form:

$$\begin{matrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 1 & -1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 1 & 0 & 0 & \ldots & -1 & 0 & \ldots & 0 \\ x_1 & x_2 & x_3 & \ldots & x_k & x_{k+1} & \ldots & x_n \end{matrix},$$

while the row $k+2$ of S has the form

$$[x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_k \quad x_{k+1} \quad \ldots \quad x_n \ ] + e_i^T - e_j^T$$

for some i and j.

From the fact that $\mathcal{Z}(S) = 1$, it follows that $x_1 \geq 0$, while $x_j \leq 0, j = 2, \ldots, k$; consequently we set $x_p = -y_p$, for $p = 2, \ldots, k$. Further, from the facts that each row sum of S is zero, $e_1^T S = 0^T$, and $\mathcal{Z}(S) = 1$, it follows that the sum of the positive elements in each row is bounded above by 1.

Suppose first that $i = 1$. Since both $x_1, x_1+1 \in [0, 1]$, we find that necessarily $x_1 = 0$. Also, by considering row $k+2$, we see that each of $x_{k+1}, \ldots, x_n$ must be nonpositive. It now follows that row $k+1$ of S is 0^T, a contradiction since S has distinct rows. Hence $i \geq 2$ and a similar argument (reversing the roles of rows $k+1$ and $k+2$) yields $j \geq 2$.

Next, suppose that $2 \leq i \leq k$ and without loss of generality we take $i = 2$. Considering row $k+2$, we find that $1 - y_2 \leq 0$, which yields $x_2 = -y_2 = -1$. Hence $x_p = 0$ for $p = 3, \ldots, k$, and further, for each $p = k+1, \ldots, n$, we have $x_p \geq 0$. All told we have that

$$2 \ \geq \ ||(e_{k+1} - e_3)^T S||_1 \ = \ 1 - x_1 + 1 + 1 + 1 - x_1,$$

which yields $x_1 \geq 1$ and hence $x_1 = 1$. It follows then that $e_{k+1}^T S = e_1^T - e_2^T$, a contradiction to the fact that S has distinct rows. A similar argument (again, reversing the roles of rows $k+1$ and $k+2$) shows that assuming that $2 \leq j \leq k$ leads to a contradiction.

The last case is then $n \geq i, j \geq k+1$. Without loss of generality we take $i = k+1$ and $j = k+2$. Note that necessarily $x_{k+1} \leq 0$ and $x_{k+2} \geq 0$, and we set $y_{k+1} = -x_{k+1}$. Fix an index l between 2 and k. We have that

$$2 \ \geq \ \left\|(e_l - e_{k+2})^T S\right\|_1 \ = \ 1 - x_1 + \sum_{p=2,\ldots,k, p\neq l} y_p + 1 - y_l + 1 - y_{k+1} + 1 - x_{k+2}$$

$$+ \ \textstyle\sum_{p=k+3,\ldots,n} |x_q| \geq 4 - x_1 - y_l - y_{k+1} - x_{k+2},$$

from which we find that for each such $l, (x_1 + x_{k+2}) + (y_l + y_{k+1}) = 2$. It then follows that $x_1 + x_{k+2} = 1$, and that $y_{k+1} + y_l = 1, l = 2, \ldots, k$. The latter condition, in conjunction with the fact that $1 \geq y_2 + \ldots + y_k + y_{k+1}$ easily yields that $y_{k+1} = 1$ and $y_p = 0$, for $p = 2, \ldots, k$. Next, by considering the fact that $2 \geq ||(e_2 - e_{k+1})^T S||_1$, it follows that

$$2 \ \geq \ 1 - x_1 + 1 + 1 + 1 - x_1,$$

from which we deduce that $x_1 = 1$. Hence $e_{k+1}^T S = e_1^T - e_{k+1}^T$ and $e_{k+2}^T S = e_1^T - e_{k+2}^T$, which is of the desired form. We conclude that each nonzero row of S is of the form $e_1^T - e_p^T$, for some suitable index p. □

We are now in a position to prove the main result of this section.

Theorem 3.3 *Let $\lambda \in \mathbb{C}$ and $n \in I\!N$. Then there is an $n \times n$ stochastic matrix A having eigenvalue λ for which $\mathcal{Z}(A) = |\lambda|$ if and only if one of the following holds:*
i) *there is a $k \in I\!N$ with $k \leq n$, a k–th root of unity ω, and an $r \in [0, 1]$ such that $\lambda = r\omega$;*
ii) *there is a smallest odd number $k_0 \in I\!N$, with $k_0 \leq n$, a k_0–th root of -1, α, and an $r \in [0, \frac{1}{k_0 - 1}]$, such that $\lambda = r\alpha$.*

Proof: Fix $k \in I\!N$ and let C_k be a $k \times k$ cyclic permutation matrix. Observe that for each $r \in [0, 1]$, the stochastic matrix

$$A \ = \ \frac{1-r}{n} J + r \left[\begin{array}{c|c} C_k & 0 \\ \hline 0 & I_{n-k} \end{array}\right]$$

satisfies $\mathcal{Z}(A) = r$ and has, for each k–th root of unity ω, the complex number $r\omega$ as an eigenvalue. Similarly, for each odd k_0, with $1 \leq k_0 \leq n$, the stochastic matrix

$$B = \frac{1-r}{n}J + r\left[\begin{array}{c|c} \frac{1}{k_0-1}(J-C_{k_0}) & 0 \\ \hline \frac{1}{k_0}J & 0 \end{array}\right]$$

satisfies $\mathcal{Z}(A) = \frac{r}{k_0-1}$ and has, for each k_0–th root of -1, α, the complex number $\frac{r}{k_0-1}\alpha$ as an eigenvalue. Thus we see that each $\lambda \in \mathbb{C}$ satisfying i) or ii) is realized as an eigenvalue of some stochastic matrix with the desired properties.

Now suppose that there is an $n \times n$ stochastic matrix A having eigenvalue λ such that $\mathcal{Z}(A) = |\lambda|$. If $\lambda = 1$, then certainly λ is of the form described in i). Henceforth, we suppose that $\lambda \neq 1$. Let v be an eigenvector for A corresponding to λ. Appealing to Lemma 3.1, we assume without loss of generality that v has distinct entries, that diameter graph of v has no isolated vertices, and that A is $m \times m$ for some $m \leq n$.

Consider the diameter graph $\Gamma(v)$. First, suppose that every vertex of $\Gamma(v)$ has degree one. Then m is even, and $\Gamma(v)$ is a collection of $\frac{m}{2}$ independent edges. If $i \sim j$ in $\Gamma(v)$, then from Lemma 2.3, there is an edge $k \sim l$ in $\Gamma(v)$ such that $(e_i - e_j)^T A = (e_k - e_l)^T$, from which it follows that $\lambda(v_i - v_j) = v_k - v_l$. Consider the directed graph D, whose vertices are ordered pairs (i,j) such that $i \sim j$ in $\Gamma(v)$, with an arc from (i,j) to (k,l) if and only if $(e_i - e_j)^T A = (e_k - e_l)^T$. Observe that D has m vertices, and that each vertex of D has outdegree 1. Letting M be the adjacency matrix of D, we find that λ is an eigenvalue of M, with an eigenvector whose entry in the position corresponding to (i,j) is $v_i - v_j$, for each i and j. Since λ is an eigenvalue of the $(0,1)$ matrix M, each row of which contains a single one, it follows readily that λ is a k–th root of unity for some $k \leq m$.

Suppose now that $\Gamma(v)$ has maximum degree at least two. We then find that $\frac{1}{\mathcal{Z}(A)}A$ satisfies the hypotheses of Lemma 4.1. Hence, $\frac{1}{\mathcal{Z}(A)}A$ can be written as $\mathbf{1}y^T \pm S$, where S is of the form described in that lemma. Since such an S can be written as $\mathbf{1}e_i^T - P$, for some index i and permutation matrix P, we see that for some vector x^T, we have either $A = \mathbf{1}x^T + \mathcal{Z}(A)P$ or $A = \mathbf{1}x^T - \mathcal{Z}(A)P$. In the former case, we find that the eigenvalues of A distinct from 1 are of the form $\mathcal{Z}(A)\omega$ where ω is a k–th root of unity for some $k \leq m$.

On the other hand, if we have $A = \mathbf{1}x^T - \mathcal{Z}(A)P$, then note that each entry of x^T is bounded below by $\mathcal{Z}(A)$, and that $x^T\mathbf{1} = 1 + \mathcal{Z}(A)$. Since A is $m \times m$, then necessarily $1 + \mathcal{Z}(A) = x^T\mathbf{1} \geq m\mathcal{Z}(A)$, so that $\mathcal{Z}(A) \leq \frac{1}{m-1}$. Further, the eigenvalues of A different from 1 are either of the form $\mathcal{Z}(A)\omega$ for some ω satisfying $\omega^k = 1$ for some $k \leq m$, (if P has an even cycle) or of the form $\mathcal{Z}(A)\alpha$, where α is a k–th root of -1 and k is odd and at most m (if P has an odd cycle). This latter case yields eigenvalues of the form described in ii). □

Corollary 3.4 *Let A be an irreducible stochastic matrix of order n, with left stationary vector π^T. We have $|\lambda| = \mathcal{Z}(A)$ for each eigenvalue $\lambda \neq 1$ of A if and only if there is some $k \in I\!N$ such that $A^k = (\mathcal{Z}(A))^k I + (1 - (\mathcal{Z}(A))^k)\mathbf{1}\pi^T$.*

Proof: Suppose that $|\lambda| = \mathcal{Z}(A)$ for each eigenvalue $\lambda \neq 1$. From Theorem 3.3 it follows that there is a $k \in I\!N$ such that $\lambda^k \geq 0$ for each eigenvalue $\lambda \neq 1$. We thus find that $\lambda^k = (\mathcal{Z}(A))^k$ for all such λ. Further, by Theorem 2.7 for each such eigenvalue λ of A, the algebraic and geometric multiplicities coincide. It now follows that the matrix A^k has just two distinct eigenvalues: 1 with algebraic multiplicity one, and $(\mathcal{Z}(A))^k$ with geometric multiplicity $n-1$. It is now straightforward to determine that $A^k = (\mathcal{Z}(A))^k I + (1 - (\mathcal{Z}(A))^k)\mathbf{1}\pi^T$. Conversely, if $A^k = (\mathcal{Z}(A))^k I + (1 - (\mathcal{Z}(A))^k)\mathbf{1}\pi^T$ for some $k \in I\!N$, we find that A^k has two distinct eigenvalues, namely 1, and $(\mathcal{Z}(A))^k$ of algebraic multiplicity $n-1$. Thus, if $\lambda \neq 1$ is an eigenvalue of A, then $\lambda^k = (\mathcal{Z}(A))^k$, yielding the desired conclusion. □

Remark 3.5 By a slight modification of the techniques in this section, the following result can be established.
Let A be an $n \times n$ matrix real with constant row sums μ such that $\mathcal{Z}(A) = 1$, and equality holds in (2.7) *for some eigenvalue $\lambda \neq \mu$. Then either λ is a k–th root of unity for some $k = 1, \ldots, n$, or λ is a k–th root of -1 for some odd k between* 1 *and n.*

4 Random Walk on a Graph

Let $\mathcal{G}$ be a connected graph on n vertices, conveniently labeled $i = 1, \ldots, n$, and let $d_1, \ldots, d_n$ be their corresponding degrees. Let M be the adjacency matrix of $\mathcal{G}$ and let $D = \mathrm{diag}(d_1, \ldots, d_n)$. Then, as explained in the introduction, the matrix $A = A(\mathcal{G}) = D^{-1}M \in \mathbb{R}^{n,n}$ is the transition matrix for a random walk on $\mathcal{G}$. In this section we shall study the structure of graphs $\mathcal{G}$ whose random walk has a transition matrix A which satisfies the equality case in the DDZ bound, namely, that $\mathcal{Z}(A) = \gamma(A)$.

We begin with the following lemma which can essentially be deduced from the proof of [17, Theorem 4] and also from work in [5, 6].

Lemma 4.1 *Suppose that $A \in \mathbb{R}^{n,n}$ is the transition matrix for the random walk on a graph $\mathcal{G}$. Let $1 \leq i, j \leq n$ be two vertices of $\mathcal{G}$, of degrees d_i and d_j, respectively, and suppose that $d_i \geq d_j$. Let N_i and N_j denote the neighbourhoods of vertices i and j, respectively. Then*

$$\frac{1}{2}\left\|(e_i - e_j)^T A\right\|_1 \;=\; \frac{|N_i \setminus N_j|}{d_i}.$$

Proof: Note that $\frac{1}{2}||(e_i - e_j)^T A||_1$ is given by the sum of the positive entries in $(e_i - e_j)^T A$. Since $d_i \geq d_j$, we see that $(e_i - e_j)^T A$ has a positive entry in position k if and only if $i \sim k$ but vertices $j \nsim k$. In that case, necessarily $(e_i - e_j)^T A e_k = \frac{1}{d_i}$. The result now follows. $\square$

Corollary 4.2 *Let A be as in Lemma 4.1. Then*

$$\mathcal{Z}(A) \;=\; \max\left\{\frac{|N_i \setminus N_j|}{d_i} | i, j \text{ are vertices in } \mathcal{G} \text{ with } d_j \geq d_j\right\}.$$

Corollary 4.3 *Let $\mathcal{G}$ be a connected graph with normalized Laplacian matrix $\mathcal{L}I - d^{-1/2}AD^{-1/2}$. If $\lambda \neq 0$ is an eigenvalue of $\mathcal{L}$, then*

$$1 - \max\left\{\frac{|N_i \setminus N_j|}{d_i} | i, j \text{ are vertices in } \mathcal{G} \text{ with } d_j \geq d_j\right\}$$

$$\leq \;\lambda\; \leq\; 1 \;+\; \max\left\{\frac{|N_i \setminus N_j|}{d_i} | i, j \text{ are vertices in } \mathcal{G} \text{ with } d_j \geq d_j\right\}.$$

In the next lemma we obtain a block structure of a transition matrix of a random walk which satisfies the equality case in the DDZ eigenvalue bound.

Lemma 4.4 *Let A be the transition matrix for the random walk on $\mathcal{G}$. Let z be an eigenvector corresponding to $\lambda_{\text{sub}}(A)$ and suppose that equality holds in (2.7). Let a and b a maximal and a minimal entries in z, respectively. If z has entries that are strictly between a and b, then A and z can be partitioned* *conformally as*

$$A = \left[\begin{array}{c|c|c|c} A_{1,1} & A_{1,2} & 0 & \frac{1}{d}J \\ \hline A_{2,1} & A_{2,2} & 0 & \frac{1}{d}J \\ \hline 0 & 0 & A_{3,3} & A_{3,4} \\ \hline A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{array}\right] \quad \text{and} \quad z = \left[\begin{array}{c} b\mathbf{1} \\ \hline a\mathbf{1} \\ \hline c_0 \\ \hline c_1 \end{array}\right], \qquad (4.12)$$

where the entries of c_0 *and* c_1 *are strictly between* a *and* b*. Further,* $A_{3,4} \geq \frac{1}{d}J$*, and each of* $A_{4,1}, A_{4,2}$ *and* $A_{4,3}$ *is positive.*

Proof: We begin by writing z as $z = \left[\begin{array}{c} b\mathbf{1} \\ \hline a\mathbf{1} \\ \hline c \end{array}\right]$, where $b\mathbf{1} < c < a\mathbf{1}$. Applying Lemma 2.3 we find that for any pair of indices i and j such that $z_i = b$ and $z_j = a$, and any k such that $b < z_k < a$, we have that $a_{i,k} = a_{j,k}$. It follows that for any index p such that $z_p = b$ or $z_p = a$, and each k such that $b < z_k < a$, there is a w_k such that $a_{p,k} = w_k$. Since $\mathcal{G}$ is connected, $w_k > 0$, for some $1 \leq k \leq n$, and since every w_k is an element in the p–th row, it follows that there is some d such that each nonzero w_k is equal to $\frac{1}{d}$.

It follows then that we may write A and z as

$$A = \left[\begin{array}{c|c|c|c} A_{1,1} & A_{1,2} & 0 & \frac{1}{d}J \\ \hline A_{2,1} & A_{2,2} & 0 & \frac{1}{d}J \\ \hline 0 & 0 & A_{3,3} & A_{3,4} \\ \hline A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{array}\right] \quad \text{and} \quad z = \left[\begin{array}{c} b\mathbf{1} \\ \hline a\mathbf{1} \\ \hline c_0 \\ \hline c_1 \end{array}\right],$$

respectively, where $A_{4,1}$ and $A_{4,2}$ are positive matrices, and the elements of c_0 and c_1 are strictly between a and b.

Let the subsets in the partitioning of A be $S_1, \ldots, S_4$, respectively, with cardinalities $m_1, \ldots, m_4$, respectively. Note that $S_4 \neq \emptyset$, since $\mathcal{G}$ is connected, but that S_3 may be empty. Suppose that $S_3 \neq \emptyset$. By considering $||(e_i - e_j)^T A||_1$ for $i \in S_1, j \in S_3$ we find that $\mathcal{Z}(A) \geq 1 - \frac{m_4}{d}$, while by considering $||(e_i - e_k)^T A||_1$ for $i \in S_1, k \in S_2$ we have, in light of Lemma 2.3, that $\mathcal{Z}(A) \leq 1 - \frac{m_4}{d}$. Hence $\mathcal{Z}(A) = 1 - \frac{m_4}{d}$, and again by considering $(e_i - e_j)^T A$ for $i \in S_1, j \in S_3$ it follows that $A_{3,4} \geq \frac{1}{d}$. The positivity of $A_{4,3}$ now follows from the combinatorial symmetry of A.

□

Corollary 4.5 *Suppose that $\mathcal{G}$ is as in Lemma 4.4 and that A and z have been partitioned as in (4.12). Denote the corresponding subsets of the partitioning (in order) as $S_1, \ldots, S_4$, with cardinalities $m_1, \ldots, m_4$, respectively. Note that necessarily $S_4 \neq \emptyset$. Then:*

i) *Suppose that $S_3 = \emptyset$. Let $\hat{A}$ denote the principal submatrix of A on the rows and columns corresponding to $S_1 \cup S_2$. Then $\hat{A}$ can be written as $\hat{A} = \frac{d-m_4}{d}\overline{A}$, where $\overline{A}$ is stochastic and yields equality in (2.7). Further, there is a $\lambda_{\text{sub}}(\overline{A})$ eigenvector for $\overline{A}$ having only two distinct entries, with all entries corresponding to indices in S_1 taking one value and all entries corresponding to indices in S_2 taking the other value.*

ii) *If $S_3 \neq \emptyset$, then $\mathcal{Z}(A) = 1 - \frac{m_4}{d}$. Further, we have that either*

$$\begin{cases} A_{1,1} &= 0, \\ A_{2,2} &= 0, \\ A_{1,2}\mathbf{1} &= \left(1 - \dfrac{m_4}{d}\right)\mathbf{1}, \\ A_{2,1}\mathbf{1} &= \left(1 - \frac{m_4}{d}\right)\mathbf{1}, \\ \lambda_{\text{sub}} &= -\left(1 - \dfrac{m_4}{d}\right), \end{cases},$$

or

$$\begin{cases} A_{1,2} &= 0, \\ A_{2,1} &= 0, \\ A_{1,1}\mathbf{1} &= \left(1 - \dfrac{m_4}{d}\right)\mathbf{1}, \\ A_{2,2}\mathbf{1} &= \left(1 - \dfrac{m_4}{d}\right)\mathbf{1}, \\ \lambda_{\text{sub}} &= \left(1 - \dfrac{m_4}{d}\right). \end{cases}$$

Proof: i) From (4.12) and Lemma 2.3, it follows that if $i \in S_1$ and $j \in S_2$, then $\frac{1}{2}||(e_i - e_j)^T A||_1 = \mathcal{Z}(A)$, from which we find that $\mathcal{Z}(A) = \mathcal{Z}(\hat{A})$. From the eigenequation, we have that $\hat{A}\begin{bmatrix} b\mathbf{1} \\ a\mathbf{1} \end{bmatrix} + \frac{1}{d}Jc_1 = \lambda_{\text{sub}}(A)\begin{bmatrix} b\mathbf{1} \\ a\mathbf{1} \end{bmatrix}$. We claim that, in fact, $\lambda_{\rm sub}(A)$ is an eigenvalue of $\hat{A}$. To see this note that if not, then it follows that $\begin{bmatrix} b\mathbf{1} \\ a\mathbf{1} \end{bmatrix} = \left(\lambda_{\rm sub} I - \hat{A}\right)^{-1} \frac{1}{d} J c_1$, a contradiction since $\hat{A}$ has constant row sums. Thus $\lambda_{\rm sub}(A)$ is an eigenvalue of $\hat{A}$. It now follows that $|\lambda_{\rm sub}(\hat{A})| = \mathcal{Z}(\hat{A}) = \mathcal{Z}(A)$.

Next let $v = \begin{bmatrix} b\mathbf{1} \\ a\mathbf{1} \end{bmatrix}$ so that $\hat{A}v + \frac{1}{d}Jc_1 = \lambda_{\rm sub} v$. If $\lambda_{\rm sub}(\hat{A}) \neq \frac{d-m_4}{d}$, set $x = -\mathbf{1}^T c_1/(d\lambda_{\rm sub} - d + m_4)$. It now follows that $v + x\mathbf{1}$ is a $\lambda_{\rm sub}(\hat{A})$ eigenvector for $\hat{A}$ having two distinct entries, with all entries corresponding to S_1 identical and all entries corresponding to S_2 identical. Finally, if $\lambda_{\rm sub}(\hat{A}) = \frac{d-m_4}{d}$, it follows that $\hat{A}$ can be written as a direct sum of two nonnegative matrices, both necessarily with constant row sums $\frac{d-m_4}{d}$, and the eigenvector conclusion now follows. Setting $\overline{A} = \frac{d}{d-m_4}\hat{A}$, the desired conclusions are now evident.

ii) As in Lemma 4.4, we have $\mathcal{Z}(A) = 1 - \frac{m_4}{d}$. Further, from (4.12) it follows that each of $A_{1,1}, A_{1,2}, A_{2,1}$, and $A_{2,2}$ has constant rows sums, and evidently we have that

$$A_{1,1}\mathbf{1} + A_{1,2}\mathbf{1} \; = \; (1 - \frac{m_4}{d})\mathbf{1} \;\; \textbf{and} \;\; A_{2,1}\mathbf{1} + A_{2,2}\mathbf{1} \; = \; \left(1 - \frac{m_4}{d}\right)\mathbf{1}.$$

Letting $x_{1,1},\ x_{1,2},\ x_{2,1}$, and $x_{2,2}$ be the row sums of $A_{1,1},\ A_{1,2},\ A_{2,1}$, and $A_{2,2}$, respectively, we find from the eigenequation that $\lambda_{\rm sub} = x_{1,1} - x_{2,1} = x_{2,2} - x_{2,1}$. Thus if $\lambda_{\rm sub} = 1 - \frac{m_4}{d}$, then $x_{1,1} = x_{2,2} = 1 - \frac{m_4}{d}$ and $x_{1,2} = x_{2,1} = 0$,. while if $\lambda_{\rm sub} = -\left(1 - \frac{m_4}{d}\right)$, then $x_{1,2} = x_{2,1} = 1 - \frac{m_4}{d}$ and $x_{1,1} = x_{2,2} = 0$. The conclusions on $A_{1,1}, \ldots, A_{2,2}$ now follow. □

Remark 4.6 Suppose that A is as in Corollary 4.5 and that $S_3 = \emptyset$. Set $|S_1 \cup S_2| = m$ and $|S_4| = k$. Let $\mathcal{G}(S_1 \cup S_2)$ and $\mathcal{G}(S_4)$ denote the induced subgraphs of $\mathcal{G}$ on the vertex sets $S_1 \cup S_2$ and S_4, respectively. From Lemma 4.4, it follows that $\mathcal{G}(S_1 \cup S_2)$ is regular, say of degree r. Let $r_1, \ldots, r_k$ denote the degree sequence for the induced subgraph $\mathcal{G}(S_4)$. Evidently $d = r + k$. In order that $\mathcal{Z}(A) = \frac{|N_i \setminus N_j|}{d}$ for some $i \in S_1$ and $j \in S_2$, all of the following conditions must hold:

i) for each $p, q \in S_4$ with $r_p \;\geq\; r_q$,

$$\frac{|N_p \setminus N_q|}{r_p + m} \;\leq\; \frac{|N_i \setminus N_j|}{r + m}$$

and

ii) for each $q \in S_4$, either $r + k - m \geq r_q \geq k - |N_i \setminus N_j|$, or

$$r_q \;\geq\; \max\left\{r + k - m, \frac{(r+k)(m-r)}{|N_i \setminus N_j|} - m\right\}.$$

Remark 4.7 Suppose that A is as in Corollary 4.5 and that $S_3 \neq \emptyset$. Denote the degrees of the vertices in the subgraph induced by S_3 by $q_i,\ i = 1, \ldots, m_3$ and the degrees of the vertices in the subgraph induced by S_4 by $r_j,\ j = 1, \ldots, m_4$. Set $p = d - m_4$. In order that $\mathcal{Z}(A) = \frac{|N_k \setminus N_l|}{d}$ for some $k \in S_1$ and $l \in S_2$, all of the following conditions must hold:

i) For each $i \in S_4$, either

$$m_4 - p \;\leq\; r_i \;\leq\; p + m_4 - m_1 - m_2 - m_3,$$

or

$$r_i \;\geq\; \max\left\{p + m_4 - m_1 - m_2 - m_3, \frac{m_4}{p}(m_1 + m_2 + m_3) - (p + m_4)\right\};$$

ii) For each $i \in S_3$ and $j \in S_4$, either $q_i + m_4 \leq r_j + m_1 + m_2 + m_3$ and

$$\frac{m_1 + m_2 + m_3 - q_i}{m_1 + m_2 + m_3 + r_j} \;\leq\; \frac{p}{p + m_4},$$

or

$$q_i + m_4 \;\geq\; r_j + m_1 + m_2 + m_3$$

and

$$\frac{m_4 - r_j}{m_4 + q_i} \;\leq\; \frac{p}{p + m_4};$$

iii) For each $i, j \in S_3$ with $q_i \geq q_j$, we have that

$$\frac{|N_i \setminus N_j|}{q_i + m_4} \;\leq\; \frac{p}{p + m_4};$$

iv) For each $i, j \in S_4$ with $r_i \geq r_j$, we have that

$$\frac{|N_i \setminus N_j|}{m_1 + m_2 + m_3 + r_i} \;\leq\; \frac{p}{p + m_4}.$$

Our next result gives the block structure of certain nonbipartite graphs which satisfy the equality case in the DDZ eigenvalue bound.

Theorem 4.8 *Suppose that $\mathcal{G}$ is a nonbipartite, connected graph for which equality holds in* (2.7). *Suppose further that there is a λ_{sub} eigenvector z having* *just two distinct entries a and b with $a > b$. Suppose also that $z_1 = a$, $z_2 = b$, that $1 \sim 2$, which can be assumed without loss of generality as $\mathcal{G}$ is connected, and that $d_1 > d_2$. Then A and z can be taken to have the following form, where the partitionings are conformal:*

$$A = \left[\begin{array}{c|c|c|c} 0 & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T \\ \hline \frac{1}{d_2}\mathbf{1} & 0 & \frac{1}{d_2}J & \frac{1}{d_2}J \\ \hline 0 & \frac{1}{d_1}J & A_{4,4} & A_{4,5} \\ \hline \frac{1}{d_1}\mathbf{1} & \frac{1}{d_1}J & A_{5,4} & A_{5,5} \end{array}\right] \quad \text{and} \quad z = \left[\begin{array}{c} b \\ \hline a\mathbf{1} \\ \hline b\mathbf{1} \\ \hline b\mathbf{1} \end{array}\right]. \tag{4.13}$$

Further, the matrix $d_1 \left[\begin{array}{c|c} A_{4,4} & A_{4,5} \\ \hline A_{5,4} & A_{5,5} \end{array}\right]$ *is the adjacency matrix of a biregular graph with degrees $d_1 - |N_1 \setminus N_2|$ and $d_1 - |N_1 \setminus N_2| - 1$.*

Proof: We partition the rows and columns of A, as well as z, as follows: $S_1 = \{1\},\ S_2 = \{2\},\ S_3 = (N_1 \setminus N_2) \setminus \{2\},\ S_4 = (N_2 \setminus N_1) \setminus \{1\},\ S_5 = N_1 \cap N_2,\ S_6 = \{i | z_i = a, i \nsim 1, 2\},$ and $S_7 = \{i | z_i = b, i \nsim 1, 2\}$. (We note that some subsets in this partitioning may be empty; however, $S_5 \neq \emptyset$, since $\mathcal{Z}(A)$ is assumed to be less than 1.) With this partitioning it follows that

$$A = \left[\begin{array}{c|c|c|c|c|c|c} 0 & \frac{1}{d_1} & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T & 0^T & 0^T \\ \hline \frac{1}{d_2} & 0 & 0^T & \frac{1}{d_2}\mathbf{1}^T & \frac{1}{d_2}\mathbf{1}^T & 0^T & 0^T \\ \hline A_{3,1} & A_{3,2} & & & \cdots & & A_{3,7} \\ \hline \vdots & & & & & & \vdots \\ \hline A_{7,1} & A_{7,2} & & & \cdots & A_{7,6} & A_{7,7} \end{array}\right] \quad \text{and} \quad z = \left[\begin{array}{c} b \\ \hline a \\ \hline a\mathbf{1} \\ \hline b\mathbf{1} \\ \hline b\mathbf{1} \\ \hline a\mathbf{1} \\ \hline b\mathbf{1} \end{array}\right]. \tag{4.14}$$

From the eigenequation $Az = \lambda_{\text{sub}} z$ it is straightforward to determine that $\lambda_{\text{sub}} = -\frac{|N_1 \setminus N_2|}{d_1}$ and that $\lambda_{\text{sub}} a = b$. In particular, we find from this last observation that if $i \in S_2 \cup S_3 \cup S_6$, then $a_{i,j} = 0$, for each $j \in S_2 \cup S_3 \cup S_6$. Applying that observation in conjunction with the combinatorial symmetry of

A, it follows that

$$A = \left[\begin{array}{c|c|c|c|c|c|c} 0 & \frac{1}{d_1} & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T & 0^T & 0^T \\ \hline \frac{1}{d_2} & 0 & 0^T & \frac{1}{d_2}\mathbf{1}^T & \frac{1}{d_2}\mathbf{1}^T & 0^T & 0^T \\ \hline A_{3,1} & 0 & 0 & A_{3,4} & A_{3,5} & 0 & A_{3,7} \\ \hline 0 & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} \\ \hline A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} \\ \hline 0 & 0 & 0 & A_{6,4} & A_{6,5} & 0 & A_{6,7} \\ \hline 0 & 0 & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} \end{array}\right].$$

Further, since

$$\lambda_{\text{sub}} b \;=\; a\frac{|N_1 \setminus N_2|}{d_1} + b\left(1 - \frac{|N_1 \setminus N_2|}{d_1}\right),$$

it follows that

$$A_{5,2}\mathbf{1} + A_{5,3}\mathbf{1} + A_{5,6}\mathbf{1} \;=\; \frac{|N_1 \setminus N_2|}{d_1}.$$

But then from the combinatorial symmetry of A, we see that $A_{5,1} > 0$, so that if $i \in S_5$ and $j \in S_6$, then

$$\frac{1}{2}||(e_i - e_j)^T A||_1 \;>\; \frac{|N_1 \setminus N_2|}{d_1},$$

a contradiction and we conclude that $S_6 = \emptyset$.

Thus we can take our matrix A to be written as

$$A \;=\; \left[\begin{array}{c|c|c|c|c|c} 0 & \frac{1}{d_1} & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T & 0^T \\ \hline \frac{1}{d_2} & 0 & 0^T & \frac{1}{d_2}\mathbf{1}^T & \frac{1}{d_2}\mathbf{1}^T & 0^T \\ \hline A_{3,1} & 0 & 0 & A_{3,4} & A_{3,5} & A_{3,7} \\ \hline 0 & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,7} \\ \hline A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,7} \\ \hline 0 & 0 & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,7} \end{array}\right].$$

Note that $A_{4,2}\mathbf{1}+A_{4,3}\mathbf{1} = \frac{|N_1 \setminus N_2|}{d_1}\mathbf{1}$ and that $A_{5,2}\mathbf{1}+A_{5,3}\mathbf{1} = \frac{|N_1 \setminus N_2|}{d_1}\mathbf{1}$. Thus by considering $j \in S_4$ or $j \in S_5$, it follows that $\frac{1}{2}||(e_2 - e_j)^T A||_1 \leq \frac{|N_1 \setminus N_2|}{d_1}$ only if $A_{4,7}$ and $A_{5,7}$ are zero matrices. Hence $A_{7,4} = 0$ and $A_{7,5} = 0$ by combinatorial symmetry. But this last is a contradiction since then for any $j \in S_7$, $e_2^T A$ and $e_j^T A$ have disjoint support. We conclude that $S_7 = \emptyset$.

Consequently, our matrix A can be written as

$$A \;=\; \left[\begin{array}{c|c|c|c|c} 0 & \frac{1}{d_1} & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T \\ \hline \frac{1}{d_2} & 0 & 0^T & \frac{1}{d_2}\mathbf{1}^T & \frac{1}{d_2}\mathbf{1}^T \\ \hline A_{3,1} & 0 & 0 & A_{3,4} & A_{3,5} \\ \hline 0 & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} \\ \hline A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} \end{array}\right],$$

with z partitioned conformally as $z = \left[\begin{array}{c} b \\ \hline a \\ \hline a\mathbf{1} \\ \hline b\mathbf{1} \\ \hline b\mathbf{1} \end{array}\right]$. By considering $\frac{1}{2}\left\|(e_1 - e_j)^T A|\right\|_1$ for any $j \in S_3$, it follows that $A_{3,5} > 0$ (assuming that $S_3 \neq \emptyset$). If $S_4 \neq \emptyset$, we note that if $A_{3,4}$ contains a zero entry, say in the column corresponding to $i \in S_4$, then it follows that the columns of $A_{4,4}$ and $A_{5,4}$ corresponding to index i must be zero columns. Hence the rows of $A_{4,4}$ and $A_{4,5}$ corresponding to index i must also be zero rows. But then two rows of A have disjoint support, namely the second row of A and the row corresponding to index $i \in S_4$, a contradiction. We conclude that if $S_4 \neq \emptyset$, then $A_{3,4} > 0$. It now follows that every row of A corresponding to an index in S_3 is the same as row 2 of A.

Collapsing S_2 and S_3 into a single set $S_{\overline{2}}$, we find that A and z can be written as

$$A \;=\; \left[\begin{array}{c|c|c|c} 0 & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T \\ \hline \frac{1}{d_2}\mathbf{1} & 0 & \frac{1}{d_2}J & \frac{1}{d_2}J \\ \hline 0 & A_{4,\overline{2}} & A_{4,4} & A_{4,5} \\ \hline A_{5,1} & A_{5,\overline{2}} & A_{5,4} & A_{5,5} \end{array}\right] \quad \textbf{and} \quad z \;=\; \left[\begin{array}{c} b \\ \hline a\mathbf{1} \\ \hline b\mathbf{1} \\ \hline b\mathbf{1} \end{array}\right].$$

From combinatorial symmetry, we see that $A_{4,\overline{2}}$ and $A_{5,\overline{2}}$ must be positive, as is $A_{5,1}$. Since $A_{4,\overline{2}}\mathbf{1} = \frac{|N_1 \backslash N_2|}{d_1}\mathbf{1}$ and $A_{5,\overline{2}}\mathbf{1} = \frac{|N_1 \backslash N_2|}{d_1}\mathbf{1}$, it follows that the vertices of S_4 and S_5 must all have degree d_1. In particular,

$$A \;=\; \left[\begin{array}{c|c|c|c} 0 & \frac{1}{d_1}\mathbf{1}^T & 0^T & \frac{1}{d_1}\mathbf{1}^T \\ \hline \frac{1}{d_2}\mathbf{1} & 0 & \frac{1}{d_2}J & \frac{1}{d_2}J \\ \hline 0 & \frac{1}{d_1}J & A_{4,4} & A_{4,5} \\ \hline \frac{1}{d_1}\mathbf{1} & \frac{1}{d_1}J & A_{5,4} & A_{5,5} \end{array}\right].$$

The conditions on $A_{4,4}, A_{4,5}, A_{5,4}$, and $A_{5,5}$ now follow readily. □

Note that in Theorem 4.8, the matrix $d_1 \left[\begin{array}{c|c} A_{4,4} & A_{4,5} \\ \hline A_{5,4} & A_{5,5} \end{array}\right]$ is the adjacency matrix of a biregular graph, say $\mathcal{H}$, with the vertices in S_4 having degree $d_1 - |N_1 \setminus N_2|$ and the vertices in S_5 having degree $d_1 - |N_1 \setminus N_2| - 1$. In order that the matrix A of (4.13) satisfies $\mathcal{Z}(A) = \frac{|N_1 \setminus N_2|}{d_1}$, the following conditions on H must hold:

i) each vertex in S_4 is adjacent to at most $|N_1 \setminus N_2|$ vertices in S_4, and each vertex in S_5 is adjacent to at most $|N_1 \setminus N_2| - 1$ vertices in S_4;

ii) for each pair of vertices $i, j \in S_4, |N_i \setminus N_j| \leq |N_1 \setminus N_2|$;

iii) for each pair of vertices $i, j \in S_5, |N_i \setminus N_j| \leq |N_1 \setminus N_2|$;
and
iv) for each pair of vertices $i \in S_4, j \in S_5, |N_i \setminus N_j| \leq |N_1 \setminus N_2| - 1$.

Finally, we also note that if $S_4 = \emptyset$, then necessarily $H = K_{d_1 - |N_1 \setminus N_2|}$.

As an example of an adjacency matrix of a graph $\mathcal{G}$ satisfying the conditions of Theorem 4.8 we give the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Here $D = \text{diag}([6,\ 4,\ 4,\ 4,\ 4,\ 6,\ 6,\ 6])$ and the transition matrix for the random walk, $A = D^{-1}M$ induced by $\mathcal{G}$ satisfies that $\gamma(A) = 2/3 = |-2/3| = \mathcal{Z}(A)$. We observe that the eigenvector of A corresponding to $\lambda_{\text{sub}} = -2/3$ is, indeed, given by

$$z = [-0.2774,\ 0.4160,\ 0.4160,\ 0.4160,\ 0.4160,\ -0.2774,\ -0.2774,\ -0.2774]^T.$$

In our next result we investigate the form of the transition matrix which satisfies the DDZ bound for a random walk induced by a regular graph.

Theorem 4.9 *Suppose that* $A = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right]$ *is a* $n \times n$ *transition matrix for a connected* d*–regular graph* $\mathcal{G}$ *that satisfies equality in* (2.7)*, with corresponding* λ_{sub}*–eigenvector* $z = \left[\begin{array}{c} b\mathbf{1} \\ \hline a\mathbf{1} \end{array}\right]$*, where* $a > b$*. Label the subsets of the partition* S_1 *and* S_2*, respectively. Fix indices* i *and* j *with* $i \in S_1$ *and* $j \in S_2$*, respectively, and with* $i \sim j$*. Then*

$$\lambda_{\mathrm{sub}} = -\frac{|N_i \setminus N_j|}{d}.$$

Set $\alpha = |N_i \cap N_j \cap S_2|$ *and* $\beta = |N_i \cap N_j \cap S_1|$*. Then*

$$A_{1,1}\mathbf{1} = \frac{\beta}{d}\mathbf{1},\ A_{1,2}\mathbf{1} = \frac{|N_i \setminus N_j| + \alpha}{d}\mathbf{1},\ A_{2,1}\mathbf{1} = \frac{|N_i \setminus N_j| + \beta}{d}\mathbf{1},\ \text{and}\ A_{2,2}\mathbf{1} = \frac{\alpha}{d}\mathbf{1}.$$

Proof: From Lemma 2.5 b), we find immediately that $\lambda_{\mathrm{sub}} = -\frac{|N_i \setminus N_j|}{d}$. From the equations $Az = \lambda_{\mathrm{sub}} z$ and $A\mathbf{1} = \mathbf{1}$, we find that each of the blocks $A_{1,1}, \ldots, A_{2,2}$ must have constant row sums, say $x_{1,1}, \ldots, x_{2,2}$, respectively. It is straightforward to see that, necessarily, the eigenvalues of the 2×2 matrix $X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$ are 1 and λ_{sub}, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} b \\ a \end{bmatrix}$, respectively.

Considering $(e_i - e_j)^T A$ and applying Lemma 2.3 we see that one of two situations holds: either $z_k = a$ for each $k \in N_i \setminus N_j$, and $z_k = b$, for each $k \in N_j \setminus N_i$, or $z_k = b$ for each $k \in N_i \setminus N_j$, and $z_k = a$, for each $k \in N_j \setminus N_i$. Suppose that the latter case occurs. It then follows that

$$X = \begin{bmatrix} \frac{|N_i \setminus N_j| + \beta}{d} & \frac{\alpha}{d} \\ \frac{\beta}{d} & \frac{|N_i \setminus N_j| + \alpha}{d} \end{bmatrix}$$

which fails to have λ_{sub} as an eigenvalue. Hence only the former case can occur, from which we find that $e_i^T A_{1,2}\mathbf{1} = |N_i \setminus N_j| + \alpha$, and hence $e_i^T A_{1,1}\mathbf{1} = \beta$. A similar argument yields $e_j^T A_{2,1}\mathbf{1} = |N_i \setminus N_j| + \beta$, and hence $e_j^T A_{2,2}\mathbf{1} = \alpha$. The conclusion now follows. □

Remark 4.10 Suppose that A is as in Theorem 4.9 and let $\mathcal{G}(S_1)$ and $\mathcal{G}(S_2)$ denote the subgraphs induced by S_1 and S_2, respectively. It follows that $\mathcal{G}(S_1)$ and $\mathcal{G}(S_2)$ are both regular, of degrees $|N_i \setminus N_j| + \alpha$ and $|N_i \setminus N_j| + \beta$, respectively. Further, in order that $\mathcal{Z}(A) = \frac{|N_i \setminus N_j|}{d}$, each of the following conditions must hold:

i) for each $k, l \in S_1, \ |N_k \setminus N_l| \le |N_i \setminus N_j|$;

ii) for each $k, l \in S_2, \ |N_k \setminus N_l| \le |N_i \setminus N_j|$;

and

iii) for each $k \in S_1, \ l \in S_2, \ |N_k \setminus N_l| = |N_i \setminus N_j|$.

According to Seneta [19, Definition 3.2], a stochastic matrix A is called *scrambling* if any two rows of A have at least one positive element in a coincident position. It is easy to see that for such matrices A, $\mathcal{Z}(A) < 1$. In a similar vein, we say that a graph $\mathcal{G}$ is *scrambling* if it has the property that each pair of vertices has a common neighbour. Evidently $\mathcal{G}$ is scrambling if and only if the transition matrix for the corresponding random walk on $\mathcal{G}$ is a scrambling stochastic matrix. This leads us to the following result:

Theorem 4.11 *Let $\mathcal{G}$ be a scrambling graph on $n \ge 4$ vertices and let A be the transition matrix for the corresponding random walk on $\mathcal{G}$. Then*

$$|\lambda_{\text{sub}}(A)| \ \le \ \frac{n-2}{n-1}.$$

Furthermore equality holds if and only if $\mathcal{G} = K_2 \vee O_{n-2}$, where "$\vee$" denotes the join of two graphs and O_k denotes the empty graph on k vertices.

Proof: From (2.7) we know that $|\lambda_{\text{sub}}(A)| \le \mathcal{Z}(A)$ and, on applying Lemma 4.1, it follows that

$$\mathcal{Z}(A) \ = \ \max\left\{ \frac{|N_i \setminus N_j|}{d_i} \mid i, j \text{ are vertices of } G \text{ and } d_i \ge d_j \right\}.$$

We thus readily find that

$$|\lambda_{\text{sub}}(A)| \ \le \ \max\left\{ \frac{d_i - 1}{d_i} \mid i = 1, \ldots, n \right\} \ \le \ \frac{n-2}{n-1}.$$

Suppose now that $|\lambda_{\text{sub}}(A)| = \frac{n-2}{n-1}$ and note that necessarily $\mathcal{G}$ must have at least one vertex of degree $n-1$. Let z be an eigenvector of A corresponding to λ_{sub}, say, with maximum entry a and minimum entry b. Suppose first that z has an entry strictly between a and b. Let i and j correspond to entries in z equal to a and b, respectively. Referring to (4.12) of Lemma 4.4, we find that vertices i and j have the same degree, say d. Since $\frac{n-2}{n-1} = \frac{d-1}{d}$, we find that $d = n-1$.

But in this case, $\frac{1}{2}||(e_i - e_j)^T A||_1 = \frac{1}{n-1}$, a contradiction. We conclude that z has no entries strictly between a and b.

Suppose next that $\mathcal{G}$ is regular, say of degree d. It follows that $\mathcal{Z}(A) \leq \frac{d-1}{d}$, from which we conclude that $d = n-1$. But then $\mathcal{G} = K_n$, the complete graph on n vertices, and again we have a contradiction. It now follows that A satisfies the hypotheses of Theorem 4.8, necessarily with $d_1 = n-1$. Referring to (4.13), we find that A can be written as follows for some d_2:

$$A = \left[\begin{array}{c|c|c} 0 & \frac{1}{n-1}\mathbf{1}^T & \frac{1}{n-1}\mathbf{1}^T \\ \hline \frac{1}{d_2}\mathbf{1} & 0 & \frac{1}{d_2}J \\ \hline \frac{1}{n-1}\mathbf{1} & \frac{1}{n-1}J & \frac{1}{n-1}(J-I) \end{array}\right]. \tag{4.15}$$

Suppose that $\mathcal{G}$ has k vertices of degree d_2 and $n-k$ vertices of degree $n-1$. We find readily from (4.15) that $\mathcal{Z}(A) = \frac{k}{n-1}$ so that necessarily we have that $k = n-2$. It follows that $d_2 = 2$ and that $\mathcal{G} = K_2 \vee O_{n-2}$.

Conversely, if $\mathcal{G} = K_2 \vee O_{n-2}$, then

$$A = \left[\begin{array}{cc|c} 0 & \frac{1}{n-1} & \frac{1}{n-1}\mathbf{1}^T \\ \frac{1}{n-1} & 0 & \frac{1}{n-1}\mathbf{1}^T \\ \hline \frac{1}{2}\mathbf{1} & \frac{1}{2}\mathbf{1} & 0 \end{array}\right],$$

which is easily seen to have eigenvalues 0 (of multiplicity $n-3$), $1, -\frac{1}{n-1}$ and $-\frac{n-2}{n-1}$. □

Theorem 4.12 yields the following corollary for the eigenvalues of the normalized Laplacian arising from scrambling graphs:

Corollary 4.12 *Let $\mathcal{G}$ be a scrambling graph on $n \geq 4$ with normalized Laplacian matrix $\mathcal{L}$. If $\lambda \neq 0$ is an eigenvalue of $\mathcal{L}$, then $\frac{1}{n-1} \leq \lambda \leq \frac{2n-3}{n-1}$. Equality holds in either of the bounds on λ if and only if $\mathcal{G} = K_2 \vee O_{n-2}$.*

Recall that $\mathcal{G}$ is a *threshold graph* on n vertices if it can be generated from a one–vertex graph by repeated applications of the following two operations: (i) addition of a single isolated vertex to the graph, and (ii) addition of a single to the graph that is connected to all other vertices. Recall further that threshold graphs are characterized by the property that they contain no induced subgraphs that are isomorphic to either P_4, C_4 or $K_2 \cup K_2$. It is not difficult to see that the only regular threshold graphs are either complete or empty.

Our final result in this paper is:

Theorem 4.13 *Let $\mathcal{G}$ be a connected threshold graph on n vertices and let A be the adjacency matrix for the corresponding random walk on $\mathcal{G}$. Then equality holds in (2.7) if and only if $\mathcal{G}$ can be written as $\mathcal{G} = O_p \vee K_{n-p}$, for some $1 \le p \le n-1$.*

Proof: Suppose first that A is of the form described in (4.13) in Theorem 4.8 and partition the rows and columns of A as $S_1, \ldots, S_4$ conformally with (4.13). Since $d_2 < d_1 \le n-1$, we find that $p \equiv |S_2| \ge 2$. If $S_3 \ne \emptyset$, then selecting vertices $u,v \in S_2$ and $w \in S_3$, we find that the subgraph of G induced by vertices $1, u, v, w$ is C_4, a contradiction. Hence $S_3 = \emptyset$, so that $d_1 = n-1$. Hence the vertices in S_4 must also have degree $n-1$ and it follows that $\mathcal{G} = O_p \vee K_{n-p}$.

Suppose next that A is of the form (4.12) described in Theorem 4.4, and partition the rows and columns of A as $S_1, \ldots, S_4$ conformally with (4.12). Since the subgraph $\mathcal{H}$ of $\mathcal{G}$ induced by the vertices of $S_1 \cup S_2$ is regular, it is either a complete subgraph or an empty subgraph. The latter case then yields $\mathcal{Z}(A) = \frac{1}{2}||(e_i - e_j)^T A||_1 = 0$, for $i \in S_1$and $j \in S_2$, a contradiction. Thus we see that $S_1 \cup S_2$ induces a complete subgraph on $d+1$ vertices from which it follows that $\mathcal{Z}(A) = \frac{1}{d}$. Observe that in this situation, necessarily $S_3 = \emptyset$ for otherwise $\mathcal{Z}(A) \ge \frac{d-1}{d}$. Hence $d = n-1$, so that $|\lambda_{\text{sub}}(A)| = \frac{1}{n-1}$. It now follows readily that in fact $\mathcal{G} = K_n$.

Conversely, it is straightforward to determine that if $\mathcal{G} = O_p \vee K_{n-p}$, then A yields equality in (2.7).

□

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