Computing Upper-bounds of the Minimum Dwell Time of Linear Switched Systems via Homogeneous Polynomial Lyapunov Functions

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Abstract— This paper investigates the minimum dwell time for switched linear systems. It is shown that a sequence of upper bounds of the minimum dwell time can be computed by exploiting homogeneous polynomial Lyapunov functions and convex optimization problems based on linear matrix inequalities (LMIs). This sequence is obtained by adopting two possible representations of homogeneous polynomials, one based on Kronecker products, and the other on the square matrix representation (SMR). Some examples illustrate the use and the potentialities of the proposed approach. It is also conjectured that the proposed approach is asymptotically non-conservative, i.e. the exact minimum dwell time is obtained by using homogeneous polynomials with sufficiently large degree.

I. INTRODUCTION

This work deals with the stability of switched linear systems under a dwell time constraint. Problems in the design of switching control systems in which switching takes place "slowly" between system matrices, arise frequently in practice, see for instance [1]–[9]. In such problems, one is faced with determining the minimum time between switching (i.e. the dwell time) such that that exponential switching is maintained. It is well known that the computation of the exact minimum dwell time is demanding. In [10], an upper bound is computed on the basis of the norm of the transition matrices associated with the system matrices, and is further discussed in [11]. More recently, an alternative method based on convex optimization problems is presented in [12]. In this paper, the authors use linear matrix inequalities (LMIs) and one-parameter search techniques to compute a guaranteed dwell time. The inequalities generate a piecewise quadratic Lyapunov function $v(x)$, discontinuous at the switching instants t_k, but such that the sequence $v(x(t_k))$, for $k = 0, \cdots, \infty$, converges uniformly to zero.

This paper extends the result in [12] by adopting homogeneous polynomial Lyapunov functions, which have been proposed in the context of time-varying systems [13]. A candidate Lyapunov function is looked for by using two possible representations, the first based on Kronecker products, and the second based on the square matrix representation (SMR) introduced in [14]. Both representations lead to sufficient conditions for stability of the switched linear system under

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a dwell time requirement via LMI feasibility tests, which are convex optimization problems with LMI constraints. A bisection search over a scalar parameter produces an upper bound of the minimum dwell time. The representation based on Kronecker products enjoys a more explicit formulation, while the one based on the SMR provides less conservative results. A number of examples illustrate the proposed approach, showing that the exact minimum dwell time can be often obtained.

The paper is organized as follows. In Section II the basic problem is formulated and some preliminary results are given. In Section III the proposed condition based on Kronecker products is derived. Section IV presents the formulation of this condition by adopting the SMR. Section V illustrates the proposed approach through a number of examples. Lastly, Section VI concludes the paper with some final remarks.

II. PRELIMINARIES

The notation used throughout the paper is as follows: $\mathbb{R}$: space of real numbers; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A': transpose of A; $A > 0$: symmetric positive definite matrix A; $A \otimes B$: Kronecker product of matrices A and B; s.t.: subject to.

We consider switched linear systems of the form

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, and $\sigma(t)$ is a switching signal taking values in a finite set $S = \{1, 2, \ldots, M\}$. All matrices A_i, $i = 1, 2, \ldots, M$, are assumed to be Hurwitz, and we characterize switching rules by saying that the signal $\sigma(t)$ orchestrates switching between the matrices $A_1, A_2, \ldots, A_M$.

In this work we impose a further restriction on the system class described by the above equation. More specifically, we impose restrictions on the set of admissible switching signals by defining the set

$$D_T = \{\sigma(t) : \ t_{k+1} - t_k \geq T\}$$

where t_k are the commutation instants and $T \geq 0$. The *minimum dwell time problem* is then to compute the minimum T ensuring exponential stability of system (1) for all possible $\sigma(t) \in D_T$. We define this time as

$$
\begin{aligned}
T_{min} \quad = \quad &\inf\{T \geq 0 : \ (1) \text{ is exponentially stable} \\
&\text{for all } \sigma(t) \in D_T\}.
\end{aligned}
$$

Our starting point in this paper is the following Theorem that was given in [12] for guaranteeing a dwell time.

Theorem 1 (see [12]): Assume that, for given $T > 0$,

$$\exists P_i : \begin{cases} P_i > 0 & \forall i \\ A_i'P_i + P_iA_i < 0 & \forall i \\ e^{A_i'T}P_j e^{A_iT} < P_i & \forall i \neq j. \end{cases} \quad (2)$$

Then, the system is exponentially stable for every $\sigma(\cdot) \in D_T$.
□

The above result deserves a few remarks.

(i) For given Hurwitz matrices A_i, $i = 1, 2, \ldots, M$, there always exist $T > 0$ such that (2) holds. Indeed, as T goes to infinity, the third inequality reduces to $P_i > 0$ and the feasibility of the second is guaranteed by Hurwitz stability of the matrices A_i.

(ii) If the inequalities are always satisfied for $T \to 0$, then, in the limit, it follows that $P_j - P_i \to 0$ so that the condition for quadratic stability is recovered, namely

$$A_i'P + PA_i < 0, \quad \forall i$$

where $P > 0$ is the limit of P_i as T goes to 0.

(iii) The function

$$v(x, t) = x'P_{\sigma(t)}x$$

is a piecewise quadratic Lyapunov function for system (1) for every $\sigma(\cdot) \in D_T$.

(iv) Associated with a given a sequence t_k, $k = 0, 1, \ldots$, it is possible to write the discrete-time switched system

$$\hat{x}(k+1) = F_{\hat{\sigma}(k)}\hat{x}(k)$$

where

$$\hat{\sigma}(k) = \sigma(t_k), \quad \hat{x}(k) = x(t_k)$$
$$F_{\hat{\sigma}(k)} = e^{A_{\sigma(t_k)}(t_{k+1}-t_k)}.$$

Then, this system is stable under arbitrary switching under the piecewise quadratic Lyapunov function

$$w(\hat{x}, k) = \hat{x}'P_{\hat{\sigma}(k)}\hat{x}.$$

(v) Theorem 1 can be easily adapted to comply with possible state jumps in the system state. Indeed, assume that at each commutation instant t_k, the system state is reset according to the rule

$$x(t_k) = S_{\sigma(t_k)}x(t_k^-). \quad (3)$$

Then stability in D_T is guaranteed if there exist positive definite matrices P_i satisfying

$$\exists P_i : \begin{cases} P_i > 0 & \forall i \\ A_i'P_i + P_iA_i < 0 & \forall i \\ e^{A_i'T}S_j'P_jS_j e^{A_iT} < P_i & \forall i \neq j. \end{cases} \quad (4)$$

(vi) If the sufficient condition stated in Theorem 1 is feasible for T, then it holds also for $T + \xi$ for all $\xi \geq 0$.

The algorithm to find an upper bound of the minimum dwell time consists in finding the minimum value of T such that (2) holds. Notice that this computation only involves the solution of a set of LMIs plus a line search over the parameter T. However, the sufficient condition stated in Theorem 1 is not necessary for stability in D_T. This means that a system can be stable in D_T and no positive definite matrices P_i exist satisfying (2). The reason is that the inequalities define a Lyapunov function $v(x) = x'P_{\sigma(t)}x$, which is piecewise quadratic, whereas for stability in D_T, more complex Lyapunov functions are required. This latter observation is characterized by the following result which can be found in [15].

Theorem 2 (see [15]): The system is exponentially stable in D_T if and only if there exist continuous functions $v_i(x)$ such that

$$\begin{cases} v_i(x) > 0 & \forall x \neq 0_n \; \forall i \\ \left.\dfrac{dv_i(x)}{dt}\right|_{\dot{x}=A_ix} < 0 & \forall x \neq 0_n \; \forall i \\ v_j(e^{A_iT}x) < v_i(x) & \forall x \neq 0_n \; \forall i \neq j. \end{cases} \quad (5)$$

□

Let us observe that each $v_i(x)$ in Theorem 2 can be chosen homogeneous due to the fact that the system is linear.

III. Conditions via Kronecker Products

The idea exploited in this paper is to adopt homogeneous polynomial Lyapunov functions, which have the form

$$v(x) = \sum_{\substack{i_1 + \ldots + i_n = 2m \\ i_1 \geq 0, \ldots, i_n \geq 0}} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \quad (6)$$

being $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$ the function variable, $2m$ the degree for a positive integer m, and $c_{i_1,\ldots,i_n} \in \mathbb{R}$ some coefficients.

One way to represent homogeneous polynomial Lyapunov functions is to use Kronecker products. Indeed, for any matrix (vector) J and positive integer i let us define the notation

$$J^{\otimes i} = \begin{cases} J \otimes J^{\otimes i-1} & \text{if } i \geq 1 \\ 1 & \text{if } i = 0. \end{cases} \quad (7)$$

Then, $v(x)$ in (6) can be rewritten as

$$v(x) = \left(x^{\otimes m}\right)' \Pi x^{\otimes m}$$

for a suitable symmetric matrix $\Pi = \Pi' \in \mathbb{R}^{n^m \times n^m}$ which contains the coefficients $c_{i_1,\ldots,i_n}$. Let $\mathcal{A}_{i,\otimes m} \in \mathbb{R}^{n^m \times n^m}$ be the matrix satisfying

$$\frac{dx^{\otimes m}}{dt} = \frac{dx^{\otimes m}}{dx}A_ix = \mathcal{A}_{i,\otimes m}x^{\otimes m} \quad \forall x.$$

It turns out that

$$\mathcal{A}_{i,\otimes m+1} = \mathcal{A}_{i,\otimes m} \oplus A_i$$

where the Kronecker sum of two matrices, say X and Y is defined as

$$X \oplus Y = X \otimes I + I \otimes Y.$$

Then, we have the following result.

Theorem 3: Assume that, for given $T > 0$ and positive integer m,

$$\exists \Pi_i : \begin{cases} \Pi_i > 0 \quad \forall i \\ \mathcal{A}'_{i,\otimes m}\Pi_i + \Pi_i \mathcal{A}_{i,\otimes m} < 0 \quad \forall i \\ e^{\mathcal{A}'_{i,\otimes m}T}\Pi_j e^{\mathcal{A}_{i,\otimes m}T} < \Pi_i \quad \forall i \neq j. \end{cases} \quad (8)$$

Then, (1) is exponentially stable for every $\sigma(\cdot) \in D_T$.
Proof. Suppose that (8) holds, and define

$$v_i(x) = \left(x^{\otimes m}\right)' \Pi_i x^{\otimes m}.$$

Since

$$\left(e^{A_i t}x\right)^{\otimes m} = e^{\mathcal{A}_{i,\otimes m}t}x^{\otimes m} \quad \forall t \geq 0$$

it follows that (5) is satisfied thus implying that the system is exponentially stable for every $\sigma(\cdot) \in D_T$. $\square$

Remark 1: Let us observe that, for $m = 1$, (8) coincides with (2).

Let us indicate with $T_{\otimes m}$ the smallest upper bound of T_{min} guaranteed by Theorem 3, i.e.

$$T_{\otimes m} = \inf\left\{T \geq 0 : \text{(8) holds}\right\}.$$

The following result provides a key property of the condition (8), which allows one to calculate $T_{\otimes m}$ via a bisection search where at each iteration the condition (8) is tested.

Theorem 4: Assume that (8) holds for some $T > 0$ and positive integer m. Then, (8) holds also for $T + \xi$ and m for all $\xi \geq 0$.
Proof. Suppose that (8) holds, and define $v_i(x) = (x^{\otimes m})' \Pi_i x^{\otimes m}$. Consider any $\xi \geq 0$. From the second inequality one has that

$$v_i(x(\xi)) \leq v_i(x(0)) \quad \forall x(0)$$

which implies that

$$e^{\mathcal{A}'_{i,\otimes m}\xi}\Pi_i e^{\mathcal{A}_{i,\otimes m}\xi} \leq \Pi_i.$$

Pre- and post-multiplying the third inequality of (8) by $e^{\mathcal{A}_{i,\otimes m}\xi}$ and $e^{\mathcal{A}'_{i,\otimes m}\xi}$ respectively, one gets that

$$e^{\mathcal{A}'_{i,\otimes m}(T+\xi)}\Pi_j e^{\mathcal{A}_{i,\otimes m}(T+\xi)} < \Pi_i.$$

Therefore, the theorem holds. $\square$

The inequalities (8) are characterized by an important property for a fixed T. Indeed, denote by $\Pi_{i,\otimes r}$ the positive definite matrices satisfying (8) for a certain T and $m = r$. Then, one can set

$$\Pi_{i,\otimes 2^{k+1}m} = \Pi_{i,\otimes 2^k m} \otimes \Pi_{i,\otimes 2^k m}, \quad \forall k.$$

This means that if (8) are feasible for $m = 2^k r$, they are also feasible for $m = 2^{k+1}r$. In conclusion, the sequence $T_{\otimes 2^k r}$, $k = 0, 1, \cdots$ is monotonically non-increasing with respect to k and the limit

$$T_{\otimes *r} = \lim_{k \to \infty} T_{\otimes 2^k r}.$$

exists. Of course, $T_{\otimes *r}$ is an upper bound of the minimum dwell time T_{min}, for each r. As such

$$T_{\otimes *} = \min_{r>0} T_{\otimes *r} \quad (9)$$

is also an upper bound of T_{min}.

Remark 2: The above result can be strengthened by proving the monotonicity with respect to k of the sequence of upper bounds indexed by $m = \zeta^k r$, for $\zeta \geq 2$.

Remark 3: The sufficient condition of Theorem 3 lends itself to be slightly modified so as to cope with the stability analysis of system (1) under the reset condition (3). As a matter of fact, it is enough to replace condition

$$e^{\mathcal{A}'_{i,\otimes m}T}\Pi_j e^{\mathcal{A}_{i,\otimes m}T} < \Pi_i \quad \forall i \neq j$$

with

$$e^{\mathcal{A}'_{i,\otimes m}T}(S_j^{\otimes m})' \Pi_j S_j^{\otimes m} e^{\mathcal{A}_{i,\otimes m}T} < \Pi_i \quad \forall i \neq j.$$

IV. CONDITIONS VIA THE SMR

Any polynomial $h(x)$ of degree $2m$ in $x \in \mathbb{R}^n$ can be written in a more compact and complete way by using the SMR which was introduced in [14] to establish whether a polynomial is sum of squares of polynomials (SOS) via LMIs.
Indeed, let $x^{\{m\}} \in \mathbb{R}^{d(n,m)}$ be a vector containing a base for the homogeneous polynomials of degree m in $x \in \mathbb{R}^n$, where

$$d(n,m) = \frac{(n+m-1)!}{(n-1)!m!}$$

and let us define the set

$$\mathcal{L}_m = \left\{L = L' : \ x^{\{m\}'} L x^{\{m\}} = 0 \ \forall x\right\}$$

whose dimension is given by

$$d_{par}(n,m) = \frac{1}{2}d(n,m)\left(d(n,m)+1\right) - d(n,2m).$$

Let $L_m : \mathbb{R}^{d_{par}(n,m)} \to \mathbb{R}^{d(n,m) \times d(n,m)}$ be any linear parametrization of the set $\mathcal{L}_m$. Then, the SMR of $h(x)$ is given by

$$h(x) = x^{\{m\}'}\left(H + L_m(\alpha)\right) x^{\{m\}}, \quad \forall \alpha \in \mathbb{R}^{d_{par}(n,m)}$$

where $H \in \mathbb{R}^{d(n,m) \times d(n,m)}$ is a suitable constant matrix. See also [16], [17] for details about the SMR, and see [18] where homogeneous polynomial Lyapunov functions and the SMR are exploited for establishing robust stability of uncertain systems with time-varying uncertainties.
Let $\mathcal{A}_{i,\{m\}} \in \mathbb{R}^{d(n,m) \times d(n,m)}$ be the matrix satisfying

$$\frac{dx^{\{m\}}}{dx}A_i x = \mathcal{A}_{i,\{m\}}x^{\{m\}} \quad \forall x$$

which can be computed with the formula given in [19], [16]. Then we have the following theorem.

Theorem 5: Assume that, for given $T > 0$ and positive integer m,

$$\exists \Pi_i, \alpha_i, \alpha_{i,j} :$$

$$\begin{cases} \Pi_i > 0 \quad \forall i \\ \mathcal{A}'_{i,\{m\}}\Pi_i + \Pi_i \mathcal{A}_{i,\{m\}} + L_m(\alpha_i) < 0 \quad \forall i \\ e^{\mathcal{A}'_{i,\{m\}}T}\Pi_j e^{\mathcal{A}_{i,\{m\}}T} < \Pi_i + L_m(\alpha_{i,j}) \quad \forall i \neq j. \end{cases}$$
$$(10)$$

Then, (1) is exponentially stable for every $\sigma(\cdot) \in D_T$.
Proof. Suppose that (10) holds, and define

$$v_i(x) = x^{\{m\}'}\Pi_i x^{\{m\}}.$$

We have that

$$x^{\{m\}'}L_m(\alpha)x^{\{m\}} = 0 \quad \forall \alpha.$$

Moreover, it can be shown that

$$\left(e^{A_i t}x\right)^{\{m\}} = e^{\mathcal{A}_{i,\{m\}}t}x^{\{m\}} \quad \forall t \geq 0.$$

Hence, it follows that (5) holds, which implies that the system is exponentially stable for every $\sigma(\cdot) \in D_T$. $\square$

Remark 4: Observe that, for $m = 1$, (10) coincides with (8) and (2).

To see that the condition (10) is not more conservative than the condition (8), the following result is noted.

Theorem 6: Assume that, for given $T > 0$ and positive integer m, (8) holds. Then, (10) holds for the same T and m.
Proof. Let Π_i be such that (8) holds for T and m. We now show that there exist $\tilde{\Pi}_i, \alpha_i, \alpha_{i,j}$ such that (10) holds for the same T and m.
Define the homogeneous polynomial Lyapunov functions of degree $2m$ $v_i(x) = (x^{\otimes m})'\Pi_i x^{\otimes m}$. We have that (5) holds with these Lyapunov functions. Now, let us define

$$\tilde{\Pi}_i = K_0'\Pi_i K_0 \tag{11}$$

where K_0 is the matrix satisfying

$$x^{\otimes m} = K_0 x^{\{m\}} \quad \forall x.$$

We have that

$$\begin{aligned} v_i(x) &= x^{\{m\}'}\tilde{\Pi}_i x^{\{m\}} \\ \tilde{\Pi}_i &> 0. \end{aligned}$$

Then, let us define

$$\Phi_i = K_0'\left(\mathcal{A}'_{i,\otimes m}\Pi_i + \Pi_i \mathcal{A}_{i,\otimes m}\right)K_0.$$

We have that

$$\begin{aligned} \left.\frac{dv_i(x)}{dt}\right|_{\dot{x}=A_i x} &= x^{\{m\}'}\Phi_i x^{\{m\}} \\ \Phi_i &< 0 \end{aligned}$$

and

$$\exists \alpha_i : \ \mathcal{A}'_{i,\{m\}}\tilde{\Pi}_i + \tilde{\Pi}_i \mathcal{A}_{i,\{m\}} + L_m(\alpha_i) = \Phi_i$$

because $\mathcal{A}'_{i,\{m\}}\tilde{\Pi}_i + \tilde{\Pi}_i \mathcal{A}_{i,\{m\}}$ and Φ_i are SMR matrices of the same homogeneous polynomial. Lastly, let us define

$$\Phi_{i,j} = K_0'\left(e^{\mathcal{A}'_{i,\otimes m}T}\Pi_j e^{\mathcal{A}_{i,\otimes m}T} - \Pi_i\right)K_0.$$

We have that

$$\begin{aligned} v_j(e^{A_i T}x) - v_i(x) &= x^{\{m\}'}\Phi_{i,j}x^{\{m\}} \\ \Phi_{i,j} &< 0 \end{aligned}$$

and

$$\exists \alpha_{i,j} : \ e^{\mathcal{A}'_{i,\{m\}}T}\tilde{\Pi}_j e^{\mathcal{A}_{i,\{m\}}T} - \tilde{\Pi}_i - L_m(\alpha_{i,j}) = \Phi_{i,j}$$

because $e^{\mathcal{A}'_{i,\{m\}}T}\tilde{\Pi}_j e^{\mathcal{A}_{i,\{m\}}T} - \tilde{\Pi}_i$ and $\Phi_{i,j}$ are SMR matrices of the same homogeneous polynomial. Therefore, the theorem holds. $\square$

Let us indicate with $T_{\{m\}}$ the smallest upper bound of T_{min} guaranteed by Theorem 5, i.e.

$$T_{\{m\}} = \inf\{T \geq 0 : (10) \text{ holds}\}. \tag{12}$$

The following result is analogous to Theorem 4 and allows one to calculate $T_{\{m\}}$ via a bisection search where at each iteration the condition (10) is tested.

Theorem 7: Assume that (10) holds for some $T > 0$ and positive integer m. Then, (10) holds also for $T + \xi$ and m for all $\xi \geq 0$.
Proof. Suppose that (10) holds, and define $v_i(x) = x^{\{m\}'}\Pi_i x^{\{m\}}$. Consider any $\xi \geq 0$. From the second inequality one has that

$$v_i(x(\xi)) \leq v_i(x(0)) \quad \forall x(0)$$

which implies that

$$e^{\mathcal{A}'_{i,\{m\}}\xi}\Pi_i e^{\mathcal{A}_{i,\{m\}}\xi} \leq \Pi_i.$$

Pre- and post-multiplying the third inequality of (10) by $e^{\mathcal{A}_{i,\{m\}}\xi}$ and $e^{\mathcal{A}'_{i,\{m\}}\xi}$ respectively, one gets that

$$e^{\mathcal{A}'_{i,\{m\}}(T+\xi)}\Pi_j e^{\mathcal{A}_{i,\{m\}}(T+\xi)}$$
$$< \Pi_i + e^{\mathcal{A}'_{i,\{m\}}\xi}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,\{m\}}\xi}.$$

Lastly, let us observe that

$$\begin{aligned} & x^{\{m\}'}e^{\mathcal{A}'_{i,\{m\}}\xi}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,\{m\}}\xi}x^{\{m\}} \\ &= \left(e^{A_i \xi}x\right)^{\{m\}'}L_m(\alpha_{i,j})\left(e^{A_i \xi}x\right)^{\{m\}} \\ &= 0 \end{aligned}$$

which implies that

$$e^{\mathcal{A}'_{i,\{m\}}\xi}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,\{m\}}\xi} \in \mathcal{L}_m.$$

Hence,

$$\exists \tilde{\alpha}_{i,j} : \ L_m(\tilde{\alpha}_{i,j}) = e^{\mathcal{A}'_{i,\{m\}}\xi}L_m(\alpha_{i,j})e^{\mathcal{A}_{i,\{m\}}\xi}$$

and, therefore, the theorem holds. $\square$

Remark 5: For the stability analysis of system (1) under the reset condition (3), it is enough to replace, in the statement of Theorem 5, the condition

$$e^{\mathcal{A}'_{i,\{m\}}T}\Pi_j e^{\mathcal{A}_{i,\{m\}}T} < \Pi_i + L_m(\alpha_{i,j}) \quad \forall i \neq j$$

with

$$e^{\mathcal{A}'_{i,\{m\}}T}\tilde{S}'_j\Pi_j\tilde{S}_j e^{\mathcal{A}_{i,\{m\}}T} < \Pi_i + L_m(\alpha_{i,j}) \quad \forall i \neq j$$

where $\tilde{S}_j$ is the matrix satisfying

$$(S_j x)^{\{m\}} = \tilde{S}_j x^{\{m\}}.$$

Similarly to $T_{\otimes m}$, the upper bound $T_{\{m\}}$ in (12) is characterized by a monotonicity property. Indeed, it can be shown that

$$T_{\{\zeta m\}} \leq T_{\{m\}} \quad \forall \zeta \geq 1.$$

Analogously to the limit $T_{\otimes *}$ defined for $T_{\otimes m}$, we define the limit $T_{\{*\}}$ for $T_{\{m\}}$, which is also given by

$$T_{\{*\}} = \lim_{s \to \infty} T(s) \tag{13}$$

where

$$T(s) = \min\left\{T_{\{1\}}, T_{\{2\}}, \ldots, T_{\{s\}}\right\}. \tag{14}$$

Lastly, it is worth discussing the conservatism of the proposed approach. Since the upper bound $T_{\{m\}}$ in (12) is obtained by exploiting the SMR for establishing positivity of homogeneous polynomials, the conservatism of $T_{\{m\}}$ is related to the possibility of expressing positive homogeneous polynomials as SOS, see for instance [20]–[22], [16].

More specifically, for a fixed value of m, $T_{\{m\}}$ is the best upper bound obtainable with a homogeneous polynomial Lyapunov function of degree $2m$ provided that the homogeneous polynomials in (5) are positive if and only if they are SOS.

In particular, it is possible to show that $T_{\{m\}}$ coincides with such a best upper bound in the case of second order systems. Moreover, since any positive homogeneous polynomial can be expressed as ratio of two SOS homogeneous polynomials, it is conjectured that $T_{\{*\}}$ coincides with T_{min}.

V. Examples

In this section, some examples are presented to illustrate the usefulness of our computational method. The obtained bounds are compared with the one provided in the pioneering paper [10], i.e.

$$T_{HM} = \max_i \inf_{\alpha > 0, \beta > 0}\left\{\frac{\alpha}{\beta} : \|e^{A_i t}\| \leq e^{\alpha - \beta t}, \quad \forall t > 0\right\}.$$

In addition, we consider

$$T_{LB} = \min\left\{T \geq 0 \text{ s.t. } \max_q \left|\lambda_q\left(\prod_{p=1}^M e^{B_p \tau}\right)\right| < 1,\right.$$

$$\left.\forall \tau > T\right\}$$

where λ_q denotes a generic eigenvalue and $\{B_1, B_2, \cdots, B_M\}$ are matrices corresponding to any

permutation among those of the set $\{A_1, A_2, \cdots, A_M\}$. Of course

$$T_{LB} \leq T_{min},$$

i.e. T_{LB} is a lower bound of the minimum dwell time. Despite all our attempts, we were not able to work out a third order example with $T_{\{*\}} > T_{min}$.

A. Example 1

Consider

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -9 & -1 \end{bmatrix}.$$

We get the following upper bounds:

m	$T_{\otimes m}$	$T_{\{m\}}$
1	0.6222	0.6222
2	0.6216	0.6079
3	0.6207	0.6073
4	0.6197	0.6073

In this example $T_{HM} = 2.2321$ and $T_{LB} = 0$. It turns out that the true minimum dwell time T_{min} coincides with the $T_{\{*\}}$. Indeed, this is confirmed by finding a switching sequence with $t_{k+1} - t_k = 0.6073 - \epsilon$ yielding a non asymptotically stable system. For instance, it can be easily verified that taking the periodic signal of period $t_1 + t_2$

$$\sigma(t) = \begin{cases} 1, & t \in [0, t_1) \\ 2, & t \in [t_1, t_1 + t_2) \end{cases}$$

with $t_1 = 0.8800$ and $t_2 = 0.6073$, the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to one). Therefore, $T_{min} = T_{\{*\}} = T_{\{4\}} = 0.6073$.

B. Example 2

Consider

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ -3 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}.$$

We get the following upper bounds:

m	$T_{\otimes m}$	$T_{\{m\}}$
1	0.6437	0.6437
2	0.6281	0.3629
3	0.4607	0.3510
4	0.3747	0.3510

In this example $T_{HM} = 2.9816$ and $T_{LB} = 0$. Analogously to Example 1, it can be easily verified that taking the periodic signal of period $t_1 + t_3$

$$\sigma(t) = \begin{cases} 1, & t \in [0, t_1) \\ 3, & t \in [t_1, t_1 + t_3) \end{cases}$$

with $t_1 = 0.3510$ and $t_3 = 0.4700$, the associated periodic system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is not asymptotically stable (the maximum modulus of the characteristic multipliers is equal to one). Therefore, $T_{min} = T_{\{*\}} = T_{\{4\}} = 0.3510$.

C. Example 3

Consider

$$A_1 = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ -2 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 6 \\ -2 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix}.$$

We get the following upper bounds:

m	$T_{\otimes m}$	$T_{\{m\}}$
1	1.9135	1.9135
2	1.9108	1.9065
3	1.9087	1.9023
4	1.9070	1.8997

In this example $T_{HM} = 15.7089$ and $T_{LB} = 1.8788$. Probably $T_{\{4\}}$ is not tight this time, nevertheless it is expected that one can reach T_{min} for values of m larger than 4. Observe that, clearly, $T_{min} \in [T_{LB}, T_{\{4\}}] = [1.8788, 1.8997]$.

D. Example 4

Consider

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1 & 0 & 6 \\ -1 & -1 & -5 \\ 0 & 1 & -1 \end{bmatrix}.$$

We get the following upper bounds:

m	$T_{\otimes m}$	$T_{\{m\}}$
1	0.3930	0.3930
2	0.2616	0.0549
3	0.0027	0.0000
4	0.0000	0.0000

Hence $T_{\{*\}} = 0$. In this example $T_{HM} = 2.2395$. Of course $T_{min} = T_{LB} = 0$.

VI. CONCLUSIONS

This paper has addressed stability of switched linear systems under a dwell time constraint. LMI conditions have been proposed to compute upper bounds of the minimum dwell time, based on the use of Kronecker products and the SMR of homogeneous polynomials. The examples show that the exact minimum dwell time can be arbitrarily approached by increasing the degree of the homogeneous polynomial. This is in accordance with our conjecture that the proposed LMI conditions are not conservative for dwell time investigations of switched linear systems. Further work will be devoted to prove this conjecture and to derive upper bounds of the degree of the homogeneous polynomial Lyapunov function required to achieve non-conservatism.

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