

Strict Positive Realness of Descriptor Systems in State Space¹

Ezra Zeheb, Robert Shorten and S. Shravan K. Sajja

Abstract— In this paper we give necessary and sufficient spectral conditions for various notions of strict positive realness for single input single output, impulse free Descriptor Systems. These conditions only require calculation of eigenvalues of a single matrix. A characterization of a KYP-like lemma for descriptor systems is also derived, and its implications for the stability of a class of switched descriptor systems are briefly discussed.

Index Terms—Descriptor systems, Kalman-Yacubovich-Popov lemma, strict positive realness, extended strict positive realness.

I. INTRODUCTION

In this paper we consider the passivity properties of single-input single-output (SISO) linear time-invariant (LTI) descriptor systems of the form:

$$\sum: \begin{cases} E\dot{x} = Ax + bu \\ y = c^T x + du \end{cases} \quad (1)$$

where $E \in \mathbb{R}^{n \times n}$ is a possibly singular matrix. Such descriptor systems appear frequently in engineering systems; for example, in the description of interconnected large scale systems, in economic systems (e.g. the fundamental dynamic Leontief model), biological systems, network analysis [13], and in a variety of control engineering problems. Descriptor systems are particularly important in the simulation and design of (Very Large Scale Integrated) VLSI circuits. Here one is often interested in obtaining reduced order models of an original large scale model, such that certain properties of the original system are preserved. One such property is passivity. In control system design, descriptor systems are useful in the description of switched systems in which states are subject to reset. In such problems, one is interested in determining conditions on the switched systems such that stability can be demonstrated. Here, also passivity is a tool that can be used with some success.

Ezra Zeheb is with the Department of Electrical Engineering, Technion - Israel Institute of Technology, Haifa, Israel and Jerusalem College of Engineering, Jerusalem, Israel (email: zeheb@ee.technion.ac.il).

Robert Shorten and S. Shravan K. Sajja are with The Hamilton Institute, NUI Maynooth, Ireland (email: Robert.Shorten@nuim.ie; surya.sajja.2009@nuim.ie).

Our objective in this paper is to obtain simple conditions to determine whether an LTI descriptor system is passive or not. Normally, passivity of descriptor system is determined by examining the properties of a transfer function over an infinite set of frequencies. Our main contribution in this paper is to show that passivity can be reduced to evaluation of the eigenvalues of an n -dimensional matrix.

Our results are important for a number of reasons.

1. We obtain very compact conditions that characterize passivity of a descriptor system. These are new and have not appeared in the literature. They are directly obtained from the state space representation of the system. These methods do not involve evaluating a transfer function at all frequencies but only involves the calculation of eigenvalues of an $n \times n$ matrix. Importantly, they are also valid for both strictly proper and proper transfer functions. An important application of these results is in determining the passivity radius of Descriptor systems, and for model order reductions. Although these applications are not given in the paper, their application is immediate; see [19].
2. Our conditions lead directly to a Kalman-Yacubovich-Popov (KYP)-like lemma for Descriptor systems. While other KYP-like lemmas have been proposed earlier for Descriptor systems, these are usually given under certain restrictive assumptions, such as extended positive realness. Our conditions on the other hand are relatively free of these assumptions and readily extend to the multiple-input multiple-output (MIMO) case.

We note also that we restrict our attention in this paper to SISO systems. This is deliberate. While conditions for MIMO transfer functions can be readily obtained, here we exploit specific properties of scalar transfer functions. In the MIMO case, one needs to define Hamiltonian matrices that are of order $2n \times 2n$, and their manipulation is considerably different to that in the SISO case. Also, in SISO case, the obtained conditions have a clear interpretation in terms of Lyapunov stability and this interpretation can be used to derive

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conditions for the stability of switched descriptor systems. The MIMO case will be reported in future publications.

We also note that the techniques presented in this paper are new, novel and have not appeared in the context of Descriptor systems elsewhere. All the derivations given, exploit properties of various reciprocal transfer functions, and a full rank decomposition of the matrix E . While one of the authors (Shorten) has used the relationship between the KYP lemma, matrix inverses and reciprocal transfer functions, in a recent paper on regular systems [14], the derivations presented are somewhat different to [14], and are completely new in the context of Descriptor systems. Furthermore, the use of full rank decompositions, and inverting the system matrix, is also a very different approach to the study of Descriptor systems, in which the Weierstrass form is the norm.

Our starting points in this paper are the necessary and sufficient conditions for strict positive realness (SPR) of a stable rational transfer function $H(s)$, given by following two conditions:

- A. $H(s)$ is real for real values of s .
- B. $\text{Re}[H(s)] > 0$ for $\text{Re}[s] \geq 0$

There exist numerous methods to test these conditions; however, many situations lead to a state space characterization of a system, rather than a transfer function characterization of the system.

$$\sum: \begin{cases} \dot{x} = Ax + bu \\ y = c^T x + du \end{cases} \quad (1)$$

where u and y are the scalar input and output of the system, respectively, $x \in \mathbb{R}^{n \times 1}$ is the state variable vector, and $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{n \times 1}$, $d \in \mathbb{R}$. Necessary and sufficient conditions for the system Σ to be SPR have been discussed in [9]-[12].

In this paper we discuss positive realness of descriptor systems namely, where the state space characterization of the system cannot be described in the form of (1), but rather in the most general characterization of a linear time-invariant continuous-time single-input single output system:

$$\sum: \begin{cases} E\dot{x} = Ax + bu \\ y = c^T x + du \end{cases} \quad (2)$$

In such situations, it is of interest to obtain compact results to ascertain the essential dynamic properties of the system directly. Passivity and positive realness are equivalent for a linear time invariant system and the KYP lemma characterizes positive realness in terms of linear matrix inequalities. Our contribution in this paper is to establish

similar facts for Descriptor systems, and to derive simple necessary and sufficient conditions for strict positive realness of a system characterized as in (2). These conditions may be viewed as natural questions that follow the work presented in [9-12, 14, 16]. The conditions that we obtain involve only eigenvalue computations of matrices derived from the given (A), (b), (c), d and (E). The need to check positivity of an expression which depends for all frequencies, as required in condition B of the basic SPR conditions above, is thus avoided. This is a significant advantage. More importantly, by relating these conditions to similar conditions for standard systems, a new KYP-like lemma for descriptor systems is obtained. This latter point is important as it gives important insights into the study of switched descriptor systems.

The structure of the paper is as follows. The basic result providing the necessary and sufficient conditions for SPR of a system characterized as in (2), is derived and proved in Section II. Here, some assumptions are made with regard to the class of systems which is considered. In Section III we derive a sufficient condition for extended strict positive realness (ESPR), which requires, in addition to SPR, that the transfer function of the system $H(s)$ be positive as $s \rightarrow \infty$ as well. This sufficient condition is also necessary except for degenerate systems where a degree reduction occurs. The way to derive the necessary and sufficient condition in these degenerate cases is also pointed out in Section III, but is too cumbersome to be formulated explicitly. In Section IV the basic result is derived in a different way, resulting in a slightly modified form, and removing two of the assumptions made in Section II, namely that $d \neq 0$ and that the matrix $M \triangleq A - \frac{1}{d}bc^T$ is invertible. The cost of this derivation is the assumption that the Descriptor system is impulse free. In Section V we derive a new KYP-like lemma for descriptor systems. This is then in turn used to give new insights into the stability of switched and nonlinear descriptor systems. Section VI includes numerical examples, and in Section VII we conclude the paper.

II. THE CLASS OF SYSTEMS AND THE BASIC RESULT

Let \mathbb{R} denote the real numbers and $\mathbb{R}^{n \times n}$ denote the real matrices.

Let

$$\sum: \begin{cases} E\dot{x} = Ax + bu \\ y = c^T x + du \end{cases}$$

be a single input-single output (SISO) descriptor system, i.e., $E \in \mathbb{R}^{n \times n}$ is a singular matrix and in addition we have $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{n \times 1}$, $d \in \mathbb{R}$. Assume that A is invertible and that $\exists j\omega_0$ ($\omega_0 \in \mathbb{R}$) such that

$\det[j\omega_0 E - A] \neq 0$, i.e., $(j\omega E - A)^{-1}$ is well defined. Assume also that $d \neq 0$ and that $\left(A - \frac{1}{d}bc^T\right)$ is invertible. Also assume that (2) is a impulse free descriptor system, i.e. $\text{rank}(E) = \text{deg}(\det[sE - A])$.

Theorem 1:

Denote

$$M \triangleq A - \frac{1}{d}bc^T. \quad (3)$$

The system (2) is strictly positive real (SPR) if, and only if,

1. $c^T(-A)^{-1}b + d > 0$.
2. All the eigenvalues of the matrix (EA^{-1}) should be in the open left half of the complex plane (OLHP), except for an eigenvalue of multiplicity at least one at the origin.
3. There are no real negative eigenvalues of the matrix $(EA^{-1}EM^{-1})$.

Proof of Theorem 1

The transfer function of the system in (2) is

$$H(s) = c^T(sE - A)^{-1}b + d. \quad (4)$$

This state space formulation of $H(s)$ ensures that $H(s)$ is a rational function which is real for real values of s , thus condition A is satisfied. Now equivalent conditions to condition B are:

(B1) All poles of the rational function $H(s)$ are in the open left half of the complex plane.

(B2) $\text{Re}[H(j\omega)] > 0 \quad \forall \omega \in \mathbb{R}$.

(5)

Following in the spirit of [9, 16]. Let us consider Eq. (4) and write down, a useful expression for $\text{Re}[H(j\omega)]$.

$$\begin{aligned} \text{Re}[H(j\omega)] &= d + \frac{1}{2} \left[c^T (j\omega E - A)^{-1} b + c^T (-j\omega E - A)^{-1} b \right] \\ &= d + \frac{1}{2} \left\{ c^T \left[(j\omega E - A)^{-1} + (-j\omega E - A)^{-1} \right] b \right\}. \quad (6) \end{aligned}$$

Use the matrix identity

$$X^{-1} + Y^{-1} = X^{-1}(X + Y)Y^{-1}$$

To obtain:

$$\frac{1}{2} \left[\underbrace{(j\omega E - A)^{-1}}_X + \underbrace{(-j\omega E - A)^{-1}}_Y \right]$$

$$\begin{aligned} &= (j\omega E - A)^{-1}(-2A)(-j\omega E - A)^{-1} \cdot \frac{1}{2} \\ &= -(j\omega E - A)^{-1}(A^{-1})^{-1}(-j\omega E - A)^{-1} \end{aligned} \quad (7)$$

Use the matrix identity

$$(X \cdot Y)^{-1} = Y^{-1}X^{-1}$$

To obtain

$$\begin{aligned} \frac{1}{2} \left[(j\omega E - A)^{-1} + (-j\omega E - A)^{-1} \right] &= - \left[A^{-1}(j\omega E - A) \right]^{-1} (-j\omega E - A)^{-1} \\ &= \left[(j\omega E + A)A^{-1}(j\omega E - A) \right]^{-1} \\ &= \left[(j\omega EA^{-1} + I)(j\omega E - A) \right]^{-1} \\ &= \left[-\omega^2 EA^{-1}E - j\omega E + j\omega E - A \right]^{-1} = - \left[\omega^2 EA^{-1}E + A \right]^{-1}. \quad (8) \end{aligned}$$

So that:

$$\begin{aligned} \text{Re}[H(j\omega)] &= d - c^T (\omega^2 EA^{-1}E + A)^{-1} b \\ &= d \left[1 - \frac{1}{d} c^T (\omega^2 EA^{-1}E + A)^{-1} b \right]. \quad (9) \end{aligned}$$

Denote the following two vectors U and V in \mathbb{R}^n :

$$U = (\omega^2 EA^{-1}E + A)^{-1} b; \quad V^T = -\frac{1}{d} c^T \quad (10)$$

and use the identity, which was also used in [16],

$$1 + V^T U = \det[I_n + UV^T]. \quad (11)$$

We obtain:

$$\text{Re}[H(j\omega)] = d \cdot \det \left[I_n - \frac{1}{d} (\omega^2 EA^{-1}E + A)^{-1} bc^T \right]$$

or

$$\text{Re}[H(j\omega)] = d \frac{\det \left[\omega^2 EA^{-1}E + A - \frac{1}{d} bc^T \right]}{\det \left[\omega^2 EA^{-1}E + A \right]}. \quad (12)$$

Since $\det(A^{-1}) \neq 0$,

$$\text{Re}[H(j\omega)] = d \frac{\det(\omega^2 EA^{-1}E + M) \cdot \det(A^{-1})}{\det(\omega^2 EA^{-1}E + A) \cdot \det(A^{-1})}$$

or

$$\text{Re}[H(j\omega)] = d \frac{\det \left[\omega^2 (EA^{-1})^2 + MA^{-1} \right]}{\det \left[\omega^2 (EA^{-1})^2 + I \right]}. \quad (13)$$

Necessity

1. If

$$c^T(-A)^{-1}b + d \leq 0 \quad (14)$$

In contradiction to condition 1 of the theorem, then

$$\text{Re}[H(j0)] \leq 0 \quad (15)$$

So that (5) is not satisfied. Thus, condition 1 of the theorem is necessary.

2. If the matrix (EA^{-1}) has eigenvalues in the closed right half of the complex plane in contradiction to condition 2, then $\exists s_0$ with $\text{Re } s_0 \geq 0$ such that

$$\det(s_0 I - EA^{-1}) = 0 \quad (16)$$

Therefore, for $s_0 \neq 0$ there exists $s_1 = 1/s_0$ with $\text{Re } s_1 \geq 0$ such that

$$\det\left(\frac{1}{s_1} I - EA^{-1}\right) = 0 \quad (17)$$

and

$$(-1)^n s_1^n \det\left(\frac{1}{s_1} I - EA^{-1}\right) = 0. \quad (18)$$

Inserting $(-1)^n s_1^n$ into the determinant, we obtain

$$\det(s_1 EA^{-1} - I) = 0 \quad (19)$$

and since $\det(A) \neq 0$, we have

$$\det(s_1 EA^{-1} - I) \cdot \det(A) = \det(s_1 E - A) = 0. \quad (20)$$

Recalling Eqn. (4) and the fact that $\text{Re } s_1 \geq 0$, Eqn. (20) violates condition (B1). Thus, condition 2 of the theorem is necessary.

REMARK 1: Note that $s = 0$ is an exception. For $s = 0$ we have, recalling that (E) is a singular matrix,

$$\det(sI - EA^{-1}) = 0 \quad (21)$$

so that the matrix (EA^{-1}) does have an eigenvalue of multiplicity at least one at $s = 0$.

3. Consider the denominator of Eqn. (13)

$$\det\left[\omega^2 (EA^{-1})^2 + I\right] = \left|\det(I + j\omega EA^{-1})\right|^2 \quad (22)$$

Thus, the denominator is positive for all $\omega \in \mathbb{R}$, unless

$$\det(I + j\omega EA^{-1}) = 0. \quad (23)$$

However, for $\omega \neq 0$,

$$\det(I + j\omega EA^{-1}) = (-j\omega)^n \cdot \det\left(j\frac{1}{\omega} I - EA^{-1}\right) \quad (24)$$

Thus, the denominator becomes zero only if the matrix (EA^{-1}) has a purely imaginary eigenvalue. However, by condition 2 which has been proven necessary, the matrix (EA^{-1}) does not have purely imaginary eigenvalues, except at the origin. Thus, the denominator of Eqn. (13) is positive $\forall \omega \in \mathbb{R}$, and sign $\text{Re}[H(j\omega)]$ is determined only by the

sign of the numerator of Eqn. (13). Turn now to the numerator of Eqn. (13). Since $\det(AM^{-1}) \neq 0$, we have

$$d \cdot \det\left[\omega^2 (EA^{-1})^2 + MA^{-1}\right] = \frac{d}{\det(AM^{-1})} \cdot \det\left[\omega^2 (EA^{-1}EM^{-1}) + I\right], \quad (25)$$

Which, for $\omega \neq 0$, is equal to

$$\frac{(-1)^n \cdot d \cdot \omega^{2n}}{\det(AM^{-1})} \det\left[-\frac{1}{\omega^2} I - EA^{-1}EM^{-1}\right]. \quad (26)$$

Now if the matrix $(EA^{-1}EM^{-1})$ has a real negative eigenvalue, in contradiction to condition 3 of the theorem, then there exists $\omega \in \mathbb{R}$ such that the determinant in expression (26) becomes zero, and thus condition (B2) for SPR-ness is violated. Hence, the necessity of condition 3 of the theorem.

REMARK 2: Similar to Remark 1, evidently the matrix $(EA^{-1}EM^{-1})$ does have an eigenvalue of multiplicity at least one at the origin.

Sufficiency

Assume that conditions 1-3 of the theorem are satisfied. We will prove that conditions (B1) and (B2) for SPR-ness must also be satisfied. The poles of $H(s)$ are the zeros of $\det(sE - A)$ (see Eqn. (4)). By condition 2 of the theorem we know that all the zeros of $\det(sI - EA^{-1})$ are in the open left half complex plane, except the zero at $s = 0$. The same considerations as in Eqns. (16) to (20) ensure that all the zeros of $\det(sE - A)$ are also in the open left half of the complex plane. Hence, condition (B1) for SPR-ness is satisfied.

Now, recalling Eqn. (4), condition 1 of the theorem ensures that

$$\text{Re}[H(0)] > 0.$$

Also, the denominator of $\text{Re}[H(j\omega)]$ as expressed in Eqn. (13) is positive $\forall \omega \in \mathbb{R}$, as proved in the necessity part of the proof, and the numerator of $\text{Re}[H(j\omega)]$ as expressed in Eqn. (26) does not change its sign $\forall \omega \in \mathbb{R}$ by condition 3 of the theorem. Thus, condition (B2) for SPR-ness is satisfied, which completes the proof of the theorem.

REMARK 3: Condition 2 of theorem 1 can be replaced by a simpler condition, as derived in the following corollary 1

Corollary 1

The eigenvalues of (EA^{-1}) are in the open left half of the complex plane if, and only if,

$$E\dot{x} = Ax \quad (27)$$

is stable, i.e. the zeros of

$$\det(sE - A) \quad (28)$$

Are in the open left half of the complex plane.

Proof of Corollary 1

The zeros of $\det(sE - A)$ are the same zeros as those of

$$(-1)^n \cdot \det(sE - A) \cdot \det(A^{-1}) = \det(I - sEA^{-1}) \quad (29)$$

For $s \neq 0$ we have

$$\det(I - sEA^{-1}) = s^n \cdot \det\left(\frac{1}{s}I - EA^{-1}\right) \quad (30)$$

Thus, the zeros of $\det\left(\frac{1}{s}I - EA^{-1}\right)$ are in the open left half complex plane if, and only if, the zeros of $\det(sE - A)$ are in the open left half complex plane. However, the zeros of $\det(sI - EA^{-1})$ are in the open left half complex plane if, and only if, the zeros of $\det\left(\frac{1}{s}I - EA^{-1}\right)$ are in the open left half complex plane

III. EXTENDED SPR

We now turn our attention to the case of systems that are Extended Strict Positive Real. Extended strict positive realness requires that in addition to the conditions for SPR, also that the transfer function at infinity be positive.

Theorem 2

A strict positive real function is also extended strict positive real if

$$c^T [\text{ad } E]b - d \cdot \text{trace}[A(\text{ad } E)] \neq 0 \quad (31)$$

where $\text{ad} \triangleq$ adjugate.

To prove Theorem 2 we need the following lemmas:

Lemma 1

Consider the expression $\det(sE - A)$. The coefficients of s^n

and s^{n-1} in

$\det(sE - A)$ are

$$\det(E) \quad (32)$$

and

$$-\text{trace}[A(\text{ad } E)] \quad (33)$$

respectively.

Proof of Lemma 1

Let

$$\det(sE - A) \triangleq \alpha_n s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_1 s + \alpha_0 \quad (34)$$

Then,

$$\begin{aligned} \frac{1}{s^n} \det(sE - A) &= \alpha_n + \alpha_{n-1} \frac{1}{s} + \dots + \alpha_1 \frac{1}{s^{n-1}} + \alpha_0 \frac{1}{s^n} \\ &= \det\left(E - \frac{1}{s}A\right) \end{aligned} \quad (35)$$

Denote $p \triangleq \frac{1}{s}$, then

$$\frac{1}{s^n} \det(sE - A) = \det(E - pA) = \alpha_n + \alpha_{n-1}p + \dots + \alpha_0 p^n \triangleq f(p) \quad (36)$$

The Taylor expansion of $f(p) = \det(E - pA)$ around $p = 0$, is

$$f(p) = \det(E - pA) = f(0) + \frac{df}{dp}(0) \cdot p + \frac{1}{2!} \frac{d^2 f}{dp^2}(0) \cdot p^2 + \dots$$

(37)

Comparing (37) and (36) we have

$$\alpha_n = f(0) = \det(E) \quad (38)$$

$$\begin{aligned} \alpha_{n-1} &= \frac{df}{dp}(0) = \frac{d}{dp} [\det(E - pA)] \Big|_{p=0} \\ &= -\text{trace}[A(\text{ad } E)] \end{aligned} \quad (39)$$

Lemma 2

Consider the expression $c^T [\text{ad}(sE - A)]b$, the coefficient of s^{n-1} in this expression is $c^T (\text{ad } E)b$

Proof of Lemma 2

For $s \rightarrow \infty$

$$c^T [\text{ad}(sE - A)]b = c^T [\text{ad}(sE)]b$$

and since $\text{ad}(sE)$ is an $(n-1) \times (n-1)$ matrix, we have for $s \rightarrow \infty$

$$c^T [\text{ad}(sE - A)]b = s^{n-1} \cdot c^T [\text{ad}(E)]b$$

Proof of Theorem 2

By continuity, if the function is SPR, then at infinity

$$\text{Re}[H(s)] = H(s) \geq 0 \quad (40)$$

Hence, for extended strict positive realness, $\text{Re}[H(s)] = 0$ at infinity should be prevented. In other words, $H(s)$ should not be a strictly proper function.

$$H(s) = c^T (sE - A)^{-1} b + d = \frac{c^T \text{ad}(sE - A)b + d \cdot \det(sE - A)}{\det(sE - A)} \quad (41)$$

by lemma 1, and since (E) is singular in our case, the degree of the denominator is at most $(n-1)$, and the coefficient of s^{n-1} in the denominator is

$$-\text{trace}[A(\text{ad } E)] \quad (42)$$

By Lemma 1 and Lemma 2, the degree of the numerator is at most $(n-1)$ and the coefficient of s^{n-1} in the numerator is

$$c^T [\text{ad}(E)]b - d \cdot \text{trace}[A[\text{ad}(E)]] \quad (43)$$

In order to prevent $H(s)$ from being a strictly proper function, it is sufficient that

$$c^T [\text{ad}(E)]b - d \cdot \text{trace}[A[\text{ad}(E)]] \neq 0$$

REMARK 4: Note that Theorem 2 is also a necessary condition for extended SPR in all cases which are not degenerate, i.e, when not both

$$c^T [\text{ad}(E)]b \quad (44)$$

and

$$\text{trace}[A[\text{ad}(E)]] \quad (45)$$

become zero simultaneously. Only in the above case, where both (44) and (45) are zero, there is a degree reduction in both the numerator and denominator of (41). In this case the condition (31) is replaced by a second order condition by similar reasoning as in the proof of Theorem 2, and so on with the cases requiring a third order condition, fourth order condition, etc.

IV. CONSISTENCY WITH THE KALMAN-YACUBOVICH-POPOV LEMMA

In the previous section it was assumed that the constant term

d was non zero, and that the matrix $\left(A - \frac{bc^T}{d} \right)$ is invertible.

In fact, both of these assumptions are non necessary and the principal result holds in situations where either or both of these assumptions fail. This can be easily established using the reciprocal system as in [21,14]. Rather than presenting this derivation now, we focus on a formulation that leads to the same result, and also leads to a new formulation of the KYP-like lemma for descriptor systems.

In the control and systems theory literature a slightly modified definition of strict positive realness is often discussed. This definition, which dates back to the book by Narendra and Taylor [1], is made so as to make the notion of a strictly positive real transfer function consistent with the classical Kalman-Yacubovich-Popov lemma. We now revisit some of our previous results with this new definition in mind. As we shall see, many of the results in Sections II and III can be recovered in a slightly different way, and some of the assumptions relaxed. To be consistent with results already in the literature [18, 19, 20], we now assume that the system (1) is impulse free; namely $\deg(\det[sE - A]) = \text{rank}(E) = p$. Further, to make this new definition of strict positive realness distinct from that discussed above we call this new version KYP strict positive realness (KYP-SPR). More formally we say that the transfer function $H(j\omega)$ is KYP-SPR if and only if:

$$\exists \varepsilon > 0 \text{ such that } H(s - \varepsilon) \quad (46)$$

is positive for some positive ε . This definition is equivalent to the following conditions on $H(j\omega)$ for proper and strictly proper transfer functions. The second condition only applies in the strictly proper case where $\text{Re}[H(j\infty)] = 0$ [1]:

$$\begin{aligned} \text{(i)} \quad & \text{Re}[H(j\omega)] > 0 \quad \forall \omega \in \mathbb{R} \\ \text{(ii)} \quad & \lim_{\omega \rightarrow \infty} \omega^2 \text{Re}[H(j\omega)] > 0 \end{aligned} \quad (47)$$

This definition of SPR is consistent with KYP lemma for a regular system. We make no such claim here for Descriptor

systems; rather we include this version of SPR for completeness.

REMARK 5: Note that the standard (frequency domain) conditions of SPR and KYP-SPR coincide everywhere except at $\omega = \infty$.

We now have the following result.

Theorem 3

Consider the stable, impulse free, descriptor system (2) with transfer function $H(s) = c^T (sE - A)^{-1} b + d$. Assume that the pencil $\det(sE - A)$ is regular and that the matrix A is invertible. Define the matrix

$$M^{-1} = A^{-1} - A^{-1}b(-d + c^T A^{-1}b)^{-1} c^T A^{-1} \quad (48)$$

Let the matrix E have rank $p < n$. Then,

(i) $H(s)$ is KYP-SPR if, and only if, $d - c^T A^{-1}b$ is positive and the matrix product $EA^{-1}EM^{-1}$ has no real negative eigenvalues, and at most $n - p + 1$ zero eigenvalues.

(ii) $H(s)$ is ESPR if, and only if, it is SPR with exactly $n - p$ zero eigenvalues.

REMARK 6: Note that if $d \neq 0$ and the matrix M which is defined in (3) is invertible, then the inverse of M is M^{-1} as defined in (48). However, the definition of M^{-1} in (48) is valid for $d = 0$ and M (or M^{-1}) singular, as well.

Proof of Theorem 3

The proof consists of four distinct parts. (A) First we show that a certain matrix, obtained from a full rank decomposition of E is Hurwitz stable. (B) Then, we replace the original representation of transfer function $H(j\omega)$ with another more convenient one. (C) Using this representation we determine spectral conditions for system to be KYP-SPR. (D) Finally, we relate these conditions back to spectral conditions on the original representation of $H(j\omega)$.

PART (A): The matrix A is invertible, and the matrix E is of rank p. Let X, Y be a full rank decomposition of E; $E = XY^T$, $X, Y \in \mathbb{R}^{n \times p}$. Then, the matrix $Y^T A^{-1} X$ is Hurwitz stable and consequently invertible. To see this note that $\deg(\det[sE - A]) = \text{rank}(E) = p$ since the descriptor system is impulse free. This implies that $\deg(\det[sEA^{-1} - I_n]) = \text{rank}(E) = p$ since the matrix A is invertible. But this implies that the matrix $XY^T A^{-1}$ has exactly $n-p$ eigenvalues at the origin and p eigenvalues in the

open left half of the complex plane. Now consider the matrix $XY^T A^{-1} \in \mathbb{R}^{n \times n}$. This matrix shares its non-zero eigenvalues with the matrix $Y^T A^{-1} X \in \mathbb{R}^{p \times p}$. Since there are exactly p of these, and since this matrix is of dimension $p \times p$, it follows that $Y^T A^{-1} X$ is Hurwitz stable (and invertible).

PART (B): We now present an alternative representation of the descriptor system transfer function. Let $H(j\omega) = c^T (j\omega E - A)^{-1} b + d$. Then, this transfer function can be written:

$$H(j\omega) = d - c^T A^{-1} b + c^T A^{-1} X \left(Y^T A^{-1} X - \frac{1}{j\omega} I_p \right)^{-1} Y^T A^{-1} b \quad (49-a)$$

Thus the corresponding reciprocal system is:

$$H\left(\frac{1}{j\omega}\right) = \tilde{d} + \tilde{c}^T (j\omega I_p - \tilde{A})^{-1} \tilde{b} \quad (49-b)$$

where $\tilde{d} = d - c^T A^{-1} b$, $\tilde{c} = c^T A^{-1} X$, $\tilde{A} = Y^T A^{-1} X$ and $\tilde{b} = -Y^T A^{-1} b$. To see this, simply apply the matrix inversion lemma to $(sE - A)^{-1}$ everywhere this matrix inverse exists. Note that this transfer function is well defined everywhere since $Y^T A^{-1} X$ is Hurwitz stable. Thus, if $H(j\omega)$ exists at $\omega = \infty$, then it is equal to $H(j\infty) = d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b$. In this case the notion of ESPR reduces to positivity of this quantity.

PART (C): Now we recall the recently derived result (which we give with proof for completeness) [14]. This result states the following. Suppose that A is a Hurwitz matrix. Then, the following statements are equivalent.

- (a) The transfer function $H(j\omega) = d + c^T (j\omega I - A)^{-1} b$ is KYP-SPR;
- (b) $d - c^T A^{-1} b > 0$ and the matrix product $A^{-1}(A^{-1} - A^{-1}b(-d + c^T A^{-1}b)^{-1} c^T A^{-1})$ has no negative eigenvalues and at most one zero eigenvalue.

The proof of these statements is given in full in [14]. Here we give merely an outline (although our statement is more general than that in [14] as there only the case where $d=0$ is considered). Note first that $H(0) = d - c^T A^{-1} b$, since $H(j\omega)$, is assumed to be KYP-strictly positive real it follows that $H(0) = d - c^T A^{-1} b$ is necessarily positive. Now we use the fact that conditions for KYP-SPR of $H(j\omega)$ can be rewritten

in terms of conditions on $H\left(\frac{1}{j\omega}\right)$, as in done in [14]. We

$$\text{have } H\left(\frac{1}{j\omega}\right) = \bar{d} + \bar{c}^T(j\omega I - \bar{A})^{-1}\bar{b}, \quad \text{with } \bar{A} = A^{-1},$$

$\bar{b} = -A^{-1}b$, $\bar{c}^T = c^T A^{-1}$, and $\bar{d} = d - c^T A^{-1}b$. This follows from the well known formula on reciprocity of transfer functions [17]. Now suppose that $H\left(\frac{1}{j\omega}\right)$ is not KYP-strictly

positive real. By continuity, either there exists (i) a finite

$$\tilde{\omega} = \frac{1}{\omega} \quad \text{such} \quad \text{that} \quad \text{Re}[H(j\tilde{\omega})] = 0 \quad \text{or} \quad \text{(ii)}$$

$\lim_{\tilde{\omega} \rightarrow 0} \tilde{\omega}^2 \text{Re}[H(j\tilde{\omega})] = 0$. Using the results in [14, 15, 16], and assuming $d - c^T A^{-1}b > 0$, we can write

$$\begin{aligned} \text{Re}[H(j\tilde{\omega})] &= 2\bar{d} \det\left[1 - \frac{1}{\bar{d}} \bar{c}^T (\omega^2 I + \bar{A}^2)^{-1} \bar{A} \bar{b}\right] \\ &= 2\bar{d} \det\left[I - \frac{1}{\bar{d}} (\omega^2 I + \bar{A}^2)^{-1} \bar{A} \bar{b} \bar{c}^T\right] \\ &= 2\bar{d} \det\left[(\omega^2 I + \bar{A}^2)^{-1}\right] \cdot \det\left[\omega^2 I + \bar{A}^2 - \frac{1}{\bar{d}} \bar{A} \bar{b} \bar{c}^T\right] \end{aligned}$$

$$\text{Thus,} \quad \text{Re}[H(j\tilde{\omega})] = \frac{2\bar{d} \det[\omega^2 I + G]}{\det[\omega^2 I + \bar{A}^2]} \quad \text{where}$$

$$G := A^{-1} \left(A^{-1} - \frac{A^{-1} b c^T A^{-1}}{-d + c^T A^{-1} b} \right). \quad \text{Since } A \text{ is Hurwitz, all the}$$

real eigenvalues of $\bar{A}^2 = A^{-2}$ are positive which implies that $\det[\omega^2 I + \bar{A}^2] \neq 0$ for all ω . Noting that $\det[\omega^2 I + \bar{A}^2] > 0$ for ω sufficiently large, it follows from continuity arguments that $\det[\omega^2 I + \bar{A}^2] > 0$ for all ω . Recalling that $\bar{d} > 0$, conditions for KYP-SPR are equivalent to $\det[\omega^2 I + G] > 0$ for all $\omega \in \mathbb{R}, \omega \neq 0$, and

$\lim_{\omega \rightarrow 0} \frac{1}{\omega^2} \det[\omega^2 I + G] > 0$ (see [14]). Note the latter condition need only be checked when $\text{Re}[H(j\infty)] = 0$. From [14] the above conditions are equivalent to $\det[\lambda I - G] \neq 0$ for all

$\lambda \in \mathbb{R}, \lambda < 0$ and $\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \det[\lambda I - G] \neq 0$. The first of these

conditions is equivalent to requiring that G has no negative eigenvalues. The second condition is equivalent to a zero eigenvalue being of maximal multiplicity one [14].

PART (D): Parts (A) and (B), KYP-SPR of $H(j\omega)$ of (2) can be checked with a spectral condition derived from (49-b). Since, by assumption, the system is impulse free, the matrix $Y^T A^{-1} X$ is Hurwitz. Consequently, from PART (C) KYP-

strict positive realness of $H(j\omega)$ is equivalent to the condition that the matrix product

$$\tilde{A} \left(\tilde{A} - \frac{1}{\bar{d}} \tilde{b} \tilde{c}^T \right) = (Y^T A^{-1} X)(Y^T A^{-1} X - Y^T A^{-1} b (-d + c^T A^{-1} b)^{-1} c^T A^{-1} X) \quad (50)$$

has no negative eigenvalues and at most one zero eigenvalue.

Now we use the fact that the non-zero eigenvalues of RS^T and $S^T R$ coincide for any two matrices of compatible dimension. This means that

$(XY^T A^{-1})(XY^T A^{-1} - XY^T A^{-1} b (-d + c^T A^{-1} b)^{-1} c^T A^{-1})$ has at most $n - p + 1$ zero eigenvalues, and no negative real eigenvalues. But $E = XY^T$. So the above product is $EA^{-1}EM^{-1}$ and the assertion of item (i) of the theorem is proven. Recall that if $H(j\omega)$ is extended SPR then,

$$\begin{aligned} H(j\infty) &= d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b > 0 \\ &\Rightarrow d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b \neq 0 \\ &\Rightarrow (d - c^T A^{-1} b) \left[1 + (d - c^T A^{-1} b)^{-1} c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b \right] \neq 0 \end{aligned}$$

as $d - c^T A^{-1} b > 0$, $H(j\infty) > 0 \Rightarrow$

$$\left[1 + (d - c^T A^{-1} b)^{-1} c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b \right] \neq 0$$

$$\Leftrightarrow \det \left[I_n + \frac{bc^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1}}{(d - c^T A^{-1} b)} \right] \neq 0$$

$$\Leftrightarrow \det \left[I_r + \frac{Y^T A^{-1} b c^T A^{-1} X (Y^T A^{-1} X)^{-1}}{(d - c^T A^{-1} b)} \right] \neq 0$$

$$\Leftrightarrow \det \left[(Y^T A^{-1} X) + \frac{Y^T A^{-1} b c^T A^{-1} X}{(d - c^T A^{-1} b)} \right] \cdot \det \left[(Y^T A^{-1} X)^{-1} \right] \neq 0$$

$$\Leftrightarrow \det \left[\left((Y^T A^{-1} X) - Y^T A^{-1} b (-d + c^T A^{-1} b)^{-1} c^T A^{-1} X \right) (Y^T A^{-1} X)^{-1} \right] \neq 0$$

So $(Y^T A^{-1} X)(Y^T A^{-1} X - Y^T A^{-1} b (-d + c^T A^{-1} b)^{-1} c^T A^{-1} X)$ cannot have any zero eigenvalues and consequently the product $EA^{-1}EM^{-1}$ must have precisely $n - p$ zero eigenvalues.

REMARK 7: Note that our condition involves the matrices EA^{-1} and EM^{-1} . By assumption $\det[sI - EA^{-1}] = 0$ has roots at the origin and otherwise in the left half of the complex plane. In view of the necessary conditions for strict positive realness, for $d > 0$, the pencil

$$\det \left[sI - E \left(A - \frac{1}{d} b c^T \right)^{-1} \right] = 0 \quad \text{cannot have any roots in the}$$

open right half of the complex plane. In fact, the polynomial cannot have roots anywhere on the imaginary axis except at the origin as the following argument demonstrates. Suppose

that the pencil $\det[sI - EM^{-1}] = 0$ has purely imaginary roots (other than at the origin). Then, the matrix EM^{-1} must also have purely imaginary eigenvalues. Call these $\pm j\delta$. Now note that EA^{-1} and EM^{-1} differ by a rank-1 matrix. Then the product $EA^{-1}EM^{-1}$ can be written $X(X - yz^T) = X^2 - Xyz^T$ where X is a matrix with purely imaginary eigenvalues. Now note that X^2 must have a real negative eigenvalue $-\delta^2$ of geometric multiplicity two (corresponding to the purely imaginary eigenspace of the matrix X). Also, Xyz^T has a kernel of dimension at least $n-1$. Thus it follows that $-\delta^2$ is an eigenvalue of $EA^{-1}EM^{-1}$ and by the main theorem the system cannot be SPR. A similar argument was used also in [14].

V. A KYP-LIKE LEMMA FOR IMPULSE-FREE DESCRIPTOR SYSTEMS AND STABILITY OF SWITCHED DESCRIPTOR SYSTEMS

We now use the results in the previous section to obtain a KYP-like lemma for SISO descriptor systems [18, 19 and 20]. For convenience we recall the classical KYP lemma for SISO systems. There are many extensions of this lemma (for example, relaxing the observability/controllability assumption) and the following arguments can be modified to obtain a relaxed version of these for descriptor systems of the form that we have considered in this paper.

Strict positive realness of a transfer function matrix and the existence of quadratic Lyapunov functions are closely related. The precise relationship is given by the Kalman-Yacubovich-Popov lemma [15]. Roughly speaking, the Meyer version of the KYP lemma can be stated as follows. Let $A \in \mathbb{R}^{n \times n}$ be a Hurwitz matrix. Let $b \in \mathbb{R}^{n \times 1}$, $c^T \in \mathbb{R}^{1 \times n}$ and d be a non negative scalar. Let $(A, b), (A, c)$ be controllable/observable pairs respectively. Then, there exists a positive definite matrix $P = P^T \in \mathbb{R}^{n \times n}$, $P > 0$ such that [17]

$$\begin{bmatrix} A & b \\ -c^T & -d \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & b \\ -c^T & -d \end{bmatrix} \leq 0,$$

$A^T P + PA < 0$, if and only if $H(j\omega) = d + c^T(j\omega I - A)^{-1}b$ is KYP-SPR.

An important alternative statement of the KYP lemma for SISO systems (b,c vectors) is that strict positive realness of $H(j\omega)$ is equivalent to the existence of $P = P^T > 0$ satisfying either:

$$(i) \quad A^T P + PA < 0 \text{ and } \left(A - \frac{1}{d}bc^T \right)^T P + P \left(A - \frac{1}{d}bc^T \right) < 0$$

when d is strictly positive;

$$(ii) \quad \text{or } (bc^T)^T P + P(bc^T) \leq 0 \text{ when } d = 0.$$

Thus KYP strict positive realness is equivalent to the existence of a positive definite matrix P that simultaneously satisfies a pair of Lyapunov equations. When such a P exists the function $V(x) = x^T P x$ is said to be a common quadratic Lyapunov function (CQLF) for the dynamic systems $\dot{x} = Ax$ and $\dot{x} = \left(A - \frac{1}{d}bc^T \right)x$ (re. $\dot{x} = -bc^T x$).

With this version of the KYP lemma in mind, and in view of the results in [14], we now consider the implications of the results in Section IV. It follows from Part (C) and Part (D) of the proof of Theorem 3, that the descriptor system is KYP-SPR iff there exists a CQLF (in the weak sense) for the following dynamic systems

$$\dot{x} = (Y^T A^{-1} X)x; \quad (51)$$

$$\dot{x} = (Y^T A^{-1} X - Y^T A^{-1} b(-d + c^T A^{-1} b)^{-1} c^T A^{-1} X)x. \quad (52)$$

provided that $(Y^T A^{-1} X, -Y^T A^{-1} b)$ is a controllable pair and $(c^T A^{-1} X, Y^T A^{-1} X)$ is observable.

REMARK 8: Our controllability and observability conditions require that two matrices must be of dimension p . By exploiting the fact that the product of two invertible matrices is itself invertible, a test for controllability/observability that is independent of (X, Y) may be obtained by multiplying the controllability matrix by the observability matrix from the left.

Suppose such a common Lyapunov function exists for (51) and (52). We have:

$$(Y^T A^{-1} X)^T P + P(Y^T A^{-1} X) < 0$$

By pre and post multiplying by Y and Y^T respectively we have

$$(A^{-1} E)^T \tilde{P} + \tilde{P} A^{-1} E \leq 0.$$

where $\tilde{P} = YPY^T$. The same operation can be carried out for the second Lyapunov equation. Thus a KYP-like lemma for Descriptor systems may be stated as follows.

Consider the impulse free and stable descriptor system (2). Let E be rank p , and $E = XY^T$ be a full rank decomposition of E . Then, the following statements are equivalent

- (i) $H(j\omega)$ is KYP-SPR.
(ii) There exists a positive semi-definite matrix $\tilde{P} \in \mathbb{R}^{n \times n}$, $\tilde{P} = YPY^T$, $P = P^T > 0$, $P \in \mathbb{R}^{p \times p}$ such that $(A^{-1}E)^T \tilde{P} + \tilde{P}A^{-1}E \leq 0$ and $(M^{-1}E)^T \tilde{P} + \tilde{P}M^{-1}E \leq 0^2$
(iii) An equivalent LMI can also be written down.

Thus strict positive realness of a descriptor system is equivalent to the joint quadratic stability of a pair of lower dimensional systems that can be defined easily from E, A, b, c and d . This observation is entirely consistent with the intuition arising from elimination of the constraint.

REMARK 9: An important consequence of the above results is that they can be used to obtain conditions for a certain SISO descriptor system to be quadratically stable. Consider the following time-varying descriptor systems.

$$E\dot{x} = A(t)x \quad A(t) \in \left\{ A, A - \frac{bc^T}{d} \right\}$$

where both systems described by (E, A) and

$\left(E, A - \frac{bc^T}{d} \right)$ are stable and impulse free and we further

assume that continuity of Ex is preserved across the switches. We wish to determine whether or not a quadratic Lyapunov function exists for the above descriptor system. Let us try to find a Lyapunov function of the form $V(x) = x^T YPY^T x$. Note that $V(x) = 0$ if x lies in the kernel of E and is positive otherwise. By taking derivatives with respect to time and by substituting for x in the Lyapunov equation we require that

$$Q_1(\dot{x}) = \dot{x}^T \left((A^{-1}E)^T \tilde{P} + \tilde{P}A^{-1}E \right) \dot{x} = 2\dot{x}^T YP(Y^T A^{-1}X)Y^T \dot{x} \leq 0;$$

$$Q_2(\dot{x}) = \dot{x}^T \left((M^{-1}E)^T \tilde{P} + \tilde{P}M^{-1}E \right) \dot{x} \leq 0$$

Note that both of the above functions are zero when the derivative of x is in the Kernel of E . Thus, the existence of such a Lyapunov function implies that both x and its derivative approach the Kernel of E , and consequently that the switched

²Note that P matrix in item (ii) should satisfy $(Y^T A^{-1}X)^T P + P(Y^T A^{-1}X) < 0$ and

$(Y^T M^{-1}X)^T P + P(Y^T M^{-1}X) < 0$ (for $\text{Re}[H(j\infty)] > 0$) or

$(Y^T M^{-1}X)^T P + P(Y^T M^{-1}X) \leq 0$ (for $\text{Re}[H(j\infty)] = 0$). This is implicitly stated in the text of Remark 8.

descriptor system is stable³. But the existence of such a P is guaranteed from the KYP-like lemma for descriptor systems given above. It should be noted from **PART (A)** of **Theorem 3** that both the matrices $(Y^T A^{-1}X)$ and $(Y^T M^{-1}X)$ are Hurwitz. For the case when one of these matrices is marginally stable with one eigenvalue at origin, the results on quadratic stability cannot be extended directly as in [14]. In such a case, the sub-system corresponding to marginally stable matrix would not be an impulse free system and arbitrary switching between such subsystems would lead to impulsive behavior. This case would be reported in future publications.

VI. NUMERICAL EXAMPLES

We now present a number of examples to illustrate our results.

Example 1 (SPR function)

$$\text{Let } A = \begin{bmatrix} -1 & 2 \\ 3 & -7 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad d = 1$$

$$E = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; \quad \text{rank}(E) = p = 1.$$

Using Theorem 1 or Theorem 3, we have

Condition 1

$$c^T (-A)^{-1} b + d = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 = 14 > 0$$

Condition 1 is satisfied.

Condition 2

$$EA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -7 & -2 \\ -3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -13 & -4 \\ -26 & -8 \end{bmatrix}$$

$$\det[sI - EA^{-1}] = s(s + 21)$$

Condition 2 is satisfied.

Alternatively, by Corollary 1, $\det(sE - A) = 21s + 1$. Thus condition 2 is satisfied.

Condition 3

$$EA^{-1}EM^{-1} = \begin{bmatrix} 18 & 7.5 \\ 36 & 15 \end{bmatrix}$$

³ Switched descriptor system is stable since the consistency space and the kernel of E intersect only at the origin. This follows from an involved argument arising from the implications of the fact that solutions of an impulse free Descriptor system $E\dot{x} = Ax$; lie in the subspace S defined by $S = \{z \in \mathbb{R}^n : Az \in \text{Im}(E)\}$ and $S \cap \text{Ker}(E) = \{0\}$. A full detailed argument is given in "On dimensionality reduction and stability of a class of switched descriptor systems" Sajja; Corless; Zeheb; Shorten. This paper was formerly titled as "Stability conditions for a class of switching descriptor systems", Provisionally accepted, Automatica, 2011.

$$\det[sI - EA^{-1}EM^{-1}] = s(s-33)$$

Condition 3 is satisfied.

Hence the system in Example 1 is SPR.

Using Theorem 3 (ii), $n-p=2-1=1$, thus the system is also ESPR.

Alternatively, using Theorem 2,

$$c^T(adE)b = [1 \ 1] \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

$$\text{trace}(AadE) = \text{trace} \begin{bmatrix} -8 & 4 \\ 26 & -13 \end{bmatrix} = -21$$

Thus

$$c^T(adE)b - d \text{trace}[AadE] = 22 \neq 0.$$

Which ensures again that the system is ESPR.

Example 2 (Non SPR function)

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

And b, c, d, E as in Example 1.

Using Theorem 1 or theorem 3, we have

Condition 1

$$c^T(-A)^{-1}b + d = [1 \ 1] \begin{bmatrix} 2 & -1 \\ -1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 = 1 > 0$$

Condition 1 is satisfied.

Condition 2

$$EA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\det[sI - EA^{-1}] = s(s-1)$$

Thus, condition 2 is violated.

Alternatively, by Corollary 1,

$$\det(sE - A) = 2(s-1)$$

Which leads again to the conclusion that condition 2 is violated. Hence, the system in Example 2 is not SPR. (Condition 3, though, is satisfied).

Example 3 ($d=0$)

Let A, b, c, E be as in Example 1, but $d=0$. The descriptor system is stable and impulse free.

Then, using (48),

$$M^{-1} = \frac{1}{13} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Using Theorem 3, we have

$$d - c^T A^{-1} b = 13 > 0.$$

and

$$EA^{-1}EM^{-1} = \frac{1}{13} \begin{bmatrix} -21 & 21 \\ -42 & 42 \end{bmatrix}$$

$$\det[sI - EA^{-1}EM^{-1}] = s \left(s - \frac{21}{13} \right)$$

Thus, the eigenvalues of the matrix product $EA^{-1}EM^{-1}$ are $s=0$ and $s=+21/13$, which leads to the conclusion that the system in Example 3 is SPR.

Also, since $n-p=2-1=1$, and there is exactly one zero eigenvalue, the system is also ESPR.

Example 4

Let E, A, b, c, d , be defined as follows:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, d = 0$$

The given system's is not ESPR because

$$H(j\infty) = d - c^T A^{-1} b + c^T A^{-1} (Y^T A^{-1} X)^{-1} c^T A^{-1} b = 0$$

$$\text{Alternatively } c^T(adE)b - d \text{trace}[AadE] =$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0.$$

However, the system is KYP-SPR and a Lyapunov function can be found for the Descriptor system as:

$$d - c^T A^{-1} b = 1.5 > 0$$

$$n-p+1 = 3-2+1 = 2$$

$$\text{eig}(EA^{-1}EM^{-1}) = \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}, EA^{-1}EM^{-1} \text{ has no real negative eigenvalues}$$

and has exactly $n-p+1$ zero eigenvalues. Hence the system is KYP-SPR

VII. CONCLUSION

In this paper we have considered SISO Descriptor systems that are strict positive real. Three types of strict positive realness was considered; finite frequency strict positive realness; extended strict positive realness; and a notion of strict positive realness that is consistent with the KYP lemma for regular systems. In each case simple, easy to check, spectral conditions are given that are both necessary and sufficient for strict positive realness. Finally, a new KYP-like lemma for SISO descriptor systems is given and this result is then used to obtain a solution to the stability problem for a class of switched Descriptor systems. Future work will focus on extending our results to the MIMO case, and to the case of uncertain Descriptor systems.

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