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Projective modules and involutions

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Abstract

Let G be a finite group, and let $\Omega := \{t \in G \mid t^2 = 1\}$. Then Ω is a G-set under conjugation. Let k be an algebraically closed field of characteristic 2. It is shown that each projective indecomposable summand of the G-permutation module $k\Omega$ is irreducible and self-dual, whence it belongs to a real 2-block of defect zero. This, together with the fact that each irreducible kG-module that belongs to a real 2-block of defect zero occurs with multiplicity 1 as a direct summand of $k\Omega$, establishes a bijection between the projective components of $k\Omega$ and the real 2-blocks of G of defect zero.

Keywords: Involutions; Blocks of defect zero; Green correspondence; Burry–Carlson–Puig theorem

Let G be a finite group, with identity element e, and let $\Omega := \{t \in G \mid t^2 = e\}$. Then Ω is a G-set under conjugation. In this note we describe the projective components of the permutation module $k\Omega$, where k is an algebraically closed field of characteristic 2. By a projective component we mean an indecomposable direct summand of $k\Omega$ that is also a direct summand of a free kG-module. We show that all such components are irreducible, self-dual and occur with multiplicity 1.

This gives an alternative proof of Remark (2) on p. 254 of [5], and strengthens Corollaries 3 through 7 of that paper. In addition, we can give the following quick proof of Proposition 8 in [5]:

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Corollary 1. *Suppose that H is a strongly embedded subgroup of G. Then $k_H{\uparrow}^G \cong k_G \oplus [\bigoplus_{i=1}^s P_i]$ where $s \geqslant 0$ and the P_i are pairwise nonisomorphic self-dual projective irreducible kG-modules.*

Proof. That H is strongly embedded means that $|H|$ is even and $|H \cap H^g|$ is odd, for each $g \in G \backslash H$. Let $t \in H$ be an involution. Then clearly $C_G(t) \leqslant H$. So $k_H{\uparrow}^G$ is isomorphic to a submodule of $(k_{C_G(t)}){\uparrow}^G$. Mackey's theorem implies that every component of $k_H{\uparrow}^G$, other than k_G, is a projective kG-module. Being projective, these modules must be components of $(k_{C_G(t)}){\uparrow}^G$. The result now follows from Theorem 8. □

Consider the wreath product $G \wr \Sigma$ of G with a cyclic group Σ of order 2. Here Σ is generated by an involution σ and $G \wr \Sigma$ is isomorphic to the semidirect product of the base group $G \times G$ by Σ. The conjugation action of σ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. The elements of $G \wr \Sigma$ will be written (g_1, g_2), $(g_1, g_2)\sigma$ or σ.

We shall exploit the fact that kG is a $kG \wr \Sigma$-module. For, as is well known, kG is an $k(G \times G)$-module via: $x \cdot (g_1, g_2) := g_1^{-1} x g_2$, for each $x \in kG$, and $g_1, g_2 \in G$. The action of Σ on kG is induced by the permutation action of σ on the distinguished basis G of kG: $g^\sigma := g^{-1}$, for each $g \in G$. Clearly σ acts as an involutary k-algebra anti-automorphism of kG. It follows that the actions of $G \times G$ and Σ on kG are compatible with the group relations in $G \wr \Sigma$.

By a *block* of kG, or a 2-block of G, we mean an indecomposable k-algebra direct summand of kG. Each block has associated to it a primitive idempotent in $Z(kG)$, a Brauer equivalence class of characters of irreducible kG-modules and a Brauer equivalence class, modulo 2, of ordinary irreducible characters of G. A block has defect zero if it is a simple k-algebra, and is real if it contains the complex conjugates of its ordinary irreducible characters. Theorem 8 establishes a bijection between the real 2-blocks of G that have defect zero and the projective components of $k\Omega$.

We could equally well work over a complete discrete valuation ring R of characteristic 0, whose field of fractions F is algebraically closed, and whose residue field $R/J(R)$ is k. So we use $\mathcal{O}$ to indicate either of the commutative rings k or R.

All our modules are right-modules. We denote the trivial $\mathcal{O}G$-module by $\mathcal{O}_G$. If M is an $\mathcal{O}G$-module, we use $M{\downarrow}_H$ to denote the restriction of M to H. If H is a subgroup of G and N is an $\mathcal{O}H$-module, we use $N{\uparrow}^G$ to denote the induction of N to G. Whenever $g \in G$, we write $\underline{g}$ for $(g, g) \in G \times G$, and we set $\underline{X} := \{\underline{x} \mid x \in X\}$, for each $X \subset G$. Other notation and concepts can be found in a standard textbook on modular representation theory, such as [1] or [4].

If B is a block of $\mathcal{O}G$, then so too is $B^o = \{x^\sigma \mid x \in B\}$. We call B a real block if $B = B^o$. Our first result describes the components of $\mathcal{O}G$ as $\mathcal{O}G \wr \Sigma$-module.

Lemma 2. *There is an indecomposable decomposition of $\mathcal{O}G$ as $\mathcal{O}G \wr \Sigma$-module*:

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_r \oplus \left(B_{r+1} + B_{r+1}^o\right) \oplus \cdots \oplus \left(B_{r+s} + B_{r+s+1}^o\right).$$

Here $B_1, \ldots, B_r$ are the real 2-blocks and $B_{r+1}, B_{r+1}^o, \ldots, B_{r+s}, B_{r+s}^o$ are the nonreal 2-blocks of G.

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Proof. This follows from the well-known indecomposable decomposition of $\mathcal{O}G$, as an $\mathcal{O}(G\times G)$-module, into a direct sum of its blocks, and the fact that $B_i^\sigma = B_i$ for $i = 1,\ldots,r$, and $B_{r+j}^\sigma = B_{r+j}^o$ for $j = 1,\ldots,s$. $\square$

An obvious but useful fact is that $\mathcal{O}G$ is a permutation module:

Lemma 3. *The $\mathcal{O}G\wr\Sigma$-module $\mathcal{O}G$ is isomorphic to the permutation module $(\mathcal{O}_{\underline{G}\times\Sigma})\uparrow^{G\wr\Sigma}$.*

Proof. The elements of G form a $G\wr\Sigma$-invariant basis of $\mathcal{O}G$. Moreover if $g_1, g_2 \in G$, then $g_2 = g_1\cdot(g_1, g_2)$. So G is a transitive $G\wr\Sigma$-set. The stabilizer of $e\in\mathcal{O}G$ in $G\wr\Sigma$ is $\underline{G}\times\Sigma$. The lemma follows from these facts. $\square$

Let C be a conjugacy class of G. Set $C^o := \{c\in G \mid c^{-1}\in C\}$. Then C^o is also a conjugacy class of G, and $C\cup C^o$ can be regarded as an orbit of $\underline{G}\times\Sigma$ on the $G\wr\Sigma$-set G. As such, the corresponding permutation module $\mathcal{O}(C\cup C^o)$ is a $\mathcal{O}\underline{G}\times\Sigma$-direct summand of $\mathcal{O}G$. If $C = C^o$, we call C a real class of G. In this case for each $c\in C$ there exists $x\in G$ such that $c^x = c^{-1}$. The point stabilizer of c in $\underline{G}\times\Sigma$ is $\underline{C_G(c)}\langle\underline{x}\sigma\rangle$. So

$$\mathcal{O}C \cong (\mathcal{O}_{\underline{C_G(c)}\langle\underline{x}\sigma\rangle})\uparrow^{\underline{G}\times\Sigma}.$$

If $C\neq C^o$, we call C a nonreal class of G. In this case the point stabilizer of $c\in C\cup C^o$ in $\underline{G}\times\Sigma$ is $\underline{C_G(c)}$. So

$$\mathcal{O}(C\cup C^o) \cong (\mathcal{O}_{\underline{C_G(c)}})\uparrow^{\underline{G}\times\Sigma}.$$

Suppose now that the real classes are $C_1,\ldots,C_t$ and that the nonreal classes are $C_{t+1}, C_{t+1}^o,\ldots,C_{t+u}, C_{t+u}^o$. Then we have:

Lemma 4. *There is a decomposition of $\mathcal{O}G$ as an $\mathcal{O}\underline{G}\times\Sigma$-permutation module*:

$$\mathcal{O}G = \mathcal{O}C_1\oplus\cdots\oplus\mathcal{O}C_t\oplus\mathcal{O}\big(C_{t+1}\cup C_{t+1}^o\big)\oplus\cdots\oplus\mathcal{O}\big(C_{t+u}\cup C_{t+u+1}^o\big).$$

Proof. This follows from Lemma 3 and the discussion above. $\square$

By a quasi-permutation module we mean a direct summand of a permutation module. Our next result is Lemma 9.7 of [1]. We include a proof for the convenience of the reader.

Lemma 5. *Let M be an indecomposable quasi-permutation $\mathcal{O}G$-module and suppose that H is a subgroup of G such that $M{\downarrow_H}$ is indecomposable. Then there is a vertex V of M such that $V\cap H$ is a vertex of $M{\downarrow_H}$. If H is a normal subgroup of G, then this is true for all vertices of M.*

Proof. Let U be a vertex of M. As $\mathcal{O}_U \mid M{\downarrow_U}$ we have $\mathcal{O}_{U\cap H} \mid (M{\downarrow_H}){\downarrow_{U\cap H}}$. But $U\cap H$ is a vertex of $\mathcal{O}_{U\cap H}$. So Mackey's theorem implies that there exists a vertex W of $M{\downarrow_H}$ such that $U\cap H\leqslant W$.

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As $M{\downarrow_H}$ is a component of the restriction of M to H, Mackey's theorem shows that there exists $g \in G$ such that $W \leqslant U^g \cap H$. Now U^g is a vertex of M. So by the previous paragraph, and the uniqueness of vertices of $M{\downarrow_H}$ up to H-conjugacy, there exists $h \in H$ such that $U^g \cap H \leqslant W^h$. Comparing cardinalities, we see that $W = U^g \cap H$. So $U^g \cap H$ is a vertex of $M{\downarrow_H}$.

Suppose that H is a normal subgroup of G. Then $U \cap H \leqslant W$ and $W = U^g \cap H = (U \cap H)^g$ imply that $U \cap H = W$. $\square$

R. Brauer showed how to associate to each block of $\mathcal{O}G$ a G-conjugacy class of 2-subgroups, its so-called defect groups. It is known that a block has defect zero if and only if its defect groups are all trivial. J.A. Green showed how to associate to each indecomposable $\mathcal{O}G$-module a G-conjugacy class of 2-subgroups, its so-called vertices. He also showed how to identify the defect groups of a block using its vertices as an indecomposable $\mathcal{O}(G \times G)$-module.

Corollary 6. *Let B be a block of $\mathcal{O}G$ and let D be a defect group of B. If B is not real then $\underline{D}$ is a vertex of $B + B^o$, as $\mathcal{O}G \wr \Sigma$-module. If B is real, then there exists $x \in N_G(D)$, with $x^2 \in D$, such that $\underline{D}\langle \underline{x}\sigma\rangle$ is a vertex of B, as $\mathcal{O}G \wr \Sigma$-module. In particular, Σ is a vertex of $B + B^o$ if and only if B is a real 2-block of G that has defect zero.*

Proof. J.A. Green showed in [2] that $\underline{D}$ is a vertex of B, when B is regarded as an indecomposable $\mathcal{O}(G \times G)$-module. Suppose first that B is not real. Then $B + B^o = (B{\downarrow_{G\times G}}){\uparrow^{G\wr\Sigma}}$, for instance by Corollary 8.3 of [1]. It follows that $B + B^o$ has vertex $\underline{D}$, as an indecomposable $\mathcal{O}G \wr \Sigma$-module.

Suppose then that $B = B + B^o$ is real. Lemma 3 shows that B is $\underline{G} \times \Sigma$-projective. So we may choose a vertex V of B such that $V \leqslant \underline{G} \times \Sigma$. Moreover, B is a quasi-permutation $\mathcal{O}G \wr \Sigma$-module, and its restriction to the normal subgroup $G \times G$ is indecomposable. Lemma 5 then implies that $V \cap (G \times G) = V \cap \underline{G}$ is a vertex of $B{\downarrow_{G\times G}}$. So by Green's result, we may choose D so that $V \cap \underline{G} = \underline{D}$. Now $G \times G$ has index 2 in $G \wr \Sigma$. So Green's indecomposability theorem, and the fact that $B{\downarrow_{G\times G}}$ is indecomposable, implies that $V \nsubseteq (G \times G)$. It follows that there exists $x \in N_G(D)$, with $x^2 \in D$, such that $V = \underline{D}\langle \underline{x}\sigma\rangle$.

If B has defect zero, then $D = \langle e\rangle$. So $x^2 = e$. In this case, $\langle \underline{x}\sigma\rangle = \Sigma^{(e,x)}$ is $G \wr \Sigma$-conjugate to Σ. So Σ is a vertex of B. Conversely, suppose that Σ is a vertex of $B + B^o$. The first paragraph shows that B is a real block of G. Moreover B has defect zero, as $\Sigma \cap \underline{G} = \langle e\rangle$. $\square$

We quote the following result of Burry, Carlson and Puig [4, 4.4.6] on the Green correspondence:

Lemma 7. *Let $V \leqslant H \leqslant G$ be such that V is a p-group and $N_G(V) \leqslant H$. Let f denote the Green correspondence with respect to (G, V, H). Suppose that M is an indecomposable $\mathcal{O}G$-module such that $M{\downarrow_H}$ has a component N with vertex V. Then V is a vertex of M and $N = f(M)$.*

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We can now prove our main result. Part (ii) is Remark (2) on p. 254 of [5], but our proof is independent of the proof given there.

Theorem 8.

(i) *Let $t \in G$, with $t^2 = e$. Suppose that P is an indecomposable projective direct summand of $(\mathcal{O}_{C_G(t)})\uparrow^G$. Then P is irreducible and self-dual and occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_G(t)})\uparrow^G$. In particular P belongs to a real 2-block of G that has defect zero.*

(ii) *Suppose that M is a projective indecomposable $\mathcal{O}G$-module that belongs to a real 2-block of G that has defect zero. Then there exists $s \in G$, with $s^2 = e$, such that M is a component of $(\mathcal{O}_{C_G(s)})\uparrow^G$. Moreover, s is uniquely determined up to conjugacy in G.*

Proof. If $t = e$ then $P = \mathcal{O}_G$. So P is irreducible and self-dual. The assumption that P is projective and the fact that $\dim_{\mathcal{O}}(P) = 1$ implies that $|G|$ is odd. So all blocks of $\mathcal{O}G$, in particular the one containing P, have defect zero.

Now suppose that $t \neq e$. Let T be the conjugacy class of G that contains t. The permutation module $\mathcal{O}T$ is a direct summand of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$. Regard P as an $\mathcal{O}\underline{G}$-module. Let $I(P)$ be the inflation of this module to $\underline{G} \times \Sigma$. Then $I(P)$ is a component of $\mathcal{O}T$. As Σ is contained in the kernel of $I(P)$, and P is a projective $\mathcal{O}G$-module, it follows that $I(P)$ has vertex Σ as an indecomposable $\mathcal{O}\underline{G} \times \Sigma$-module.

By Lemma 2, and the Krull–Schmidt theorem, there exists a 2-block B of G such that $I(P)$ is a component of the restriction $(B + B^o)\downarrow_{\underline{G} \times \Sigma}$. An easy computation shows that $N_{G \wr \Sigma}(\Sigma) = \underline{G} \times \Sigma$. It then follows from Lemma 7 that $(B + B^o)$ has vertex Σ and also that $I(P)$ is the Green correspondent of $(B + B^o)$ with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$. We conclude from Corollary 6 that B is a real 2-block of G that has defect zero.

Let $\hat{B}$ be the 2-block of $G \wr \Sigma$ that contains B. Then $\hat{B}$ is real and has defect group Σ. Let $\hat{A}$ be the Brauer correspondent of $\hat{B}$. Then $\hat{A}$ is a real 2-block of $\underline{G} \times \Sigma$ that has defect group Σ. Now $\hat{A} = A \otimes \mathcal{O}\Sigma$, where A is a real 2-block of $\mathcal{O}\underline{G}$ that has defect zero. In particular A has a unique indecomposable module, and this module is projective, irreducible and self-dual. Corollary 14.4 of [1] implies that $I(P)$ belongs to $\hat{A}$. So P belongs to A. We conclude that P is irreducible and self-dual and belongs to a real 2-block of G that has defect zero.

Now B occurs with multiplicity 1 as a component of $\mathcal{O}G$, and $I(P)$ is the Green correspondent of B with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$. So $I(P)$ has multiplicity 1 as a component of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$. It follows that P occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_G(t)})\uparrow^G$, and with multiplicity 0 as a component of $(\mathcal{O}_{C_G(r)})\uparrow^G$, for $r \in G$ with $r^2 = e$, but r not G-conjugate to t. This completes the proof of part (i).

Let R be a real 2-block of G that has defect zero. Then R has vertex Σ as indecomposable $\mathcal{O}G \wr \Sigma$-module. So its Green correspondent $f(R)$, with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$, is a component of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$ that has vertex Σ. Lemma 4 and the Krull–Schmidt theorem imply that $f(R)$ is isomorphic to a component of $\mathcal{O}(C \cup C^o)$, for some conjugacy class C of G. Now Σ is a central subgroup of $\underline{G} \times \Sigma$. So Σ must be a subgroup of the point stabilizer of $C \cup C^o$ in $\underline{G} \times \Sigma$. It follows that $s^2 = e$, for each $s \in C$.

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Let N denote the restriction of $f(R)$ to $\underline{G}$, and consider N as an $\mathcal{O}G$-module. We have just shown that N is a component of $(\mathcal{O}_{C_G(s)})\uparrow^G$. Arguing as before, we see that N is an indecomposable projective $\mathcal{O}G$-module that belongs to a real 2-block of G that has defect zero.

The last paragraph establishes an injective map between the real 2-blocks of G that have defect zero and certain projective components of $\mathcal{O}\Omega$. As each block of defect zero contains a single irreducible $\mathcal{O}G$-module, this map must be onto. It follows that the module M in the statement of the theorem is a component of some permutation module $(\mathcal{O}_{C_G(s)})\uparrow^G$, where $s \in G$ and $s^2 = e$. The fact that s is determined up to G-conjugacy now follows from the last statement of the proof of part (i). This completes the proof of part (ii). □

It is possible to simplify the above proof by showing that if B is a real 2-block of G that has defect zero, then its Green correspondent, with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$ is M^{Fr}, where M^{Fr} is the Frobenius conjugate of the unique irreducible $\mathcal{O}G$-module that belongs to B.

Suppose that R is a complete discrete valuation ring and that L is an $RC_G(t)$-module, where L has R-rank 1 and $O^2(C_G(t))$ acts trivially on L. Then the 2-modular reduction of L is the trivial $kC_G(t)$-module, although L is not necessarily the trivial $RC_G(t)$-module. Now each projective irreducible kG-module lifts to a projective irreducible RG-module. So the conclusions of part (i) of the above theorem apply to $L\uparrow^G$: all of its projective components are irreducible and self-dual. We thank the referee for pointing out this extension of our result.

The proof of Theorem 8 hints at the fact that we have some 2-local control over all the components of $(\mathcal{O}_{C_G(t)})\uparrow^G$. The investigation of special properties of such components is continued in [3].

Corollary 9. *Let $\Omega = \{t \in G \mid t^2 = e\}$. Then there is a bijection between the real 2-blocks of G that have defect zero and the projective components of $\mathcal{O}\Omega$.*

Here is a sample application. It was suggested to me by G.R. Robinson.

Corollary 10. *Let $n \geqslant 1$ and let t be an involution in the symmetric group Σ_n. If $n = m(m+1)/2$ is a triangular number, and t is a product of $\lfloor (m^2+1)/4 \rfloor$ commuting transpositions, then there is a single projective irreducible $\mathcal{O}\Sigma_n$-module, and this module is the unique projective component of $(\mathcal{O}_{C_{\Sigma_n}(t)})\uparrow^{\Sigma_n}$. For all other values of n or nonconjugate involutions t, the modules $(\mathcal{O}_{C_{\Sigma_n}(t)})\uparrow^{\Sigma_n}$ are projective free.*

Proof. We give a proof of the following result in [3, Corollary 8.4]: Let G be a finite group, let B be a real 2-block of G of defect zero, and let χ be the unique irreducible character in B. Then there exists a 2-regular conjugacy class C of G such that $C = C^o$, $|C_G(c)|$ is odd, for $c \in C$, and $\chi(c)$ is nonzero, modulo a prime ideal containing 2. Moreover, there exists an involution $t \in G$ such that $c^t = c^{-1}$, and for this t we have $\langle \chi_{C_G(t)}, 1_{C_G(t)} \rangle = 1$. The existence of t was shown in [5]. The identification of t using the class C was first shown by R. Gow (in unpublished work).

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Suppose that $(\mathcal{O}_{C_{\Sigma_n}(t)})\uparrow^{\Sigma_n}$ has a projective component. Then Σ_n has a 2-block of defect zero, by Theorem 8. The 2-blocks of Σ_n are indexed by triangular partitions $\mu = [m, m-1, \ldots, 2, 1]$, where m ranges over those natural numbers for which $n - m(m+1)/2$ is even. Moreover, the 2-block corresponding to μ has defect zero if and only if $n = m(m+1)/2$. In particular, we can assume that $n = m(m+1)/2$, for some $m \geqslant 1$.

Let B be the unique 2-block of Σ_n that has defect zero, let χ be the unique irreducible character in B and let $g \in \Sigma_n$ have cycle type $\lambda = [2m-1, 2m-5, \ldots]$. Then $|C_{\Sigma_n}(g)|$ is odd. As the parts of λ are the "diagonal hooklengths" of μ, the Murnaghan–Nakayama formula shows that $\chi(g) = 1$. Now λ has $\lfloor (m-1)/2 \rfloor$ nonzero parts. So g is inverted by an involution t that is a product of $(n - \lfloor (m-1)/2 \rfloor)/2 = \lfloor (m^2+1)/4 \rfloor$ commuting transpositions. It follows from Theorem 8 and the previous paragraph that the unique irreducible projective B-module occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_{\Sigma_n}(t)})\uparrow^{\Sigma_n}$. The last statement of the corollary now follows from Theorem 8. □

References

[1] J.L. Alperin, Local Representation Theory, Cambridge Stud. Adv. Math., vol. 11, 1986.
[2] J.A. Green, Blocks of modular representations, Math. Z. 79 (1962) 100–115.
[3] J. Murray, Extended defect groups and extended vertices, Osaka J. Math, in press.
[4] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, 1989.
[5] G.R. Robinson, The Frobenius–Schur indicator and projective modules, J. Algebra 126 (1989) 252–257.