POINTWISE MULTIPLIERS FOR REVERSE HÖLDER SPACES

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Abstract. We classify weights which map reverse Hölder weight spaces to other reverse Hölder weight spaces under pointwise multiplication. We also give some fairly general examples of weights satisfying weak reverse Hölder conditions.

1. Introduction and Examples

In this paper our main task (Section 2) will be to classify those weights f for which fw is in some reverse Hölder weight space for all w in some other reverse Hölder weight space. In most cases, we will find that it is necessary and sufficient for f to be in some related weight space. The weight spaces with which we shall be concerned are RH_p $(0 < p \le \infty)$, and larger spaces which we shall denote as WRH_p. The RH_p condition, first examined by Gehring [G], is quite useful in many areas of analysis, particularly in the theory of quasiconformal mappings. It is intimately related to the A_q condition of Muckenhoupt [Mu] and their theory has in fact been developed together (notably in [C-F]). If one tries to develop the theory of quasiregular mappings as for quasiconformal mappings (see [B-I]), one is forced to consider a reverse Hölder condition weaker than RH_p, leading to the class of weights which we denote as WRH_p. As this condition is not as well understood as RH_p, we shall give some fairly general examples of WRH_p weights.

Let us first introduce some terminology and notation. Let $\Omega \subseteq \mathbf{R}^n$ be a fixed open set. By a weight on Ω, we mean any non-negative function on Ω, which is not identically zero. Since we are concerned with integrals throughout, a "set" will mean a measurable set, and sets of measure zero do not concern us. A "cube" will always refer to a cube in Ω whose faces are perpendicular to coordinate axes. The sidelength of a cube Q will be denoted by $l(Q)$. We say two cubes are adjacent if their closures intersect, but their interiors are disjoint. For any set E and weight w, we write $|E|$ for the Lebesgue measure of E, $w(E) = \int_E w$, and

$$\|w\|_{p,E} = \left(\frac{1}{|E|} \int_E w^p(x)\, dx \right)^{1/p}, \qquad p \in \mathbf{R}\backslash\{0\}$$

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We also write $\|w\|_{\infty,Q} = \operatorname*{ess\,sup}_{x\in Q} w(x)$. We shall be concerned with reverse Hölder conditions of either of the two following forms

$$\|w\|_{p,Q} \le K\|w\|_{q,\sigma Q}, \qquad \text{whenever } \sigma Q \subseteq \Omega. \tag{1.1}$$

$$\|w\|_{p,Q} \le K\|w\|_{q,Q}, \qquad \text{whenever } \sigma Q \subseteq \Omega. \tag{1.2}$$

Here $0 < q < p \le \infty$, $K > 1$, and σQ is the concentric dilate of a cube Q by a factor $\sigma \ge 1$. If $\sigma > 1$ and $\sigma Q \subseteq \Omega$, we say Q is "σ-allowable" (or "allowable"). We denote the Hardy-Littlewood maximal operator by M and, for any exponent $1 < p < \infty$, we shall write $p' = p/(p-1)$.

We denote the class of weights satisfying (1.1) by $WRH^{\Omega}_{p,q;\sigma}$ if $\sigma > 1$, and by $RH^{\Omega}_{p,q}$ if $\sigma = 1$. For $\sigma > 1$, we denote by $RH^{\Omega,\mathrm{loc}}_{p,q;\sigma}$ the class of weights satisfying (1.2). Given $w \in WRH^{\Omega}_{p,q}$, we define $WRH^{\Omega}_{p,q;\sigma}(w)$ to be the smallest constant K for which (1.1) is true; a similar notation is employed for all other weight spaces (we shall term this best constant the "norm" of the weight in the weight space). If A, B are positive quantities, we shall write $A \prec B$ to indicate that A is bounded above by a constant dependent only on B; if the bound for A depends on a set S of quantities, we write $A \prec S$.

We shall use the following basic facts about the $WRH^{\Omega}_{p,q;\sigma}$, $RH^{\Omega,\mathrm{loc}}_{p,q;\sigma}$, and $RH^{\Omega}_{p,q}$:

(A) All three types of weight spaces are independent of q (for $0 < q < p$) and the first two are independent of $\sigma > 1$. Therefore, we shall usually drop references to q and σ in future, assuming $q = p/2$ and $\sigma = 20$ (this choice of σ simplifies the proof of Theorem 2.6). Also, $WRH^{\Omega}_{p,q;\sigma}(w) \prec \{p, q, \sigma, n, WRH^{\Omega}_{p,p/2;20}(w)\}$, and the corresponding control statements for RH^{Ω}_{p} and $RH^{\Omega,\mathrm{loc}}_{p}$ are also true.

(B) If $w \in RH^{\Omega}_{p}$ then $w \in RH^{\Omega}_{p+\epsilon}$ for some ϵ, where $1/\epsilon \prec \{p, n, RH^{\Omega}_{p}(w)\}$ (for any $0 < p < \infty$). The corresponding results for the other spaces are also true.

(C) If $w \in RH^{\Omega}_{p}$ for some $p > 1$, then $w \in A^{\Omega}_{s}$ for some $1 < s < \infty$; conversely, if $w \in A^{\Omega}_{p}$ for some $1 \le p$, then $w \in RH^{\Omega}_{s}$ for some $1 < s < \infty$. This result is also true for $RH^{\Omega,\mathrm{loc}}_{p}$ and $A^{\Omega,\mathrm{loc}}_{p}$.

The space A^{Ω}_{p} mentioned in (C) is the space of weights satisfying

$$\|w\|_{1,Q} \le K\|w\|_{1-p',Q} \qquad \text{whenever } Q \subseteq \Omega.$$

This is the well-known weight condition of Muckenhoupt. $A^{\Omega,\mathrm{loc}}_{p}$ is defined by the obvious modification to the scope of this inequality. For RH^{Ω}_{p}, (A) is trivial; (B) and (C) are contained in [G] and [C-F]. For WRH^{Ω}_{p}, (A) is due to Iwaniec and Nolder [I-N], and (B) can be found in [B-I]. Using (A)–(C) above, it is easy to show that $RH^{\Omega}_{1} = \bigcup_{p<\infty} A^{\Omega}_{p} \equiv A^{\Omega}_{\infty}$ and also that $w \in RH^{\Omega}_{p}$ if and only if $w^p \in A^{\Omega}_{\infty}$ for any $0 < p < \infty$. It also follows that if $w \in RH^{\Omega}_{p}$ for some $p > 0$, then $1/w \in RH^{\Omega}_{q}$ for some $q > 0$.

Most of the versions of (A)–(C) for $RH^{\Omega,\mathrm{loc}}_{p}$ follow immediately from the corresponding version for RH^{Ω}_{p}, since $w \in RH^{\Omega,\mathrm{loc}}_{p}$ if and only if $w \in RH^{Q}_{p}$, for all allowable Q (with a uniform RH_p norm), and $A_p^{\Omega,\mathrm{loc}}$ is "local" in a similar sense. The one exception is the fact that $RH_p^{\Omega,\mathrm{loc}}$ is independent of $\sigma > 1$. To see that this is true, one simply dissects an arbitrary $(1+\sigma)/2$-allowable cube into 2^n σ-allowable subcubes. By combining the defining inequality over all the subcubes, it follows easily that $RH^{\Omega,\mathrm{loc}}_{p,q;\sigma} \subseteq RH^{\Omega,\mathrm{loc}}_{p,q;\frac{1+\sigma}{2}}$. Since the reverse implication is trivial, iteration gives the required result.

We say a positive Borel measure μ is doubling (on the set Ω) if there is some $C < \infty$ for which $\mu(\sigma Q) < C\mu(Q)$, for all σ-allowable cubes Q and some fixed $\sigma > 1$. We denote this class of measures by $\mathbf{D}^\Omega_\sigma$, or simply $\mathbf{D}^\Omega$. If f is a weight, we shall write $f \in \mathbf{D}^\Omega$ in place of the more awkward $f(x)\,dx \in \mathbf{D}^\Omega$. It is well-known that $RH_p^{\Omega,\mathrm{loc}} \subset \mathbf{D}^\Omega$ (this result is essentially contained in [C-F]). The following lemma is useful in dealing with the $\mathbf{D}^\Omega$ condition.

Lemma 1.3. *$\mathbf{D}^\Omega_\sigma$ is independent of $\sigma > 1$. Moreover, for any $r > 0$, $\mu \in \mathbf{D}^\Omega$ if and only if there is some constant $C = C_r$ such that $\mu(Q') \le A\mu(Q)$ whenever Q, Q' are adjacent, $l(Q') \le r \cdot l(Q)$, and $(1+2r)Q \subseteq \Omega$.*

Proof. Suppose $1 < \tau < \sigma$ and let $\sigma' = (1+\sigma)/2$. If $\mu \in \mathbf{D}^\Omega_\sigma$ then, by slicing an arbitrary σ'-allowable cube Q into subcubes Q_k ($1 \le k \le 2^n$), each of sidelength $l(Q)/2$, we see that

$$\mu(\sigma' Q) \le 2^n \sum_{k=1}^{2^n} \mu(\sigma Q_k) \le C \sum_{k=1}^{2^n} \mu(Q_k) = C\mu(Q).$$

Thus $\mathbf{D}^\Omega_{\sigma'} \subseteq \mathbf{D}^\Omega_\sigma$. By induction, we get $\mathbf{D}^\Omega_{1+2^{-m}\delta} \subseteq \mathbf{D}^\Omega_\sigma$ for every $m > 0$, where $\delta = \sigma - 1$. Let $\tau_1 = 1 + 2^{-m_0}\delta$, where m_0 is the smallest integer m for which $1 + 2^{-m}\delta < \tau$; also let $\tau_2 = \tau/\tau_1$. Then $\tau_2 < \tau_1$, since $\tau_1\tau_2 = \tau < \tau_1^2$. If Q is τ-allowable, then Q and $\tau_2 Q$ are both τ_1-allowable (and $\mu \in \mathbf{D}^\Omega_{\tau_1}$). Therefore,

$$\mu(\tau Q) \le C\mu(\tau_2 Q) \le C\mu(\tau_1 Q) \le C^2\mu(Q).$$

Conversely, iteration of the defining inequality for $\mathbf{D}^\Omega_\tau$ gives $\mathbf{D}^\Omega_{\tau^n} \subseteq \mathbf{D}^\Omega_\tau$ for all $n > 0$. Choosing n so large that $\tau^n > \sigma$ gives $\mathbf{D}^\Omega_\sigma \subseteq \mathbf{D}^\Omega_{\tau^n} \subseteq \mathbf{D}^\Omega_\tau$, as required.

Suppose that $\mu \in \mathbf{D}^\Omega$ and that Q, Q' are as in the statement of the lemma. Then $Q' \subset (1+2r)Q$, and so $\mu(Q') \le \mu((1+2r)Q) \le C\mu(Q)$. Conversely, the annulus $((1+2r)Q)\backslash Q$ can be covered with a finite number (dependent on n, r) of cubes adjacent to Q, with sidelength r times that of Q. Thus $\mu((1+2r)Q) \le C\mu(Q)$, and so $\mu \in \mathbf{D}^\Omega$, as required. □

Let us add the following to our list of basic facts about weight spaces:

(D) $w \in WRH_p^\Omega$ if and only if $w^p \in WRH_1^\Omega$. In fact, for any $0 < q < p$, $WRH^\Omega_{p,q;\sigma}(w) = (WRH^\Omega_{1,q/p;\sigma}(w^p))^{1/p}$. The corresponding statements for RH_p^Ω and $RH_p^{\Omega,\mathrm{loc}}$ are also true.

(E) If $w \in WRH_p^\Omega$ and $w^\epsilon \in \mathbf{D}^\Omega$ for some $\epsilon > 0$, then $w \in RH_p^{\Omega,\mathrm{loc}}$.

The first statement in (D) follows from the second statement and (A) above; the second statement is simple to verify. (E) is trivial: if $w^\epsilon \in \mathbf{D}^\Omega$, then $\|w\|_{p,Q} \le K\|w\|_{\epsilon,2Q} \le 2^n K\|w\|_{\epsilon,Q}$ for all allowable Q.

The weight conditions RH_p^Ω and A_p^Ω have been extensively studied ([G-R] is a good source for their theory), and are much better understood than WRH_p^Ω. Therefore, we shall begin by giving some fairly general examples of WRH_p^Ω weights.

It is known that non-negative subharmonic functions on Ω (and more generally non-negative subsolutions in Ω of any self-adjoint elliptic partial differential equations [Mo]) satisfy (1.1) with $p = \infty$; therefore such fuctions are WRH_∞^Ω weights if they are not identically zero. Note that, in contrast to RH_p^Ω weights, WRH_p^Ω weights can grow arbitrarily fast (for example, $f \in WRH_\infty^{\mathbf{R}^2}$, if $f(z) = |e^{e^z}|$ for all $z \in \mathbf{C} \equiv \mathbf{R}^2$). Convex functions are subharmonic, and so if f is convex, non-negative, and not identically zero, then $f \in WRH_\infty^{\mathbf{R}^n}$. We shall weaken the notion of convexity to produce more examples of weights in WRH_p^Ω $(0 < p \le \infty)$. First, let us state the following easy geometrical lemma.

Lemma 1.4. *Suppose a cube Q is sliced into 3^n subcubes of equal size. Let $Q_0 = (1/3)Q$ be the central subcube and let $\{Q_i\}_{i=1}^{2^n}$ be the corner subcubes, i.e. those which include a vertex of Q. Furthermore, suppose $x_i \in Q_i$ $(i = 1, \dots, 2^n)$. Then $\operatorname{co}\left(\{x_i\}_{x=1}^{2^n}\right) \supseteq Q_0$, where $\operatorname{co} S$ indicates the convex hull of the set S.*

Proof. We may assume without loss of generality that $Q_0 = \prod_{i=1}^{n}[-1,1]$ (so that $Q = \prod_{i=0}^{n}[-3,3]$). The result is obviously true for $n = 1$, so we assume inductively that it is true for all dimensions $n \le k$, where $k \ge 1$. For dimension $n = k+1$, let us order the corner subcubes Q_i so that, for all $1 \le i \le 2^k$, $Q_i = P_i \times [-3,-1]$ and $Q_{i+2^k} = P_i \times [1,3]$, where $\{P_i\}_{i=1}^{2^k}$ are the corner subcubes of the $\mathbf{R}^k$-cube $P = \prod_{i=1}^{k}[-3,3]$. For each $1 \le i \le 2^k$, the convex hull of $\{x_i, x_{i+2^k}\}$ is a line segment which includes points $y_{i,t} = (u_{i,t}, t)$ for all $-1 \le t \le 1$, where $u_{i,t} \in P_i$. It follows from the inductive hypothesis that $\operatorname{co}\left(\{y_{i,t}\}_{i=1}^{2^k}\right) \supseteq P_0 \times \{t\}$, where $P_0 = (1/3)P$. The inductive step now follows easily, so we are done. $\square$

Definition 1.5. If g is a real-valued function on $\mathbf{R}^n$, and $\{\, x \in \mathbf{R}^n \mid g(x) < \alpha \,\}$ is convex for every $\alpha \in \mathbf{R}$, we say g is convex-contoured.

It is easy to see that convex functions are convex-contoured, as is any radially increasing function. On the other hand, $g(x) = \arctan|x|$ is an example of a convex-contoured function on $\mathbf{R}^n$ which is not convex (or even subharmonic). There are also, of course, subharmonic functions which are not convex-contoured (for example, $g(z) = |\cos z|$ for $z \in \mathbf{C}$).

Proposition 1.6. *If $u \in RH_p^\Omega$ for some $0 < p \le \infty$, and g is a convex-contoured weight, then $w \equiv ug \in WRH_p^\Omega$.*

Proof. Note first that any convex-contoured weight g is locally bounded. In fact, if Q_0 is any cube, then g attains its maximum value over Q_0 at one of the vertices, v. Writing $\alpha = g(v)$, Lemma 1.4 ensures that the set $\{g(x) \geq \alpha\}$ includes one of the corner subcubes Q_i of $Q = 3\,Q_0$. Let us fix an exponent q such that $0 < q < p$, and assume that Q is allowable. Since $RH_p^\Omega \subset \mathbf{D}^\Omega$,

$$\|w\|_{p,Q_0} \leq \alpha \|u\|_{p,Q_0} \leq K\alpha \|u\|_{q,Q_i} \leq K\|w\|_{q,Q_i} \leq 3^{n/q} K\|w\|_{q,3Q_0}$$

where K depends on $RH_p^\Omega(w)$. $\square$

The simple geometrical assumption in Proposition 1.6 that g is convex-contoured is not crucial; it is easy to alter the above proof to handle certain weaker conditions. For example, it suffices to assume only that there exist $C, \epsilon > 0$ and $\sigma > 1$, such that for any σ-allowable Q, there is a subset S of σQ for which $|S|/|Q| > \epsilon$ and

$$\operatorname*{ess\,sup}_{x\in Q} g(x) \leq C \operatorname*{ess\,inf}_{x\in S} g(x).$$

In particular, it is easy to see that $u \cdot \chi_S \in WRH_p^\Omega$ for all $u \in RH_p^\Omega$, if $S \subset \mathbf{R}^n$ is the "checkerboard" set for which $x \in S$ if and only if the sum of the integer parts of the coordinates of x is even.

Obviously, $RH_p^\Omega \subseteq RH_p^{\Omega,\mathrm{loc}} \subseteq WRH_p^\Omega$. Using Proposition 1.6, it is easy to see that the second containment is always strict. For example, if σQ is an allowable cube, then $w \equiv \chi_{\Omega\setminus Q} \in WRH_\infty^\Omega$, but $w \notin RH_p^{\Omega,\mathrm{loc}}$ for any $p > 0$, since $w^p \notin \mathbf{D}^\Omega$. On the other hand, if $w \in RH_p^{\Omega,\mathrm{loc}}$, then $w \in RH_q^\Omega$ for some $q < p$, where q depends only on p, $RH_p^{\Omega,\mathrm{loc}}(w)$, and the dimension n; this fact follows from Corollary 3.17 of [Sta]). For $\Omega = \mathbf{R}^n$, $RH_p^\Omega = RH_p^{\Omega,\mathrm{loc}}$ but, if $\Omega \neq \mathbf{R}^n$, these spaces are distinct. For example, it is easy to see that $w_r(x) = (\mathrm{dist}(x, \mathbf{R}^n\setminus\Omega))^{-r} \in RH_\infty^{\Omega,\mathrm{loc}}$, for all $r > 0$. However, $w_r \notin RH_\infty^\Omega$. In fact, if we choose a cube Q such that $\partial Q \cap \partial\Omega$ is non-empty, then $w_r \notin L^{n/r}(Q)$, and so $w \notin RH_{n/r}^\Omega$. It follows that, if $\Omega \neq \mathbf{R}^n$, then $RH_p^{\Omega,\mathrm{loc}} \not\subseteq RH_q^\Omega$ for all $0 < p, q \leq \infty$.

2. Pointwise Multipliers

The examples of WRH_p^Ω weights given in the first section lead us to ask what conditions on a weight f guarantee that, for all $0 < p < \infty$, $f \cdot RH_p^\Omega \equiv \{\, fw \mid w \in RH_p^\Omega \,\}$ is a subset of WRH_p^Ω. More generally, one can ask when it is true that $f \cdot S \subseteq T$, where S, T are reverse Hölder spaces. In this section, we shall show (Theorem 2.9) that a quantitative version of such a containment can only occur if $S \subseteq T$. Thus, the only possible cases are $f \cdot RH_p^\Omega \subseteq RH_q^\Omega$, $f \cdot RH_p^\Omega \subseteq WRH_q^\Omega$, $f \cdot WRH_p^\Omega \subseteq WRH_q^\Omega$, and local versions of the first two (for some particular indices $0 < q \leq p \leq \infty$ in each case). It is not hard to classify f in the first case (Theorem 2.3), but the third case (Theorem 2.6) presents considerably more difficulties. In the second case, we can only give a partial answer (Theorem 2.4). We need a couple of preliminary lemmas, the first of which is the version of the Whitney covering lemma found in [Sa].

Lemma 2.1. *Given $R \geq 1$, there is a dimensional constant C_R such that if G is an open subset of $\mathbf{R}^n$, then $G = \bigcup_k Q_k$, where the cubes Q_k are disjoint, $\sum_k \chi_{RQ_k} \leq C_R \chi_G$ and*

$$5R \leq \frac{\operatorname{dist}(Q_k, G^c)}{\operatorname{diam}(Q)} \leq 15R.$$

The next lemma shows that certain weak versions of the A_∞ condition are equivalent to the WRH_1^Ω condition. We shall only need the equivalence of (ii) and (iv), but we include (i), as it is interesting for its own sake.

Lemma 2.2. *For any fixed $\sigma > 1$, the following conditions on a weight w are equivalent.*

(i) *There exist constants $0 < \alpha < 1$ and $0 < \beta < 1/C_2$, such that if E is a subset of a σ-allowable cube Q, and $|E|/|Q| \leq \alpha$, then $w(E)/w(\sigma Q) \leq \beta$. C_2 is the constant in Lemma 2.1, for $R = 2$.*
(ii) *There exist constants $C, \epsilon > 0$ such that if E is a subset of an σ-allowable cube Q, then $w(E)/w(\sigma Q) \leq C\left(|E|/|Q|\right)^\epsilon$.*
(iii) *$w \in WRH_p^\Omega$ for some $p > 1$.*
(iv) *$w \in WRH_1^\Omega$.*

Proof. For the sake of simplicity, we shall assume $\sigma = 2$. To see that (i) implies (ii), let us first write $C = (500\sqrt{n})^n$. It suffices to show that, for all positive integers k, $w(E)/w(2Q) \leq \beta^k C_2^{k-1}$ whenever $|E|/|Q| \leq \alpha^k/C^{k-1}$. The statement is true for $k = 1$, so we assume inductively that it is true for $k = k_0 \geq 1$. If $|E|/|Q| \leq \alpha^{k_0+1}/C^{k_0}$, we apply Lemma 2.1, with $R = 2$, to the set $G = \{\, x \in Q \mid M\chi_E(x) > \alpha/(100\sqrt{n})^n \,\} \supset E$ to get that $G = \bigcup_k Q_k$, where the cubes Q_k are disjoint, $\sum_k \chi_{2Q_k} \leq C_2 \chi_G$ and

$$10 \leq \frac{\operatorname{dist}(Q_k, G^c)}{\operatorname{diam}(Q)} \leq 30.$$

It follows that $100\sqrt{n}Q_k$ intersects G^c and that $|E_k|/|Q_k| < \alpha$, where $E_k = E \cap Q_k$. Therefore $w(E_k)/w(2Q_k) \leq \beta$, and so

$$w(E) = \sum_k w(E_k) = \sum_k \frac{w(E_k)}{w(2Q_k)} w(2Q_k) \leq \beta C_2 w(G).$$

But by a standard weak-type estimate on M (see [Ste, p. 5]),

$$|G| \leq \frac{(500\sqrt{n})^n}{\alpha} |E| \leq \frac{\alpha^{k_0}}{C^{k_0-1}} |Q|$$

and so $w(E) \leq \beta C_2 \cdot \beta^{k_0} C_2^{k_0-1} w(2Q)$, which completes the inductive step.

Let us now prove that (ii) implies (iii). We work with an arbitrary but fixed cube Q and normalise so that $w(2Q) = |Q|$. Letting $E_k = \{x \in Q \| 2^k \le w(x) < 2^{k+1}\}$, it is clear that $|E_k| \le 2^{-k} w(Q) \le 2^{-k}|Q|$, and so $w(E_k) < C2^{-\epsilon k} w(2Q)$. Thus,

$$\begin{aligned}\int_Q w^{1+\epsilon/2} &\le w(2Q) + \sum_{k=0}^{\infty} 2^{\epsilon(k+1)/2} w(E_k) \\ &\le w(2Q)\left(1 + 2^{\epsilon/2}\sum_{k=0}^{\infty} C2^{-\epsilon k/2}\right) \\ &\le C' w(2Q) = C'|Q|,\end{aligned}$$

where $C' = C'(C,\epsilon)$. Thus $\|w\|_{1+\epsilon/2,Q} \le C'' = 2^n C'' \|w\|_{1,2Q}$, and so $w \in WRH^{\Omega}_{1+\epsilon/2}$, as required.

Since trivially (iii) $\implies$ (iv) and (ii) $\implies$ (i), and we know from Section 1 that (iv) $\implies$ (iii), we need only prove that (iii) $\implies$ (ii) to finish the proof. If $E \subseteq Q$, and $w \in WRH^{\Omega}_p$ for some $p > 1$, then

$$\frac{1}{|Q|}\int_E w = \|w\chi_E\|_{Q,1} \le \|w\|_{p,Q} \|\chi_E\|_{p',Q} \le C\|w\|_{1,2Q}\left(\frac{|E|}{|Q|}\right)^{1/p'},$$

which proves (ii) with $\epsilon = 1/p'$. □

It is important for our purposes to note that, in proving that (iv) implies (ii), we can choose C and ϵ to depend only on n, σ and $WRH^{\Omega}_1(w)$. We are now ready to state and prove the first, and easiest, of our pointwise multiplier theorems. The case $p = q$ of this theorem was previously answered by Johnson and Neugebauer [J-N].

Theorem 2.3.

(i) *If* $0 < q \le p < \infty$, *then* $f \cdot RH^{\Omega}_p \subseteq RH^{\Omega}_q$ *if and only if* $f \in \bigcap_{r<s} RH^{\Omega}_r$, *where* $s = pq/(p-q)$ ($s = \infty$ *if* $p = q$).

(ii) *If* $0 < q \le \infty$, *then* $f \cdot RH^{\Omega}_\infty \subseteq RH^{\Omega}_q$ *if and only if* $f \in RH^{\Omega}_q$.

Proof. We shall first prove (i). (D) allows us to reduce our task to the case $q = 1$, since $f \cdot RH^{\Omega}_p \subseteq RH^{\Omega}_q$ if and only if $f^q \cdot RH^{\Omega}_{p/q} \subseteq RH^{\Omega}_1$, and $f \in \bigcap_{r<s} RH^{\Omega}_r$ if and only if $f^q \in \bigcap_{r<(p/q)'} RH^{\Omega}_r$.

Suppose that $f \in \bigcap_{r<p'} RH^{\Omega}_r$, and that $w \in RH^{\Omega}_p$. Thus $w \in RH^{\Omega}_t$ for some $t > p$; in fact, $\|w\|_{t,Q} \le C\|w\|_{-\epsilon,Q}$ for some $0 < \epsilon < 1$, and all $Q \subseteq \Omega$. Since $t' < p'$,

$$\begin{aligned}\|wf\|_{1,Q} &\le \|w\|_{t,Q}\|f\|_{t',Q} \le C'\|w\|_{t,Q}\|f\|_{\epsilon/2,Q} \\ &\le C'\|w\|_{t,Q}\|wf\|_{\epsilon,Q}/\|w\|_{-\epsilon,Q} \le C\cdot C'\|wf\|_{\epsilon,Q},\end{aligned}$$

where the first and third inequalities are by Hölder's inequality. Thus, $w \in RH^{\Omega}_{1,\epsilon} = RH^{\Omega}_1$.

Conversely, suppose that $f \cdot RH^{\Omega}_p \subseteq RH^{\Omega}_1$ for some $1 < p < \infty$. In particular, $f \cdot 1 = f \in RH^{\Omega}_1$, and so $f^{1/p} \in RH^{\Omega}_p$, which in turn implies that $f^{1+1/p} \in RH^{\Omega}_1$. Continuing this iteration, we see that $f \in RH^{\Omega}_{r_m}$, where $r_m = \sum_{k=0}^{m} 1/p^k$. But $r_m \to p'$ $(m \to \infty)$, and so $f \in \bigcap_{r<p'} RH^{\Omega}_r$.

The proof of (ii) is quite similar. Choosing $w = 1$, we see that $f \cdot 1 \in RH^{\Omega}_q$ is a necessary condition. To prove the converse, suppose first that $q < \infty$. If $f \in RH^{\Omega}_q$, then $f \in RH^{\Omega}_{tq}$ for some $t > 1$. Also, $\|w\|_{t'q,Q} \le C\|w\|_{-\epsilon,Q}$, for some $0 < \epsilon < q$ and all $Q \subseteq \Omega$. Thus,

$$\begin{aligned} \|wf\|_{q,Q} &\le \|w\|_{t'q,Q}\|f\|_{tq,Q} \le C\|w\|_{t'q,Q}\|f\|_{\epsilon/2,Q} \\ &\le C\|w\|_{t'q,Q}\|wf\|_{\epsilon,Q}/\|w\|_{-\epsilon,Q} \le C'\|wf\|_{\epsilon,Q}, \end{aligned}$$

and so $wf \in RH^{\Omega}_q$. If $q = \infty$, the proof follows in the same manner, except that the first use of Hölder's inequality is replaced by the inequality $\|wf\|_{\infty,Q} \le \|w\|_{\infty,Q}\|f\|_{\infty,Q}$. □

It is easily seen from the above proof that if, for some weight f, $f \cdot RH^{\Omega}_p \subseteq RH^{\Omega}_q$, then a quantitative version of the same statement is true, namely $RH^{\Omega}_p(fw) \prec RH^{\Omega}_p(w)$. Similarly, containment leads to quantitatively controlled containment in Theorem 2.6; in Theorem 2.4, one obtains controlled containment, as long as f satisfies the stated sufficient condition.

We now state an analog of the above theorem for the case $f \cdot RH^{\Omega}_p \subseteq WRH^{\Omega}_q$; we omit the proof which is easily obtained by a few minor modifications to the above proof ("Q" becomes "σQ" in a few places). The one part which cannot be carried over is the iteration in the proof of the converse part of (i); this is why we cannot give a full-strength analog of (i) (although it seems likely that such an analog is true).

Theorem 2.4.

(i) *If $0 < q \le p < \infty$, a necessary condition for $f \cdot RH^{\Omega}_p \subseteq WRH^{\Omega}_q$ is that $f \in WRH^{\Omega}_q$; a sufficient condition is that $f \in \bigcap_{r<s} WRH^{\Omega}_r$, where $s = pq/(p-q)$ ($s = \infty$ if $p = q$).*

(ii) *If $0 < q \le \infty$, then $f \cdot RH^{\Omega}_\infty \subseteq WRH^{\Omega}_q$ if and only if $f \in WRH^{\Omega}_q$.*

Various other analogs of Theorems 2.3 and 2.4 could be stated. For example, it follows as an easy corollary to Theorem 2.3 that if we replace every RH^{Ω}_* space with the corresponding $RH^{\Omega,\mathrm{loc}}_*$ space, the statement of Theorem 2.3 remains valid. Also, one can prove the version of Theorem 2.4 where $RH^{\Omega,\mathrm{loc}}_*$ replaces WRH^{Ω}_* in exactly the same fashion as the original proof.

The following special case of Theorem 2.4 is interesting, as it answers the question posed at the beginning of this section; it also sheds some light on the checkerboard set example given after Proposition 1.6. We omit the obvious proof.

Corollary 2.5. *$u \cdot RH_p^\Omega \subseteq WRH_p^\Omega$ for all $0 < p \le \infty$ if and only if $u \in WRH_\infty^\Omega$. In particular, $u = \chi_S$ has this property if and only if there exists some $\epsilon > 0$ for which $|S \cap (2\,Q)| > \epsilon |Q|$ for all cubes Q for which $|S \cap Q| > 0$.*

Theorem 2.6.

(i) *If $0 < q \le p < \infty$, then $f \cdot WRH_p^\Omega \subseteq WRH_q^\Omega$ if and only if $f \in \bigcap_{r<s} RH_r^{\Omega,\mathrm{loc}}$, where $s = pq/(p-q)$ ($s = \infty$ if $p = q$).*

(ii) *If $0 < q \le \infty$, then $f \cdot WRH_\infty^\Omega \subseteq WRH_q^\Omega$ if and only if $f \in RH_q^{\Omega,\mathrm{loc}}$.*

Most of the statement of this final theorem can be proved by modifying the proof of Theorem 2.3. There is however one major obstacle to be overcome: we must show that if $f \cdot WRH_p^\Omega \subseteq WRH_q^\Omega$, then $f \in \mathbf{D}^\Omega$. If we assume that $WRH_q^\Omega(wf) \prec WRH_p^\Omega(w)$, this is not difficult to prove. Let us consider, for example, the case $q = 1$. If $f \notin \mathbf{D}^\Omega$, then Lemma 1.3 implies that, for each positive integer k, there are adjacent allowable cubes Q_k, Q_k', for which $l(Q_k') < l(Q_k)/4k$ but $f(Q_k) < f(Q_k')/k$. Letting $S_k = (\mathbf{R}^n \backslash 3\,Q_k) \cup Q_k \cup Q_k'$, it is easy to see that $\{WRH_\infty^\Omega(\chi_{S_k})\}_{k=1}^\infty$ is a bounded sequence of numbers (this is in fact Lemma 2.7 for a sequence of length 1). Letting $Q = (3/2)Q_k$ and $E = Q_k'$, we see that $E \subseteq Q$, $|E|/|Q| < (6k)^{-n}$, and that

$$\int_E f\chi_{S_k} = f(E) > \frac{k}{k+1}\left(\int_{2\,Q} f\chi_{S_k}\right).$$

By Lemma 2.2, the sequence $\{WRH_1^\Omega(f\chi_{S_k})\}_{k=1}^\infty$ must be unbounded, which contradicts our additional assumption.

To eliminate this quantitative control, we must essentially find a single weight w which does the work of all the weights χ_{S_k} above. If the cubes can be chosen so that the dilates $4\,Q_k$ are disjoint, our task is easy: we let $w = \chi_S$, where $S = \bigcup_{i=1}^\infty S_k$. Arguing as before, it follows that $wf \notin WRH_1^\Omega$. Because of disjointness, it is not difficult to see that $w \in WRH_\infty^{\mathbf{R}^n}$, contradicting our hypothesis.

This argument does not extend to the case where the cubes Q_k intersect, or if they are too close together, because the cubes will then "interfere" with each other. Some of the more general cases can be handled by more elaborate versions of this argument; the task of altering the cubes and the weight w, so that the cubes do not interfere with each other will necessitate some extra technicalities. The more elaborate weights we shall construct will be associated with certain sequences of quadruples $\{(P_k, P_k', A_k, d_k)\}_{k=k_1}^{k_2}$, where $-\infty \le k_1 \le k_2 \le \infty$, P_k and P_k' are adjacent cubes, A_k is a cube containing the dilates $50\,P_j$ for all $j > k$, and $2.9 \le d_k \le 3$. These quadruples will be such that $l(P_k')$ and $l(A_k)$ are less than $l(P_k)$ (in particular, the sidelengths $l(P_k)$ form a decreasing sequence), and A_k is k-conditioned, where we say a set A is "k-conditioned" (or "conditioned with respect to (P_k, P_k', d_k)") if A is fully contained in one of the sets P_k, P_k', $S_k \equiv d_k P_k \backslash (P_k \cup P_k')$, $\mathbf{R}^n \backslash d_k P_k$, which partition $\mathbf{R}^n$. Letting $B_k = \bigcup_{j>k} 3\,P_j$, the associated weights will have the form

$$w(x) = \begin{cases} a_k, & x \in \mathbf{R}^n \backslash (S_k \cup B_k) \\ b_k, & x \in S_k \backslash B_k. \end{cases}$$

where $0 \le b_k \le a_k$. Furthermore, we have the "continuity condition"

$$a_{k+1} = \begin{cases} b_k, & \text{if } B_k \subset S_k \\ a_k, & \text{if } B_k \subset \mathbf{R}^n \backslash S_k. \end{cases}$$

It follows that $\max_{x \in S_k} w(x) = b_k$ and $\max_{x \in \mathbf{R}^n \backslash S_k} w(x) = a_k$. We shall denote by W the class of all such weights. For our purposes, b_k will be very small compared with a_k.

The weights χ_{S_k} previously considered are of this type ($k_1 = k_2 = 0$, $a_0 = 1$, $b_0 = 0$). Since the weights in W generalize these weights, and the k-conditioning of the sets A_k is designed to stop the cubes interfering with each other, the following result should come as surprise.

Lemma 2.7. *If $w \in W$, then $w \in WRH_\infty^{\mathbf{R}^n}$. In fact, $WRH_\infty^{\mathbf{R}^n}(w) \le C_n$, where C_n depends only on n.*

Proof. We will prove the lemma with $C_n = 5^n / l_n$, where $l_n = 1 - \left(\frac{3}{4}\right)^n - \left(\frac{1}{40}\right)^n$. Without loss of generality, we assume $k_1 = -\infty$ and $k_2 = \infty$ (we can choose $a_k = b_k$, when k is outside a given range). Fixing a cube Q, we have $|P_{k-1}| \ge |Q| > |P_k|$ for some integer k. Now, $20\,Q$ is j-conditioned for all except possibly one integer $j < k$. To see this, note that if $l < k$ is the largest exceptional integer, then $20\,Q$ intersects $3\,P_l$, and so $20\,Q \subset 50\,P_l$ (since $|Q| \le |P_l|$). Thus $20\,Q \subset A_j$ for any $j < l$, and so $20\,Q$ is j-conditioned for all $j < l$.

We therefore assume that $20\,Q$ is j-conditioned for all $j < k$, $j \ne l$. If $4\,Q$ is l-conditioned, then the assumptions on w imply that $w(x) \le c$ for all $x \in 4Q$, with equality for $x \in 4\,Q \backslash B_{k-1}$ (c is either a_k or b_k). It follows from our hypotheses that $|B_k| < 10^{-n}|P_k|$, and so $|B_{k-1}| < |3\,P_k| + |B_k| \le (3^n + 10^{-n})|Q|$. Therefore, $|\{x \in 4\,Q \mid w(x) = c\}| \ge (4^n - 3^n - 10^{-n})|Q|$, and so $\|w\|_{1,4Q} \ge l_n \|w\|_{\infty,Q}$.

We must now take care of the alternative case when $4\,Q$ is not l-conditioned. Let us first show that for any cube Q_0 which is not l-conditioned, $|(5\,Q_0)\backslash S_l| > |Q_0|$. This is easy to see if $|Q_0| \ge |P_l|$, so suppose $|Q_0| < |P_l|$. If Q_0 intersects $\mathbf{R}^n \backslash (3\,P_l)$, then $|(5\,Q_0)\backslash(3\,P_l)| > |Q_0|$, whereas if Q_0 intersects P_l, then $|(5\,Q_0) \cap P_l| > |Q_0|$. The last way that Q_0 can fail to be l-conditioned is if Q_0 intersects P'_l. In this case, it is clear that $|(5\,Q_0) \cap P'_l| > |Q_0|$ if $|P'_l| \ge |Q_0|$, while $|(5\,Q_0) \cap P_l| > |Q_0|$ if $|P'_l| < |Q_0|$. Letting $Q_0 = 4\,Q$, we see that $|(20\,Q) \cap (\mathbf{R}^n \backslash S_l)| > |4\,Q|$. Proceeding as in the previous case, we see that $\|w\|_{1,\,20Q} \ge 5^{-n} l_n \|w\|_{\infty,Q}$, which finishes the proof of the lemma. □

We are now ready to prove the main theorem. In this proof, a dilate of a cube Q will refer to rQ for any $r > 0$ (not just $r > 1$); when we need to be more precise, we refer to rQ as the r-dilate of Q.

Proof of Theorem 2.6. We shall first prove (i). As in Theorem 2.3, it suffices to do so in the case $q = 1$. If $f \in \bigcap_{r<p'} RH_r^{\Omega,\mathrm{loc}}$, it follows that for some $0 < \epsilon < 1$, all $r < p'$, all allowable Q, and some constant $C = C_r$, $\|f\|_{r,Q} \le C\|f\|_{-\epsilon,2Q}$. Suppose also that $w \in WRH_p^\Omega$, and so $w \in WRH_t^\Omega$ for some $t > p$. Since $t' < p'$,

$$\|wf\|_{1,Q} \le \|f\|_{t',Q}\|w\|_{t,Q} \le C'\|f\|_{t',Q}\|w\|_{\epsilon/2,2Q}$$
$$\le C'\|f\|_{t',Q}\|wf\|_{\epsilon,2Q}/\|f\|_{-\epsilon,2Q} \le C\cdot C'\|wf\|_{\epsilon,2Q},$$

where the first and third inequalities are by Hölder's inequality. Thus $w \in WRH_1^\Omega$.

Conversely, suppose that $f \cdot WRH_p^\Omega \subseteq WRH_1^\Omega$ for some $1 < p < \infty$. Modifying the iteration argument of Theorem 2.3, we see that $f \in \bigcap_{r<s} WRH_r^\Omega$. Because of (E), the desired result will follow if we can show that $f \in \mathbf{D}^\Omega$.

Let us first show that $f(Q) > 0$ for all allowable cubes Q. If not, then there exists an allowable Q for which $f(Q) = 0$, but $f(tQ) > 0$ for all $t > 1$. We inductively construct a sequence of cubes $\{C_k\}_{k=1}^\infty$, with associated parameters

$$a_k = \inf\{r \mid C_k \subset (1+r)Q\}$$
$$b_k = \sup\{r \mid C_k \text{ and } (1+r)Q \text{ are disjoint}\}$$

satisfying

(1) $a_1 = 1/2$,
(2) $0 < b_k < a_k/2$,
(3) $a_{k+1} < b_k/(k+1)$,
(4) $f(((1+a_{k+1})Q)\backslash Q) < f(C_k)/k$.

C_1 is easily constructed since $f(2\,Q) > 0$. Having chosen C_j, we choose $a_{j+1} > 0$ so small that (3) and (4) are satisfied. Since $f((1+a_j)Q) > 0$, it is clear that we can choose a cube C_{j+1} which has positive f-measure and for which (2) is satisfied (for $k = j+1$). Letting $S = Q \cup (\bigcup_{i=1}^\infty C_k) \cup (\mathbf{R}^n \backslash 3\,Q)$, it follows easily from (2) that $w = \chi_S \in WRH_\infty^\Omega$. Letting $E_k = C_k$ and $Q_k = (k/2)C_k$, we see that

$$\frac{|E_k|}{|Q_k|} = \left(\frac{2}{k}\right)^n \to 0 \quad (k \to \infty).$$

However, $2\,Q_k \subset b_{k-1}Q_k$ and so, by (2),

$$\int_{E_k} fw = f(E_k) > \frac{k}{k+1}\int_{2\,Q_k} fw.$$

By Lemma 2.2, $f \notin WRH_1^\Omega$, contradicting our hypothesis.

Let us assume that $f \notin \mathbf{D}^\Omega$ and arrive at a contradiction. By Lemma 1.3 there are, for every $k > 0$, adjacent 20-allowable cubes Q_k, Q'_k, for which $l(Q'_k) < l(Q_k)/2k$ but $f(Q_k) < f(Q'_k)/k$. In the discussion after the statement of Theorem 2.6, we saw that this leads to a contradiction if the dilates $4Q_k$ are disjoint. There is of course, nothing special about the dilation factors 3 and 4 in this argument. If the cubes Q_k can be chosen so that, for some $r > 1$, their r-dilates are disjoint, simple modifications to the above argument will give the required contradiction.

Therefore, we shall assume that $\{Q_k\}$ cannot be chosen to have disjoint r-dilates for any $r > 1$. We shall need to consider separately three types of cube sequences: cubes which stay about the same size, cubes whose sidelengths tend to 0, and cubes whose sidelengths grow without bound (by selecting a subsequence, all cube sequences reduce to one of these three types).

Suppose first that $0 < r < l(Q_k) < R < \infty$ for all $1 \le k$. Since no subsequence of the cubes has disjoint dilates, it follows that the sequence $\{Q_k\}$ is compactly supported in Ω. By choosing a subsequence if necessary, we can assume that $Q_k \to Q$ and $Q'_k \to \{x\}$, for some allowable cube Q and some $x \in \Omega$ (by which we mean that the vertices of Q_k converge to the corresponding vertices of Q, and the vertices of Q'_k all converge to x). But now, $f(Q_k) < f(Q'_k)/k \to 0$ as $k \to \infty$, and so $f(Q) = 0$ (by Lebesgue's dominated convergence theorem) which, as we have already seen, leads to a contradiction.

Suppose next that the cubes Q_k, Q'_k can be chosen so that $l(Q_k) \to 0$ $(k \to \infty)$. Again, we can assume the cubes Q_k are compactly supported in Ω and so, by choosing a subsequence if necessary, we can assume that $Q_k \to \{x\}$ for some $x \in \Omega$. Let A_k be the smallest cube containing $50\,Q_j$ for all $j > k$, and so $A_k \to \{x\}$. By taking a subsequence if necessary, we can assume that for all $k \in \mathbf{N}$, $2\,A_k$ is a subset of Ω and that

(a) $1000\,l(A_k) < l(Q'_k)$,
(b) $2f(A_k) < f(Q'_k)$,
(c) $f(Q_k) < 2^{-k} f(Q'_k)$,
(d) $|Q'_k| < 2^{-n(k+1)}|Q_k|$.

For $1 \le k < \infty$, we now construct adjacent cubes P_k, P'_k and a dilation factor $2.9 \le d_k \le 3$ such that A_k is k-conditioned. These cubes will be constructed by modest dilations of the cubes Q_k, Q'_k. More precisely, it will be true that

$$\frac{99}{100} Q_k \subset P_k \subseteq Q_k, \qquad Q'_k \subseteq P'_k \subset \frac{102}{100} Q'_k. \tag{2.8}$$

In particular, A_k contains $50\,P_j$ for all $j > k$.

We need to consider several cases for this construction. If A_k is conditioned with respect to $(Q_k, Q'_k, 3)$, we let $(P_k, P'_k, d_k) = (Q_k, Q'_k, 3)$. Otherwise, if A_k intersects $(101/100)Q'_k$, let P'_k be the smallest dilate of Q'_k which contains A_k, let P_k be the dilate of Q_k which is adjacent to P'_k, and let $d_k = 3$. Otherwise, if A_k intersects Q_k, we let P_k be the dilate of Q_k which is adjacent to A_k, let P'_k be the dilate of Q'_k adjacent to P_k, and let $d_k = 3$ (note that P'_k does not intersect A_k, because of the previous case). Finally, if A_k is only partially contained in $3\,Q_k$, the triple $(Q_k, Q'_k, 2.9)$ will suffice. In each case, it follows from (a) that our new cubes satisfy (2.8). The new cubes satisfy conditions very similar to (a)–(d). Specifically,

(a′) $1000\,l(A_k) < l(P'_k)$,
(b′) $2f(A_k) < f(P'_k)$,
(c′) $f(P_k) < 2^{-k} f(P'_k)$,
(d′) $|P'_k| < 2^{-nk}|P_k|$.

Now let

$$w_k(x) = \begin{cases} \dfrac{f(P'_k)}{2^k f(3\,P_k)}, & x \in S_k \\ 1, & \text{otherwise} \end{cases}$$

and $w(x) = \prod_{k=1}^{\infty} w_k(x)$. Clearly, $w \in W \subset WRH_\infty^\Omega$. We get the desired contradiction by showing that $wf \notin WRH_1^\Omega$. First note that

$$\frac{\int_{3\,P_k \setminus P_k'} f w_k}{\int_{P_k'} f w_k} = \frac{f(P_k) + \int_{S_k} f w_k}{f(P_k')} < 2^{-k} + 2^{-k} = 2^{-k+1}.$$

By construction, $w(x) \le w_k(x)$ for $x \in 3\,P_k$, with equality if $x \notin A_k$. Thus

$$\int_{3\,P_k \setminus P_k'} f w \le \int_{3\,P_k \setminus P_k'} f w_k.$$

Since w_k is constant on P_k', it follows from (b$'$) that

$$\int_{P_k'} f w \ge \frac{1}{2} \int_{P_k'} f w_k,$$

and so

$$\frac{\int_{3\,P_k \setminus P_k'} f w}{\int_{P_k'} f w} < 2^{-k+2}.$$

Since $P_k' \subset 2P_k$, it follows from (d$'$) and Lemma 2.2, that $fw \notin WRH_1^\Omega$.

Finally, we need to consider the case when $|Q_k| \to \infty$ $(k \to \infty)$. Since we are assuming that no subsequence of these cubes can be produced to have disjoint r-dilates for any $r > 1$, we can inductively produce a subsequence of these cubes whose 3/2-dilates are pairwise-intersecting. We redefine Q_k to be the k-th term of this subsequence. We can assume, in addition, that $l(Q_{k+1}) > 6\; l(Q_k)$, from which it follows that $2\,Q_k \subset 2\,Q_{k+1}$ (for all $k \in \mathbf{N}$), and hence that $\bigcup_{k=1}^{\infty} (5/2)Q_k = \mathbf{R}^n$.

We define $A_k = 50\,Q_{k-1}$ (and so $A_k \supseteq 50\,Q_j$ for all $j < k$). We can also assume that these new cubes Q_k, Q_k', and A_k satisfy conditions (a)–(d). As before, we can construct (P_k, P_k', d_k) so that A_k is k-conditioned, and

$$\frac{99}{100} Q_k \subset P_k \subseteq Q_k, \qquad Q_k' \subseteq P_k' \subset \frac{102}{100} Q_k'.$$

We can actually choose $d_k \equiv 3$, since (a) implies that $A_k \subset (5/2)Q_k$.

We shall inductively define weights u_k for $k > 0$, and then define $w(x) = \lim_{k\to\infty} u_k(x)$. This limit will exist for all $x \in \mathbf{R}^n$, because the weights u_k will be defined so that $u_k(x) = u_j(x)$ for all $j > k$, $x \in 3\,P_k$. We define $u_0 \equiv 1$, to start the induction. If $k = 1$ or if $A_k \subset P_k \cup P_k'$, we define

$$u_k(x) = \begin{cases} \dfrac{f(P_k')u_{k-1}(x)}{2^k f(3P_k)}, & x \in S_k \\ u_{k-1}(x), & \text{otherwise.} \end{cases}$$

Otherwise (i.e., if $k > 1$ and $A_k \subset S_k \equiv d_k P_k \backslash (P_k \cup P'_k)$), we define

$$u_k(x) = \begin{cases} u_{k-1}(x), & x \in S_k \\ \dfrac{2^k f(3P_k) u_{k-1}(x)}{f(P'_k)}, & \text{otherwise.} \end{cases}$$

We also write

$$w_k(x) = \begin{cases} \dfrac{f(P'_k)}{2^k f(3P_k)}, & x \in S_k \\ 1, & \text{otherwise.} \end{cases}$$

Notice that, for each $k < 0$, u_k is a constant times $\prod_{i=1}^{k} w_i$, but is normalized to ensure that $w \not\equiv 0$. Clearly, $w \in W \subset WRH_\infty^\Omega$ (our sequence of quadruples is indexed in the reverse order to that in the definition of W, but this does not matter). The proof that $fw \notin WRH_1^\Omega$ is similar to the case $|Q_k| \to 0$. First,

$$\int_{3P_k \backslash P'_k} f w_k < 2^{-k+1} \int_{P'_k} f w_k.$$

Next, there exists a constant $c_k > 0$ such that $w(x) \leq c_k w_k(x)$ for all $x \in 3P_k$, with equality if $x \notin A_k$. Since $2f(A_k) < f(P'_k)$, it follows as before that

$$\int_{3P_k \backslash P'_k} f w < 2^{-k+2} \int_{P'_k} f w,$$

and so $fw \notin WRH_1^\Omega$. This finishes the proof of (i).

For (ii), it is obvious that $f \in RH_q^{\Omega,\mathrm{loc}}$ is a necessary condition for containment. For the converse, the case $q < \infty$ can be reduced to the case $q = 1$, which can then be proved by straightforward modifications to the proof of (i) (the main task, proving that $f \in \mathbf{D}^\Omega$, has essentially been proven already, since all of the weights w we constructed are WRH_∞^Ω weights). The case $q = \infty$ also follows easily. If $f \cdot WRH_\infty^\Omega \subseteq WRH_\infty^\Omega \subset WRH_2^\Omega$, then $f \in RH_2^{\Omega,\mathrm{loc}} \subset \mathbf{D}^\Omega$ (by the case $q < \infty$), and also $f \cdot 1 = f \in WRH_\infty^\Omega$. It follows that $f \in RH_\infty^{\Omega,\mathrm{loc}}$. $\square$

The following theorem shows more or less that if S and T are reverse Hölder spaces, f is a weight, and $f \cdot S \subseteq T$, then $S \subseteq T$.

Theorem 2.9. *Suppose f is a weight and $0 < p, q \leq \infty$.*

(i) *If $f \cdot RH_p^\Omega \subseteq WRH_q^\Omega$, and $WRH_q^\Omega(fw) \prec RH_p^\Omega(w)$, then $q \leq p$.*
(ii) $f \cdot WRH_p^\Omega \not\subseteq RH_q^{\Omega,\mathrm{loc}}$.
(iii) *If $\Omega \neq \mathbf{R}^n$, then $f \cdot RH_p^{\Omega,\mathrm{loc}} \not\subseteq RH_q^\Omega$.*

Proof. Let us prove (i). We can assume without loss of generality that $p = 1$. Suppose, for the purposes of contradiction, that f is a weight for which $WRH_q^\Omega(fu) \prec RH_1^\Omega(u)$, for some $q > 1$. Thus, $f = f \cdot 1 \in WRH_q^\Omega$. Let us fix a 3-allowable cube Q, normalize f so that $\|f\|_{1,\,2Q} = 1$, and fix s satisfying $1/q < s < 1$. For any $a \in \mathbf{R}^n$, $p_a(x) = |x-a|^{-sn} \in A_1 \subset RH_1$, and $RH_1(p_a)$ is independent of a. We shall denote by C any constant independent of a. If $Q_a \subseteq Q$ is a cube centered at a, and $u_a = fp_a$, then

$$u_a(Q_a) \le |Q_a| \cdot \|u_a\|_{q,Q_a} \le |Q_a|^{1-1/q}|Q|^{1/q}\|u_a\|_{q,Q}$$

But, by hypothesis, $\{WRH_q^\Omega(u_a)\}_{a\in Q}$ is bounded, and so

$$\|u_a\|_{q,Q} \le C\|u_a\|_{1/2,\,2Q} \le C\|p_a\|_{1,\,2Q}\|f\|_{1,\,2Q} \le C|Q|^{-s}.$$

Therefore

$$u_a(Q_a) \le C\left(\frac{|Q_a|}{|Q|}\right)^{1/q'}|Q|^{1-s}.$$

Since $p_a(x) > c_n|Q_a|^{-s}$ for $x \in Q_a$,

$$f(Q_a) \le C\left(\frac{|Q_a|}{|Q|}\right)^{s+1/q'}|Q|.$$

If we split Q into subcubes P_k $(1 \le k \le N^n)$, each of sidelength $l(Q)/N$, this last inequality implies that

$$f(Q) = \sum_{k=1}^{N^n} f(P_k) \le N^n \cdot C|Q|/N^{n(s+1/q')} = C|Q|/N^{n(s-1/q)},$$

where C is independent of N. Letting $N \to \infty$, this implies that $f(Q) = 0$ for all allowable cubes, a contradiction since $f \not\equiv 0$.

To prove (ii), suppose that $f \cdot WRH_p^\Omega \subseteq RH_q^{\Omega,\mathrm{loc}}$ for some weight f. Thus, $f \in RH_q^{\Omega,\mathrm{loc}}$, and so $1/f \in RH_s^{\Omega,\mathrm{loc}}$ for some $s > 0$. It follows from Theorem 2.3 that

$$WRH_p^\Omega = f^{-1}\cdot f \cdot WRH_p^\Omega \subseteq RH_s^{\Omega,\mathrm{loc}}RH_q^{\Omega,\mathrm{loc}} \subseteq RH_t^{\Omega,\mathrm{loc}},$$

where $t = sq/(s+q)$ if $q < \infty$, and $t = s$ if $q = \infty$. Since $WRH_\infty^\Omega \not\subseteq RH_r^{\Omega,\mathrm{loc}}$ for any $r > 0$, this gives us the required contradiction.

We saw at the end of Section 1 that if $\Omega \ne \mathbf{R}^n$, then $RH_p^{\Omega,\mathrm{loc}} \not\subseteq RH_t^\Omega$ for any $p, t > 0$. The proof of (iii) now follows in a similar fashion to that of (ii), so we omit it. □

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