Fuzzy Complex Grassmannian Spaces
and their Star Products

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Abstract

We derive an explicit expression for an associative star product on non-commutative versions of complex Grassmannian spaces, in particular for the case of complex 2-planes. Our expression is in terms of a finite sum of derivatives. This generalises previous results for complex projective spaces and gives a discrete approximation for the Grassmannians in terms of a non-commutative algebra, represented by matrix multiplication in a finite-dimensional matrix algebra. The matrices are restricted to have a dimension which is precisely determined by the harmonic expansion of functions on the commutative Grassmannian, truncated at a finite level. In the limit of infinite-dimensional matrices we recover the commutative algebra of functions on the complex Grassmannians.

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1 Introduction

There has recently been much interest in non-commutative geometry, [1, 2], both as a novel direction in string theory [3] and as a new tool in quantum field theory [4]. In the latter approach the theory is formulated on a "fuzzy" space, a series of discrete approximations to a continuous space-time manifold $\mathcal{M}$. The space of fields on each approximation is finite-dimensional, so the fuzzy space serves as a regulator, similar to a lattice. In contrast to the latter, the truncation enjoys all symmetries of the approximated continuous manifold. This has several useful implications including the absence of a doubling problem for chiral fermions [5, 6]. Topologically non-trivial field configurations can be included and the chiral anomaly emerges naturally in this approach [7, 8].

In more detail, a fuzzy space is given by a series of finite-dimensional algebras A_L that approximate the commutative algebra of functions on $\mathcal{M}$ in the following way: each A_L is identified with a subset of functions on $\mathcal{M}$ and the product in A_L induces a (non-commutative) product of functions in this subset, the so-called star product; in the limit $L \to \infty$, the subset should exhaust the set of all functions and the star product go over to the (commutative) pointwise product of functions. In the present case, the algebras A_L are full matrix algebras. The fuzzy spaces are thus matrix geometries, which go over to the usual continuous manifold as the size of the matrix is taken to infinity.

Non-commutative star products are known to exist for every Poisson manifold, in particular for symplectic manifolds [9]. Star products realised by finite-dimensional matrix algebras have been constructed in [10] for all homogeneous Kähler manifolds provided a certain quantisation condition on the metric is satisfied. These algebras can also be obtained using generalised coherent states [11] or the method of orbits for irreducible representations of Lie groups [12, 13]. The relation between these approaches has been discussed in [14] and that to deformation quantisation in [15, 16]. For the relation to Fourier transformation on group space see [17]. In terms of functions on the manifold, the above formulations provide an expression of the star product as an integral over the analytical continuation of the functions. This is not very convenient for explicit calculations in non-commutative field theory. An explicit, local formula in terms of a finite number of derivatives is so far only known for complex projective spaces [18] including the case of the fuzzy 2-sphere [19] already treated in [20].

In this paper, we derive an analogous formula for fuzzy complex Grassmannians, that is finite matrix geometry approximations to the usual Grassmannians, $G_k^N \cong U(N)/[U(k) \times U(N-k)]$, which are homogeneous spaces isomorphic to the space of complex k-planes in $\mathbf{C}^N$. The matrix geometries consist of matrices acting on the irreducible representation of $SU(N)$ which is given by the L-fold Young product of the representation on antisymmetric k-tensors. As L is increased the continuous Grassmannian is recovered. The special case $k = 1$ requires the L-fold symmetric product of the fundamental representation of $SU(N)$, as in [18]. The star product for the case $k = 2$ is constructed explicitly, using globally well-defined but over-complete co-ordinates on the Grassmannian. The result is expressed as a finite sum over multiple derivatives of the functions, which are decomposed into

irreducible representations of the stability group $S[U(k) \times U(N-k)]$ acting on the tangent space. It is shown that the star product reduces to the usual commutative product as $L \to \infty$. The particular case G_2^4 may be of some interest in field theory as it is a symplectic manifold which has S^4 as a Lagrangian sub-manifold. The corresponding matrix geometries can be viewed as non-commutative versions of T^*S^4, the co-tangent bundle to S^4 [21]. A star product on the complex Grassmannians as an infinite sum over derivatives is known [22, 23]. However, this formula cannot be restricted to finite-dimensional sub-algebras and therefore cannot serve as a star product on a fuzzy approximation to the manifold.

The layout of the paper is as follows: in section 2 we describe the complex Grassmannians, G_k^N, in terms of projection operators acting in $\mathbf{C}^N$, we introduce a set of global co-ordinates which are over-complete and satisfy a set of quadratic constraints which ensures that they indeed describe G_k^N; in section 3 we analyse the algebra of functions in terms of representations of $SU(N)$; in section 4 we describe the finite matrix geometries that define the fuzzy Grassmannians $G_{k,\mathrm{F}}^N$; section 5 gives the construction of the star product for the special case $k = 2$, $G_{2,\mathrm{F}}^N$, while section 6 presents a conjecture on the possible form for $k > 2$ and some observations on the construction; section 7 gives a summary and conclusions and some technical details are relegated to the appendices.

2 Complex Grassmannian spaces

The complex Grassmannian space $G_k^N \cong U(N)/[U(k) \times U(N-k)] = SU(N)/S[U(k) \times U(N-k)]$ can be represented as the space of all Hermitean rank-k projectors $\mathcal{P}$ acting on $\mathbf{C}^N$. This is easily seen since any such projector can be diagonalised by an element of $U(N)$ and there is still a residual adjoint action of $U(k) \times U(N-k)$ which leaves it invariant. It will be convenient to describe G_k^N using a redundant set of globally well-defined co-ordinates, ξ^A, $A = 1, \ldots, N^2 - 1$. First we introduce orthonormal Hermitean generators t_A of the Lie algebra of $SU(N)$ satisfying

$$t_A t_B = \tfrac{1}{N}\delta_{AB} + \tfrac{1}{\sqrt{2}}(d_{AB}{}^C + i f_{AB}{}^C) t_C , \tag{1}$$

where $f_{AB}{}^C$ are the structure constants of $SU(N)$ and $d_{AB}{}^C$ the components of the usual symmetric traceless tensor. The t_A are normalised so that $\mathrm{tr}(t_A t_B) = \delta_{AB}$. This allows us to parameterise $\mathcal{P}$ in terms of $N^2 - 1$ real parameters ξ^A,

$$\mathcal{P} = \tfrac{k}{N} + \xi^A t_A . \tag{2}$$

The condition that $\mathcal{P}$ is as projector translates into

$$\xi^A \xi^A = \frac{k(N-k)}{N} \qquad \text{and} \qquad \frac{1}{\sqrt{2}} d_{AB}{}^C \xi^A \xi^B = \frac{N-2k}{N}\, \xi^C , \tag{3}$$

and ξ^A now parameterise G_k^N when these conditions are imposed. This construction embeds the Grassmannian in the space of traceless Hermitian matrices, which we identify with $\mathbf{R}^{N^2-1}$ parameterised by the unrestricted ξ^A.

As discussed in detail in reference [18] for the case of $\mathbf{CP}^{N-1} \equiv G_1^N$, the complex structure and metric are encoded in a Hermitean projector

$$K^A{}_B \equiv \mathrm{tr}\left[\mathcal{P}t^A(1-\mathcal{P})t_B\right] = \tfrac{1}{2}\left(P^A{}_B + iJ^A{}_B\right) \tag{4}$$

where $J^A{}_B = \sqrt{2}f^A{}_{BC}\xi^C$ is the complex structure on G_k^N and $P^A{}_B = P_B{}^A = -(J^2)^A{}_B$ (indices are raised and lowered with the flat metric δ^A_B of $\mathbf{R}^{N^2-1}$). The matrix $K^A{}_B$ projects derivatives with respect to ξ^A onto the holomorphic tangent space of the Grassmannian when acting on the right and onto the anti-holomorphic tangent space when acting on the left; so $\nabla_A \equiv K_A{}^B \partial/\partial\xi^B$ is a holomorphic derivative and $\bar{\nabla}_B \equiv K^A{}_B \partial/\partial\xi^A$ an anti-holomorphic one. To see that $K^A{}_B$ is a projector we use the completeness relation for the generators,

$$(t_A)^i{}_j(t^A)^k{}_l = \delta^i{}_l\delta^k{}_j - \tfrac{1}{N}\delta^i{}_j\delta^k{}_l, \tag{5}$$

to show that

$$K^A{}_B t^B = \mathcal{P}t^A(1-\mathcal{P}) \qquad \text{and} \qquad t_B K^B{}_A = (1-\mathcal{P})t_A\mathcal{P}. \tag{6}$$

(These relations will prove very useful in the ensuing analysis.) Examining the real and imaginary parts of $K^A{}_B$ separately reveals that $P = -J^2$ and $PJ = JP = J$. This means that P itself is a projector onto the tangent space of G_k^N, with rank $2k(N-k)$. An important observation for the following analysis is that the differential operators $K_A{}^B \partial/\partial\xi^B$ commute with the constraints (3), as they must do, since K projects onto the tangent space. This can also be proven using (6). It is this fact that allows us to use the global co-ordinates, rather than local co-ordinates, in the final differential expression for the star product.

Covariant derivatives can be constructed by projecting derivatives with respect to the flat coordinates ξ^A to the tangent space. Multiple covariant holomorphic derivatives are thus defined as

$$\nabla_{A_1}\cdots\nabla_{A_n}f(\xi) \equiv \left(K_{A_1}{}^{B_1}\cdots K_{A_n}{}^{B_n}\right)\left(\partial_{B_1}K_{B_2}{}^{C_2}\partial_{C_2}\cdots K_{B_n}{}^{C_n}\partial_{C_n}f(\xi)\right) \tag{7}$$

where $\partial_A = \partial/\partial\xi^A$. In our case there is a simplification because

$$K^{AB}K^{CD}(\partial_B K_D{}^E) = 0 \tag{8}$$

which follows from the definition (4) of K and the completeness relation (5). It implies

$$\nabla_{A_1}\cdots\nabla_{A_n}f(\xi) = \left(K_{A_1}{}^{B_1}\cdots K_{A_n}{}^{B_n}\right)\left(\partial_{B_1}\cdots\partial_{B_n}f(\xi)\right). \tag{9}$$

3   Harmonic analysis

In order to obtain a finite-dimensional truncation of the space of functions on the Grassmannian, which is compatible with the symmetries, we decompose the space of functions into irreducible representations of the isometry group $G = SU(N)$.

We think of G_k^N as the space G/H of (right) cosets in G with respect to $H = S[U(k)\times U(N-k)]$, the subset of matrices in $U(k)\times U(N-k)$ with unit determinant.

Functions on G/H can thus be considered as functions on G that are invariant under the left action of H. They transform under G according to the right action.

The full space of functions on G is spanned by the matrix elements $D^J_{MM'}$ of all irreducible unitary representations J of G. Decomposing into irreducible representations of H, we may write the first component index as $M = (n, j, m)$ where j labels the irreducible representations of H, m the corresponding components and n distinguishes copies of equivalent representations. The left action of H is then given by

$$D^J_{(n,j,m)\,M'}(h^{-1}g) = \sum_{m''} D^J_{(n,j,m)\,(n,j,m'')}(h^{-1})\, D^J_{(n,j,m'')\,M'}(g)\,. \tag{10}$$

The H-invariant matrix elements are those for which the first index corresponds to the trivial representation of H, labelled for instance by $j = m = 0$. The space of functions on G/H is thus spanned by the matrix elements $D^J_{(n,0,0)\,M'}$.

Under the right action of G,

$$D^J_{(n,0,0)\,M'}(gg') = \sum_{M''} D^J_{(n,0,0)\,M''}(g)\, D^J_{M''\,M'}(g')\,, \tag{11}$$

so, for fixed J and n, the $D^J_{(n,0,0)\,M'}$ span the vector space of the representation J of G. The space of functions on G^N_k thus contains all irreducible representations of $SU(N)$ that contain the trivial representation upon restriction to $S[U(k) \times U(N-k)]$. The multiplicities are given by the multiplicity of the trivial representation in the restriction.

We will describe representations of $SU(N)$ by Young diagrams. These will be denoted by their symbols $J = [j_1, j_2, \ldots, j_{N-1}]$ where j_i denotes the number of *columns* of height i of the diagram.[3] The fundamental representation, for instance, has $J = [1, 0, \ldots]$ and the adjoint one $J = [1, 0, \ldots, 0, 1]$. Note that the complex conjugate representation is given by $J^* = [j_{N-1}, \ldots, j_1]$. In appendix C, we show that a representation J of $SU(N)$ contains the trivial representation of $S[U(k) \times U(N-k)]$ if and only if it appears in the direct product

$$M_L \equiv [\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}] \otimes [\mathbf{0}_{N-k-1}, L, \mathbf{0}_{k-1}] \tag{12}$$

for $L \geq n/N$ where $n = \sum_i j_i$ is the number of boxes in the diagram J and $\mathbf{0}_{k-1}$ stands for $k - 1$ zero entries. The multiplicity in the restriction is the same as that in the direct product. The M_L satisfy $M_1 \subset M_2 \subset \cdots$, so they provide a hierarchy of truncations of the space of functions on the Grassmannian.
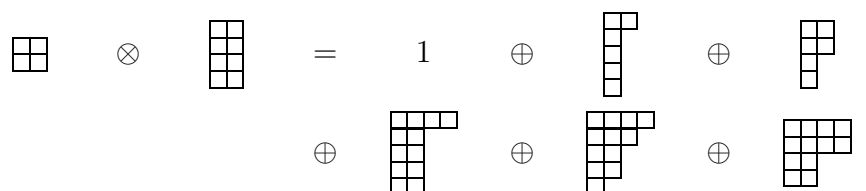
The representation $[\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}]$ is the Young product of L anti-symmetric k-tensors, for instance ▦▦▦▦▦ for $k = 2$, $L = 5$. As an example, for $N = 6$ and $k = 2$, the first truncation is

$$M_1 = [0, 1, 0, 0, 0] \otimes [0, 0, 0, 1, 0] = [0, 0, 0, 0, 0] \oplus [1, 0, 0, 0, 1] \oplus [0, 1, 0, 1, 0]\,,$$

$$\tag{13}$$

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[3]Note that this symbol is different from the highest weight vector sometimes used to describe a diagram, the entries of the latter being the number of boxes in each row.

the second is

$$M_2 = [0,2,0,0,0] \otimes [0,0,0,2,0] = [0,0,0,0,0] \oplus [1,0,0,0,1] \oplus [0,1,0,1,0]$$
$$\oplus [2,0,0,0,2] \oplus [1,1,0,1,1] \oplus [0,2,0,2,0] .$$

 (14)

Although not needed in the following, we would like to state for illustrational purposes that the full harmonic analysis on G_k^N (the "union" of all M_L) is given by the representations

$$J = \begin{cases} [m_1, \ldots, m_k, 0, \ldots, 0, m_k, \ldots, m_1] & \text{if } 2k < N , \\ [m_1, \ldots, m_{k-1}, 2m_k, m_{k-1}, \ldots, m_1] & \text{if } 2k = N , \end{cases} \quad (15)$$

with $m_i = 0, 1, \ldots$, each representation occurring once. The case $2k > N$ can be obtained by replacing k by $N - k$, since $G_k^N \cong G_{N-k}^N$. The representations (15) correspond to Young diagrams which can be obtained by putting a diagram with at most k rows next to its conjugate (which has at least $N - k \geq k$ boxes in any column), as can be verified for the examples given above. This result is also derived in appendix C.

4 Matrix geometry

In the previous section, we have obtained a series of truncations of the space of functions on the Grassmannian. We will now see that these carry a natural product. The representation content M_L of a given truncation can in fact be realised as an algebra of matrices in a representation J of $SU(N)$: since such matrices transform under $SU(N)$ by conjugation, they form the representation space of $J \otimes J^*$; so M_L, as introduced in eq. (12), is equivalent to the space of matrices in the representation $[\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}]$. Since the matrix product respects the action of $SU(N)$, the algebra M_L has the same symmetries as G_k^N.

In order to obtain the corresponding product of (truncated) functions, we shall now construct an injective map from M_L to the space of functions on the Grassmannian which also respects the group action (an equivariant map). This map automatically provides the notions of differentiation and integration needed for the construction of actions: one just has to map the corresponding notions for functions back to matrices. Equivariance guarantees that they are compatible with the truncation. The map will also provide a non-commutative product for functions in the image of M_L, the star product. If the star product tends to the point-wise product in the limit $L \to \infty$, we have succeeded in constructing a fuzzy G_k^N. Since we will restrict ourselves to $k = 2$ in the following sections, we present the map only for this case. The generalisation to other values of k should be obvious.

For G_2^N the basic building block will be the anti-symmetric representation ⊟ , corresponding to $L = 1$. The first non-trivial truncation of functions therefore requires using $[N(N-1)/2] \times [N(N-1)/2]$ matrices. A function on G_2^N is associated with such a matrix $\hat{F}$ by restricting the tensor product of the fundamental projector (2) to the anti-symmetric representation ⊟,

$$\rho \equiv (\mathcal{P} \otimes \mathcal{P})_a \, , \tag{16}$$

and constructing

$$F_1(\xi) = \mathrm{tr}[\rho(\xi)\hat{F}] \, . \tag{17}$$

Since $\mathcal{P}$ has rank 2, ρ has rank 1: let the plane onto which $\mathcal{P}$ projects be spanned by the vectors $\vec{v}$ and $\vec{w}$; ρ then projects onto the 1-dimensional subspace of the representation space of ⊟ spanned by the anti-symmetric product of $\vec{v}$ and $\vec{w}$ (an explicit proof is given in appendix B). A general truncation requires taking the L-fold (Young) product $[0, L, 0, \dots] = $ ⊞⋯⊟ of ⊟ , which has dimension $n_L^N = \frac{(N+L-1)!(N+L-2)!}{(N-1)!L!(N-2)!(L+1)!}$, and using $n_L^N \times n_L^N$ matrices. Equations (13) and (14), for instance, show the decomposition of 15×15 and 105×105 matrices as harmonics for G_2^6. In the following we shall drop trailing zeros in symbols of representations, so the above representations will be denoted by $[0, L]$. The Young product can be obtained as a component of the symmetric tensor product, so a projector can be constructed by restricting the L-fold tensor product of ρ to the representation $[0, L]$,

$$\rho_L = (\overbrace{\rho \otimes \cdots \otimes \rho}^{L \text{ times}})_{[0,L]} \, , \tag{18}$$

(of course $\rho_1 = \rho$) and a function can be associated with any $n_L^N \times n_L^N$ matrix $\hat{F}$ by

$$F_L(\xi) = \mathrm{tr}[\rho_L(\xi)\hat{F}] \, . \tag{19}$$

Appendix B contains a proof that the map (19) is injective.

The matrix geometries introduced here coincide with those obtained from complex line bundles [10] or generalised coherent states [11], see [14, 15]. F_L is usually called the covariant symbol of the operator $\hat{F}$ in these formulations, and injectivity is well known and follows from an analyticity argument. The relation with coherent states will be discussed in some more detail in appendix B.

5   Star product on G_2^N

Multiplication of truncated functions on the Grassmannian can now be defined using matrix multiplication. The star product of two functions, $F_L = \mathrm{tr}(\rho_L \hat{F})$ and $G_L = \mathrm{tr}(\rho_L \hat{G})$, is obtained from the matrix product through the map

$$(F_L \star G_L)(\xi) = \mathrm{tr}\big[\rho_L(\xi)\hat{F}\hat{G}\big] \, . \tag{20}$$

By construction this is an associative product and it keeps within the class of functions truncated at level L. Our aim is to find an explicit expression for this star

product, purely in terms of F_L and G_L and their derivatives, thus eliminating the explicit reference to matrices.

By orthonormality of the matrix elements in the representation $[0, L]$, $\int \mathrm{d}\mu(g)\, D^{[0,L]}_{M_1 M_2}(g^{-1}) D^{[0,L]}_{M_3 M_4}(g) = (1/n_L^N)\delta_{M_1 M_4}\delta_{M_2 M_3}$, $\hat{F}$ can be expanded as

$$\hat{F} = \int \mathrm{d}\mu(g)\, \tilde{F}(g) D^{[0,L]}(g) \tag{21}$$

with

$$\tilde{F}(g) \equiv n_L^N \operatorname{tr}\left[D^{[0,L]}(g^{-1})\hat{F}\right] . \tag{22}$$

Inserting this into (19), we obtain

$$F_L(\xi) = \int \mathrm{d}\mu(g)\, \omega_L(\xi, g)\tilde{F}(g) \tag{23}$$

with

$$\omega_L(\xi, g) \equiv \operatorname{tr}\left[\rho_L(\xi) D^{[0,L]}(g)\right] \tag{24}$$

and the star product can be expressed as

$$(F_L \star G_L)(\xi) = \int \mathrm{d}\mu(g) \int \mathrm{d}\mu(g')\, \omega_L(\xi, gg')\tilde{F}(g)\tilde{G}(g') . \tag{25}$$

We seek an expression for the star product in terms of derivatives acting on $F_L(\xi)$ and $G_L(\xi)$. By eqs. (25) and (23), this can be achieved by deriving an expression for $\omega_L(\xi, gg')$ in terms of derivatives of $\omega_L(\xi, g)$ and $\omega_L(\xi, g')$ with respect to ξ. The latter is greatly facilitated by the observation that ω_L can be expressed in terms of ω_1,

$$\omega_L(\xi, g) = [\omega_1(\xi, g)]^L , \tag{26}$$

because, by eq. (18), ρ_L factorises into rank-1 projectors ρ and $D^{[0,L]}$ acts as a direct product as well. The reason behind this relation is that the representation $[0, L]$ when projected to $S[U(2) \times U(N-2)]$ by ρ_L factorises as $[0, 1]^L$ since all other irreducible components of the product involve tensors that are anti-symmetric in 3 or more indices and therefore vanish in $SU(2)$.

As a first step, we have to find an expression for $\omega_1(\xi, gg')$. This is a straightforward but somewhat lengthy exercise. It is deferred to appendix A and yields

$$\omega_1(\xi, gg') = \omega_1(\xi, g) \left(1 + \overleftarrow{\partial}_A K^{AB} \overrightarrow{\partial}_B + \tfrac{1}{4}\overleftarrow{\partial}_A \overleftarrow{\partial}_B K^{AC} K^{BD} \overrightarrow{\partial}_C \overrightarrow{\partial}_D\right) \omega_1(\xi, g') \tag{27}$$

where $\partial_A = \partial/\partial\xi^A$ and K is the projector onto the holomorphic tangent space introduced in eq. (4). Substituting (27) in (25) and interchanging differentiation with respect to ξ with integration over g and g', we now have the star product at level one,

$$(F_1 \star G_1)(\xi) = F_1(\xi) \left(1 + \overleftarrow{\partial}_A K^{AB} \overrightarrow{\partial}_B + \tfrac{1}{4}\overleftarrow{\partial}_A \overleftarrow{\partial}_B K^{AC} K^{BD} \overrightarrow{\partial}_C \overrightarrow{\partial}_D\right) G_1(\xi) . \tag{28}$$

7

For higher L we have to consider $\omega_L = (\omega_1)^L$. Equation (27) implies

$$
\omega_L(\xi, gg') = \sum_{n+m\leq L} \frac{L!}{n!\, m!\, (L-n-m)!} (\omega\omega')^{L-n-m} \left[(\partial_A\omega)K^{AB}(\partial_B\omega')\right]^n
$$
$$
\times \left[\tfrac{1}{4}(\partial_C\partial_D\omega)K^{CE}K^{DF}(\partial_E\partial_F\omega')\right]^m \tag{29}
$$

where we have used the abbreviations $\omega \equiv \omega_1(\xi, g)$ and $\omega' \equiv \omega_1(\xi, g')$. The right-hand side of this equation has to be expressed in terms of multiple derivatives acting on $\omega_L(\xi, g)$ and $\omega_L(\xi, g')$. It contains several different terms with a given number of derivatives. This means that we have to distinguish components of multiple derivatives of ω_L.

To this end, we decompose multiple holomorphic derivatives $\nabla_{A_1} \cdots \nabla_{A_n} \omega_L$ as defined in eq. (9) with respect to irreducible representations of the stability group H which acts on the tangent space. It will be sufficient to consider the subgroup $H_0 = SU(2) \times SU(N-2)$ of H. Representations of H_0 will be denoted by (J, J') where J is a representation of the first factor and J' one of the second. To find the representation content of a single holomorphic derivative $\nabla_A = K_A{}^B \partial_B$, note that the fundamental representation [1] of $SU(N)$ decomposes as

$$
[1]\big|_{H_0} = ([1], [0]) \oplus ([0], [1]) \tag{30}
$$

into the fundamental representations of $SU(2)$ and $SU(N-2)$ upon restriction to H_0. The two components can be obtained by projection with $\mathcal{P}$ and $1 - \mathcal{P}$. Now use eqs. (4) and (5) to write ∇_A in terms of (anti-)fundamental indices,

$$
(t^A)^i{}_j \nabla_A f(\xi) = \left[(1-\mathcal{P})t^B\mathcal{P}\right]^i{}_j \partial_B f(\xi) . \tag{31}
$$

The matrix $t^B\partial_B f$ transforms like the traceless component of $[1] \times [1]^*$ under $SU(N)$. Since the index i is projected by $1 - \mathcal{P}$ to $([0], [1])$ while j is projected by $\mathcal{P}$ to $([1], [0])^*$, we find that the holomorphic derivative transforms like $([0], [1]) \otimes ([1], [0])^* = ([1]^*, [1])$. Note that tracelessness is guaranteed by the projections in eq. (31). By the same reasoning, an anti-holomorphic derivative $\bar{\nabla}_A = K^B{}_A \partial_B$ transforms like $([1], [1]^*)$. A multiple holomorphic derivative transforms like the symmetric tensor product of n copies of the representation $([1]^*, [1])$. In order to obtain an explicit expression for the decomposition of this product, it turns out to be useful to first decompose the tensor product of n copies of the fundamental representation of $SU(N)$.

This can be done by considering the action of the symmetric group S_n, whose elements permute the factors in the tensor product [28]. The latter can then be decomposed into irreducible representations of $SU(N) \times S_n$ with the help of character projection operators. They provide the following decomposition of unity,

$$
1 = \sum_{|J|=n} P_J \qquad \text{where} \qquad P_J \equiv \frac{d_J}{n!} \sum_{\pi \in S_n} \chi_J(\pi)\pi . \tag{32}
$$

Here, the sum is over all Young diagrams with n boxes, χ_J is the character of the symmetric group in the representation J and d_J the dimension of that representation.

We will only need

$$d_{[l,m]} = \frac{(l+2m)!(l+1)}{(l+m+1)!m!} \; . \tag{33}$$

The projectors P_J are orthogonal, $P_J P_{J'} = \delta_{J,J'} P_J$. They are discussed in more detail in appendix D. When acting on tensors, P_J projects onto a direct product of the representations of $SU(N)$ and the symmetric group given by J. Consequently the image of P_J contains d_J copies of the $SU(N)$ representation J.

This decomposition can be used to write the projector onto symmetric tensors with *adjoint* indices as

$$\frac{1}{n!} \delta_{A_1}^{\{B_1} \cdots \delta_{A_n}^{B_n\}} = \frac{1}{n!} \sum_{|J|=n} \mathrm{tr}\big[\big(t_{A_1} \otimes \cdots \otimes t_{A_n} \big) \big(t^{\{B_1} \otimes \cdots \otimes t^{B_n\}} \big) P_J \big] \tag{34}$$

where curly brackets denote symmetrisation. The projector from general adjoint tensors to symmetric holomorphic tensors (or anti-holomorphic tensors when acting to the left) can now be decomposed as

$$\frac{1}{n!} K_{A_1}{}^{\{B_1} \cdots K_{A_n}{}^{B_n\}} = \sum_{l+2m=n} K_{[l,m]\,A_1 \ldots A_n}{}^{B_1 \ldots B_n} \tag{35}$$

with projection operators

$$\begin{aligned}
K_J^{A_1 \ldots A_n, B_1 \ldots B_n} &= \tfrac{1}{n!} \mathrm{tr}\big[\big(t^{A_1} \otimes \cdots \otimes t^{A_n} \big) \big(t^{\{C_1} \otimes \cdots \otimes t^{C_n\}} \big) P_J \big] K_{C_1}{}^{B_1} \cdots K_{C_n}{}^{B_n} \\
&= \tfrac{1}{n!} \mathrm{tr}\big[\big(t^{A_1} \otimes \cdots \otimes t^{A_n} \big) P_J (1 - \mathcal{P})^{\otimes n} \big(t^{\{B_1} \otimes \cdots \otimes t^{B_n\}} \big) \mathcal{P}^{\otimes n} P_J \big].
\end{aligned} \tag{36}$$

The second expression has been obtained by inserting eq. (6) for the contraction of $K_C{}^B$ with t^C and generating a second projector using the fact that $P_J = P_J^2$ commutes with symmetric tensors. To see that K_J is a projector, one has to use the completeness relation (5) for the generators t^A and the fact that P_J and $K_A{}^B$ are projectors. The first expression shows that, when acting to the right, K_J projects onto the holomorphic tangent space. The second expression reveals that the holomorphic tensors thus obtained are projected to a sum of equivalent irreducible representations of $H_0 = SU(2) \times SU(N-2)$: the first indices of the n matrices t^{B_i} are projected to $[1]^{\otimes n}$ of $SU(N-2)$ and then symmetrised according to the Young diagram J, the second indices are projected to $[1]^{*\otimes n}$ of $SU(2)$ and symmetrised in the same way; the result is a sum of representations equivalent to (J^*, J) of H_0. Since P_J projects onto d_J copies of J corresponding to different symmetrisations and the symmetrisation on both sides of the symmetric $t^{\{B_1} \otimes \cdots \otimes t^{B_n\}}$ has to be the same, K_J projects onto d_J copies of (J^*, J). When acting to the left, K_J projects onto the anti-holomorphic tangent space since the factors K commute with the trace. The result is projected to d_J copies of the representation (J, J^*).

These findings imply that $K_{[j_1, j_2, \ldots]}$ vanishes if $j_i \neq 0$ for any $i > 2$ as can be seen explicitly by noting that the second P_J is multiplied by $\mathcal{P}^{\otimes j}$ in eq. (36) and $\mathcal{P}$ is a rank-2 projector. This fact has been used to restrict the sum in eq. (35). Note

9

that $K^{A,B}_{[1]} = K^{AB}$ and

$$
\begin{aligned}
K^{AB,CD}_{[0,1]} = \tfrac{1}{4}\big\{\, &\mathrm{tr}\big[\mathcal{P}t^A(1-\mathcal{P})t^C\big]\,\mathrm{tr}\big[\mathcal{P}t^B(1-\mathcal{P})t^D\big] \\
&- \mathrm{tr}\big[\mathcal{P}t^A(1-\mathcal{P})t^C\mathcal{P}t^B(1-\mathcal{P})t^D\big]\big\} \,+\, (C \leftrightarrow D)\,.
\end{aligned}
\tag{37}
$$

A useful observation is that $K_{[l,m]}$ factorises when contracted with arbitrary symmetric tensors S and T,

$$
\begin{aligned}
&S_{\{A_1\ldots A_n\}}\, K^{A_1\ldots A_n,B_1\ldots B_n}_{[l,m]}\, T_{\{B_1\ldots B_n\}} \\
&= d_{[l,m]}\, S_{\{A_1\ldots A_n\}}\, K^{A_1\ldots A_l,B_1\ldots B_l}_{[l]}\, K^{A_{l+1}A_{l+2},B_{l+1}B_{l+2}}_{[0,1]} \cdots K^{A_{n-1}A_n,B_{n-1}B_n}_{[0,1]}\, T_{\{B_1\ldots B_n\}}\,, \quad (38)
\end{aligned}
$$

or, in index-free notation, $SK_{[l,m]}T = d_{[l,m]}S(K_{[l]} \otimes K^{\otimes m}_{[0,1]})T$ where $d_{[l,m]}$ is given in eq. (33). This equation is proved in appendix D.

The irreducible components of the derivatives of $\omega_L = \omega^L$ are calculated in appendix E; in index-free notation, the result is

$$
\begin{aligned}
&(\partial^{l+2m}\omega^L)\, K_{[l,m]}\, (\partial^{l+2m}\omega'^L) \\
&\qquad = c^{(L)}_{l,m}\, (\omega\omega')^{L-l-m}\big[(\partial\omega)^l K_{[l]}(\partial\omega')^l\big]\big[\tfrac{1}{4}(\partial\partial\omega)K_{[0,1]}(\partial\partial\omega')\big]^m \quad (39)
\end{aligned}
$$

where

$$
c^{(L)}_{l,m} = \left(\frac{L!(L+1)!}{(L-l-m)!(L+1-m)!}\right)^2 d_{[l,m]}\,.
\tag{40}
$$

The crucial ingredient in the derivation is the following relation between single and double derivatives of ω,

$$
K^{AC}K^{BD}\omega\partial_C\partial_D\omega = K^{AB,CD}_{[0,1]}\omega\partial_C\partial_D\omega = 2K^{AB,CD}_{[0,1]}\partial_C\omega\partial_D\omega\,,
\tag{41}
$$

which can be obtained by use of eqs. (5) and (60) in a somewhat tedious calculation. In addition, since ω is only quadratic in ξ^A, one has $\partial_A\partial_B\partial_C\omega = 0$.

With the tools provided, it is easy to bring the terms in the expression (29) for $\omega_L(\xi, gg)$ into the form (39). Owing to eq. (41), we have

$$
\tfrac{1}{4}\partial_C\partial_D\omega K^{CE}K^{DF}\partial_E\partial_F\omega' = \partial_C\partial_D\omega K^{CE,DF}_{[0,1]}\partial_E\partial_F\omega'\,.
\tag{42}
$$

Now we use the decomposition (35) together with the factorisation property (38) to write

$$
(\partial_A\omega K^{AB}\partial_B\omega')^n = \sum_{l+2i=n} d_{[l,i]}\big[(\partial\omega)^l K_{[l]}(\partial\omega')^l\big]\big[(\partial\partial\omega)K_{[0,1]}(\partial\partial\omega')\big]^i (\omega\omega')^i\,.
\tag{43}
$$

Inserting this into eq. (29) and combining equivalent terms, we obtain

$$
\omega_L(\xi, gg') = \sum_{l+m\leq L} \tilde{c}^{(L)}_{l,m}\,(\omega\omega')^{L-l-m}\big[(\partial\omega)^l K_{[l]}(\partial\omega')^l)\big]\big[(\partial\partial\omega)K_{[0,1]}(\partial\partial\omega')\big]^m
\tag{44}
$$

10

with

$$\tilde{c}_{l,m}^{(L)} = \sum_{i=0}^{\min(m,L-l-m)} \frac{L!}{(l+2i)!\,(m-i)!\,(L-l-m-i)!}\,d_{[l,i]} \, . \tag{45}$$

The sum can be performed using the identity

$$\sum_{i=0}^{\min(p,n)} \binom{p}{i} \binom{q}{n-i} = \binom{p+q}{n} \, , \tag{46}$$

one finds

$$\tilde{c}_{l,m}^{(L)} = \frac{L!(L+1)!}{(L-l-m)!(L+1-m)!(l+2m)!}\,d_{[l,m]} \, . \tag{47}$$

Now the terms have exactly the form (39) and we can write

$$\omega_L(\xi, gg') = \sum_{l+m \leq L} a_{l,m}^{(L)} \left(\partial^{l+2m} \omega_L(\xi, g) \right) K_{[l,m]}(\xi) \left(\partial^{l+2m} \omega_L(\xi, g') \right) \tag{48}$$

with

$$a_{l,m}^{(L)} = \frac{(L-l-m)!(L+1-m)!}{L!(L+1)!(l+2m)!} \, . \tag{49}$$

Inserting eq. (48) into eq. (25) and interchanging differentiation with respect to ξ with integration over g and g', we obtain the final expression for the star product,

$$(F_L \star G_L)(\xi) =$$
$$\sum_{l+m \leq L} a_{l,m}^{(L)} \left(\partial_{A_1} \cdots \partial_{A_{l+2m}} F_L(\xi) \right) K_{[l,m]}^{A_1...A_{l+2m},B_1...B_{l+2m}}(\xi) \left(\partial_{B_1} \cdots \partial_{B_{l+2m}} G_L(\xi) \right) \, . \tag{50}$$

Recall that $K_{[l,m]}$ projects the derivatives on its right to holomorphic ones in the irreducible representation $[l,m]^* \otimes [l,m]$ of the subgroup $SU(2) \times SU(N-2)$ of the stability group and those on its left to anti-holomorphic ones in the complex conjugate representation.

Since the star product (50) is ultimately derived from a matrix product it is guaranteed to be associative, by construction. For large L, $a_{l,m}^{(L)} \sim L^{-l-2m}$, so all terms except $l = m = 0$ are suppressed and the star product tends to the point-wise product. We conclude that the series of matrix geometries introduced in section 4 indeed constitutes a fuzzy version of the complex Grassmannians G_2^N. As a consistency check note that restricting the sum to $m = 0$ reduces this to the known result for $\mathbf{CP}^{N-1}$, [18].

6   Observations on the construction and generalisation to $k > 2$

The above results suggest a natural generalisation to the case $k > 2$, although we have not verified it: the definition of matrix geometries in section 4 generalises immediately to Young products of antisymmetric k-tensors; in eq. (27), we expect

analogous terms with up to k derivatives acting on each side, and eq. (50) is expected to contain projections $K_{[j_1 \ldots j_k]}$ with $\sum_i j_i \leq L$; a natural extension of the combinatorial factor (49) would be

$$a_J^{(L)} = \frac{1}{|J|!} \prod_i \frac{1}{L + r_i - c_i} \tag{51}$$

where the product runs over all boxes of the Young diagram J and r_i and c_i are the row and column of box i.

Returning to the particular case $k = 2$, it might be more convenient, for practical purposes, to express $K_{[l,m]}$ in terms of K and $K_{[0,1]}$ alone by using the inverse of the decomposition (35): we can add factors of $K_{[0,1]}$ on both sides of (35) and use (38) to write

$$S\big(K^{\otimes(p-2n)} \otimes K_{[0,1]}^{\otimes n}\big)T = \sum_{m=n}^{[p/2]} M_{n,m}^{(p)} \, S\big(K_{[p-2m]} \otimes K_{[0,1]}^{\otimes m}\big)T \tag{52}$$

where S and T are symmetric tensors and $M_{n,m}^{(p)} = d_{[p-2m,m-n]}$. For fixed p, $M^{(p)}$ can be considered as a triangular matrix with $M_{n,m}^{(p)} = 0$ if $n > m$. It can be inverted by use of the relation

$$\sum_{m=n}^{n'} d_{[p-2m,m-n]} \, e_{p-2n',n'-m} = \delta_{n,n'} \qquad \text{with} \qquad e_{i,j} = (-1)^j \binom{i+j}{j} \tag{53}$$

which can be derived by standard techniques. We find $(M^{(p)})_{m,n}^{-1} = e_{p-2n,n-m}$ for $m \leq n$ and 0 otherwise. The inverse of eq. (52) therefore is

$$S \, K_{[p-2m,m]} \, T = d_{[p-2m,m]} \sum_{n=m}^{[p/2]} e_{p-2n,n-m} \, S\big(K^{\otimes(p-2n)} \otimes K_{[0,1]}^{\otimes n}\big)T \tag{54}$$

where the factorisation property (38) has been used on the left-hand side. Inserting this into eq. (50) and re-expanding indices, we obtain an alternative formula for the star product,

$$(F_L \star G_L)(\xi) = \sum_{n+m \leq L} b_{n,m}^{(L)} \, (\partial_{A_1} \cdots \partial_{A_{n+2m}} F_L(\xi)) \, \big(K^{A_1 B_1}(\xi) \cdots K^{A_n B_n}(\xi)$$

$$\times K_{[0,1]}^{A_{n+1}A_{n+2},B_{n+1}B_{n+2}}(\xi) \cdots K_{[0,1]}^{A_{n+2m-1}A_{n+2m},B_{n+2m-1}B_{n+2m}}(\xi)\big) \, (\partial_{B_1} \cdots \partial_{B_{n+2m}} G_L(\xi)) \tag{55}$$
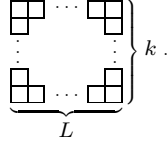
where (33), (49) and (53) have been combined into

$$b_{n,m}^{(L)} = \sum_{i=0}^{\min(m,L-n-m)} (-1)^i \frac{(L-n-m-i)! \, (L+1+i-m)! \, (n+i)! \, (n+2i+1)}{L! \, (L+1)! \, (n+m+i+1)! \, (m-i)! \, i! \, n!} . \tag{56}$$

K and $K_{[0,1]}$ are explicitly given by

$$K^{AB} = \tfrac{1}{2}(-J^{AC} J_C{}^B + i J^{AB}) \qquad \text{with} \qquad J^{AB} = \sqrt{2} f^{AB}{}_C \xi^C , \tag{57}$$

$$K_{[0,1]}^{AB,CD} = \left(\frac{N-2}{4N} \left(\delta_E^A \delta_F^B + \delta_F^A \delta_E^B\right) - \frac{1}{4} \left(d^{AG}{}_E \, d^B{}_{GF} + d^{AG}{}_F \, d^B{}_{GE}\right)\right) K^{EC} K^{FD} .$$

12

7 Conclusions

We have extended an explicit, local description of fuzzy matrix geometries beyond the known examples of $\mathbf{CP}_{\mathrm{F}}^{N-1}$ to complex Grassmannians $G_k^N \cong U(N)/[U(k) \times U(N-k)]$. The geometry of the Grassmannian is conveniently described in terms of $N^2 - 1$ globally defined co-ordinates ξ^A, satisfying the quadratic constraints (3), and a rank-k projector $\mathcal{P}$ defined in (2). The fuzzy Grassmannians $G_{k,\mathrm{F}}^N$ are defined by algebras of $n_L^N \times n_L^N$ matrices, $\hat{F}$, where n_L^N is the dimension of the irreducible representation of $SU(N)$ which is the L-fold Young product of the k-fold anti-symmetric representation and has Young diagram

Such matrices are mapped injectively to functions using a rank-1 projector constructed from the k-fold anti-symmetric tensor product of $\mathcal{P}$, $\rho = (\mathcal{P} \otimes \cdots \otimes \mathcal{P})_a$ as the L-fold product $\rho_L = (\rho \otimes \cdots \otimes \rho)_{[\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}]}$. The map is given by

$$F_L(\xi) = \mathrm{tr}\big[\rho_L(\xi)\hat{F}\big] \ .$$

The set of functions thus obtained constitute a truncation of the harmonic expansion of a general function on G_k^N.

An associative star product between two such functions can then be defined using matrix multiplication as

$$(F_L \star G_L)(\xi) = \mathrm{tr}\big[\rho_L(\xi)\hat{F}\hat{G}\big] \ .$$

The right-hand side can be re-expressed as a (bi-)differential operator on $F_L(\xi)$ and $G_L(\xi)$ and is given explicitly, for $k = 2$, in eq. (50),

$$(F_L \star G_L)(\xi) =$$
$$\sum_{l+m \leq L} a_{l,m}^{(L)} \left(\partial_{A_1} \cdots \partial_{A_{l+2m}} F_L(\xi)\right) K_{[l,m]}^{A_1 \ldots A_{l+2m}, B_1 \ldots B_{l+2m}}(\xi) \left(\partial_{B_1} \cdots \partial_{B_{l+2m}} G_L(\xi)\right)$$

where $a_{l,m}^{(L)}$ is given by eq. (49) and $K_{[l,m]}$ by eq. (36). $K_{[l,m]}$ projects the multiple derivatives on F_L and G_L to irreducible representations of the stability group $H = S[U(2) \times U(N-2)]$. It includes a projection of the derivatives to the tangent space of G_2^N in the embedding space parameterised by ξ. Tangent vectors transform under H and the multiple tangential derivative forms a direct product representation which is reducible. The derivatives on G_L are projected by $K_{[l,m]}$ to the component equivalent to the irreducible representation $[l,m]^* \otimes [l,m]$ of $SU(2) \times SU(N-2) \subset H$. Here, $[l,m]$ stands for a Young product of m copies of the anti-symmetric representation $\boxvert$ and l copies of the fundamental representation $\square$, for instance $[4,3] = \boxvert\boxvert\boxvert\square\square\square\square$. The derivatives on F_L are projected to the complex conjugate representation. The projection implies that all derivatives on G_L are holomorphic while those on F_L are

anti-holomorphic. They can in fact be replaced by covariant derivatives in local co-ordinates [18]. When the sum is restricted to $m = 0$ this star product reduces to the $k = 1$ case of $\mathbf{CP}^{N-1}$, [18].

An even more explicit expression for the star product, possibly better suited for practical purposes, is given in eq. (55).

Explicit, local formulas for the star product prove very useful in perturbative calculations in non-commutative field theory [30] and might also provide new insights in string theory where fuzzy versions of co-adjoint orbits appear as world volume geometries of D-branes in WZW models [31, 32].

8 Acknowledgements

A Kernel at level 1

We derive an expression for the kernel $\omega_1(\xi, gg')$ in terms of derivatives of $\omega_1(\xi, g)$ and $\omega_1(\xi, g')$ with respect to ξ. Inserting eq. (16) and the definition of the anti-symmetric representation, $D^{[0,1]}(g) = (g \otimes g)_a$, into eq. (24) and abbreviating ω_1 as ω, we obtain

$$\omega(\xi, g) = \text{tr}\big[(\mathcal{P} \otimes \mathcal{P})_a (g \otimes g)_a\big] \qquad \text{with} \qquad \mathcal{P} = \tfrac{2}{N} + \xi^A t_A \tag{58}$$

The trace of a tensor product in the anti-symmetric representation can be expressed as

$$\text{tr}\big[(A \otimes B)_a\big] = \tfrac{1}{2}\big[\text{tr}(A)\,\text{tr}(B) - \text{tr}(AB)\big] . \tag{59}$$

A crucial identity, $\mathcal{P}^i{}_{[l}\mathcal{P}^j{}_m\mathcal{P}^k{}_{n]} = 0$, follows from the fact that $\mathcal{P}$ is a rank-2 projector. It implies

$$\begin{aligned}
\text{tr}(\mathcal{P}A)\,\text{tr}(\mathcal{P}B)\,\text{tr}(\mathcal{P}C) &- \text{tr}(\mathcal{P}A\mathcal{P}B)\,\text{tr}(\mathcal{P}C) - \text{tr}(\mathcal{P}A\mathcal{P}C)\,\text{tr}(\mathcal{P}B) \\
&- \text{tr}(\mathcal{P}B\mathcal{P}C)\,\text{tr}(\mathcal{P}A) + \text{tr}(\mathcal{P}A\mathcal{P}B\mathcal{P}C) + \text{tr}(\mathcal{P}B\mathcal{P}A\mathcal{P}C) = 0 .
\end{aligned} \tag{60}$$

Since

$$\partial_A \omega = \text{tr}\big[(t_A \otimes \mathcal{P})_a (g \otimes g)_a\big] + \text{tr}\big[(\mathcal{P} \otimes t_A)_a (g \otimes g)_a\big] \tag{61}$$

we can use (6) to show

$$\partial_A \omega(\xi, g)\, K^{AB}\, \partial_B \omega(\xi, g') = 2\,\text{tr}\big[(\mathcal{P} \otimes \mathcal{P})_a (g \otimes g)_a \big(\mathcal{P} \otimes (1 - \mathcal{P})\big)_a (g' \otimes g')_a\big] . \tag{62}$$

Similarly, for the second derivative

$$\begin{aligned}
\partial_A \partial_B \omega(\xi, g)\, &K^{AC} K^{BD}\, \partial_C \partial_D \omega(\xi, g') \\
&= 4\,\text{tr}\big[(\mathcal{P} \otimes \mathcal{P})_a (g \otimes g)_a \big((1 - \mathcal{P}) \otimes (1 - \mathcal{P})\big)_a (g' \otimes g')_a\big] .
\end{aligned} \tag{63}$$

Since $\rho_1 = (\mathcal{P} \otimes \mathcal{P})_a$ is a rank-1 projector, the simple product can be written as

$$\omega(\xi, g)\omega(\xi, g') = \text{tr}\left[(\mathcal{P} \otimes \mathcal{P})_a (g \otimes g)_a (\mathcal{P} \otimes \mathcal{P})_a (g' \otimes g')_a\right] . \tag{64}$$

Using $1 = (\mathcal{P} \otimes \mathcal{P})_a + (\mathcal{P} \otimes (1 - \mathcal{P}))_a + ((1 - \mathcal{P}) \otimes \mathcal{P})_a + ((1 - \mathcal{P}) \otimes (1 - \mathcal{P}))_a$, eqs. (62), (63) and (64) can be combined to

$$\omega(\xi, gg') = \omega(\xi, g) \left(1 + \overleftarrow{\partial}_A K^{AB} \overrightarrow{\partial}_B + \tfrac{1}{4} \overleftarrow{\partial}_A \overleftarrow{\partial}_B K^{AC} K^{BD} \overrightarrow{\partial}_C \overrightarrow{\partial}_D\right) \omega(\xi, g') , \tag{65}$$

which is the desired expression.

B Coherent states

We show that the projector ρ_L as given in eq. (18) has rank 1, and we present a simple argument for why the map from matrices to functions is injective.

To this end, we require a more explicit representation for ρ_L. The vector space of the irreducible representation of $SU(N)$ with symbol $J = [0, L]$ can be realised as a sub-space with certain symmetry properties of the space of $2L$-index tensors. We construct it as the image of a Young symmetriser. We first assign tensor indices to the boxes in the Young diagram of the representation by putting the numbers $1, 2, \ldots, 2L$ in ascending order into one column after the other, for instance $\begin{smallmatrix}1&3&5&7\\2&4&6&8\end{smallmatrix}$ for $[0, 4]$. The Young symmetriser is now defined as

$$Y_{[0,L]} = \frac{2^L}{L+1} \mathcal{A}_L \mathcal{S}_L ,$$

$$\mathcal{A}_L = \prod_{i=1}^{L} \left[\tfrac{1}{2}(1 - \tau_i)\right] , \qquad \mathcal{S}_L = \frac{1}{L!} \sum_{\pi_1 \in R_1} \pi_1 \frac{1}{L!} \sum_{\pi_2 \in R_2} \pi_2 \tag{66}$$

where τ_i interchanges the two boxes of the ith column of the diagram and R_i denotes the set of permutations that permute the boxes of row i. So $\mathcal{S}_L$ symmetrises the rows of the diagrams, while $\mathcal{A}_L$ anti-symmetrises the columns. Both are symmetric projectors. The Young symmetriser is a projector, $Y_{[0,L]}^2 = Y_{[0,L]}$, but not symmetric.

Operators in the vector space of $[0, L]$ can be unambiguously described as operators $\hat{F}$ on $2L$-tensors that satisfy

$$\hat{F} Y_{[0,L]} = Y_{[0,L]} \hat{F} = \hat{F} . \tag{67}$$

Now we can prove that ρ_L has rank 1 by expressing the rank-2 projector $\mathcal{P}$ in terms of an orthonormal basis $|\varphi\rangle$, $|\psi\rangle$ of the complex plane onto which it projects,

$$\mathcal{P} = |\varphi\rangle \langle\varphi| + |\psi\rangle \langle\psi| .$$

The level-1 projector ρ was defined in (16) as the projection of the tensor product of $\mathcal{P}$ with itself to the anti-symmetric representation $[0, 1]$. Since $Y_{[0,1]}$ reduces to a single anti-symmetrisation,

$$\rho \equiv (\mathcal{P} \otimes \mathcal{P})_a = (\mathcal{P} \otimes \mathcal{P}) Y_{[0,1]} = |\varphi\psi\rangle \langle\varphi\psi| \tag{68}$$

where
$$|\varphi\psi\rangle \equiv \tfrac{1}{\sqrt{2}}\big(|\varphi\rangle\,|\psi\rangle - |\psi\rangle\,|\varphi\rangle\big)\;. \tag{69}$$

For the projector at level L, we obtain

$$\rho_L = (\rho \otimes \cdots \otimes \rho)Y_{[0,L]} = |\varphi\psi\rangle^L\,\langle\varphi\psi|^L\,Y_{[0,L]} \tag{70}$$

The state $|\varphi\psi\rangle$ completely characterises the plane that corresponds to a point in G_2^N. It is therefore natural that it occurs as a fundamental object in the construction. The states $|\varphi\psi\rangle^L$ coincide, up to a conventional phase, with the generalised coherent states discussed in [11]. Since $\hat{F}Y_{[0,L]} = Y_{[0,L]}\hat{F}$,

$$F_L(\xi) = \langle\varphi\psi|^L\,\hat{F}\,|\varphi\psi\rangle^L\;. \tag{71}$$

So F_L is the covariant symbol, as defined in [11], of the operator $\hat{F}$.

This expression can be used to show that the map is injective. We have to show that $\hat{F}$ can be reconstructed from F_L. Since $|\varphi\psi\rangle^L = 2^{L/2}\mathcal{A}_L(|\varphi\rangle\,|\psi\rangle)^L$,

$$F_L(\xi) = 2^L\big(\langle\varphi|\,\langle\psi|\big)^L\mathcal{A}_L\hat{F}\mathcal{A}_L\big(|\varphi\rangle\,|\psi\rangle\big)^L\;.$$

Due to the anti-symmetrisation between $|\varphi\rangle$ and $|\psi\rangle$ this function can be homogeneously extended to general (non-orthonormal) $|\varphi\rangle$ and $|\psi\rangle$. We choose $|\varphi\rangle = \sum_{n=1}^{N} a_n\,|n\rangle$ and $|\psi\rangle = \sum_{n=1}^{N} b_n\,|n\rangle$ with canonical basis vectors $|n\rangle$. By differentiating with respect to a, b and their complex conjugates, all matrix elements of $\mathcal{S}_L\mathcal{A}_L\hat{F}\mathcal{A}_L\mathcal{S}_L$ can be obtained. Using the symmetry (67), we obtain $\mathcal{S}_L\hat{F}$ and thus also $\frac{2^L}{L+1}\mathcal{A}_L\mathcal{S}_L\hat{F} = \hat{F}$.

C   Restrictions and direct products

We shall derive the relation between the restriction of representations of $G = SU(N)$ to $H = S[U(k) \times U(N-k)]$ and the direct product of certain representations used in section 3. In this appendix, we will allow for columns of height N, $J = [j_1, j_2, \ldots, j_N]$, in diagrams describing representations of $SU(N)$. These do not lead to new representations, since representations differing only by j_N are unitarily equivalent, but this generalisation will make formulas much simpler. We embed H into $SU(N)$ as

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}(N-k)\varphi}U' & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k\varphi}U'' \end{pmatrix} \tag{72}$$

where $U' \in SU(k)$ and $U'' \in SU(N-k)$ and $\mathrm{e}^{\mathrm{i}\varphi} \in U(1)$. This shows that $H = [SU(k) \times SU(N-k) \times U(1)]/Z_n$ where n is the least common multiple of k and $N-k$. Representations of H can thus be considered as representations of $SU(k) \times SU(N-k) \times U(1)$ that represent Z_n trivially. This fixes the charge q of the $U(1)$ factor $\mathrm{e}^{\mathrm{i}q\varphi}$ of the representation modulo n. We will denote these representations as $(J', J'')_q$ where J' and J'' are symbols of $SU(k)$ and $SU(N-k)$ representations, respectively, and q is the charge of the $U(1)$ representation.

The restriction of an $SU(N)$ representation to H can be written as

$$J|_H = \bigoplus_{J',J''} m^J_{J',J''} (J', J'')_{(N-k)|J'|-k|J''|} . \tag{73}$$

Here, $|J| = \sum_i j_i$ is the number of boxes in the diagram J and we assume that the diagrams have been chosen such that the total number of boxes in J' and J'' is the same as in J,

$$|J| = |J'| + |J''| . \tag{74}$$

Note that the $U(1)$ representation is determined by the $SU(k) \times SU(N-k)$ representation, so the multiplicities $m^J_{J',J''}$ are the same as for the restriction from $SU(N)$ to the latter. It is known ([24, 25], also see [26, 27]) that these can be obtained from the decomposition of the direct product of the $SU(N)$ representations with diagrams J' and J'',

$$J' \otimes J'' = \bigoplus_{J} m^J_{J',J''} J \qquad \text{in } SU(N). \tag{75}$$

Here, again, the restriction (74) on the number of boxes applies. Note that in eq. (75) J' and J'' are interpreted as $SU(N)$ representations while they are interpreted as $SU(k)$ respectively $SU(N-k)$ representations in eq. (73).

We are interested in the case where the trivial representation of H appears on the right-hand side of (73). This means that J' and J'' only have columns of height k and $N-k$, respectively, $J' = [\mathbf{0}_{k-1}, L', \mathbf{0}_{N-k-1}]$ and $J'' = [\mathbf{0}_{N-k-1}, L'', \mathbf{0}_{k-1}]$ where $\mathbf{0}_{k-1}$ stands for $k-1$ zero entries, etc. In addition, the $U(1)$ charge has to vanish, $(N-k)|J'| = k|J''|$. Since $|J'| = L'k$ and $|J''| = L''(N-k)$, this implies $L' = L'' \equiv L$, so that J' and J'' are complex conjugate representations of $SU(N)$. We conclude that a representation of $SU(N)$ contains the trivial representation of $S[U(k) \times U(N-k)]$ if and only if it appears in the decomposition of the direct product

$$M_L \equiv [\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}] \otimes [\mathbf{0}_{N-k-1}, L, \mathbf{0}_{k-1}] \tag{76}$$

for some L. The multiplicities are given by the multiplicities $m^J_{[\mathbf{0}_{k-1},L,\mathbf{0}_{N-k-1}],[\mathbf{0}_{N-k-1},L,\mathbf{0}_{k-1}]}$ in the product. Note that the multiplicity does not depend on the number j_N of columns of height N in the diagram J chosen for a given representation. This means that a representation appears in M_L if and only if $NL \geq |J|$ where J is the minimal diagram ($j_N = 0$) of the representation. Therefore $M_L \subset M_{L'}$ if $L < L'$.

Decomposition of the direct product

For illustrational purposes, we will explicitly perform the decomposition of the direct product (76) into irreducible representations. We can assume $k \leq N - k$ since $G^N_k \cong G^N_{N-k}$. The decomposition is achieved by Young diagram techniques. Recall the rules for decomposing the direct product of two irreducible representations of $SU(N)$ [28]:

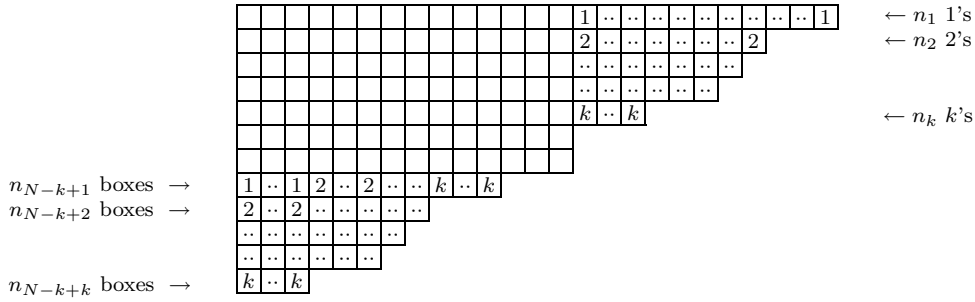1. Label each box in the second diagram by its row number.

Figure 1: Decomposition of the tensor product $[\mathbf{0}_{N-k-1}, L, \mathbf{0}_{k-1}] \otimes [\mathbf{0}_{k-1}, L, \mathbf{0}_{N-k-1}]$.

2. Attach all boxes with 1's to the first diagram, then all boxes with 2's and so on, such that

   (a) at all stages the intermediate diagram corresponds to an irreducible representation of $SU(N)$, i.e. all columns start in the first row and are connected and the height of the columns monotonically decreases from left to right,

   (b) no column contains any number more than once, and

   (c) when counted from the right, the n-th i does not appear before the n-th $i-1$.

Applying these rules to our case, we have to attach boxes with L copies of each of the numbers $1, \ldots, k$ to a rectangle of height $N-k$ and width L. This is indicated in figure 1. We have already anticipated some facts about the resulting distribution of boxes, which we will explain now. The number of 1's in the first row has been denoted by n_1. The remaining $L - n_1$ 1's have to be in the $N - k + 1$-st row. Denoting the number of 2's in the second row by n_2, there can be at most $n_1 - n_2$ 2's in row $N - k + 1$, because of rule 2c. There must thus be at least $L - n_1$ 2's in row $N - k + 2$. However, owing to rule 2b, the number of 2's in that row is at most $L - n_1$. Therefore, there have to be exactly $n_1 - n_2$ 2's in row $N - k + 1$ and $L - n_1$ 2's in row $N - k + 2$. By the same reasoning, we find that the number of i's in row $N - k + l$ for $l > 0$ is equal to the number of $i - l + 1$'s in the row $N - k + 1$ which is in turn equal to $n_{i-l} - n_{i-l+1}$ if $i \geq l$ and 0 if $i < l$ (we have put $n_0 \equiv L$). Adding up, we find for the number of boxes in row $N - k + l$,

$$n_{N-k+l} = \sum_{i=l}^{k} (n_{i-l} - n_{i-l+1}) = L - n_{k-l+1} \quad \text{for} \quad l = 1, \ldots, k \ . \tag{77}$$

Graphically, this means that the subdiagram of columns $L + 1, L + 2, \ldots$ combines, after a rotation by π, with the remainder of the diagram to a rectangle of width L and height N. In terms of symbols $J = [j_1, \ldots, j_N]$, where j_i is the number of columns of height i, this implies

$$J = \begin{cases} [m_1, \ldots, m_k, 0, \ldots, 0, m_k, \ldots, m_1] & \text{if } 2k < N \ , \\ [m_1, \ldots, m_{k-1}, 2m_k, m_{k-1}, \ldots, m_1] & \text{if } 2k = N \end{cases} \tag{78}$$

18

with $m_i = n_i - n_{i+1}$ where $n_{k+1} \equiv 0$. All non-negative values of m_i satisfying $\sum_{i=1}^{k} m_i = n_1 \le L$ occur. Furthermore, each diagram can be obtained in only one way, since it is determined by the numbers n_i ($i = 1, \ldots, k$). So all multiplicities equal 1.

D Symmetric group

In this appendix, we shall provide a proof of the factorisation property (38). On the way, we will recall some facts about representations of the symmetric group and the associated projectors P_J used in the text. These projectors can be considered as elements of the group algebra $\mathbf{R}[S_j] \equiv \mathrm{span}\, S_n = \{A = \sum_{\pi \in S_n} A_\pi \pi | A_\pi \in \mathbf{R}\}$, the set of formal linear combinations of group elements. The vector space $\mathbf{R}[S_j]$ carries a representation of the algebra $\mathbf{R}[S_j]$ whose action is given by left multiplication. This representation restricts to a representation of the subgroup S_n of $\mathbf{R}[S_j]$. It is usually called the regular representation. Sometimes it is convenient to identify $\mathbf{R}[S_j]$ with the algebra $F(S_n)$ of functions on the group S_n by setting $A(\pi) \equiv A_\pi$. The action of a group element π is then given by $(\pi A)(\sigma) = A(\pi^{-1}\sigma)$. $F(S_n)$ can in fact be considered as the dual vector space of $\mathbf{R}[S_j]$ since each function on the group can be linearly and uniquely extended to a function on $\mathbf{R}[S_j]$. The identification of $\mathbf{R}[S_j]$ with its dual space can be obtained from the inner product

$$\langle A, B \rangle \equiv \sum_{\pi \in S_n} A_\pi B_\pi = \frac{1}{n!} \mathrm{tr}(A^\mathrm{T} B) \tag{79}$$

by putting $A(B) = \langle A, B \rangle$. The trace in eq. (79) is over the regular representation and we have set $A^\mathrm{T} = \sum_\pi A_\pi \pi^{-1}$.

Of particular importance are the central elements of $\mathbf{R}[S_j]$, that are invariant under conjugation with any group element $\pi \in S_n$, $A = \pi A \pi^{-1}$. They correspond to class functions, i.e. functions that depend only on the conjugacy class of their argument. An orthogonal basis in the subspace of class functions is given by the characters χ_J associated with the irreducible representations J of S_n,

$$\langle \chi_J, \chi_{J'} \rangle = n!\, \delta_{JJ'} \,. \tag{80}$$

So every class function can be expanded as

$$A = \sum_J A_J \chi_J \qquad \text{with} \qquad A_J = \frac{1}{n!} \langle \chi_J, A \rangle \,. \tag{81}$$

To each $A \in \mathbf{R}[S_j]$, one can associate a central element $\overline{A}$ by averaging with respect to conjugation,

$$\overline{A} \equiv \frac{1}{n!} \sum_{\pi \in S_n} \pi A \pi^{-1} = \frac{1}{n!} \sum_J \langle \chi_J, A \rangle \chi_J \tag{82}$$

where we have used (81) and the invariance of χ_J under conjugation.

The regular representation is in general reducible. It contains each irreducible representation J with a multiplicity that is given by the dimension d_J of the representation. The component containing all copies of an irreducible representation J

can be obtained as the image of the symmetric projection operator introduced in eq. (32), in the dual picture,

$$P_J = \frac{d_J}{n!}\chi_J .\tag{83}$$

The decomposition of unity

$$1 = \sum_J P_J \tag{84}$$

provides a decomposition of $\mathbf{R}[S_j]$ into orthogonal subspaces [29].

Now we will show how the averaged tensor product of two projectors P_{J_1} and P_{J_2} onto irreducible representations of S_{n_1} and S_{n_2} can be expressed in terms of irreducible projectors. $P_{J_1} \otimes P_{J_2}$ can be extended to $\mathbf{R}[S_j]$ where $n = n_1 + n_2$. By (82), we have

$$\overline{P_{J_1} \otimes P_{J_2}} = \sum_J a_J \chi_J \tag{85}$$

with

$$a_J = \frac{1}{n!}\langle \chi_J, P_{J_1} \otimes P_{J_2}\rangle .\tag{86}$$

The restriction of χ_J to $(\pi, \sigma) \in S_{n_1} \times S_{n_2}$ decomposes into irreducible characters as

$$\chi_J(\pi, \sigma) = \sum_{J_1, J_2} c^J_{J_1 J_2} \chi_{J_1}(\pi) \chi_{J_2}(\sigma) \tag{87}$$

where $c^J_{J_1 J_2} \in \mathbf{Z}$ are multiplicities or Clebsch-Gordan coefficients. With (80) we get

$$a_J = \frac{1}{n!}\sum_{J_1', J_2'} c^J_{J_1' J_2'} \langle \chi_{J_1'}, P_{J_1}\rangle \langle \chi_{J_2'}, P_{J_2}\rangle = \frac{d_{J_1} d_{J_2}}{n!} c^J_{J_1 J_2} \tag{88}$$

and therefore

$$\overline{P_{J_1} \otimes P_{J_2}} = \sum_J \frac{d_{J_1} d_{J_2}}{n!} c^J_{J_1 J_2} \chi_J = \sum_J \frac{d_{J_1} d_{J_2}}{d_J} c^J_{J_1 J_2} \overline{P_J} .\tag{89}$$

By iteration, this result can be generalised to multiple products,

$$\frac{1}{d_{J_1} \cdots d_{J_m}}\overline{P_{J_1} \otimes \cdots \otimes P_{J_m}} = \sum_J c^J_{J_1 \ldots J_m} \frac{1}{d_J}\overline{P_J} .\tag{90}$$

Note that symmetrisation with respect to S_n implies symmetrisation with respect to $S_{n'} \subset S_n$,

$$\overline{A \otimes B \otimes C} = \overline{A \otimes \overline{B \otimes C}} .\tag{91}$$

Now we can prove eq. (38). The right-hand side of this equation can be written as a single trace like in eq. (36) but with P_J replaced by $P_{[l]} \otimes P^{\otimes m}_{[0,1]}$. Owing to the symmetric tensors S and T all factors in the trace except $P_{[l]} \otimes P^{\otimes m}_{[0,1]}$ are symmetric under conjugation, so $P_{[l]} \otimes P^{\otimes m}_{[0,1]}$ can be replaced by its symmetrised version $\overline{P_{[l]} \otimes P^{\otimes m}_{[0,1]}}$. Now we can insert eq. (90). The only term on the right-hand side that does not vanish when projected by $\mathcal{P}^{\otimes n}$ to a representation of $SU(2)$ is $J = [l, m, 0, \ldots]$ with multiplicity 1. The dimensions of the representations $[l]$ and $[0, 1]$ (of the symmetric group) are 1, while the dimension of $[l, m]$ appearing in the denominator of (90) just cancels that on the right-hand side of (38), so we obtain the left-hand side.

20

E   Projection of multiple derivatives

We compute the product of multiple (anti-)holomorphic derivatives of ω_L and ω'_L projected to the representation $([l,m]^*, [l,m])$ of the stability group,

$$X_{l,m}^{(L)} \equiv (\partial_{A_1} \cdots \partial_{A_{l+2m}} \omega^L) \, K_{[l,m]}^{A_1 \dots A_{l+2m}, B_1 \dots B_{l+2m}} \, (\partial_{B_1} \cdots \partial_{B_{l+2m}} \omega'^L) \,. \tag{92}$$

By eq. (9) and since $K_{[l,m]}$ contains the projector K, the derivatives in this equation are really covariant derivatives, holomorphic ones acting on ω' and anti-holomorphic ones on ω. Equation (38) implies that $K_{[l,m]}$ can be replaced by $d_{[l,m]} K_{[l]} \otimes K_{[0,1]}^{\otimes m}$,

$$X_{l,m}^{(L)} = d_{[l,m]} \, (\partial^{l+2m} \omega^L) \left(K_{[l]} \otimes K_{[0,1]}^{\otimes m} \right) (\partial^{l+2m} \omega'^L) \tag{93}$$

where we have introduced an index-free notation. Using the second equality of eq. (41), we find

$$K_{[0,1]}^{AB,CD} \partial_C \partial_D \omega^L = \frac{L(L+1)}{2} \omega^{L-1} K_{[0,1]}^{AB,CD} \partial_C \partial_D \omega \tag{94}$$

which iterates to

$$(K_{[0,1]} \partial\partial)^m \omega^L = \frac{L!(L+1)!}{(L-m)!(L+1-m)!} \, \omega^{L-m} \left(\tfrac{1}{2} K_{[0,1]} \partial\partial\omega \right)^m \tag{95}$$

because the triple derivative of ω vanishes. The first equality of eq. (41) implies that $K_{[l]} \partial^l \omega^n$ contains only single derivatives of ω, whence

$$\left(K_{[l]} \otimes K_{[0,1]}^{\otimes m} \right) \partial^{l+2m} \omega^L = \left(K_{[l]} \otimes K_{[0,1]}^{\otimes m} \right) \left[\partial^l (K_{[0,1]} \partial\partial)^m \omega^L \right]$$

$$= \frac{L!(L+1)!}{(L-l-m)!(L+1-m)!} \, \omega^{L-l-m} \left[K_{[l]} (\partial\omega)^l \right] \left[\tfrac{1}{2} K_{[0,1]} (\partial\partial\omega) \right]^m \tag{96}$$

where, in the first step, we have used eq. (8). Since a similar equality holds for anti-holomorphic derivatives, and $K_{[l]}$ and $K_{[0,1]}$ are projectors, we find

$$X_{l,m}^{(L)} = d_{[l,m]} \left(\frac{L!(L+1)!}{(L-l-m)!(L+1-m)!} \right)^2 (\omega\omega')^{L-l-m}$$

$$\times \left[(\partial\omega)^l K_{[l]} (\partial\omega')^l \right] \left[\tfrac{1}{4} (\partial\partial\omega) K_{[0,1]} (\partial\partial\omega') \right]^m \,. \tag{97}$$

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