DISTINGUISHING PROPERTIES OF WEAK SLICE CONDITIONS II

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Abstract: The slice condition and the more general weak slice conditions are geometric conditions on Euclidean space domains which have evolved over the last several years as a tool in various areas of analysis. This paper examines some of their finer distinctive properties.

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1. Introduction

The *slice condition* is a metric-geometric condition for domains in Euclidean spaces $\mathbb{R}^n$. It is a very weak condition which, in particular, is satisfied by every simply connected planar domain, and was introduced by the first author and Koskela [6] to obtain a set of geometric classifications of domains in Euclidean spaces which support any of the Sobolev imbeddings, $p \geq n$. In later research, variations of the slice condition, including the weaker conditions known as *weak slice conditions* were used to refine these results and also to investigate questions in other areas of analysis; see [7], [8], [9], [3], [4], [1]. In particular, it is shown in [1] that in many metric measure spaces, including Euclidean space, one version of the slice condition is equivalent to Gromov hyperbolicity. This version implies all other slice-type conditions in the literature, so we may think of all slice-type conditions as weak versions of Gromov hyperbolicity.

With this range of applications, it should be useful to have a solid understanding of (weak) slice conditions, and in particular whether and how they differ from one another. Many properties and examples of these conditions were obtained in [8] and [9] but some fundamental questions remained, including a few that were listed in Section 6 of [9] as open problems. A couple of these questions were answered in [10]. In this paper, we construct examples to answer two of the remaining open problems in [9].

After some basics in Section 2, we define and briefly discuss the weak slice conditions in Section 3. Our first example is given in Section 4: it shows that there are 0-wslice domains (i.e. weak slice domains with a certain parameter α equal to zero), which are not slice domains, resolving Open Problem C in [9]. In Section 5, we show that for any pair of distinct numbers $\alpha, \beta \in [0, 1)$, there is a domain which is an α-wslice domain but not a β-wslice domain, thereby resolving Open Problem B in [9]. When $\alpha \geq \beta$, this is not hard to deduce from the results in [9], but it is somewhat surprising that the same is true when $\alpha < \beta$. In fact we prove the following result (and generalizations of it).

Theorem 1.1. *For each $0 < \alpha_0 < 1$, there are bounded Euclidean domains Ω_1 and Ω_2 such that Ω_i is an α-wslice domain, $0 \leq \alpha < 1$, if and only if $\alpha \leq \alpha_0$ (if $i = 1$) or $\alpha \geq \alpha_0$ (if $i = 2$).*

2. Notation and Terminology

Throughout this paper we will consistently employ the following notation. Note that certain parameters are optional in the sense that they are omitted from the notation when understood or when the exact choice is unimportant.

(Ω, d) is a rectifiably connected incomplete metric space possibly subject to additional restrictions (it is often just a domain in Euclidean space), $\overline{\Omega}$ is its metric completion (viewed as a superset of Ω), and $\partial\Omega := \overline{\Omega} \setminus \Omega$. For points $x, y \in \Omega$, a set $E \subset \Omega$, positive numbers r, s, we let:

$r \vee s$ and $r \wedge s$ denote the maximum and minimum, respectively, of r and s;

$\lceil r \rceil$ and $\lfloor r \rfloor$ denote the smallest integer $m \geq r$, and the largest integer $m \leq r$, respectively;

$\mathrm{len}(E) \equiv \mathrm{len}_d(E)$ denotes the Hausdorff 1-dimensional measure of E with respect to the metric d (so if E is an arc, $\mathrm{len}_d(E)$ is just its d-arclength);

$\mathrm{diam}(E) \equiv \mathrm{diam}_d(E)$ denotes the d-diameter of E;

$\delta(x) \equiv \delta_\Omega(x)$ denotes the distance from x to $\partial\Omega$,

$B(x,r) \equiv B_{d,\Omega}(x,r) := \{y \in \Omega \,:\, d(x,y) < r\}$,

$B_x := B_d(x, \delta_\Omega(x))$, and

$\Gamma_\Omega(x,y)$ denotes the class of all rectifiable paths $\lambda : [0,t] \to \Omega$ for which $\lambda(0) = x$ and $\lambda(t) = y$. We do not distinguish notationally between paths and their images. Whenever E is an (open or closed) ball, tE denotes its concentric dilate by a factor $t > 0$.

For $\alpha \in [0,1]$ we will also make extensive use of *subhyperbolic lengths* and the corresponding *metrics*. Denoting arclength measure by ds, we define these quantities by

$$\mathrm{len}_\alpha(\gamma) \equiv \mathrm{len}_{\alpha,\Omega}(\gamma) := \int_\gamma \delta_\Omega^{\alpha-1}(z)\, ds(z)$$

whenever γ is a rectifiable path in Ω, and

$$d_{\alpha,\Omega}(x,y) := \inf_{\gamma \in \Gamma_\Omega(x,y)} \mathrm{len}_{\alpha,\Omega}(\gamma),$$

We note that if Ω is a domain in Euclidean space, or in an imbedded k-manifold in $\mathbb{R}^n$, then $\mathrm{len}_{0,\Omega}$ and $d_{0,\Omega}$ are the well-known *quasihyperbolic length* and *quasihyperbolic distance*, and $d_{1,\Omega}$ is the *inner metric* with respect to Ω. For brevity, we shall denote the inner metric on Ω as d_Ω and the corresponding inner diameter of a subset E of Ω as $\mathrm{diam}_\Omega(E)$ in such cases. We shall also write $k(x,y) \equiv k_\Omega(x,y)$ in place of $d_{0,\Omega}(x,y)$.

Let us call $\gamma \in \Gamma_\Omega(x,y)$ $(\alpha; C_1, C_2)$-*efficient*, or simply α-*efficient*, if

$$\mathrm{len}_{\alpha,\Omega}(\gamma) \leq (1 + C_1) d_{\alpha,\Omega}(x,y) + C_2$$

We say that $\gamma \in \Gamma_\Omega(x,y)$ is an (α, C_1, C_2)-*quasigeodesic* for x, y if γ and all its subpaths are $(\alpha; C_1, C_2)$-efficient, while we say that γ is an α-*geodesic* if it is $(\alpha; 0, 0)$-efficient (or equivalently an $(\alpha; 0, 0)$-quasigeodesic). Obviously, efficient paths always exist, with (C_1, C_2) as close to $(0,0)$ as we wish, but α-geodesics might not exist. For instance in the Euclidean case, α-geodesics exist if $\alpha = 0$, but might not if $\alpha > 0$; see [12] and [8, Example 1.1].

Let $C \geq 1$, $x, y \in \Omega$, and let $\gamma \in \Gamma_\Omega(x,y)$ be a path of length l which is parametrized by arclength. We say that γ is a *C-uniform path* for $x, y \in \Omega$ if $l \leq Cd(x,y)$ (*bounded turning condition*) and $t \wedge (l-t) \leq C\delta_\Omega(\gamma(t))$ (*cigar condition*). In this case, we get the following estimates

$$(2.1) \quad d_{\alpha,\Omega}(x,y) \leq \begin{cases} 4C^2 \log\left(1 + \dfrac{d(x,y)}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right), & \alpha = 0, \\ C'[\delta_\Omega(x) \vee \delta_\Omega(y) \vee d(x,y)]^\alpha, & 0 < \alpha \leq 1. \end{cases}$$

where $C' = C'(C, \alpha)$. The $\alpha > 0$ case follows by an easy integration, estimating distance to the boundary by the triangle inequality for the initial and final parts of the path that are close to x and y, respectively, and by uniformity for the rest of the path. The case $\alpha = 0$ is Lemma 2.14 of [2].

3. Weak Slice and Slice Conditions

In this section we define, and briefly discuss, weak slice conditions; throughout we assume that $0 \leq \alpha < 1$. For more details, we refer the reader to [8], [9], and [10]. We also define the slice condition.

Suppose $C \geq 1$. A finite collection $\mathcal{F}$ of pairwise disjoint open subsets of Ω is a *set of C-wslices* for $x, y \in \Omega$ if

$$\text{(WS-1)} \quad \forall\, S \in \mathcal{F} \;\; \forall\, \lambda \in \Gamma_\Omega(x,y): \quad \operatorname{len}(\lambda \cap S) \geq d_S/C,$$

$$\text{(WS-2)} \quad \forall\, S \in \mathcal{F}: \; S \cap B(x, \delta(x)/C) = S \cap B(y, \delta(y)/C) = \emptyset,$$

where $d_S \geq \operatorname{diam}(S)$ is some finite number associated with each wslice S. We refer to such a set of data $\{(S, d_S) \mid S \in \mathcal{F}\}$ as being *C-admissible* for the pair $x, y \in \Omega$. Next, we define $\mathrm{WS}_\alpha(x,y;\Omega;C)$ by

$$\mathrm{WS}_\alpha(x,y;\Omega;C) := \sup\Big\{\, \delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y) + \sum_{S \in \mathcal{F}} d_S^\alpha : \\ \{(S, d_S) \mid S \in \mathcal{F}\} \text{ is } C\text{-admissible for } x, y \in \Omega \,\Big\}$$

Note that $\mathrm{WS}_\alpha(x,y;\Omega;C) \geq \delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y)$, since the empty set is trivially C-admissible. *A priori*, $\mathrm{WS}_\alpha(x,y;\Omega;C)$ could possibly be infinite, but, at least in the Euclidean context, it is bounded. In fact, Lemma 2.3 of [8] implies that there exists a constant $C' = C'(C,\alpha)$ such that

$$\mathrm{WS}_\alpha(x,y;\Omega;C) \leq C'[\delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y) + d_{\alpha,\Omega}(x,y)].$$

We use subscript notation such as $\mathcal{F} := \{S_i\}_{i=1}^m$ and $d_i := d_{S_i}$ in cases where we know that $\mathcal{F}$ is nonempty.

We define an α-wslice space essentially by reversing this last inequality for large subhyperbolic distance. More precisely, we say that the pair x, y satisfy an (α, C)-*wslice condition*, $C \geq 1$, if

$$\text{(WS-3)} \qquad d_{\alpha,\Omega}(x, y) \leq C\,\mathrm{WS}_\alpha(x, y; \Omega; C),$$

and we say that Ω is a (two-sided) (α, C)-*wslice space* if all pairs of points in Ω satisfy an (α, C)-wslice condition[1]. When $\alpha = 0$, (WS-3) simply says that $k(x, y) \leq C(2 + \mathrm{card}(\mathcal{F}))$, where $\mathcal{F}$ is a C-wslice collection of maximal cardinality. Note that in light of (WS-1), each of the slices S must separate x from y in Ω. It is also convenient to say that a C-admissible set $\{(S, d_S) \mid S \in \mathcal{F}\}$ for $x, y \in \Omega$ is an (α, C)-*wslice dataset* for x, y if we additionally have the following condition:

$$d_{\alpha,\Omega}(x, y) \leq C \left(\delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y) + \sum_{S\in\mathcal{F}} d_S^\alpha \right)$$

If the numbers d_S are not specified, it is assumed that $d_S := \mathrm{diam}_d(S)$.

Oftentimes the value of the constant C is unimportant and so we will on such occasions refer simply to "α-wslice conditions and/or domains". Modulo a possible augmentation of C, condition (WS-2) can actually be dropped in case $\alpha > 0$, but it is essential in case $\alpha = 0$, lest every domain be a $(0, C)$-wslice domain; see [9, Theorem 5.1].

In working with the weak slice conditions, the following additional hypotheses have often turned out to be useful:

(WS-4)
$$\forall\, S \in \mathcal{F} \;\; \exists\, (\alpha; C_1, 0)\text{-efficient } \gamma \in \Gamma_\Omega(x, y) : \quad \mathrm{len}_{\alpha,\Omega}(\gamma \cap S) \leq C d_S^\alpha,$$

(WS-5)
$$\forall\, S \in \mathcal{F} \;\; \exists\, z_S \in S : \qquad B_d(z_S, d_S/C) \subset S,$$

(WS-1$^+$)
$$\forall\, S \in \mathcal{F} \;\; \forall\, \lambda \in \Gamma_\Omega(x, y) : \qquad \mathrm{diam}_d(\lambda_S) \geq d_S/C,$$

where λ_S denotes a component of $\lambda \cap S$ of maximal diameter. We refer to (α, C)-wslice domains which satisfy (WS-4), (WS-5), and (WS-1$^+$) as (α, C)-wslice$^+$ domains. Of these extra conditions, only

[1] In [8] and [9], the labels (WS-2) and (WS-3) were reversed, but that does not suit our more general discussion here.

(WS-1^+) is significant if we do not care about the exact value of C, since, modulo a possible quantitative change in the value of C, (WS-4) and (WS-5) can be assumed without loss of generality; see [10, Section 2]. The choice of $C_1 > 0$ and γ in (WS-4) is unimportant; we can even take γ to be an α-geodesic (and so $C_1 = 0$) if one exists. We suspect (at least in a Euclidean or inner Euclidean context, and modulo a controlled increase in the value of C and a change in the wslice dataset) that (WS-1^+) also follows from the (α, C)-wslice condition, but we cannot prove this.

If $\Omega \subsetneq \mathbb{R}^n$ is a domain, we call Ω an (α, C)-wslice, or inner (α, C)-wslice, domain if it is an (α, C)-wslice space with respect to the Euclidean or inner Euclidean metric, respectively. Notice that the difference between Euclidean and inner Euclidean α-wslice domains is rather minor since distance to the boundary, the associated subhyperbolic metrics and the Hausdorff 1-dimensional measure are unchanged, and so there is no difference in any of (WS-1) through (WS-5). The only change is in the requisite lower bound in the size of the d_S (from $\operatorname{diam}(S)$ to $\operatorname{diam}_\Omega(S)$). Nevertheless, Example 3.1 of [10] shows that there are wslice domains that are not inner wslice domains.

We say that the pair $x, y \in \Omega$ satisfy the *C-slice condition*, $C \geq 1$, if there exists $\mathcal{F} := \{(S_i, d_i)\}_{i=1}^m$, with $d_i \equiv \operatorname{diam}_d(S_i)$, and an $(0; C-1, 0)$-efficient path $\gamma \in \Gamma_\Omega(x, y)$ such that:

(a) $\mathcal{F}$ is an (α, C)-wslice dataset for x, y;
(b) (WS-4) and (WS-5) hold for each $1 \leq i \leq m$, $\alpha = 0$;
(c) $\forall\, 1 \leq i \leq m,\ z \in \gamma \cap S_i :\ 1/C \leq \delta_\Omega(z)/d_i \leq C$;
(d) $\gamma \subset B_{k_\Omega}(x, C) \cup B_{k_\Omega}(y, C) \cup \left(\bigcup_{i=1}^m \overline{S_i}\right)$.

Slice spaces and domains are then defined in the same manner as their weak slice equivalents.

This definition of a slice condition is different from the original (inner) Euclidean definition in [6], but is equivalent to it in the Euclidean and inner Euclidean settings (modulo a quantitatively controlled change in C). For the interested reader, we note that the original definition implies (a) by [8, Lemma 2.4], while (b)–(d) are easy to deduce from the original definition. In the original definition, the path γ is not assumed to be 0-efficient, but this follows from the previously mentioned estimate $\mathrm{WS}_\alpha(x, y; \Omega; C) \lesssim \delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y) + d_{\alpha,\Omega}(x, y) + 1$. Proving that the original definition follows from the new one is routine. We point out that we still do not know whether (WS-1^+) holds for slice spaces (see Open Problem A in Section 6 of [9]).

In the (inner) Euclidean setting, we point out that the (important) upper bound of (c) is redundant. Indeed, Lemma 2.2 of [8] tells us that if $\{S_i, d_i\}_{i=1}^m$ is a $(0, C)$-wslice dataset for points x, y in a Euclidean domain Ω, and d is either the Euclidean or inner Euclidean metric, then $\delta_\Omega(w) < C \operatorname{diam}_d(S_i)$ for all $w \in S_i$, $1 \le i \le m$.

The point of our new definition is that it emphasizes the distinction between slice and 0-wslice conditions. Since (WS-4) and (WS-5) follow quantitatively from any α-wslice condition, it seems that the crucial distinction is the existence of a path γ which is covered by the closure of the slices and quasihyperbolic balls around x, y. Intuitively, this means that we are able to "slice up nicely all of the region between x and y", whereas in a 0-wslice condition, we merely assume that we can "slice up nicely a reasonably large part of the region between x and y".

4. 0-wslice but not slice

Here we give an example of a 0-wslice domain that is not a slice domain, thereby resolving Open Problem C in [9]. Simpler examples with related properties can be found elsewhere. Specifically, Proposition 4.5 of [8] allows one to construct examples of α-wslice domains, $\alpha > 0$, that are not slice domains; in fact, they are not even 0-wslice domains. A one-sided 0-wslice domain (meaning that (WS-3) is assumed for arbitrary x and a fixed y) that is not a one-sided slice domain is given in [3, Example 4.9]. However, examples similar to these cannot lead to a (two-sided) 0-wslice domain that is not a slice domain.

Example 4.1. Our domain $G \subset \mathbb{R}^2$ is $(0,1)^2 \cup \left(\bigcup_{j=1}^\infty D_j\right)$, where the sets D_j are "decorations" attached to the right-hand side of the unit square, centered at $(1, a_j)$. To define D_j, we begin with a pair of rectangles of length R_j and width $4r_j$ glued together via a pair of bent strips of vertical width $2r_j$ that border an omitted square of sidelength $\sqrt{2}R_j$ with sides at angle 45 degrees to the x_1-axis, as in the diagram above. We then remove two closed sets. The first removed set is the horizontal midline segment L_j that begins at the right-hand side of our decoration and ends at a distance $2r_j$ from the left-hand side; this effectively makes the set into a

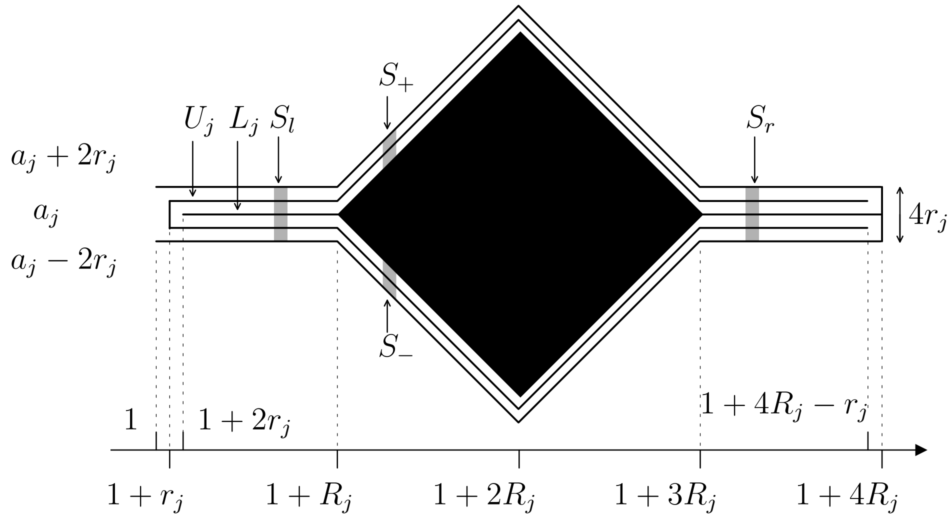

FIGURE 4.1. The decoration D_j

union of an upper and a lower corridor, both with three 45 degree bends. Finally, we remove a bent U-shaped set U_j which follows the (horizontal and diagonal) midlines of the upper and lower corridors and whose points have x_1-coordinates between $1+r_j$ and $1+4R_j-r_j$. Thus in our final decoration D_j, there are four long bent corridors, each of vertical width r_j and with x_1-coordinates between $1+r_j$ and $1+4R_j$; we call these the *first, second, third, and fourth corridors* in order of increasing y-values. The exact values of a_j, R_j, and r_j are irrelevant as long as the decorations are pairwise disjoint and $4 \le R_j/r_j \to \infty$ as $j \to \infty$; we could for instance pick $a_j = 2^{-j}$, $R_j = 4^{-j-1}$, and $r_j = 8^{-j-1}$.

The proof that G is a $(0, 10)$-wslice$^+$ domain is a rather lengthy case analysis similar to those in [10, Section 3], [3, Section 4.7], and [4, Theorem 3.6], so we merely mention the distinctive features of the proof. The most interesting pairs of points $y = (y_1, y_2)$ and $z = (z_1, z_2)$ are those that are fully contained in a single decoration D_j, and which do not lie close to the boundary of the domain in the sense that $\delta_G(y), \delta_G(z) \ge r_j/4$. For such points, the slices we use are one of the following four types of slices that we collectively refer to as *corridor slices*. Letting $N_j = \lfloor R_j/r_j \rfloor$, we split the part of D_j between the coordinate values $x_1 = 1+2r_j$ and $x_1 = 1+R_j$ into N_j *left slices* like S_l all of equal width in the first coordinate (see Figure 3.1). We similarly split the part of D_j between $x_1 = 1 + R_j$ and

$x_1 = 1+3R_j$ into $2N_j$ *upper slices* like S_+, and $2N_j$ *lower slices* like S_-. Finally, we similarly split the part of D_j between $x_1 = 1+3R_j$ and $x_1 = 1+4R_j - r_j$ into N_j *right slices* like S_r.

A $(0,10)$-wslice$^+$ inequality trivially holds when $k(y,z) \le 20$, so we may assume that $k(y,z) > 20$. Suppose y,z both lie in the fourth corridor, and by symmetry we assume that $y_1 \le z_1$. Then we take as our admissible set all left, upper, and right slices that lie in the set $[y_1 + r_j, z_1 - r_j] \times \mathbb{R}$, accompanied by their diameters. Note that since $k(y,z) \ge 20$ and $\delta_G(y), \delta_G(z) \ge r_j/4$, it follows that $z_1 \ge y_1 + 9r_j$, and it is readily verified that the chosen slices form a $(0,10)$ wslice$^+$ dataset. As a hint note that the horizontal line segment that runs through a left or right slice along the middle of a corridor has quasihyperbolic length 1. The same argument works for the other corridors, except that we use lower slices in place of upper slices for the first and second corridors.

If y,z lie in the third and fourth corridors, respectively, we similarly get a $(0,10)$ wslice$^+$ dataset by taking all right slices that lie in the set $[y_1 \wedge z_1, \infty) \times \mathbb{R}$ and that do not contain points within a distance r_j of y or z. If y,z lie in the second and fourth corridors, respectively, then we know that $k(y,z) \approx N_j$, and we get a $(0,10)$-wslice$^+$ dataset by taking all left and all right slices that do not contain points within a distance of r_j of y or z. All other possibilities are like one or the other of these last two cases.

Note that some or all upper and lower slices can be added to the wslice$^+$ dataset for certain choices of pairs y,z, but not if y,z are positioned badly. For instance if y,z lie in the third and fourth corridors, respectively, and $y_1 \vee z_1 \le 1 + R_j$, then for every upper or lower slice S, there is a path from y to z that avoids S. This problem with "slicing up" the middle part of D_j is precisely what makes every slice condition fail, an argument that we now make more precise.

Suppose $\{S_i\}_{i=0}^m$ is a set of C-slices for the pair of points $z := (1+R_j, a_j + r_j/2)$, $y := (1+R_j, a_j + 3r_j/2)$, with $\gamma \in \Gamma_G(y_j, z_j)$ being the associated path. Then γ has to contain a point u with first coordinate $1+2R_j$ in either the first or fourth corridor. Since $k(u,\{y,z\})$ tends to infinity as j tends to infinity, $u \in \bigcup_{i=1}^m \overline{S_i}$ if j is sufficiently large. Suppose therefore that $u \in \overline{S_i}$. Since there is a path from y to z that stays a distance greater than R_j from u, it follows from (WS-1) that $d_i > R_j$. The slice property now ensures that $\delta_G(u) \geq d_i/C > R_j/C$, contradicting the fact that $\delta_G(u) < r_j$ when j is sufficiently large. Thus G is not a slice domain.

Open Problem A. Find a domain $\Omega \subsetneq \mathbb{R}^n$ which is an inner 0-wslice domain, but not a slice domain. More generally, one could ask for any example of a length space which is an inner 0-wslice space but not a slice space.

The above problem is posed because the authors feel that slice-type conditions, and the relationships between them, are more subtle when the underlying metric is a length metric. Note that the previous example does not work since the corridor slices almost all have inner Euclidean diameter much larger than their Euclidean diameter, and so inequality (WS-1) of the inner $(0, C)$-wslice fails when j is sufficiently large.

5. β-wslice but not α-wslice

Suppose $0 \leq \alpha < \beta < 1$. Theorem 4.1 of [9] tells us that for domains of product type, the inner β-wslice$^+$ property is equivalent to the so-called inner β-mCigar property. By taking the product of an interval with Lappalainen's rather complicated examples of domains that are β-mCigar but not α-mCigar [14, 6.7], we therefore get domains that are (inner) β-wslice$^+$, but not (inner) α-wslice$^+$, whenever $0 \leq \alpha < \beta < 1$. In Open Problem B of [9, Section 6], the authors ask if an α-wslice$^+$ domain must necessarily be a β-wslice$^+$ domain if $0 \leq \alpha < \beta < 1$.

In this section, we answer this open problem by means of a counterexample similar to Example 4.1. Another variation of this construction will give a domain that is β-wslice$^+$, but not α-wslice$^+$, and is much simpler than the product type domains mentioned above.

Our first two counterexamples have the form $G := (0,1)^2 \cup \left(\bigcup_{j=2}^{\infty} D_j\right)$, where each attached decoration D_j is similar to the ones in Example 4.1, the only essential difference being that the horizontal rectangular parts of D_j are of width $4r'_j$ and length R'_j. These altered parts are either longer and fatter, or shorter and thinner, than before, while the diagonal parts have the same dimensions as before. The wider corridors are pinched using linear interpolation near where they meet narrower corridors. The following pair of diagrams of the leftmost part of D_j should suffice to make more precise what we mean.

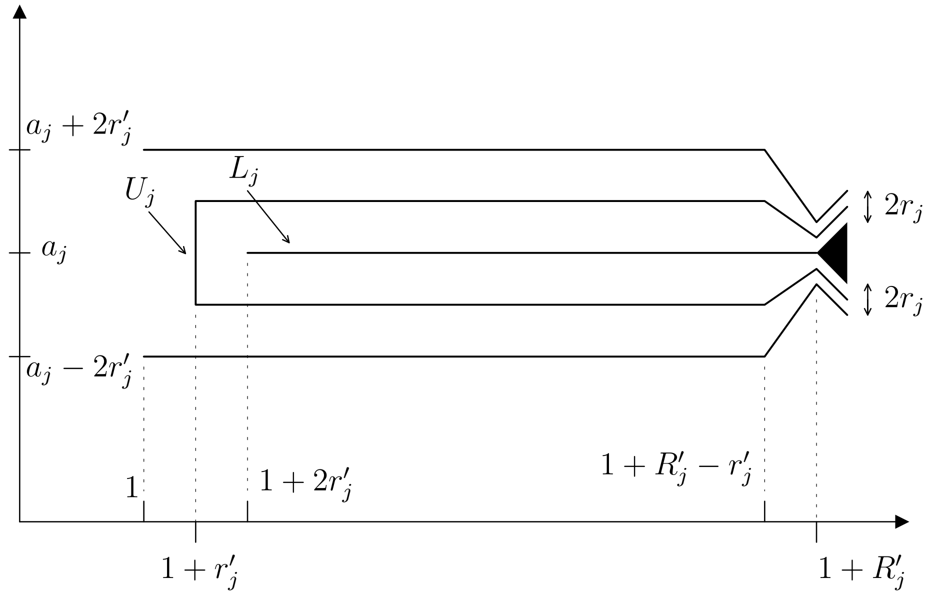

FIGURE 5.1. The left part of D_j when $R'_j > R_j$ and $r'_j > r_j$.

Let us take $R_j := 2^{-jp}$, $R'_j := 2^{-jp'}$, $r_j := 2^{-jq}$, $r'_j := 2^{-jq'}$, where the quadruple (p, q, p', q') is *allowable* if $0 < p \le q - 2$, $0 < p' \le q' - 2$, $p \ge 2$, and $q' \ge 2$; the last two bounds are assumed merely to ensure that we can attach all these decorations to one side of the unit square without overlap. The exact locations of the decorations, i.e. the values of a_j, are irrelevant as long as they do not overlap.

Theorem 5.1. *Given $0 < \alpha_0 < 1$, any allowable choice of p, p', q, q' with $p' = p + 1 - \alpha_0$ and $q' = q + 1$ gives a domain G which is an α-wslice$^+$ domain for $\alpha \le \alpha_0$, but not an α-wslice domain for $\alpha > \alpha_0$.*

Sketch of proof. Writing $N'_j = \lfloor R'_j / r'_j \rfloor$ and $N_j = \lfloor R_j / r_j \rfloor$, we define corridor slices as in Example 4.1, so that there are N'_j left slices between $x_1 = 1 + 2r'_j$ and $x_1 = 1 + R'_j - r'_j$, $2N_j$ upper and $2N_j$ lower slices between $x_1 = 1 + R'_j$ and $x_1 = 1 + R'_j + 2R_j$, and N'_j right slices between $x_1 = 1 + R'_j + 2R_j + r'_j$ and $x_1 = 1 + 2R'_j + 2R_j - r'_j$. In this proof, $A \ll B$ means that $A/B \to 0$ as $j \to \infty$.

Note that the $d_{\alpha,G}$-length of the (horizontal or diagonal) line segment given by the intersection of a single corridor slice with the midline of that corridor is comparable with r_j^α for an upper or

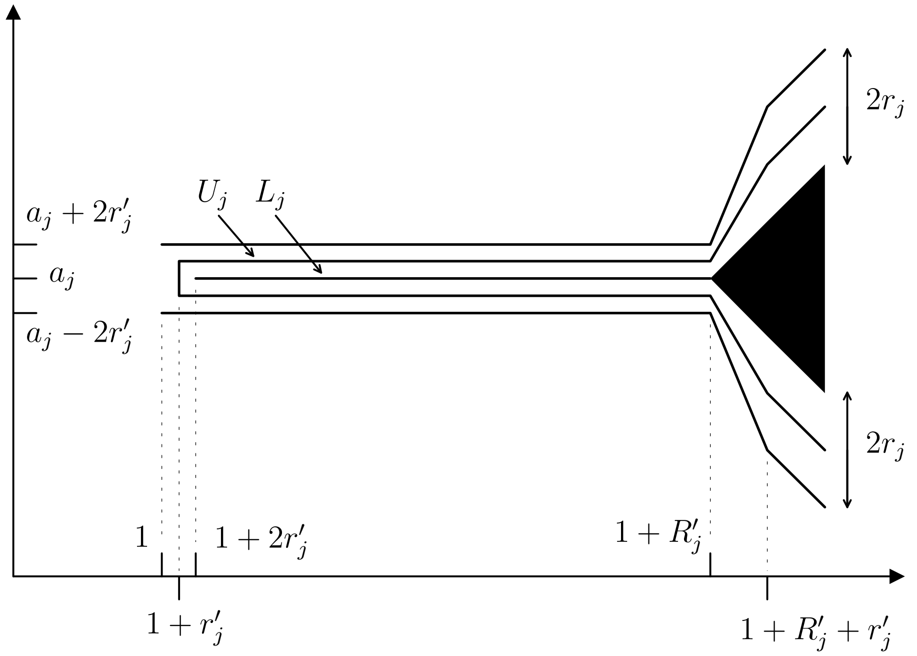

FIGURE 5.2. The left part of D_j when $R'_j < R_j$ and $r'_j < r_j$.

lower slice S, and $(r'_j)^\alpha$ for a left or right slice S. For some pairs of points y, z, the (α, C)-wslice$^+$ defining inequality holds using a similar argument to that in Example 4.1 once we pick $C = C(\alpha)$ to be large enough. However this method fails in other cases. The basic obstacle is revealed by taking $y = (1 + R'_j/2, a_j + 3r'_j/2)$ and $z = (1 + R'_j/2, a_j + r'_j/2)$. Then all paths from y to z have to go through complete horizontal and diagonal parts of at least two corridors, and so it follows that $d_\alpha(y, z) \approx L_{j,\alpha} + L'_{j,\alpha}$, where $L_{j,\alpha} := R_j/r_j^{1-\alpha}$ and $L'_{j,\alpha} := R'_j/(r'_j)^{1-\alpha}$. We cannot use upper or lower slices in any admissible set for y, z since there always exist connecting paths that avoid any given set of this type, but the set of all right slices S (paired with their diameters d_S) always gives a 10-admissible set. Denoting by $\mathcal{F}$ the set of such right slices, we see that $\sum_{S \in \mathcal{F}} d_S^\alpha \approx L'_{j,\alpha}$. Since

$$\frac{1}{j} \cdot \log_2 \left(\frac{L_{j,\alpha}}{L'_{j,\alpha}} \right) = p' - p - (1 - \alpha)(q' - q) = \alpha - \alpha_0,$$

we see that $\mathcal{W}_0 := \{(S, \mathrm{diam}(S) \mid S \in \mathcal{F}\}$ is an (α, C)-wslice$^+$ dataset for appropriate $C = C(\alpha)$ as long as $\alpha \leq \alpha_0$, as required.

However, $\mathcal{W}_0$ fails to be an (α, C)-wslice$^+$ dataset when $\alpha > \alpha_0$ since then $L'_{j,\alpha} \ll L_{j,\alpha}$.

Given $\alpha \in (\alpha_0, 1)$, it remains to show that there are no (α, C)-wslice datasets for the pair y, z, assuming that j is sufficiently large. Suppose for the sake of contradiction that $\mathcal{W} := \{(S, d_S) \mid S \in \mathcal{F}\}$ is some (α, C)-wslice dataset and write $\Sigma_\alpha := \sum_{S \in \mathcal{F}} d_S^\alpha$. Since $\delta^\alpha(y) + \delta^\alpha(z) \approx r_j^\alpha \ll L_{j,\alpha}$, it follows that $\Sigma_\alpha \approx d_\alpha(y, z) \approx L_{j,\alpha}$. Using (WS-1) and the geometry of the domain, we see that any slice that includes points outside D_j has diameter larger than $R'_j/2$. Furthermore, if m_i is the number of such slices S for which $d_S \in (2^{i-1}R'_j, 2^i R'_j]$, then (WS-1) and the fact that there are paths from y to z of length comparable to R_j together imply that $m_i \lesssim 2^{-i}R_j/R'_j$. By summing the resulting series over the index i, we see that the contribution of all such slices to Σ_α is at most comparable with $A_j := R_j/(R'_j)^{1-\alpha}$, and $A_j \ll L_{j,\alpha}$ because $p' = p + 1 - \alpha_0 < q$. We can therefore delete these slices from our dataset and our redefined set $\mathcal{W}$ is still an (α, C)-wslice dataset (if we suitably redefine C).

Consider next from the remaining slices those that do not enter into any diagonal corridor by a distance more than R'_j from the base (meaning the left and right ends of the diagonal corridors of D_j). We let λ temporarily denote the path in $\Gamma_G(y, z)$ that runs along the U-shaped mid-corridor path on the right. Since the intersection of λ with the slices under present consideration can have length at most comparable to R'_j, it follows that the number m_i of such slices S for which $d_S \in (2^i r'_j, 2^{i+1} r'_j]$ is at most comparable to $2^{-i}R'_j/r'_j$. Since any such slice has diameter at least comparable to r'_j, it follows that the contribution of such slices is at most comparable to $L'_{j,\alpha} \ll L_{j,\alpha}$.

It remains to consider the slices which lie in D_j, and enter into at least one diagonal corridor by a distance exceeding R'_j from the base. Let m_i be the number of such slices S for which $d_S \in (2^i R'_j, 2^{i+1} R'_j]$. Now such slices must include points that are a distance at most comparable with $2^i R'_j$ from the base, since if all points in such a slice are much further than this from the base then the slice cannot contain points in both the upper and lower pair of corridors and so cannot separate the pair y, z, contradicting (WS-1). We deduce that such slices are fully contained within a distance comparable to $2^i R'_j$ of the base, and so the intersection of λ with such slices can have length at most comparable to $2^i R'_j$. It follows that $m_i \lesssim 1$. In order to accommodate all such slices, the index i need only run up to the value $\log_2(R_j/R'_j)$. Consequently, we may estimate the contributions of these remaining slices with the upper bound: $\sum_{i=0}^{\log_2(R_j/R'_j)} 1 \cdot (2^i R'_j)^\alpha \approx (R'_j)^\alpha \sum_{i=0}^{\log_2(R_j/R'_j)} 2^{\alpha i} \approx (R'_j)^\alpha (R_j/R'_j)^\alpha \approx R_j^\alpha$. But this is much smaller than $L_{j,\alpha}$ when j is large and so we get a contradiction. □

The above construction is quite flexible: it can be varied to give examples with various other types of behavior. We content ourselves below with three variants, but first let us define $\alpha(D)$, the *α-set of a domain* $D \subsetneq \mathbb{R}^n$, to be the set of all $\alpha \in [0, 1)$ for which a given domain D is an α-wslice domain. Theorem 5.1 shows that there are domains G with $\alpha(G) = [0, \alpha_0]$ for each $0 < \alpha_0 < 1$. By varying some of the details in the definition of G, we now get some other α-sets. We omit the details of the proofs which are all similar to that of Theorem 5.1.

The first of our three examples allows us to get the same α-sets as the product-type examples mentioned at the beginning of this section, but is much simpler. Our second example shows that the endpoint of our α-set can be omitted, and the third shows that α-sets need not be intervals.

Example 5.2. If $0 < \alpha_0 < 1$, then any allowable choice of p, p', q, q' with $p = p' + 1 - \alpha_0$ and $q = q' + 1$ gives a domain G with $\alpha(G) = [\alpha_0, 1)$.

Example 5.3. If we redefine $r_j := j2^{-jq}$ in Theorem 5.1, but leave everything else unchanged, then $\alpha(G) = [0, \alpha_0)$. The key fact is that when $\alpha = \alpha_0$, we now have $L_{j,\alpha}/L'_{j,\alpha} \to \infty$ as $j \to \infty$.

Example 5.4. Consider a domain with decorations D_j similar to those of Theorem 5.1, but with two rectangular parts on both sides of the diagonal part. The diagonal part and the innermost pair of rectangular parts of D_j are identical in shape to the full decoration D_j of Theorem 5.1 with the exception that we must alter U_j and L_j so that they also pass through the outer rectangular parts, which have length $R''_j := 2^{-jp''}$ and width $4r''_j := 4 \cdot 2^{-jq''}$. These outer parts are chosen to be longer and fatter than the inner rectangular parts and are connected by linear interpolation to the inner parts as before. By choosing $p' = p + 1 - \alpha_0$ and $q' = q + 1$, and $p'' = p - 1 + \alpha_1$ and $q'' = q - 1$, $0 < \alpha_0 < \alpha_1 < 1$, it follows that $\alpha(G) = [0, \alpha_0] \cup [\alpha_1, 1)$.

By taking $(p,q) = (3,6)$ and $(p',q') = (p,q) \pm (1-\alpha_0, 1)$ for some $\alpha_0 \in [0,1)$, it is clear that (p,q,p',q') is always allowable. This allows us to consider domains consisting of a sequence of decorations D_j joined to the unit square that generalize the above constructions. Each D_j has diagonal corridors specified by the dimensional parameters $R_j := 2^{-3j}$ and $r_j := 2^{-6j}$, and D_j also has one or more horizontal corridors on each side of these diagonal corridors, symmetrically distributed around the center of the diagonal corridor: if the ith horizontal corridor on the left counting outwards from the diagonal corridor has dimensional parameters $R_{j,i} := 2^{-jp_{j,i}}$ and $r_{j,i} := 2^{-jq_{j,i}}$, then the ith horizontal corridor on the right is defined by these same parameters. Here $(p_{j,i}, q_{j,i}) - (3,6)$ is always $\pm(1-\alpha_{j,i}, 1)$ for some $0 < \alpha_{j,i} < 1$. The corridors are joined by linear interpolation as before. We call these *corridor decorations* and we call the domain Ω obtained by joining a sequence of such corridor decorations a *decorated square (with corridor decorations $(D_j)_{j=1}^\infty$)*.

It is not hard to show that the sets $\alpha(\Omega)$ for the set of decorated squares Ω, are closed under countable intersections and finite unions. Let us justify this first for intersections. Suppose $(\Omega_k)_{k=1}^\infty$ is a sequence of decorated squares with corridor decorations $(D_{k,j})_{j=1}^\infty$. It is routine to show that we can define a decorated square Ω with corridor decorations $(D_i)_{i=1}^\infty$, where $D_i := D_{k_i,j_i}$ for some appropriate choice of k_i, j_i, such that $\alpha(\Omega) = \bigcap_{k=1}^\infty \alpha(\Omega_k)$.

As for finite unions, if we have a finite set of decorated squares Ω_k, $k = 1, \dots, k_0$, with corridor decorations $(D_{k,j})_{j=1}^\infty$, then we take our cue from Example 5.4: for fixed j, we join together the horizontal corridors of each $D_{k,j}$, $k = 1, \dots, k_0$, as we did in Example 5.4 to get a new decoration D_j. The decorated square Ω with corridor decorations $(D_j)_{j=1}^\infty$ then has the property that $\alpha(\Omega) = \bigcup_{k=1}^{k_0} \alpha(\Omega_k)$.

The above constructions suggest that every Borel subset of $[0,1)$ may well be of the form $\alpha(G)$ for some bounded domain $G \subset \mathbb{R}^n$. However we do not know if this is so.

As pointed out at the start of this section, there are domains in $\mathbb{R}^n$ which are inner β-wslice$^+$ but not inner α-wslice$^+$ whenever $0 \le \alpha < \beta < 1$. However none of our decorated examples above are inner α-wslice domains, so they cannot answer the following problem.

Open Problem B. Given $0 \le \alpha < \beta < 1$, is there a domain in $\mathbb{R}^n$ which is inner α-wslice$^+$ (or even just α-wslice) but not β-wslice$^+$? More generally, one could ask for any example of a length space which is inner α-wslice$^+$ (or even just α-wslice) but not β-wslice$^+$.

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