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Ollscoil na hÉireann Má Nuad

Piecewise Smooth Systems and Monotonicity

by

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Masters Thesis

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Date: 16^{th} of March, 2011.

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I would like to express thanks to my parents, Patrick and Jacqueline and my sister Fiona, for their unwavering support (and for feeding and housing me!). Finally I would like to thank my friends for putting up with me!

Abstract

In this thesis we study piecewise smooth and switched positive systems and investigate the monotonicity properties of such systems. We describe many examples of such systems, particulary drawing from the mathematical biology literature, in order to motivate the work in later chapters. We describe the mathematical theory behind our work in Chapters 3 and 4. In particular we review the theory of LTI systems, positive LTI systems and monotone systems, indicating how monotonicity can be used to determine the asymptotic behaviour of positive LTI systems. We also discuss issues which arise in the study of piecewise smooth and switched linear systems and review solution concepts for such systems. In Chapter 5, we extend the Kamke conditions for smooth monotonic systems to piecewise smooth systems, and in certain cases show that they are equivalent to the monotonicity of the system.

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Chapter 1

Introduction and Overview

In this Chapter, we introduce the topics under consideration in this thesis and we provide an overview of the remainder of the work.

1.1 Introductory Remarks

Many mathematical models of physical processes entail non-smooth dynamical systems. Non-smoothness can occur in a variety of ways. For example as impulse effects (such as models for a ball bouncing with instantaneous impact dynamics), and also in the switching between different modes in hybrid and switched systems [47]. Piecewise systems are often used to make approximations of nonlinear systems [13], [14]. Non-smooth and discontinuous dynamics are found in many applications including but not limited to

- the modelling of rigid bodies, for example in robotics [32], [51]
- models of genetic regulatory networks [14]
- the modelling of cardiac arrhythmias [29]
- DNA replication [31].

Different paradigms have been developed for the study of non smooth dynamical systems. These have been described in survey papers such as [20] and [9]. They include impulsive differential equations and discontinuous differential equations, as well as various hybrid system formalisms including switched sytems, which we will discuss extensively in this thesis.

Hybrid systems are dynamical systems which exhibit both continuous and discrete dynamic behaviour and are of great practical importance. They are characterised by periods of smooth evolution interrupted by discrete state transitions, e.g. the impact of a bouncing ball, the switching of gears in a car, or even in certain models in biology. They typically involve a continuous variable $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and a discrete variable $q \in Q$ where Q is a finite or countable set. The values of the discrete variable represent modes of operation. Within each mode the system dynamics are typically given by a continuous flow defined by a differential or difference equation.

The analysis of hybrid systems is generally more difficult than that of a purely continuous or discrete system as the discrete dynamics can affect the continuous evolution and vice versa. In fact, it is the interaction between discrete and continuous dynamics that gives the subject its special flavour. For example, one major concern in the study of dynamical systems is the stability of the system. In a hybrid system, even when the stability of component systems is easy to verify, it is far from trivial to determine the stability of the system as a whole. Discontinuities and impulses play a central role in the properties of such systems. The bouncing ball in Example 2.1.1 of Chapter 2 is a simple example of a hybrid system with single discrete state and a continuous state of dimension two. Even such a simple system can display peculiar behaviour. In Chapter 4, we discuss potential issues with hybrid systems, one of which is the appearance of Zeno behaviour, which Example 2.1.1 exhibits.

Zeno behaviour is, loosely, an infinite number of discrete state transitions occuring in a finite amount of time. Of course, in practice, the ball will stop bouncing after a finite number of bounces since the impacts with the ground are neither instantaneous nor perfectly elastic.

A subclass of hybrid systems that is of particular relevance to us is the class of "state-dependent switched systems". State-dependent switched systems are ones in which the state space is partitioned into different regions, with dynamics differing according to the region. Much work has been done in formulating solution concepts for such systems, most notably by the Russian mathematician Filippov, whose work we shall be drawing from in Chapters 4 and 5. Filippov first introduced the idea of replacing the system of differential equations with a differential inclusion, [17]. In such systems one has to be careful at the boundary, and sometimes the only solutions which make sense are "Filippov solutions". These can give rise to interesting behaviour, such as the appearance of sliding modes, in which the trajectory "slides" along the boundary. Such sliding modes are of importance in Control Theory.

Two properties we shall be concerned with are positivity and monotonicity. Positive systems are those in which the state variables only take on non negative values. Many examples in biology are positive to ensure that the variables are physically meaningful. Monotonicity means that the ordering of initial states is preserved. This is a very powerful property as it allows us to obtain results concerning the asymptotic behaviour of the system, see for example [5], [4]. Monotone dynamical systems have long been studied, and the theory in its modern form was developed by M.W. Hirsch in a series of papers called "Systems of differential equations that are competitive or cooperative". Monotonicity appears in many practical examples, and for this reason its study is particularly important; see [49] for theoretical results concerning monotone systems and an application to modelling the control of protein synthesis in the cell. We are interested in investigating monotonicity for state-dependent positive switched systems.

1.2 Overview

We shall begin the thesis by setting the context for much of the later work. The topics under consideration in this thesis are relevant to a number of different application domains. For this reason, in Chapter 2, numerous practical applications of switched, piecewise smooth and hybrid systems are provided in order to motivate our later discussion, including models from the biological and medical sciences.

In Chapters 3 and 4 we discuss the mathematical background which is needed for our later discussion. Chapter 3 contains well known definitions and results pertaining to linear time invariant (LTI) and positive LTI systems, and we describe the link between positivity and monotonicity in LTI systems here.

In Chapter 4 the discussion is centred on piecewise smooth and switched linear systems in particular. Some basic definitions are given. We then go on to discuss some issues which arise in the study of piecewise smooth and switched systems. It is at this point that we give more precise solution concepts for such systems, introducing differential inclusions and the notion of a Filippov solution. The discussion then goes on to recall results in the stability of switched linear systems under arbitrary switching, and in particular results which are specific to switched positive linear systems. This includes recent work [30], [37] on the problem of the existence of a common linear copositive Lyapunov function for a positive switched system and what this means for the stability of such as a system.

In Chapter 5, we introduce new results in the monotonicity of piecewise smooth and positive switched systems for the simple case where $\mathbb{R}^n_+$ is partitioned into two regions by means of a hyperplane through the origin. We reformulate the Kamke conditions for this new situation and, using similar arguments as in [49], we show that the Piecewise-Kamke conditions are equivalent to monotonicity in certain cases.

Chapter 2

Non-Smooth Systems - a Practical Motivation

In this chapter, we set the scene for the later work by describing numerous practical examples of piecewise smooth, hybrid and positive switched systems, drawing extensively from the mathematical biology literature, where these systems are often used as a modelling tool.

2.1 Simple Examples of Hybrid and Piecewise Smooth Systems

Example 2.1.1. *A bouncing ball*

In Chapter 1, we mentioned that impacting systems often give rise to non-smooth dynamics [26]. The simplest example of such a system is a bouncing ball. Newton's laws of motion govern the continuous dynamics of the dropping ball, which is dropped from some initial height h. We consider the ball as a point mass. The continuous state is represented by

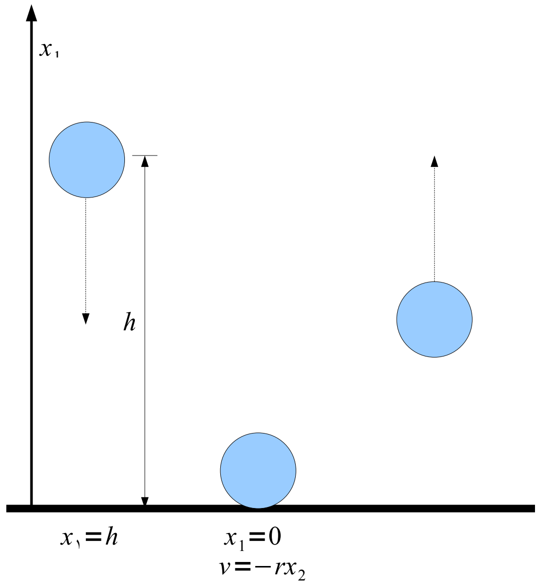

Figure 2.1: A bouncing ball

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

where x_1 denotes the vertical position of the ball and $x_2 = v$ its vertical velocity. Between each bounce it displays continuous dynamics while at each impact its velocity undergoes an abrupt change. Since we are assuming Newton's laws apply, the reversed velocity is a coefficient $0 \leq r \leq 1$ times the incoming velocity.

So, when $x_1 > 0$, the dynamics for this system are given by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -g, \end{aligned} \tag{2.1}$$

where g is acceleration due to gravity. When $x_1 = 0$, corresponding to the ball hitting the ground, the state vector is reset abruptly according to:

$$\begin{aligned} x_1^+ &= x_1 \\ x_2^+ &= -rx_2, \end{aligned} \tag{2.2}$$

i.e. when the height of the ball is zero, its velocity is reversed and decreased by a factor of r. This is a simple example of a system with state reset, or an impulse effect.

Example 2.1.2. *Gear shifting in a motor vehicle*

A simple and well known example of a hybrid system is the longitudinal motion of a car. Denote its position and velocity by x_1 and x_2 respectively, and the selected gear by $q \in Q := \{-1, 0, 1, 2, 3, 4, 5\}$, where -1 is reverse and 0 is neutral. Position and velocity are both continuous state variables while the engaged gear q is discrete. We also have a control input $a \in [a_{min}, a_{max}]$ which represents the position of the accelerator pedal. The dynamics for this system are given by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= f(a, q), \end{aligned}$$

where $f : \mathbb{R} \times Q \to \mathbb{R}$ defines the continuous dynamics in each mode. Typically we will have

$$\begin{aligned} \frac{\partial f}{\partial a} &< 0,\ f(a,q) < 0 \text{ for } q = -1 \\ \frac{\partial f}{\partial a} &= 0,\ f(a,q) = 0 \text{ for } q = 0 \\ \frac{\partial f}{\partial a} &> 0,\ f(a,q) > 0 \text{ for } q \geq 1. \end{aligned}$$

2.2 Four Examples from Mathematical Biology

Positive and monotone systems arise frequently in biology. In this section, we describe several examples to show that systems with non-smooth dynamics also arise in this context.

Example 2.2.1. *Cardiac Arrhythmia*

A cardiac arrhythmia is a term for any of a large and heterogeneous group of conditions in which there is abnormal electrical activity in the heart. The heartbeat may be too fast or too slow, and may be regular or irregular. We will consider arrhythmias which occur when cells act out of sequence, either by firing autonomously, or by refusing to respond to a stimulus from other cells such as the atrioventricular node (AV node).

The AV node is a part of the electrical control system of the heart that coordinates heart rate. It is an area of specialized tissue between the atria and the ventricles of the heart which conducts the normal electrical impulse from the atria to the ventricles.

We will now describe a piecewise smooth model of the heart in which discontinuities correspond to skipped heartbeats [15], [29]. Our presentation closely follows that of Keener and Sneyd, [29]. In these works, the authors view the AV node as a collection of cells subjected to a periodic signal $\phi(t)$ arriving from the atria, with period T. These cells are excitable and once their potential reaches a certain threshold $\theta(t)$ they fire electrical impulses into the ventricles. Immediately after firing, the cells enter a *refractory* period but then gradually recover. The threshold is dramatically increased in order to facilitate the recovery of the cells, but then decreases back to its steady state as recovery proceeds.

Once the input signal reaches its threshold, firing occurs, so that at the n^{th} firing time, denoted by t_n, we have that

$$\phi(t_n) = \theta(t_n).$$

Denote the instances before and after firing by t^- and t^+ respectively. So $\theta(t^+) - \theta(t^-)$ denotes the jump in the threshold caused by the firing of an action potential (a short-lasting event in which the electrical membrane potential of a cell rapidly rises and falls). So, we assume that after firing at time t_n,

$$\theta(t_n^+) = \theta(t_n^-) + \Delta\theta$$

where $\Delta\theta$ is some constant. In other models, it is possible to consider $\Delta\theta$ as a decreasing funtion of $\theta(t_n^-)$, but for our purposes we will be content to leave it constant, as in [29].

After firing, the threshold slowly relaxes according to

$$\theta(t) = \theta_0 + (\theta(t_n^+) - \theta_0)e^{-\gamma(t-t_n)},\ t > t_n,$$

where θ_0 denotes the base threshold and γ represents the decay rate. (Note that $\theta(t) \to \theta_0$ as $t \to \infty$).

To find the next firing time, t_{n+1}, we find the smallest solution of

$$\phi(t_{n+1}) = \theta_0 + (\theta(t_n^+) - \theta_0)e^{-\gamma(t_{n+1}-t_n)}.$$

We can rearrange this as

$$F(t_{n+1}) = F(t_n) + \Delta\theta e^{\gamma t_n} = G(t_n),$$

where

$$F(t) = (\phi(t) - \theta_0)e^{\gamma t}.$$

Discontinuity in the map $t_n \mapsto t_{n+1}$ arises because firing occurs later and later in the cycle until a beat is skipped and the subsequent firing occurs in the next

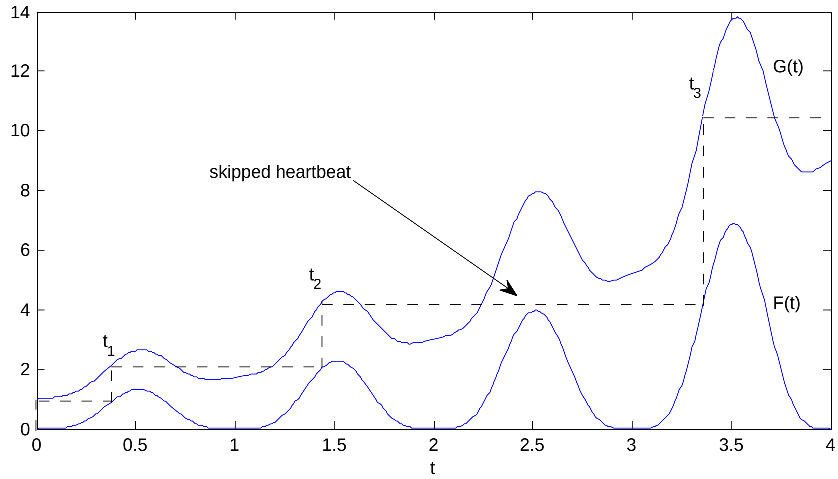

Figure 2.2: Plot of $F(t)$ and $G(t)$ with $\Delta\theta = 1$, $\gamma = 0.55$ and $\phi(t) = sin^4(\pi t)$.

cycle, see Figure 2.2. For t to be a firing time, it must be the smallest t such that $F(t) = G(t_n)$. At such a t, we have $F'(t) > 0$. Hence we can discount times t for which $F'(t) < 0$ as possible firing times. Figure 2.2 shows a plot of $F(t)$ and $G(t)$ using $\phi(t) = sin^4(\pi t)$ as an example. Given t_n, we find the next time t such that $F(t) = G(t_n)$ as in the diagram. Notice that the input is subthreshold in the interval $[2, 3]$, so the AV node fails to fire an electrical impulse into the ventricles and we get a *skipped* heartbeat.

Even though the map $t_n \mapsto t_{n+1}$ is only implictly defined, it is possible to represent it graphically using the rescaled firing time variable

$$\psi_n := \frac{t_n - k_n T}{T},\ 0 \leq \psi_n \leq 1,$$

where k_n is the largest integer less than $\frac{t_n}{T}$. So we can rewrite (10) as

$$f(\psi_{n+1}) = (f(\psi_n) + \Delta\theta e^{\gamma T \psi_n})e^{\gamma T \Delta k_n}$$

where

$$f(\psi) = (\Phi(\psi) - \theta_0)e^{\gamma T \psi},$$

$$\Phi(\psi) = \phi(T\psi)$$

and

$$\Delta k_n = k_{n+1} - k_n.$$

We can represent the dynamics of the map $H : \psi_n \mapsto \psi_{n+1}$ with a cobweb diagram. Given an initial value ψ_0, compute $H(\psi_0)$ and reflect in the main diagonal to get ψ_1. Repeat the process to find subsequent values of ψ_n. We present plots of the map H for varying values of the parameter γ in Figures 2.3-2.7. Following Keener and Sneyd, we restrict ourselves to the attracting range of the map on the unit interval. For large values of γ, the recovery from inhibition is fast, and there is a unique fixed point corresponding to a regular heartbeat in which the AV node fires everytime it receives a stimulus. All initial values ψ_0 are eventually attracted to it via the cobweb process. This is shown in Figure 2.3.

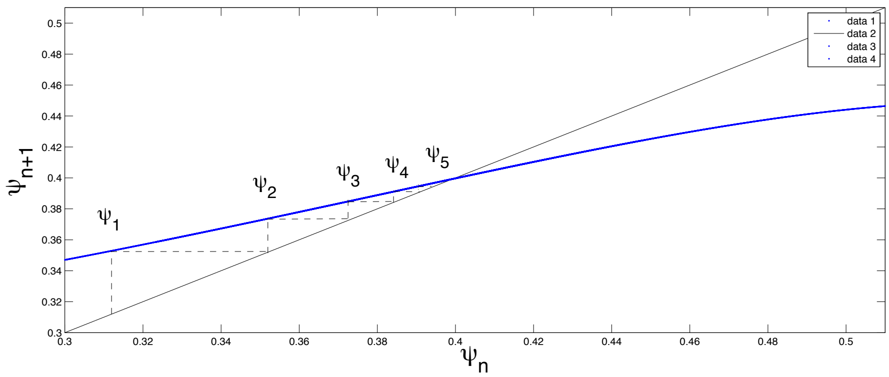

Figure 2.3: Plot of the map $\psi_n \mapsto \psi_{n+1}$ with $\gamma = 0.8$ and $\Delta\theta = 1$.

As we decrease γ, a second branch to the map appears, yet we retain our fixed point. This is the case in Figure 2.4 where $\gamma = 0.693$. Note the discontinuity in the

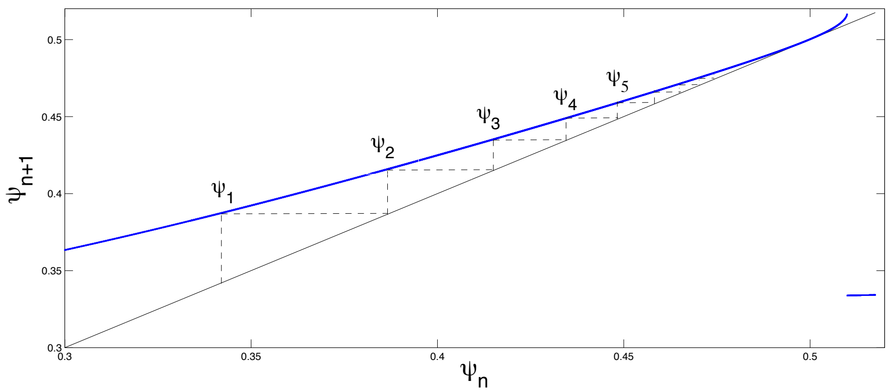

Figure 2.4: Plot of the map $\psi_n \mapsto \psi_{n+1}$ with $\gamma = 0.693$ and $\Delta\theta = 1$.

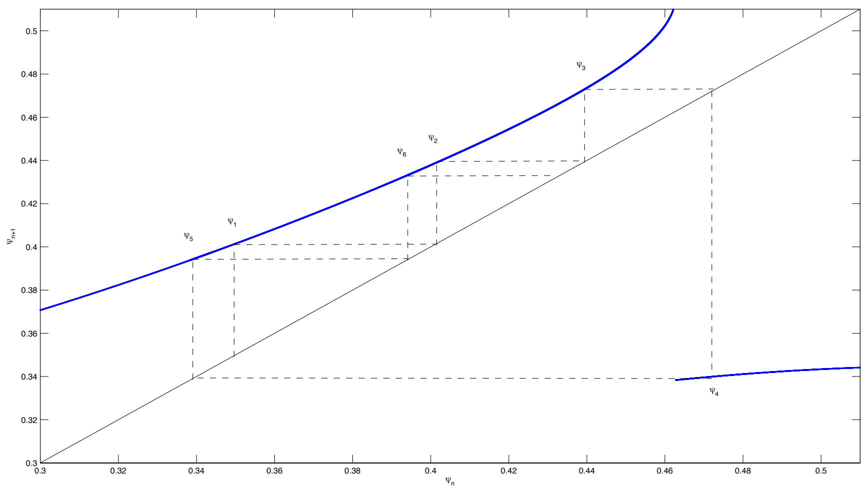

Figure 2.5: Plot of the map $\psi_n \mapsto \psi_{n+1}$ with $\gamma = 0.65$ and $\Delta\theta = 1$.

map H.

Decreasing γ even further, the second branch grows and we lose our fixed point. Subsequent firings occur later and later in the period until one beat is skipped and the next firing after the skipped beat occurs relatively early in the cycle. See Figures 2.5 and 2.6. The maps in these Figures are clearly discontinuous.

However, if we decrease γ even further, the second branch crosses the main

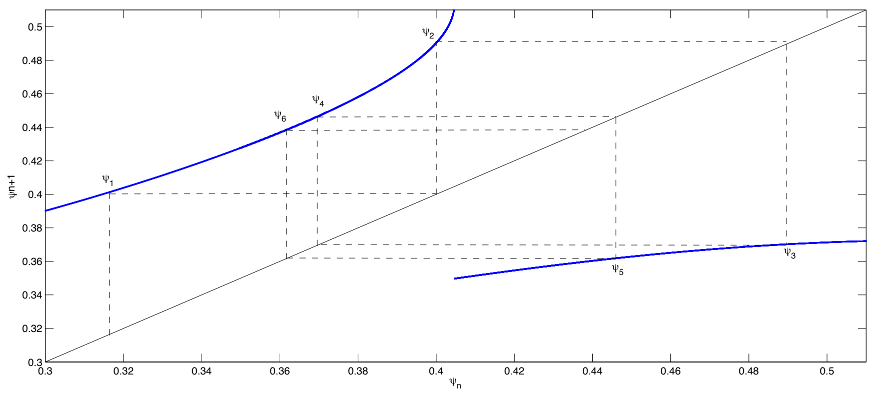

Figure 2.6: Plot of the map $\psi_n \mapsto \psi_{n+1}$ with $\gamma = 0.55$ and $\Delta\theta = 1$.

diagonal and we get another fixed point. In Figure 2.7, there is a discontinuity corresponding to a skipped heartbeat after the first firing, but then all further iterations of the map are attracted to the fixed point, and we get a regular heartbeat.

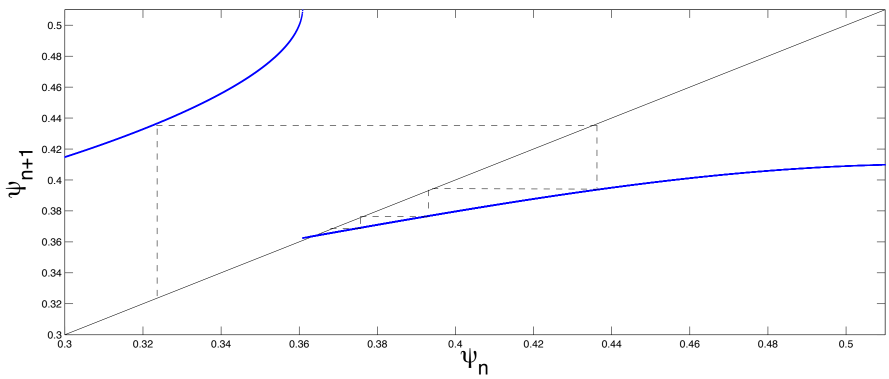

Figure 2.7: Plot of the map $\psi_n \mapsto \psi_{n+1}$ with $\gamma = 0.45$ and $\Delta\theta = 1$.

For this model, the pattern of skipped beats is sensitive to changes in γ but, according to the theory of discontinuous maps in [15], it will in general be periodic for all values of γ.

Note: This is an example of a discontinuous positive dynamical system on a subset of $[0, 1]$. ψ_n tells us at what point in the period the n^{th} firing occurs. The discontinuity occurs since the map $t_n \mapsto t_{n+1}$ is discontinuous.

Example 2.2.2. *Fitzhugh-Nagumo model and piecewise linear systems*

Before talking about the Fitzhugh-Nagumo model and its variations, we need to mention the famous model on which they are based. A model of huge importance in mathematical biology is the Hodgkin-Huxley model. This model was first described in landmark work in 1952 by Alan Lloyd Hodgkin and Andrew Huxley in order to describe how action potentials in neurons are initiated [23]. The model is comprised of a set of nonlinear coupled ordinary differential equations that approximate the electrical characteristics of excitable cells such as neurons and cardiac myocytes (muscle cells).

A modern description of the Hodgin-Huxley model is given in [29]. This gives rise to a 4 dimensional nonlinear coupled system of ordinary differential equations. However, it is quite a complicated model due to all the nonlinearities, and various simpler models have been proposed which capture its essential features. The most famous of these is the Fitzhugh-Nagumo model.

Fitzhugh and Nagumo found that they were able to reduce the model from a 4 dimensional model to a 2 dimensional one. This leads to the following model with only two variables which retains many of the qualitative features of the Hodgkin-Huxley model which are observed experimentally:

$$\begin{aligned} \frac{dv}{dt} &= f(v) - w - I \\ \frac{dw}{dt} &= \beta v - \gamma w, \end{aligned}$$

where β and γ are constants. $f(v)$ is typically chosen to be the cubic polynomial

$$f(v) = v(v - \alpha)(1 - v)$$

where $0 < \alpha < 1$.

However this is not the only possible choice for $f(v)$. Many piecewise linear choices have been used to approximate the classical cubic polynomial. The piecewise linear models preserve the essential features of the original model. This gives them certain advantages in that it allows for explicit calculations in the linear parts using standard techniques, which may not be possible with the original model. It is then possible to connect the solutions at the boundaries. One such model was proposed by McKean in 1970 in which

$$f(v) = \begin{cases} -v & \text{if } v < \frac{\alpha}{2}, \\ v - \alpha & \text{if } \frac{\alpha}{2} < v < \frac{1+\alpha}{2}, \\ 1 - v & \text{if } v > \frac{1+\alpha}{2}. \end{cases}$$

McKean's variants were introduced in order to study nerve conduction.

Yet another piecewise linear variant of the Fitzhugh Nagumo model is the Pushchino model, so called because it has been developed in Pushchino, Russia. The Pushchino model was originally proposed as a model for the ventricular action potential. In this model we have:

$$\begin{aligned} \frac{dv}{dt} &= f(v) - w \\ \frac{dw}{dt} &= \frac{1}{\tau(v)}(v - w), \end{aligned}$$

where

$$f(v) = \begin{cases} -30v & \text{if } v < v_1, \\ \gamma v - 0.12 & \text{if } v_1 < v < v_2, \\ -30(v-1) & \text{if } v > v_2, \end{cases}$$

$$\tau(v) = \begin{cases} 2 & \text{if } v < v_1, \\ \\ 16.6 & \text{if } v > v_1, \end{cases}$$

with $v_1 = \frac{0.12}{30+\gamma}$ and $v_2 = \frac{30.12}{30+\gamma}$, and γ a constant.

Later on we shall be investigating some of the theoretical properties of this kind of piecewise linear system.

Example 2.2.3. *Gene regulatory networks*

Within each organism there is a series of complex interactions between genes and their products, proteins and RNA, and a variety of small signalling molecules. The basic functions of the cell are tightly linked to the dynamics of this network of interactions. These networks, *gene regulatory networks*, have been well studied in mathematical biology. Many models involve nonlinear terms which can be approximated by piecewise linear or affine terms. We shall describe one such model here.

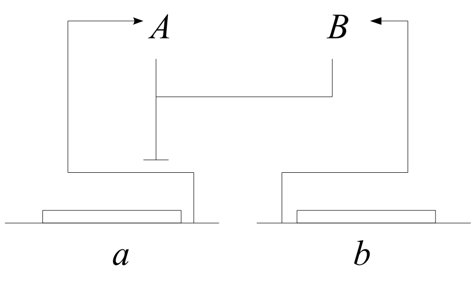

Figure 2.8: Example of a gene regulatory network with two proteins, A and B.

Following [14], [13], [43], [12] let $\mathbf{x} = (x_1, x_2, \ldots, x_n)' \in \mathbb{R}^n$ represent a vector of cellular protein concentrations. We will assume that $\mathbf{x}$ takes its values in a bounded hyperrectangular region $\Omega \in \mathbb{R}^n$, where $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. Furthermore each Ω_i is given by $\Omega_i = [0, \max_i]$ where $\max_i$ denotes the maximum concentration for x_i. For each protein i we associate threshold concentrations $\theta_i^{k_i} \in \Omega_i$, $k_i \in \{1, 2, \ldots, p_i\}$, $1 \leq i \leq n$. We will assume that the $\theta_i^{k_i}$ are ordered as follows: $\theta_i^1 < \theta_i^2 < \ldots < \theta_i^{p_i}$. When the concentration of protein i crosses a threshold, the mode of regulation of the synthesis or degradation of the other proteins or of protein i itself may change, and this change can be abrupt.

Example 2.2.4. *In Figure 2.8 we have a schematic diagram of a simple example of a gene regulatory network with two genes, a and b, coding for proteins A and B. Protein A inhibits the expression of gene b above the threshold concentration θ_a^1 and inhibits gene a above the threshold concentration θ_a^2. Protein B activates gene b above the concentration θ_b^1 and inhibits gene a above the concentration θ_b^2.*

The dynamics of the system are governed by the system of ordinary differential equations

$$\dot{x_i} = f_i(\mathbf{x}) - \nu_i x_i, \tag{2.3}$$

for $1 \leq i \leq n$. These equations define the rate of change of each concentration x_i as the difference of the rate of synthesis $f_i(\mathbf{x})$ and rate of degradation $\nu_i x_i$ of the protein. We can rewrite equations (38) in vector format as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x}) - \nu \mathbf{x}$$

where $\mathbf{f} = (f_1, f_2, ..., f_n)'$ and $\nu = diag(\nu_1, \nu_2, ..., \nu_n)$.

The key observation is that rate of activation of a gene often follows a steep sigmoidal curve; the activity of a gene changes in a switch-like manner once the concentration of a regulatory protein i reaches a certain threshold level. This allows us to approximate the f_i, $1 \leq i \leq n$ using step functions,

$$\begin{aligned} s^+(x_i, \theta_i^{k_i}) &= \begin{cases} 1 & \text{if } x_i > \theta_i^{k_i} \\ 0 & \text{if } x_i < \theta_i^{k_i}, \end{cases} \\ s^-(x_i, \theta_i^{k_i}) &= 1 - s^+(x_i, \theta_i^{k_i}). \end{aligned}$$

This approximation generates a piecewise affine system which we describe below.

Notice that the step functions $s^+(x_i, \theta_i^{k_i})$ and $s^-(x_i, \theta_i^{k_i})$ are not defined at threshold concentrations, i.e. when $x_i = \theta_i^{k_i}$ for some $k_i \in \{1, ..., p_i\}$.

We now describe the piecewise linear approximation to (2.3) in the state space Ω. The first step is to partition Ω into hyper-rectangular regions, which we call domains, using the (n-1) dimensional hyperplanes defined by

$$x_i = \theta_i^{k_i}, \tag{2.4}$$

with $1 \leq i \leq n$ and $k_i \in \{1, ..., p_i\}$, see figure 9.

As in [14], we will distinguish between two distinct classes of domain; on the one hand we will call a domain $D \in \mathcal{D}$ a regulatory domain if none of the variable assumes a threshold value. Otherwise we will call D a switching domain. Each switching domain lies in the boundary of some set of regulatory domains. More formally, we define regulatory and switching domains as follows:

Definition 2.2.1. *Regulatory domain: $D \in \mathcal{D}$ is a regulatory domain if given any $x \in D$ we have $x_i \neq \theta_i^{k_i}$ for all $i \in \{1, ..., n\}$ and for all $k_i \in \{1, ..., p_i\}$. Denote the set of regulatory domains by $\mathcal{D}_r \subset \mathcal{D}$.*

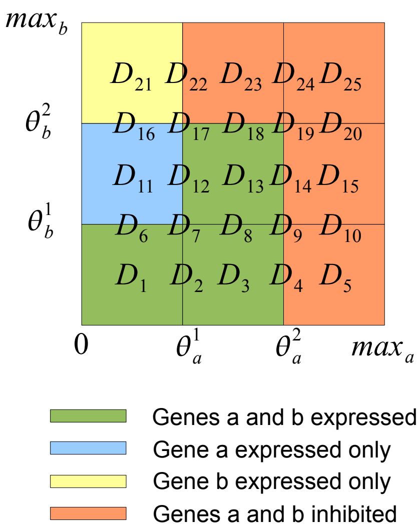

Figure 2.9: State space Ω.

Definition 2.2.2. *Switching domain: $D \in \mathcal{D}$ is a switching domain if for some $x \in D$ there exist $i \in \{1, ..., n\}$ and $k_i \in \{1, ..., p_i\}$ such that $x_i = \theta_i^{k_i}$. Denote the set of switching domains by $\mathcal{D}_s \subset \mathcal{D}$.*

Figure 2.9 shows the state space Ω partitioned into regulatory and switching domains for Example 2.2.4.

In a regulatory domain, $D \in \mathcal{D}_r$, the rate of synthesis $f_i(\mathbf{x})$ reduces to some constant. Hence, the state equations simplify to linear, uncoupled differential equations

$$\dot{x_i} = \mu_i^D - \nu_i^D x_i \tag{2.5}$$

for $1 \leq i \leq n$. These can be rewritten in vector form as

$$\dot{\mathbf{x}} = \mu^{\mathbf{D}} - \nu^{\mathbf{D}} \mathbf{x} \tag{2.6}$$

where $\mu^{\mathbf{D}} = (\mu_1^D, \mu_2^D, ..., \mu_n^D)'$ and $\nu^{\mathbf{D}} = diag(\nu_1^D, \nu_2^D, ..., \nu_n^D)$.

Suppose $\mathbf{x}_0 \in D$ is an initial condition for (2.6) and $\psi(t)$ is a continuously differentiable function such that $\psi(0) = \mathbf{x}_0$. Then $\psi(t)$ is a solution to (2.6) on some time interval $[0, \tau)$, $\tau > 0$, if

$$\begin{aligned} \psi(t) &\in D, \\ \dot{\psi}(t) &= \mu^{\mathbf{D}} - \nu^{\mathbf{D}}\psi(t) \end{aligned}$$

for all $t \in [0, \tau)$. In fact given any initial $\mathbf{x}_0 \in D$ and $\tau > 0$ there exists a unique $\psi(t)$ that is the unique solution to (2.6) on $[0, \tau]$ and this solution is given by

$$\psi(t) = \phi(D) + e^{\nu(t-t_0)}(\psi(t_o) - \phi(D)),$$

where ϕ is the function defined by

$$\begin{aligned} &\phi : \mathcal{D}_r \mapsto \Omega \\ &\phi(D) = (\phi_1(D), \phi_2(D), ..., \phi_n(D)), \end{aligned}$$

and $\phi_i(D) = \mu_i^D / \nu_i^D$ for $1 \leq i \leq n$ (equilibrium points of (2.4), i.e. when $\dot{x}_i = 0$).

$\mathbf{x} = \phi(D)$ is known as a *target equilibrium.* Solutions $\psi(t)$ to (2.6) monotonically converge towards $\phi(D)$. If $\phi(D) \in D$ then it is a stable equilibrium of the system and as $t \to \infty$ all solutions starting in D will approach it and remain in D. However, a potential problem arises when $\phi(D) \notin D$. As is shown in [14], the feedback structure of the regulatory network often tends to drive the concentrations toward a threshold level, i.e. toward a switching domain. Solutions will eventually leave D. The issue is that (2.6) is not defined in the switching domains. If a solution trajectory arriving at a switching domain from some regulatory domain can be continued to an adjacent regulatory domain then the problem can be overcome easily. However if this is not

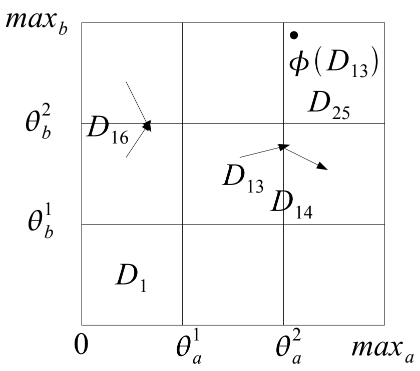

Figure 2.10: Potential problems at the threshold boundaries.

the case, one way of describing the dynamics is to extend the differential equations to differential inclusions, using techniques first proposed by Filippov. We will be discussing Filippov solutions and the circumstances in which they are used in detail in the next chapter.

These ideas are best described with a simple example. In Figure 2.10 the focal point $\phi(D_{13})$ for the regulatory domain D_{13} lies in D_{25}. A solution trajectory in D_{13} will eventually cross a threshold concentration on its path toward $\phi(D_{13})$. In this example, the solution arriving at D_{14} from D_{13} can easily be continued into D_{15}, even though (2.6) is not defined in D_{14}. A problem arises in D_{16} since the vector fields in D_{11} and D_{21} both point toward D_{16}. Filippov solutions are required to overcome this problem.

Example 2.2.5. *DNA Replication*

Another practical application of the theory of discontinuous / non smooth systems is in the mathematical modelling of DNA replication, one of the most fundamental processes in the life of a cell. DNA has a double-stranded structure, with the two strands intertwined toegether to form the characteristic double-helix. During DNA replication, each strand acts as a template for the reproduction of the complementary strand and after replication, two identical copies of the original DNA molecule have been created.

At the beginning of the replication, the two strands are forced apart when the hydrogen bonds holding them together are broken. This results in two separate antiparallel strands. This process of unwinding is initiated at particular points in the DNA known as *origins*, and the two strands form a *replication fork.* The replication forks move along the genome, and the DNA is replicated. It is possible to model the movement of the replication forks and the replication of the DNA using a hybrid system as described in [31].

The continuous variables of the system are:

$$\begin{aligned} X_i &\in \mathbb{R} - \quad \text{position of origin } i \text{ in genome,} \\ L_i &\in \mathbb{R} - \quad \text{position of left fork of origin } i, \\ R_i &\in \mathbb{R} - \quad \text{position of right fork of origin } i, \end{aligned}$$

where $i \in \{1, ..., N\}$ refers to one of the N origins along the genome.

We also have a discrete variable $q_i(t)$ for which there are six possible values.. $q_i(t) \in \{PreR, PassR, PostR, RLF, RF, LF\}$, $i \in \{1, ..., N\}$.
The six discrete states are

$$\begin{array}{rcl} PreR & - & \text{pre-replicative state (intitial value of the variable } q_i \text{ before replication begins)} \\ PassR & - & \text{passive replication,} \\ PostR & - & \text{post replicative state,} \\ RLF & - & \text{right and left fork active,} \\ RF & - & \text{only right fork active,} \\ LF & - & \text{only left fork active.} \end{array}$$

Each fork moves with velocity $v(x)$, which depends on the current position of the fork within the genome. The continuous dynamics of the system are given by

$$\begin{aligned} \dot{R}_i(t) &= \begin{cases} v(X_i(t) + R_i(t)) & \text{if } q_i(t) \in \{RLF, RF\} \\ 0 & \text{if } q_i(t) \notin \{RLF, RF\}, \end{cases} \\ \dot{L}_i(t) &= \begin{cases} v(X_i(t) - L_i(t)) & \text{if } q_i(t) \in \{RLF, LF\} \\ 0 & \text{if } q_i(t) \notin \{RLF, LF\}. \end{cases} \end{aligned}$$

Before describing the discrete dynamics let us define $LN(i)$ and $RN(i)$ as follows:

$$\begin{aligned} LN(i) &= max\{j < i : q_j \notin \{PreR, PassR, PostR\}\} \\ RN(i) &= min\{j > i : q_j \notin \{PreR, PassR, PostR\}\}. \end{aligned}$$

$LN(i)$ refers to the nearest origin to the left which has either left, right or both forks active in the replication process. $RN(i)$ is analogously defined but refers to the nearest origin to the right with active fork(s).

The rules governing the discrete dynamics of the system are now given by

$$\begin{aligned}
PreR \rightarrow PassR: \quad & X_{LN(i)} + R_{LN(i)} && \geq X_i \text{ or } X_{RN(i)} + L_{RN(i)} \leq X_i, && (2.7)\\
RLF \rightarrow RF: \quad & X_{LN(i)} + R_{LN(i)} && \geq X_i - L_i, && (2.8)\\
RLF \rightarrow LF: \quad & X_{RN(i)} - L_{RN(i)} && \leq X_i + R_i, && (2.9)\\
RF \rightarrow PostR: \quad & X_{RN(i)} - L_{RN(i)} && \leq X_i + R_i, && (2.10)\\
LF \rightarrow PostR: \quad & X_{LN(i)} + R_{LN(i)} && \geq X_i - L_i, && (2.11)\\
PreR \rightarrow RLF: \quad & \quad t \geq T_i, && && (2.12)
\end{aligned}$$

where T_i is the firing time of origin i. It is assumed in [31] to follow an exponential distribution.

In other words, (2.7) says that the origin i will go from a pre-replicative state to a passive replicative state when the position of the right fork of the previous active origin reaches the position of the origin i, or else when the position of the left fork of the previous active origin reaches the position of the origin i. By active origin, we mean one in which either the right, left or both forks of that origin are active.

Similarly, (2.8) says that the origin i goes from a state where both forks are active to a state when only the right fork is active when the right fork of the previous active origin reaches the position of the left fork of origin i. (2.9)-(2.12) are similiar to (2.8).

A thorough analysis of this system can be found in [31].

2.3 Concluding Remarks

We have described several models which are non smooth in their description of practical situations. We have seen how impacting systems can give rise to non smooth systems in Example 2.1.1 and in Examples 2.2.2 and 2.2.3, we saw how nonlinear systems can be approximated by piecewise linear sytems. In the forthcoming Chapters we shall describe the aspects of the mathematical theory behind these motivational examples.

Chapter 3

LTI Systems, Positivity and Monotonicity

In this chapter, we review the theory of LTI systems and some key stability results for such systems. We also discuss positive systems and monotonicity.

3.1 Brief Review of Linear Time Invariant (LTI) Systems

The theory of LTI systems is well-developed and has been applied extensively in control engineering [27], [45]. We shall now briefly describe such systems and then talk about some of their key properties, such as stability.

First of all, let us consider the first order linear differential equation

$$\dot{x} = ax \tag{3.1}$$

where $a \in \mathbb{R}$. The general solution to (3.1) is given by

$$x(t) = x_0 e^{at} \tag{3.2}$$

where $x_0 = x(0)$ is an initial condition for (3.1).

The natural generalisation of (3.1) is what is what we call a linear time invariant (LTI) system. Standard references for linear systems include [45] and [41].

Definition 3.1.1. *Let $A \in \mathbb{R}^{n\times n}$ be given. The linear system associated with A is given by*

$$\dot{x}(t) = Ax(t). \tag{3.3}$$

$\dot{x}(t)$ represents the derivative of the state vector $x(t)$ where $x(t) \in \mathbb{R}^n$. The components of the state vector, $x_1(t), x_2(t), ...x_n(t)$ are known as the state variables. So we may write

$$\dot{x} = \frac{dx}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}. \tag{3.4}$$

We now recall the definition of the matrix exponential which is fundamental in the study of (3.3).

Definition 3.1.2. *Suppose $A \in \mathbb{R}^{n\times n}$. The matrix exponential e^A is given by*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{3.5}$$

e^A is well-defined for all matrices $A \in \mathbb{R}^{n\times n}$ - see for example Theorem 5.6.13, page 300 of [25].

Definition 3.1.3. *Suppose $A \in \mathbb{R}^{n\times n}$. Then for $t \in \mathbb{R}$,*

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}. \tag{3.6}$$

The power series in Definition (3.1.3) is convergent everywhere so the matrix exponential is well defined for all $t \in \mathbb{R}$.

The form of the solution to (3.1) extends to (3.3) using the matrix exponential. In fact we have the following theorem:

Theorem 3.1.1. (The Fundamental Theorem for Linear Systems) *Suppose $A \in \mathbb{R}^{n\times n}$. Then given any $x_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{aligned} \dot{x} &= Ax \\ x(0) &= x_0 \end{aligned}$$

has a unique solution defined on $(-\infty, \infty)$ given by

$$x(t) = e^{At}x_0. \tag{3.7}$$

Henceforth we shall write $x(t, x_0)$ to denote the solution of (3.7) if we are talking about a specific initial condition x_0 for our LTI system.

An important concept in the study of any dynamical system is stability. $x_e \in \mathbb{R}^n$ is an equilibrium point of (3.3) if $Ax_e = 0$. Loosely, we say that x_e is a stable equilibirum point (in the sense of Lyapunov) if any solution starting near x_e stays near x_e for all time t. Note that the LTI system $\dot{x}(t) = Ax(t)$ always has an equilibrium point at the origin.

More concretely, given the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) \qquad (3.8) \\ x(0) &= x_0 \end{aligned}$$

we have the following definitions of stability: [48]

Definition 3.1.4. *Given the LTI system (3.8) , the origin is a stable equilibrium point (in the sense of Lyapunov) if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $||x_0|| < \delta$, then $||x(t, x_0)|| < \epsilon$ for all $t \geq 0$.*

However, stability in the sense of Lyapunov is a weak condition. While it requires that an initial state which starts "near enough" to the origin stays "near enough" for all time, it does not imply that solutions will tend to the origin.

Definition 3.1.5. *(Global Asymptotic Stability) Given the LTI system (3.8), the origin is a globally asymptotically stable equilibrium point if it is stable, and in addition,* $\lim_{t\to\infty} x(t, x_0) = 0$ *for all* $x_0 \in \mathbb{R}^n$*.*

Loosely, asymptotic stability says that any solution starting near enough to the origin will not only stay near to it, but will eventually converge to it. The third type of stability we shall look at is exponential stability. An exponentially stable equilibrium point is asymptotically stable and solutions will converge to it at a rate at least as fast as a decaying exponential function.

Definition 3.1.6. *(Global Exponential Stability) Given the LTI system (3.8), the origin is a globally exponentially stable equilibrium point if there exist* $\alpha,\ \beta \in \mathbb{R}$*,* $\beta < 0$*, such that* $||x(t, x_0)|| \leq \alpha ||x_0|| e^{\beta t}$*, for* $t \geq 0$ *for all* $x_0 \in \mathbb{R}^n$*.*

The infimum of the values β which satisfy Definition 3.1.5 is often called the rate of exponential convergence.
We call (3.8) globally asymptotically stable, or say that the origin is globally asymptotically stable if (3.7) is stable and given any initial condition $x(0) = x_0$ we have that $\lim_{t\to\infty} x(t, x_0) = 0$. For finite dimensional LTI systems, global asymptotic stability and global exponential stability are equivalent. [45].

The following classical result characterises the asymptotic stability of (3.3) in terms of A.

Theorem 3.1.2. *An LTI system* $\dot{x}(t) = Ax(t)$ *is asymptotically stable if and only if* $Re(\lambda) < 0$ *for all eigenvalues* λ *of* A*.*

A matrix A in which every eigenvalue has negative real part is known as a *Hurwitz* matrix. Theorem 3.1.2 shows that (3.3) is globally asymptotically stable if and only if A is Hurwitz.

3.2 Lyapunov Stability

Very important results in the stability of dynamical systems were established by the Russian mathematician and physicist Aleksandr Mikhailovich Lyapunov. The concepts we discuss in the following paragraphs come from his PhD thesis "*The general problem of the stability of motion*" which he successfully defended on the 12th of September 1892 in Moscow. It is now that we will introduce the concept of a *Lyapunov* function and then state without proof a result known as Lyapunov's theorem. The idea is to determine the stability of a system by examining the time evolution of a single scalar function which is usually denoted $V(x(t))$, along any trajectory $x(t)$ of the system. In general, it is not an easy task to find such a function, but it turns out that it is much more straightforward for LTI systems, which is all we are interested in for the moment.

$V(x(t))$ represents an implicit function of time. Assuming that $V(x)$ is differentiable, along trajectories of (3.3) we have that

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}Ax,$$

where we have made use of the chain rule. Lyapunov showed that we can determine the stability of a time invariant system by finding a function $V(x)$, with certain properties, now known as a Lyapunov function. Lyapunov functions can be used to establish stability for general nonlinear systems. To highlight the flavour of these results, we present the following Theorem for LTI systems.

Theorem 3.2.1. *Suppose we have an LTI system $\dot{x} = Ax$ where $A \in \mathbb{R}^{n\times n}$. If we* *can find a function continuously differentiable* $V : \mathbb{R}^n \mapsto \mathbb{R}$ *which satisfies*

$$\begin{aligned} V(x) &> 0 \text{ for } x \neq 0 \\ V(0) &= 0 \\ \dot{V}(x) &\leq 0, \end{aligned} \tag{3.9}$$

then the system is stable.

If in addition, we have that

$$\dot{V}(x) < 0$$

for all $x \neq 0$ *then the system is asymptotically stable.*

In fact, for an LTI system $\dot{x}(t) = Ax(t)$, we may choose a quadratic Lyanunov function of the form

$$V(x) = x^T P x$$

where $P = P^T > 0$ is a positive definite matrix, meaning that $V(x) > 0$ for all $x \neq 0$. Differentiating $V(x) = x^T P x$ along solutions of $\dot{x} = Ax$, we get

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

from which it follows that

$$\dot{V}(x) = x^T (A^T P + PA) x.$$

So the condition

$$V(x) < 0, x \neq 0$$

becomes

$$A^T P + PA < 0,$$

i.e. if we can find a positive definite matrix P such that

$$A^T P + PA < 0,$$

then the system is asymptotically stable. The following theorem [25], [45] is due to Lyapunov and was first published in his famous doctoral thesis.

Theorem 3.2.2. (Lyapunov's Theorem)

Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz.

Then for all $Q = Q^T \in \mathbb{R}^{n\times n}$ where $Q > 0$ (Q is positive definite), there exists $P \in \mathbb{R}^{n\times n}$ with $P = P^T > 0$ such that $A^T P + PA = -Q$.

Furthermore, P is given by

$$P = \int_0^\infty (e^{At})^T Q e^{At} dt.$$

Conversely, if for some $Q > 0$ there exists $P = P^T > 0$ where Q,$P \in \mathbb{R}^{n\times n}$ such that $A^T P + PA = -Q$ then A is Hurwitz.

Let us now take a look at an example to illustrate this result. Suppose we have a system given by

$$\dot{x}(t) = Ax(t)$$

where A is given by

$$A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}.$$

The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -3 \end{aligned}$$

so this system is asymptotically stable, which we shall now verify using Lyapunov's theorem. Let

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0.$$

Can we find a matrix $P \in \mathbb{R}^{2\times 2}$ with

$$P = P^T > 0$$

that satisfies

$$A^T P + PA = -Q?$$

Suppose P is given by

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}.$$

If we solve the equation

$$A^T P + PA = -Q$$

for P we get that

$$\begin{aligned} p_1 &= \frac{1}{2} \\ p_2 &= \frac{1}{4} \\ p_3 &= \frac{1}{3} \end{aligned}$$

which gives

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix} > 0.$$

Hence, we have verified that A is indeed Hurwitz.

3.3 Positive LTI Systems

We shall now describe a class of linear systems known as *positive linear time invariant systems*. In the remainder of this thesis, positive systems will play a central role. Positive systems are, by definition, systems in which the state variables take on only non-negative values. Because of this, they appear often in the modelling of many systems in biology and economics, such as Example 2.2.3 in Chapter 2, which considered gene regulatory networks. It makes sense in that example to restrict ourselves to non-negative values since the concentration of a protein cannot be negative. Several aspects of the theory of positive systems have been considered. These include the Positive Realisation problem, the question of positive stabilisation and issues related to reachability and controllability [44], [33], [3], [46]. Positive systems possess many strong stability properties. In particular, these systems are very robust with respect to the introduction of time-delays. This is shown in [22]. Extensions of the results in this paper to classes of nonlinear positive systems can be found in [38], [4]. We now give a brief review of these systems, beginning with a brief discussion on non-negative matrices.

3.3.1 Perron-Frobenius Theory

The theory of positive LTI systems has its roots in Perron-Frobenius theory of non-negative matrices so it is worth having a brief discussion on this theory [39]. According to Carl D. Meyer, "In addition to saying something useful, the Perron-Frobenius theory is elegant. It is a testament to the fact that beautiful mathematics eventually tends to be useful, and useful mathematics eventually tends to be beautiful". This is especially true in the case of positive systems, whose theory is underpinned by Perron-Frobenius theory. It deals with positive and nonnegative matrices, and the key issue is to investigate the spectral properties of these matrices.

A matrix $A \in \mathbb{R}^{n\times n}$ is nonnegative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$), $1 \leq i, j \leq n$. For $A, B \in \mathbb{R}^{n\times n}$ we write $A \geq B$ if $a_{ij} \geq b_{ij}$, and we write $A > B$ if $a_{ij} > b_{ij}$,$1 \leq i, j \leq n$.

Definition 3.3.1. *The nonnegative cone (or positive orthant) in $\mathbb{R}^n$, denoted by $\mathbb{R}^n_+$, is the set of all n-tuples with non-negative coordinates, i.e.* $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$.

This cone generates a partial ordering on $\mathbb{R}^n$ given by $y \leq x$ if $x - y \in \mathbb{R}^n_+$. This is true if and only if $y_i \leq x_i$ for all i. We will write $y < x$ if $y \leq x$ and $y \neq x$ and we will write $y \ll x$ if $y_i < x_i$ for all i.

An important result in the theory of positive matrices, due to Perron, is the following, known as the Perron Theorem. Before stating the theorem, we first recall the definition of the spectral radius of a matrix.

Definition 3.3.2. *Suppose $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. The spectral radius $\rho(A)$ of A is given by*

$$\rho(A) = \max_{1\leq i\leq n} |\lambda_i|.$$

Theorem 3.3.1. (Perron Theorem for positive matrices)
Suppose $A \in \mathbb{R}^{n\times n}$ and A is positive. Then the following are true:
(i) $\rho(A) \in \sigma(A)$ where $\sigma(A)$, the spectrum of A is the set of all eigenvalues of A.
(ii) The algebraic multiplicity of $\rho(A)$ is 1, i.e. $\rho(A)$ is a simple root of the characteristic polynomial of A, and the eigenspace associated with $\rho(A)$ therefore has dimension one.
(iii) There exists a unique vector p, known as the Perron vector, which satisfies

$$p \gg 0 \tag{3.10}$$

$$Ap = \rho(A)p \tag{3.11}$$

$$||p|| = 1 \tag{3.12}$$

(iv) Except for positive multiples of p, there does not exist any other nonnegative eigenvector of A. Any other eigenvector must have one negative or non-real component.
(v) $\rho(A) = \max_{x\in\mathcal{N}} f(x)$ where $f(x) = \min_{1\leq i\leq n,\ x_i\neq 0} \frac{(Ax)_i}{x_i}$ and $\mathcal{N} = \{x : x \geq 0$ with $x \neq 0\}$. This is known as the Collatz-Wielandt formula.

However, some of these results break down if we try and generalise to general nonnegative matrices. Frobenius was successful in extending these results to irreducible matrices, a subclass of nonnegative matrices, [19]. This extension of the Perron Theorem is what we now call the Perron-Frobenius theorem, a celebrated result in the theory of nonnegative matrices.

Definition 3.3.3. *A matrix $A \in \mathbb{R}^{n\times n}$ is called block upper triangular if it is of the form*

$$\begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where $A_1, ..., A_m$ are square matrices lying along the diagonal, the entries below $A_1, ..., A_m$ are 0 and $*$ denotes arbitrary entries lying above $A_1, ..., A_m$.

An example of a block upper matrix is the matrix

$$\begin{pmatrix} 1 & 2 & 6 & 7 & 8 \\ 3 & 4 & 2 & 7 & 3 \\ 0 & 0 & 5 & 1 & 2 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{pmatrix}.$$

In this case, we have $A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A_2 = \begin{pmatrix} 5 \end{pmatrix}$ and $A_3 = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}$.

Definition 3.3.4. *A permutation matrix is a matrix that has exactly one entry 1 in each row and each column and zeros elsewhere.*

Definition 3.3.5. *A matrix $A \in \mathbb{R}^{n\times n}$ is reducible if there exists a permutation matrix P [25] such that the matrix $P^T AP$ is block upper triangular. If A is not reducible, then A is called irreducible.*

We are now ready to state the Perron-Frobenius theorem:

Theorem 3.3.2. (Perron-Frobenius Theorem for nonnegative matrices) *Suppose $A \in \mathbb{R}^{n\times n}$ and A is nonnegative and irreducible. Then the following are true:*

(i) $\rho(A) \in \sigma(A)$.

(ii) The algebraic multiplicity of $\rho(A)$ *is 1.*

(iii) There is a unique vector p *which satisfies*

$$\begin{aligned} p &\gg 0 && (3.13)\\ Ap &= \rho(A)p && (3.14)\\ ||p|| &= 1 && (3.15) \end{aligned}$$

(iv) There are no nonnegative eigenvectors for A *except for positive multiples of* p*, regardless of the eigenvalue.*

(v) The Collatz-Wielandt formula holds.

3.3.2 Positive Systems And Stability

Definition 3.3.6. *The linear system*

$$\dot{x}(t) = Ax(t) \tag{3.16}$$

is said to be positive if for any initial condition x_0*, with* $x_0 \geq 0$*, we have* $x(t, x_0) \geq 0$ *for all* $t \geq 0$*.*

Definition 3.3.7. *A matrix* $A = (a_{ij})$ *is said to be a Metzler matrix if its off-diagonal elements are nonnegative, i.e. if* $a_{ij} \geq 0$ *if* $i \neq j$*.*

To determine whether a system given by $\dot{x}(t) = Ax(t)$ is positive or not is straightforward and can now be summarised in the following well-known theorem [16]:

Theorem 3.3.3. *A linear system* $\dot{x}(t) = Ax(t)$ *is positive if and only if* A *is a Metzler matrix.*

Proof. Suppose $\dot{x}(t) = Ax(t)$ is a positive system. Let $x(0) = e_j$, e_j is the vector with one in the j^{th} position and zeros elsewhere. So $\dot{x}(0) = Ae_j$ (the j^{th} column of

A). Since $\dot{x}(t) = Ax(t)$ is positive and the trajectory of a positive system cannot leave $\mathbb{R}^n_+$, we must have that $\dot{x}_i(0) \geq 0$ for $i \neq j$. It follows that A is Metzler.

Conversely, suppose A is Metzler. Suppose $x_i(t) = 0$ for some $t \geq 0$ and $x(t) \geq 0$. Since $a_{ij} \geq 0$ for $i \neq j$, we have that $\dot{x}_i(t) \geq 0$, i.e. we have that the vector $\dot{x}(t)$ does not point outside of $\mathbb{R}^n_+$ whenever $x(t)$ is on the boundary of $\mathbb{R}^n_+$.
Hence $\dot{x}(t) = Ax(t)$ is a positive system.

□

The following result summarises a number of equivalent properties for positive LTI systems [36].

Proposition 3.3.1. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler. The following statements are equivalent:*
(a) The LTI system (3.16) is stable;
(b) A is Hurwitz;
(c) There exists $P > 0$ such that $A^T P + PA < 0$;
(d) There exists a diagonal matrix $D > 0$ such that $A^T D + DA < 0$;
(e) There exists a vector $v \in \mathbb{R}^n_+$ with $Av \ll 0$;
(f) $A^{-1} < 0$;
(g) For any diagonal matrix $D > 0$, the system $\dot{x}(t) = DAx(t)$ is stable.

The property expressed in condition (g) in Proposition (3.3.1) is known as D-stability [36]. This shows that the stability of positive LTI systems is robust with respect to parameter variations. Specifically, if A is scaled by a positive diagonal matrix, then stability is preserved. D-stability has been considered in [2], [28]. [2] provides a neat characterisation of diagonal stability in general. However, unfortunately the conditions provided in the paper are far from easy to test in practice.
In addition, positive LTI systems are robust with respect to delay [22].

3.3.3 Lyapunov Functions

Since a positive LTI system is characterised by the fact that trajectories starting in the nonnegative orthant must remain in the nonnegative orthant for all time, it is natural to consider functions which satisfy the requirements of Lyapunov functions in $\mathbb{R}^n_+$ when considering positive systems. These functions are known as copositive Lyapunov functions. We now introduce the class of linear copositve Lyapunov functions, which will appear extensively in the next chapter when we will be giving results concerning the stability of switched linear systems.

Definition 3.3.8. *A function $V(x) = v^T x$ is a linear copositive Lyapunov function for (3.16) if and only if the vector $v \in \mathbb{R}^n$ satisfies:*

$$\begin{aligned} v &\gg 0 \\ A^T v &\ll 0. \end{aligned}$$

Under these conditions,

$$\begin{aligned} V(x) &> 0 \text{ for all } x \in \mathbb{R}^n_+, x \neq 0, \\ \dot{V}(x) &< 0 \text{ for all } x \in \mathbb{R}^n_+, x \neq 0. \end{aligned}$$

The matrix A is Metzler if and only if A^T is Metzler. Proposition 3.3.1 implies that (3.16) is asymptotically stable if and only if there exists a vector v in $\mathbb{R}^n$ with $A^T v \ll 0$. It follows from this that (3.16) is asymptotically stable if and only if it has a linear copositive Lyapunov function.

3.4 Monotonicity

A key property of positive LTI systems is monotonicity. The concept of monotonicity is also important in many nonlinear extensions and it will be central to much of our later work in this thesis. A monotone system is essentially an order preserving system. A concrete definition is now given:

Definition 3.4.1. *Suppose we have a dynamical system given by* $\dot{x}(t) = f(x(t))$ *where* $f : D \to \mathbb{R}^n$ *is* C^1 *and* $x(0) = x_0 \in D$*, where* $D \subset \mathbb{R}^n$ *is open.* $x(t, x_0)$ *denotes the solution of this system satisfying* $x(0, x_0) = x_0$*. We say that this system is a monotone system if given any* $x_0, y_0 \in D$ *such that* $x_0 \leq y_0$ *we have that* $x(t, x_0) \leq x(t, y_0)$ *for all* $t \geq 0$*, for which both solutions exist.*

Theorem 3.4.1. *If* $A \in \mathbb{R}^{n \times n}$ *is Metzler then the system given by* $\dot{x}(t) = Ax(t)$ *is monotone.*

Proof. Suppose we have $x_0, y_0 \in \mathbb{R}^n$ with $x_0 \leq y_0$.
Then $y_0 - x_0 \geq 0$ so that

$$x(t, y_0 - x_0) \geq 0$$

for all $t \geq 0$.

But

$$x(t, y_0 - x_0) = x(t, y_0) - x(t, x_0)$$

by linearity. Hence

$$x(t, x_0) \leq x(t, y_0) \tag{3.17}$$

for all $t \geq 0$. □

This result states that any positive LTI system is automatically monotone. An important generalisation of positive LTI systems is the class of positive cooperative systems. These are important because of their applications in economics, biology and ecology [49]. Cooperative systems generate monotone flows in the forward time direction. References [1], [42], [34] are concerned with extending significant aspects of the theory of positive LTI systems to cooperative systems

Definition 3.4.2. *A* C^1 *vector field* $f : D \to \mathbb{R}^n$ *is said to be cooperative on* $W \subset D$ *if the Jacobian* $\frac{\partial f}{\partial x}(a)$ *is a Metzler matrix for all* $a \in W$*.*

Note that in general, a cooperative system need not be positive.

Let us now look at a set of conditions known as the Kamke conditions, after the German mathematician Erich Kamke (1890-1961) who specialised in the theory of differential equations. Given a function f, with associated dynamical system, it can be shown that the monotonicity of the system is equivalent to the Kamke conditions being satisfied. The Kamke conditions are as follows:

Definition 3.4.3. *If $f : D \to \mathbb{R}^n$ is continuously differentiable on some open set $D \subset \mathbb{R}^n$, we say that the Kamke conditions are satisfied if*
$x, y \in D$, $x \geq y$ and $x_i = y_i$ for some $i \in \{1, ..., n\}$ implies $(f(x))_i \geq (f(y))_i$.

Proposition 3.4.1. *Let $f : D \to \mathbb{R}^n$ be C^1. The system $\dot{x}(t) = f(x(t))$ is monotone if and only if it satisfies the Kamke conditions. [49]*

We will be dealing exclusively in the positive orthant, i.e. when D is a neighbourhood of $\mathbb{R}^n_+$. This is a natural choice for dealing with applications, for example, in the study of population dynamics (due to the positive invariance of a population's density). Part of the later work will be extending the Kamke conditions to the piecewise smooth case and investigating their relation to monotonicity in this setting.

It is worth noting that monotonicity can be used to establish conditions for asymptotic stability [4], [5], [10]. These papers consider nonlinear and switched systems. We now indicate how monotonicity can be used to establish asymptotic stability by outlining the argument for positive LTI systems.

Suppose we have a system given by $\dot{x}(t) = Ax(t)$ where A is Metzler and non-singular (so that the origin is its only equilibrium). Suppose we have a vector $v \in \mathbb{R}^n_+$ with

$$v \gg 0$$

such that

$$Av \ll 0.$$

It can be shown that this implies that the trajectory $x(t, v)$ is strictly decreasing for all t. This together with

$$0 \leq x(t, v) \leq v$$

for all t, and the fact that A is non-singular implies

$$x(t, v) \to 0 \text{ as } t \to \infty.$$

Linearity now implies

$$x(t, \lambda v) \to 0 \text{ as } t \to \infty$$

for all $\lambda > 0$.

Now for any $x_0 \in \mathbb{R}^n_+$, choose λ such that $\lambda v \geq x_0$.

Finally, monotonicity implies

$$x(t, x_0) \to 0 \text{ as } t \to \infty$$

for all $x_0 \in \mathbb{R}^n_+$.

Chapter 4

Properties of Piecewise Smooth and Switched Linear Systems

In this Chapter we review some fundamental facts from the the theory of piecewise smooth systems [15], paying particular attention to switched linear systems, and discuss some basic issues relating to them, such as the definition of solutions and stability of such systems.

4.1 Basic Definitions

In Chapter 2, we described a variety of non-smooth systems. Here, we specialise to the class of switched linear systems that shall be our primary concern for the remainder of this thesis. Broadly speaking, the class of switched linear systems, and more generally, switched systems, may be divided into two subclasses; a switched system can be *time-dependent* or *state-dependent* [35], [47].

Given a finite collection of matrices $\mathcal{A} = \{A_1, ..., A_k\}$, a time-dependant switched linear system is a system of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ t \geq 0, \tag{4.1}$$

$$\sigma : \mathbb{R}_+ \to \{1, ..., k\}.$$

We call σ the switching signal, and the points of discontinuity, $t_1, t_2, ...$ of σ are known as the switching instants.

The second class of switched systems is the class of state-dependent switched systems. Loosely speaking, these take the general form

$$\dot{x}(t) = A_{\sigma(x(t))}x(t),\ t \geq 0, \tag{4.2}$$

where σ maps states to indices in $\{1, ..., k\}$. These shall play a significant role in our later discussions. We now give an informal introduction, with a more formal treatment in Section 4.2. In a system with state-dependent switching, we partition the state space into a finite or infinite number of operating regions using a family of switching surfaces. There may also be a reset map which assigns a new value to the state at each switching instance. For the most part, we shall not consider impulse effects here. Thus we are only interested in continuous solutions. It is the value of the state variable at any time instant that determines which subsystem is active.

In both cases, a switched linear system is obtained from k LTI systems of the form

$$\dot{x} = A_i x,$$

$1 \leq i \leq k$, where we have only one system active at any one time instant.

A solution of (4.1), in the time-dependent case is a function $x : \mathbb{R}_+ \mapsto \mathbb{R}^n$, with $x(0) = x_0$, which is piecewise continuously differentiable and such that there is a

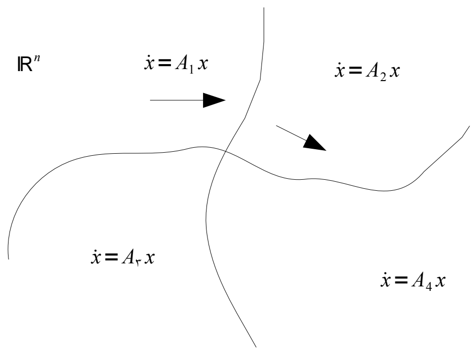

Figure 4.1: Partitioned state space - State-dependent switched system

switching signal σ which satisfies

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$

for all t except at the switching instances of σ. Because of the fact that in between successive switching instances, (4.1) behaves like an LTI system, we have for each switching signal σ, and each initial condition $x_0 = x(0)$, the existence of a unique continuous, piecewise differentiable solution $x(t)$. This solution is given by

$$x(t) = e^{A(t_k)(t-t_k)}e^{A(t_{k-1})(t_k-t_{k-1})}...e^{A(t_1)(t_2-t_1)}e^{A(0)(t_1)}x_0,$$

where $t_1 < t_2 < ...$ is the sequence of switching instances and t_k is the largest switching instant smaller than t.

4.2 Issues with State-Dependent Switching

A number of issues may arise in state-dependent switched systems which do not occur in time-dependent switched systems. It is worth describing some of these issues in detail as we shall be concerned with state-dependent switched systems in Chapter 5. Example 2.1.1 from Chapter 2, where we discussed the example of the bouncing ball can be viewed as an example of such a system. In this example, we meet a particular type of phenonemon, known as Zeno behaviour, where the state crosses the switching surface infinitely often in a finite amount of time. Further on in this discussion we shall revisit Example 2.1.1 to illustrate the occurence of Zeno behaviour. Except for our discussion of this example, we shall not consider state resets.

A key issue with state-dependent switching is that the differential equation defining the system may not be continuous. In such cases, classical C^1 solutions may not exist [9]. This leads us to consider alternative solution concepts. Of critical importance in this area is the notion of a Filippov solution, named after the Russian mathematician, Vladimir Filippov. Differential equations are replaced with differential inclusions, and so single solutions are replaced with a set of possible solutions. We shall briefly describe solutions of state-dependent switched systems, differential inclusions and then link them together with the notion of a Filippov solution.

4.2.1 Caratheodory Solutions

Suppose we are given the ordinary differential equation

$$\dot{x}(t) = f(x(t)) \tag{4.3}$$

where $x(0) = x_0$. A classical solution to this ODE on $[0, T]$ is a function

$$x : [0, T] \mapsto \mathbb{R}^n$$

which is continuously differentiable (C^1) and satisfies (4.3) for all $t \in [0, T]$. Existence and uniqueness of classical solutions is guaranteed under a variety of conditions, including f being C^1 or satisfying a Lipschitz condition [7], [8]. An important question in the context of this work is whether or not there exists a solution to (4.2) when f is discontinuous.

Example 4.2.1. *(A discontinuous vector field with nonexistence of classical solutions) [9]*

Consider the vector field $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x > 0 \\ \\ 1 & \text{if } x \leq 0. \end{cases} \tag{4.4}$$

Suppose there exists a continuously differentiable function $x : [0, t_1] \to \mathbb{R}$ such that $\dot{x}(t) = f(x(t))$ on $[0, t_1]$ and $x(0) = 0$. Then $\dot{x}(0) = f(x(0)) = 1$. So there exists $\delta > 0$ such that $x(t) > 0$ for $t \in (0, \delta)$. For such t, $\dot{x} = f(x(t)) = -1$ which contradicts the fact that $\dot{x}$ is continuous at 0. So no classical solution to (16) starting at zero exists.

Here we have an example of a discontinuous dynamical system which does not have a classical solution. It is natural to ask under what conditions a solution would exist. A first step towards addressing this is the concept of a Caratheodory solution. A Caratheodory solution of (4.3) defined on $[0, T]$ is an absolutely continuous map

$$x : [0, T] \mapsto \mathbb{R}^n$$

that satisfies (4.3) for almost all $t \in [0, T]$, i.e. for all $t \in [0, T]$ except for a set of Lebesgue measure zero. Equivalently, it is an absolutely continuous map

$$x : [0, T] \mapsto \mathbb{R}^n$$

which satifies $x(0) = x_0$ and

$$x(t) = x_0 + \int_0^t f(x(s))ds$$

for $t \in [0, T]$.

Example 4.2.2. *(A system with Caratheodory solutions but no classical solutions) [9]*

Consider the vector field $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \textit{if } x > 0 \\ \frac{1}{2} & \textit{if } x = 0 \\ -1 & \textit{if } x < 0. \end{cases} \tag{4.5}$$

The associated system $\dot{x}(t) = f(x(t))$ has no classical solution with initial condition $x(0) = 0$. This can be shown similarly to the previous example. However, it has two Caratheodory solutions $x(t) = t$ and $x(t) = -t$ defined on $[0, \infty)$. They both violate the differential equation when $t = 0$, i.e. on a set of measure zero.

4.2.2 Differential Inclusions

A key concept in the study of discontinuous differential equations is the notion of a differential inclusion. A differential inclusion is a generalisation of an ordinary differential equation and takes the form

$$\begin{aligned} \dot{x}(t) &\in F(x(t)) \\ t &\in [0, T] \\ x(0) &= x_0. \end{aligned} \tag{4.6}$$

The function F is an example of what is known as a set valued map. Given a point, a set-valued map assigns a set to that point. In general, a differential inclusion will have several solutions for a given initial condition. Note that if the set $F(x)$ consists of a single point for all x then the differential inclusion becomes an ordinary differential equation. F is sometimes known as a multivalued function.

4.2.3 Filippov Solutions

Consider a state-dependent switched system with switching surface Ω. Recall that we assume the state does not jump at switching instants. If the vector fields in adjacent regions point in the same direction relative to the switching surface, then once the continuous trajectory hits Ω it will continue on to the other side, and we obtain a solution in the sense of Caratheodory. See Figure 4.2.

However, on the other hand, if the vector fields both point toward Ω, a potential problem arises. This is due to the fact that once the trajectory reaches the switching surface Ω, it is confined there by the orientation of the vector fields. This was resolved by the Russian mathematician Vladimir Filippov, when he introduced the notion of what is now known as a Filippov solution.

The key idea behind Filippov solutions is the replacement of ODEs with differential inclusions of the form (4.6). We associate a differential inclusion to the discontinuous O.D.E. as follows. Given a point $x \in \mathbb{R}^n$, we set:

$$F(x) = \bigcap_{\mu(N)=0} \cap_{\epsilon>0} \overline{co(f(x + B_\epsilon \backslash \{N\}))}. \tag{4.7}$$

where B_ϵ is the ball about 0 of radius ϵ, co denotes the convex hull and $\mu(N)$ is Lebesgue measure. Filippov solutions of the ODE with right hand side given by f are solutions of the differential inclusion (4.6) with F given by (4.7). Formally, a Filippov solution is an absolutely continuous function

$$x : [0, T] \mapsto \mathbb{R}^n,$$

which satisfies the differential inclusion (4.6) for almost all $t \in [0, T]$.

If f is continuous at x, then

$$F(x) = \{f(x)\}.$$

On the other hand, at points of discontinuity, F is defined by looking at the closed convex hull of the values of f as we approach x.

Using the theory of differential inclusions, it can be shown [17] that solutions to the differential inclusion defined in (4.7) exist under relatively mild conditions on f.

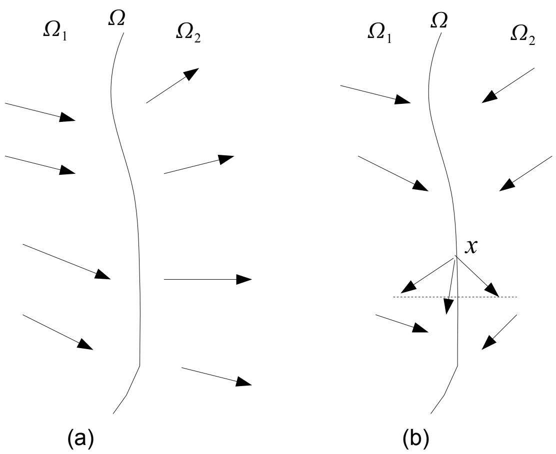

Figure 4.2: (a) Vector field crossing Ω (b) A sliding mode

To illustrate the idea of a Filippov solution, let us divide $\mathbb{R}^n$ into two open regions, Ω_1 and Ω_2, by means of a switiching surface Ω (which will be assumed to be a smooth manifold).

If $x \in \Omega_1$ or $x \in \Omega_2$ then $\dot{x} = f_1(x)$ or $\dot{x} = f_2(x)$ respectively. Assume that f_1 and f_2 extend continuously to $\bar{\Omega}_1$ and $\bar{\Omega}_2$ respectively.
According to Filippov, on Ω, $x(.)$ is a solution of the switched system if it satifies the differential inclusion

$$\dot{x} \in F(x)$$

where

$$F(x) := \{\alpha f_1(x) + (1-\alpha) f_2(x) : \alpha \in [0,1]\} \text{if } x \in \Omega. \tag{4.8}$$

If the vector fields on both sides of the switching surface point towards Ω, then the solution must be confined to Ω and so the only possibility is that it will 'slide' on Ω. This is the reason that this kind of solution is known as a *sliding mode*. In this example, there is a unique convex combination of $f_1(x)$ and $f_2(x)$ that is tangent to Ω at the point x, which determines the instantaneous velocity of the trajectory starting at x.

Sliding modes may or may not be desirable in practice. On the one hand, they can be interpreted as infinitely fast switching between systems, which may lead to excessive wear of equipment. On the other hand, they can be created to solve control problems which may not be solvable otherwise. [35]

To illustrate the concept of sliding modes, we present the following example from [35].

Example 4.2.3. *Consider the following state-dependent switched linear system:*

$$\dot{x} = \begin{cases} Ax & \text{if } x_2 \geq x_1 \\ Bx & \text{if } x_2 < x_1 \end{cases} \tag{4.9}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ *and* $\lambda \in \mathbb{R}$.

Let us now determine for what values of λ a sliding mode occurs. Our main focus in this regard is the behaviour of the system on the bounding line

$$x_1 = x_2$$

between the two operating regions Ω_A and Ω_B. Let us take the convex combination

$$\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} -1 & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\alpha - 1 \\ -\lambda + \alpha(\lambda - 1) \end{pmatrix} \tag{4.10}$$

of Ax and Bx where $x \in \Omega$, $0 \leq \alpha \leq 1$, and for convenience we choose

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In order for this convex combination to lie on Ω, we must have that

$$2\alpha - 1 = -\lambda + \alpha(\lambda - 1),$$

which means that

$$\alpha = \frac{1-\lambda}{3-\lambda}.$$

Suppose $\lambda > 1$. Then $\alpha \notin [0,1]$ which means that no convex combination of Ax, Bx lies on Ω in this case. Also, if $x_2 = x_1 + \epsilon$, trajectories point away from the surface. If $\lambda < 1$, $\alpha \in [0,1]$ which ensures that our convex combination (4.10) lies on the bounding line Ω. So a sliding mode occurs in the first quadrant if $\lambda < 1$. For $\lambda > -1$, the corresponding trajectory approaches the origin along the switching line, while for $\lambda < -1$ it goes away from the origin. The former is known as a stable sliding mode, and the latter, an unstable sliding mode.

4.2.4 Zeno Behaviour

Zeno behaviour, named after the well-known Zeno paradox, has been characterised, in the context of hybrid systems, as an infinite number of discrete state transitions occuring in a finite amount of time.. The simplest example is that of the bouncing ball, which we encountered in Chapter 1. Following [35] we normalise the gravitational constant so that (2.1) becomes

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -1. \end{aligned} \tag{4.11}$$

Integrating (4.11) results in

$$\begin{aligned} x_2(t) &= -(t-t_0) + x_2(t_0) \\ x_1(t) &= -\frac{(t-t_0)^2}{2} + x_2(t_0)(t-t_0) + x_1(t_0). \end{aligned}$$

Letting initial conditions be $t_0 = 0$, $x_1(0) = 0$ and $x_2(0) = 1$ we get

$$\begin{aligned} x_2(t) &= -t + 1 \\ x_1(t) &= -\frac{t^2}{2} + t. \end{aligned}$$

When $t = 2$, we get $x_1 = 0$ so $t = 2$ is the first switching time. Using (2.2) we get $x_2(2) = r$, where r is the coefficient of restitution.

If we repeat this using $t_0 = 2$, $x_1(2) = 0$ and $x_2(2) = r$ as initial conditions we get

$$\begin{aligned} x_2(t) &= -t + 2 + r \\ x_1(t) &= -\frac{(t-2)^2}{2} + (t-2)r. \end{aligned}$$

The next switching time is then $t = 2 + 2r$ and using (2.2) again, we get that $x_2(2 + 2r) = r^2$.

If we continue in this fashion we get the sequence of switching times

$$2, 2 + 2r, 2 + 2r + 2r^2, 2 + 2r + 2r^2 + 2r^3, ...$$

It is easy to show that the sequence of times has a finite accumulation point, given by

$$\sum_{k=0}^{\infty} 2r^k = \frac{2}{1-r}.$$

Prior to this time, the ball makes an infinite number of bounces, i.e. an infinite number of switching events occur in a finite amount of time. In practice, however, the ball will only make a finite number of bounces before stopping.

4.3 Stability for Switched Linear Systems under Arbitrary Switching

Stability is a big issue in the study of switched linear systems. Switching between stable component systems can render the overall system unstable. We now discuss some issues and results for arbitrary switched systems.

Suppose we are given a family of stable subsystems

$$\dot{x}(t) = A_i x(t), \tag{4.12}$$

$1 \leq i \leq m$. Several questions about stability arise. A fundamental problem in the study of switched linear systems is to determine whether the system (4.1), composed of the family (4.12), is stable for all switching signals σ. Notions of stability, analogous to those for LTI systems, may be defined for switched linear systems.

Definition 4.3.1. *The origin is a uniformly stable equilibrium of (4.1) if given any $\epsilon > 0$, there is some $\delta > 0$ such that $||x_0|| < \delta$ implies $||x(t, x_0)|| < \epsilon$ for $t \geq 0$, for all solutions $x(t, x_0)$ of the system.*

Definition 4.3.2. *The origin is a uniformly exponentially stable equilibrium of (4.1) if there exists $M, \beta \in \mathbb{R}$, with $M \geq 1$, $\beta > 0$ such that*

$$||x(t, x_0)|| \leq Me^{-\beta t}||x_0||$$

for $t \geq 0$, and for all solutions $x(t, x_0)$ of the system.

Note that, in the previous definitions, δ, and the constant β must be *independent of switching rule.*

As we have seen in the previous chapter, quadratic Lyapunov functions play a vital role in the study of the stability of LTI systems, and their role is well understood.

It is therefore logical to begin our study of the stability of switched linear systems with a discussion on the common quadratic Lyapunov function (CQLF) existence problem. Suppose we are given a collection $\{A_1, ..., A_m\}$ of Hurwitz matrices, with associated stable LTI systems $\dot{x}(t) = A_i x(t)$, $1 \leq i \leq m$, the aim is to discover whether or not this collection has a common quadratic Lyapunov function. That is, can we find a function $V(x) = x^T P x$, where P is symmetric and postive definite, and $PA_i + A_i^T P$ is negative definite for all i. If this is possible, then V(x) is a quadratic Lyapunov function for each individual subsystem.

Theorem 4.3.1. *Suppose we have a family $\mathcal{M} = \{A_1, ..., A_m\}$ of Hurwitz matrices, with associated switched linear system (4.1). If there exists a CQLF for $\mathcal{M}$ then (4.1) is exponentially stable under arbitrary switching.*

Given a two-dimensional switched linear system, there exist simple necessary and sufficient conditions for the existence of a CQLF [35]:

Proposition 4.3.1. *The linear systems $\dot{x}(t) = Ax(t)$ and $\dot{x}(t) = Bx(t)$, with $x(0) = x_0$, $x \in \mathbb{R}^2$ and $A, B \in \mathbb{R}^{2\times 2}$ have a CQLF if and only if all pairwise convex combinations of the matrices A, B, A^{-1} and B^{-1} are Hurwitz.*

In general, the existence of a CQLF is not a necessary condition for the exponential stability of a switched linear system [11]. In addition, there is no simple algebraic condition to determine whether or not there exists a CQLF for a family of LTI systems. There do exist partial results for special cases, some which we shall now describe.

Following [47] we view the mapping $P \mapsto PA + A^T P$ as a linear function on $S^{n\times n}$, the space of real symmetric $n \times n$ matrices. Formally we have a map $\mathcal{L}_A$ defined by the real $n \times n$ matrix A as:

$$\mathcal{L}_A : S^{n\times n} \to S^{n\times n}$$

$$\mathcal{L}_A(P) = PA + A^T P.$$

The map $\mathcal{L}_A$ has the following properties:

(i) If A has eigenvalues $\{\lambda_i\}$ with associated eigenvectors $\{v_i\}$, then $\mathcal{L}_A$ has eigenvalues $\{\lambda_i + \lambda_j\}$ with eigenvectors $\{v_i v_j^T + v_j v_i^T\}$ for all $i \leq j$. Since $\lambda_i + \lambda_j \neq 0$ for a Hurwitz matrix A, it follows that $\mathcal{L}_A$ is invertible for a Hurwitz matrix A.

(ii) A is Hurwitz if and only if there exists $P > 0$ such that $\mathcal{L}_A(P) < 0$.

Next, define $\mathcal{P}_A$ to be the set

$$\mathcal{P}_A = \{P > 0 : \mathcal{L}_A(P) < 0\}.$$

It follows that the function $V(x) = x^T P x$ is a quadratic Lyapunov function for A if and only if $P \in \mathcal{P}_A$. Note that $\mathcal{P}_A$ is an open convex cone since if $P, Q \in \mathcal{P}_A$, then $aP + bQ \in \mathcal{P}_A$, where $a, b > 0$, since

$$aP + bQ > 0$$

and

$$\begin{aligned}
\mathcal{L}_A(aP + bQ) &= (aP + bQ)A + A^T(aP + bQ) \\
&= a(PA + A^T P) + b(QA + A^T Q) \\
&< 0,
\end{aligned}$$

by linearity and that fact that the positive definite matrices form a convex cone. The following result follows from Theorem 3.2.2 (Lyapunov's Theorem).

Proposition 4.3.2. *Suppose $A \in \mathbb{R}^{n \times n}$. Then $\mathcal{P}_A$ is nonempty if and only if A is Hurwitz.*

The problem of finding a CQLF for the collection of matrices $\{A_1, ..., A_m\}$ is equivalent to determining whether or not $\mathcal{P}_{A_1} \cap ... \cap \mathcal{P}_{A_m}$ is nonempty. Observe now that if $A \in \mathbb{R}^{n \times n}$ is invertible, the cones $\mathcal{P}_A$ and $\mathcal{P}_{A^{-1}}$ are identical. For if $P \in \mathcal{P}_A$, then

$$PA + A^T P < 0$$

which implies that

$$(A^T)^{-1}P + PA^{-1} = (A^{-1})^T(PA + A^TP)A^{-1} < 0,$$

by congruence, i.e. $\mathcal{P}_A \subset \mathcal{P}_{A^{-1}}$. The result then follows by symmetry. Also observe that if $R \in \mathbb{R}^{n\times n}$ is nonsingular we have

$$\mathcal{P}_{R^{-1}AR} = R^T \mathcal{P}_A R \equiv \{R^T P R : P \in \mathcal{P}_A\}.$$

These two observations result in the following:

Proposition 4.3.3. *Suppose $\mathcal{M} = \{A_1, ..., A_m\}$ is a family of Hurwitz matrices. Then the following are equivalent:*
(i) There exists a CQLF for the systems given by the elements of $\mathcal{M}$
(ii) There exists a CQLF for the systems given by the the elements of $\{A_1^{-1}, ..., A_m^{-1}\}$.
(iii) There exists a CQLF for the systems given by the elements of $\{R^{-1}A_1R, ..., R^{-1}A_mR\}$.

The third condition in the last proposition says that CQLF existence is invariant under a change of coordinates.

It is possible in certain special cases to guarantee the existence of a CQLF for a set of systems generated by $\mathcal{M} = \{A_1, ..., A_m\}$. It has been shown [47] that if all matrices in $\mathcal{M}$ are in upper triangular form, that there exists a CQLF for the systems generated by $\mathcal{M}$. In addition, the matrix P which defines the CQLF can be chosen to be diagonal. We also have the following similar result: [40]

Theorem 4.3.2. *The set of systems generated by $\mathcal{M}$ has a CQLF if there exists a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ such that every $U^{-1}A_iU$ is upper (lower) triangular for $A_i \in \mathcal{M}$.*

Note that the matrix U in this theorem can be complex.

Recall that the matrix $A \in \mathbb{R}^{n\times n}$ is normal if $AA^T = A^T A$. The matrix $S \in \mathbb{R}^{n\times n}$ is skew-symmetric if $S^T = -S$.

Now, the system $\dot{x}(t) = Ax(t)$ has the Lyapunov function $V(x) = x^T x$ if

$$\mathcal{L}_A(I) = A^T + A < 0, \tag{4.13}$$

where I is the $n \times n$ identity matrix. So if the collection $\mathcal{M}$ is composed of matrices which satisfy (4.13), then the function $V(x) = x^T x$ is a CQLF for the switched system generated by $\mathcal{M}$. (4.13) is satisfied if A is normal and Hurwitz. Also, given a matrix $A \in \mathbb{R}^{n\times n}$ which satisfies (4.13), we have that $A + S$ must also satisfy (4.13) where $S \in \mathbb{R}^{n\times n}$ is skew-symmetric.

4.4 Numerical Methods

Given a family $\{A_1, ..., A_m\}$ of Hurwitz matrices, one advantage of considering common quadratic Lyapunov functions $V(x) = x^T Px$ in which the matrix $P = P^T > 0$ satisfies

$$A_i^T P + PA_i < 0 \tag{4.14}$$

is that there are efficient numerial methods for solving such inequalities. (4.14) is an example of a system of linear matrix inequalities (LMI). We call the system (4.14) feasible if a solution P exists and infeasible otherwise. So if one wants to check whether or not a set of Hurwitz matrices possesses a CQLF, it amounts to checking whether or not a system of LMIs is feasible. To do this, there are solvers for LMIs built on convex optimisation algorithms which are capable of solving this kind of problem [47], [35]. Conversely the following result can be used to verify the non-existence of a CQLF for a system of LTIs.

Proposition 4.4.1. *Suppose $\{A_1, ..., A_m\}$ is a family of Hurwitz matrices. A CQLF does not exist for the LTI systems generated by the matrices A_i, $1 \le i \le m$ if and only* *if there exist positive semi definite matrices R_i, $1 \leq i \leq m$, not all zero, satisfying*

$$\sum_{i=1}^{m}(A_i^T R_i + R_i A_i) \geq 0$$

There are two main disadvantages associated with using a numerical approach. First of all, LMIs do not provide much insight into why a CQLF may or may not exist for a set of LTI systems. Secondly, methods based on solving LMIs are not very effective if m is very large, or if there are an infinite number of subsystems.

4.5 Linear Copositive Lyapunov Functions and the Stability of Switched Positive Systems.

Our discussion to date has been on general switched systems. Let us now discuss results which are specific to positive systems. Recently, several authors have studied copositive Lyapunov functions for such systems [30], [37]. Most of this has been on linear or quadratic functions [6], [18], [21] . Let us recall the definition of a common linear copositive Lyapunov function [30].

Definition 4.5.1. *Suppose $A_1, ..., A_m \in \mathbb{R}^{n\times n}$ is Metzler. $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = v^T x$. $V(x)$ is a common linear copositive Lyapunov function for the positive LTI system $\dot{x} = Ax$, where $A \in \{A_1, ..., A_m\}$, if and only if*
(i) $v \gg 0$;
(ii) $A_i^T v \ll 0$, $1 \leq i \leq m$.

Now we shall consider a set of linear positive systems, and quote necessary and sufficient conditions for the existence of a common linear copositive Lyapunov function. Let us first look at state dependent switching systems. Assume that the state space may be partitioned using simplicial cones, of which we now provide the definition:

Definition 4.5.2. *A simplicial cone C in $\mathbb{R}^n$ is a cone generated by a non singular generating matrix $Q \in \mathbb{R}^{n\times n}$ as follows:*

$$C := \{x : x = \sum_{i=1}^{n} \alpha_i Q^{(i)}, \alpha_i \geq 0, i = 1, ...n\}, \tag{4.15}$$

where $Q^{(i)}$ is the i^{th} column of Q.

We now consider a set of such cones C_j with nonnegative generating matrices Q_j, $j = 1, ...N$. We quote the following result from [30], which leads directly to elegant conditions for the existence of a common linear copositive Lyapunov function. This result may be applied to state dependent systems of the form

$$\dot{x}(t) \in A(x),$$

where

$$A(x) = \{A_j x : x \in C_j\}.$$

Theorem 4.5.1. *Suppose we are given m Metzler and Hurwitz matrices, $A_1, ...A_m \in \mathbb{R}^{n\times n}$ and m closed simplicial cones C_j, as defined in (4.15), such that*

$$\mathbb{R}^n_+ = \bigcup_{j=1}^{m} C_j,$$

Then precisely one of the following statements is true:

(i) There is a positive vector $v \in \mathbb{R}^n$ such that $v^T A_j x < 0$ for all non-zero $x \in Cj$ and $j = 1, ..., m$.

(ii) There are vectors $w_j \geq 0$ not all zero such that $\sum_{j=1}^{m} B_j w_j \geq 0$, where $B_j := A_j Q_j$.

The authors of [30] then move on from state dependent switching to discuss arbitrarily switching systems. A special case of the previous result is when Q_j is the identity matrix for $1 \leq j \leq m$, which is the case when we are looking for a common linear copositive Lyapunov function for a finite set of positive LTI systems. They provide the following lemma which is used in the proof of the main result of the paper [30].

Lemma 4.5.1. *Given m Metzler and Hurwitz matrices $A_1, ..., A_m \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) There is a non-zero $v \geq 0$ such that $v^T A_j \leq 0$ for all $j = 1, ..., m$.

(ii) There are no $w_j \gg 0$ such that $\sum_{j=1}^{m} A_j w_j = 0$.

Before stating the next result we need some additional notation. Denote the set of all possible mappings

$$\sigma : \{1, ...n\} \to \{1, ..., m\}$$

by $S_{n,m}$ for all $n, m \in \mathbb{N} \setminus \{0\}$. Next given matrices A_j, $1 \leq j \leq m$, construct the following matrices:

$$A_\sigma(A_1, ..., A_m) := \begin{pmatrix} A_{\sigma(1)}^{(1)} & A_{\sigma(2)}^{(2)} & \dots & A_{\sigma(m)}^{(m)} \end{pmatrix},$$

where $\sigma \in S_{n,m}$ and $A_{\sigma(i)}^{(i)}$ is the i^{th} column of the matrix $A_{\sigma(i)}$.

Theorem 4.5.2. *Given a finite number of Hurwitz and Metzler matrices $A_1, ..., A_m \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

(i) There is a strictly positive vector $v \in \mathbb{R}^n$ such that $v^T A_j \ll 0$, $1 \leq j \leq m$.

(ii) $A_\sigma(A_1, ..., A_m)$ is Hurwitz for all $\sigma \in S_{n,m}$.

This result tells us when a switched system formed by m subsystems has a common linear copositive Lyapunov function. Given m positive LTI systems, they will have a common linear copositive Lyapunov function $V(x) = v^T x$ if and only if $A_\sigma(A_1, ..., A_m)$ is Hurwitz for all $\sigma \in S_{n,m}$. Note that $V(x) = v^T x$ decreases everywhere.

Example 4.5.1. *To illustrate this result, we include the following numerical example* *from [30]. Suppose we have three Metzler and Hurwitz matrices given by*

$$
\begin{aligned}
A_1 &= \begin{pmatrix} -12 & 6 & 6 \\ 1 & -10 & 2 \\ 5 & 3 & -10 \end{pmatrix} \\
A_2 &= \begin{pmatrix} -12 & 4 & 0 \\ 6 & -10 & 9 \\ 4 & 3 & -13 \end{pmatrix} \\
A_3 &= \begin{pmatrix} -9 & 2 & 8 \\ 6 & -10 & 4 \\ 3 & 0 & -11 \end{pmatrix}.
\end{aligned}
$$

$A_\sigma(A_1, ..., A_3)$ *is Hurwitz for all* $\sigma \in S_{3,3}$ *so we can conclude that a switched linear postive system composed of* A_1*,* A_2 *and* A_3 *will be asymptotically stable under arbitrary switching.*

Chapter 5

Some new results in piecewise monotone systems.

In Chapter 2, we discussed monotone systems and their importance in many applications. Monotonicity is a key property of positive LTI systems and in this chapter we investigate monotonicity for positive systems that are piecewise linear.

5.1 Introductory Remarks

Monotone systems are systems in which the flow preserves an ordering of the initial states. As we have already seen in Chapter 3, monotonocity is a key property of positive LTI systems and can be used to derive stability results for such systems. The concept of monotonicity comes into its own in the case of nonlinear systems and there exist many powerful theoretical results in this setting [49]. Monotonicity has been used to determine the asymptotic behaviour of certain nonlinear models in biology, for example to model the control of protein synthesis in the cell [49], and in epidemiology and population dynamics, for example in the modelling of mutualistic interactions [24]. We aim to extend monotonicity results pertaining to positive LTI systems to piecewise systems with state-dependent switching.

As has been mentioned, a monotone system is an order preserving system. More formally, recall the system given by

$$\dot{x}(t) = f(x(t))$$

is monotone if

$$x_0 \leq y_0$$

implies

$$x(t, x_0) \leq x(t, y_0)$$

for all $t \geq 0$ for which both solutions are defined.

The aim of the work in this chapter is to investigate when a piecewise linear system is monotone. We consider switched systems composed of monotone subsystems. However, such switched systems are not necessarily monotone, something which we shall elaborate on in the following discussion. We first reformulate the Kamke conditions for a special class of piecewise smooth systems. Following on from this, we investigate the relationship between the Kamke conditions and monotonicity.

5.2 Piecewise Monotone Systems

We will now introduce the class of piecewise monotone systems. In general, this class is obtained by partitioning $\mathbb{R}^n_+$ by open regions Ω_i, $i = 1, ..., p$, such that

$$\begin{aligned}
\mathbb{R}^n_+ &= \bigcup_{i=1}^{p} \bar{\Omega}_i \\
\Omega_i \cap \Omega_j &= \emptyset
\end{aligned}$$

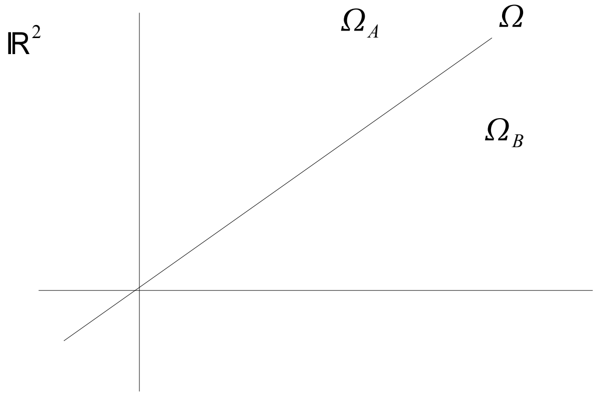

Figure 5.1: Bimodal system in $\mathbb{R}^2$.

for all $i \neq j$.

Dynamics in Ω_i are given by

$$\dot{x} = f_i(x)$$

where the system generated by each f_i is monotone and positive. So, in particular, each f_i is cooperative in the sense described in Chapter 3. As we have seen in Chapter 4, such systems often give rise to discontinuous vector fields and care is needed in defining solutions.

5.2.1 A simple class of piecewise linear systems

We are now going to focus our attention on a simple special class of such systems. We shall be considing linear bimodal systems in which we partition $\mathbb{R}^n_+$ into two regions by means of a hyperplane through the origin, see Figure 5.1. Formally we are considering the following type of system:

Given $A, B \in \mathbb{R}^{n\times n}$, with A and B Metzler, and a vector $c \in \mathbb{R}^n$, define Ω, Ω_A

and Ω_B as follows:

$$\begin{aligned}\Omega &= \{x \in \mathbb{R}^n_+ : c^T x = 0\},\\ \Omega_A &= \{x \in \mathbb{R}^n_+ : c^T x < 0\},\\ \Omega_B &= \{x \in \mathbb{R}^n_+ : c^T x > 0\}.\end{aligned}$$

Ω is a hyperplane through the origin. The system dynamics are governed by the differential equation

$$\dot{x}(t) = f(x(t)) = \begin{cases} Ax(t) & \text{if } x \in \Omega_A \\ Bx(t) & \text{if } x \in \bar{\Omega}_B. \end{cases} \tag{5.1}$$

where $A, B \in \mathbb{R}^{n\times n}$, and $\bar{\Omega}_B$ is the closure of Ω_B, i.e.

$$\bar{\Omega}_B = \{x \in \mathbb{R}^n_+ : c^T x \geq 0\}.$$

As the system (5.1) may well be discontinuous on Ω, it is important at this point to be precise with our solution concepts.

5.3 Solution Concepts

In Chapter 4, we discussed solution concepts for state dependent switched systems and piecewise smooth systems. In the regions Ω_A and Ω_B, solutions to (5.1) are simply given by the solution to $\dot{x} = Ax$ as long as the trajectory stays in Ω_A, and the solution to $\dot{x} = Bx$ for as long as the solution remains in Ω_B. However, we must be careful when talking about solutions on the switching surface Ω. As discussed above we replace (5.1) with the differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{5.2}$$

where

$$F(x) = \cap_{\epsilon>0}\overline{co(f(x + B_\epsilon \backslash \{x\}))}. \tag{5.3}$$

In our case, this becomes

$$F(x) = \overline{co\{Ax, Bx\}},$$

the closed convex hull of Ax, Bx, where $x \in \Omega$, and

$$F(x) = \begin{cases} \{Ax\} \text{ if } x \in \Omega_A \\ \{Bx\} \text{ if } x \in \Omega_B. \end{cases}$$

For our setup, this means that solutions are defined as follows:

(i) If our initial conditions are in either Ω_A or Ω_B, then we use the standard definition for the solution of an LTI system, i.e. if $x_0 \in \Omega_A$ or $x_0 \in \Omega_B$, then $x(t, x_0) = e^{At}x_0$ or $x(t, x_0) = e^{Bt}x_0$, for as long as we stay in Ω_A or Ω_B.

(ii) We must be more careful on the bounding hyperplane Ω.
(a) If solutions approach Ω from one side and leave Ω from the other side, then the trajectory continues across the switching surface and carries on into the next region.
(b) If the vector fields on both sides of Ω are pointing towards Ω, in the sense that

$$\begin{aligned} c^T Ax &> 0 \\ c^T Bx &< 0, \end{aligned}$$

then in this case, our solution takes the form of a sliding mode. We take the convex combination of the vector fields given by $\dot{x}(t) = Ax(t)$ and $\dot{x}(t) = Bx(t)$ which lies on Ω and the trajectory continues along, confined to Ω, thus generating a sliding mode. Dynamics on the surface are given by

$$\dot{x} = \alpha Ax + (1 - \alpha)Bx$$

where we have solved

$$c^T(\alpha Ax + (1 - \alpha)Bx) = 0$$

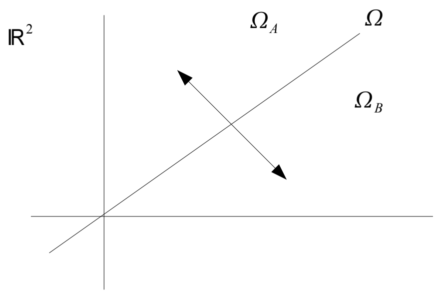

Figure 5.2: Situation (c)

for α.

(c) Suppose the vector fields on either side of Ω are pointing away from Ω, in the sense that

$$\begin{aligned} c^T Ax &< 0 \\ c^T Bx &> 0 \end{aligned}$$

In this case, solutions are not unique. For this reason we do not consider this situation in this thesis.

We now present a simple example of the occurence of sliding modes for our system class in $\mathbb{R}^2$.

Example 5.3.1. *Sliding Modes in* $\mathbb{R}^2$.

Let $c^T = (1 - 1)$. For a sliding mode to occur, we need

$$\begin{aligned} c^T Ax &> 0, \\ c^T Bx &< 0, \end{aligned}$$

i.e.

$$
\begin{aligned}
(1\ -1)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &> 0 \\
(1\ -1)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &< 0.
\end{aligned}
\tag{5.4}
$$

Hence we obtain a sliding mode if

$$
\begin{aligned}
a_{11} + a_{12} &> a_{21} + a_{22}, \\
b_{11} + b_{12} &< b_{21} + b_{22}.
\end{aligned}
$$

We could, for example, choose A and B as follows:

$$
A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 3 \\ 5 & -2 \end{pmatrix}.
$$

5.3.1 Monotonicity for (5.1)

Now that we have discussed our solution concepts for (5.1), we can consider monotonicity for this system. The fact that A and B are Metzler matrices implies that the system is monotone in each individual operating region in the following sense. Suppose we have $x_0, y_0 \in \Omega_A$ (Ω_B) with

$$x_0 \leq y_0,$$

then

$$x(t, x_0) \leq x(t, y_0)$$

for as long as $x(t, x_0), x(t, y_0) \in \Omega_A$ (Ω_B). This follows from the fact that in Ω_A (Ω_B), solutions starting at x_0 are given by $x(t, x_0) = e^{At}x_0$ ($x(t, x_0) = e^{Bt}x_0$).

However we have no guarantee that the piecewise system (5.1) will be monotone. In particular, it can happen that $x_0 \in \Omega_A$, $y_0 \in \Omega_B$ with

$$x_0 \leq y_0$$

but

$$x(t, x_0) \nleq x(t, y_0),$$

for $t \geq 0$. The following example shows that if we choose two ordered initial points, one in Ω_A and the other in Ω_B, the ordering is not necessarily preserved.

Example 5.3.2. *Given $c = (1, -1) \in \mathbb{R}^2$, let Ω be given as above, i.e. $\Omega = \{x \in \mathbb{R}^n_+ : c^T x = 0\}$. Define $A, B \in \mathbb{R}^{2\times 2}$ as follows:*

$$A = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

Let the dynamics be governed by

$$\dot{x}(t) = \begin{cases} Ax(t) & \text{if } x \in \Omega_A \\ Bx(t) & \text{if } x \in \bar{\Omega}_B. \end{cases} . \tag{5.5}$$

Since A and B are Metzler, the two subsystems given by $\dot{x}(t) = Ax(t)$ on Ω_A, $\dot{x}(t) = Bx(t)$ on Ω_B are locally monotone in the sense we described earlier. Now, choose $x_0 = (1, \frac{5}{4}) \in \Omega_A$ and $y_0 = (1, \frac{3}{4}) \in \Omega_B$ as our initial conditions. We have $x_0 \geq y_0$. However, $Ax_0 = (-2, -\frac{1}{2})$ and $By_0 = (1, \frac{3}{2})$, and it is clear from the direction of the vector field at these two points that the system (5.5) cannot be monotone. See Figure 5.3.

The above discussion gives sense to the following question. Under what conditions will the piecewise smooth system (5.1) be monotone? The remainder of this chapter will be centred on investigating this question.

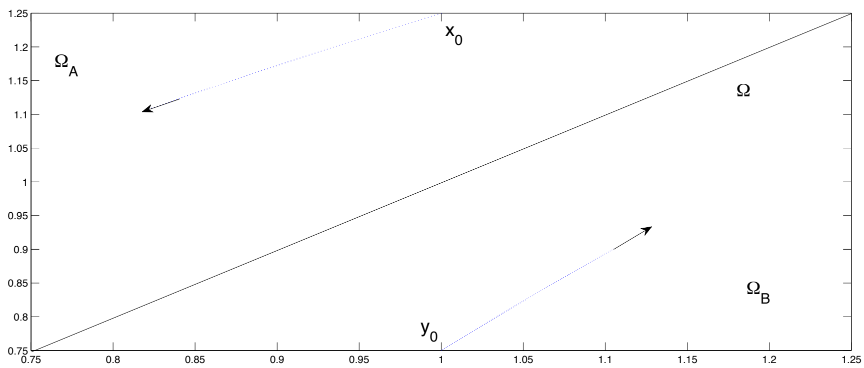

Figure 5.3: Matlab simulation for Example 5.3.2.

5.4 The Kamke Conditions

Let us now divert our attention to the Kamke conditions. For smooth systems, the Kamke conditions are equivalent to monotonocity [49]. For the convenience of the reader, we now recall the formal definition of the Kamke conditions for smooth systems.

Definition 5.4.1. *If $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable on some open set $D \subset \mathbb{R}^n$, we say that the Kamke conditions are satisfied if for all $x, y \in D$ with $x \geq y$ and $x_i = y_i$ for some $i \in \{1, ..., n\}$ we have $(f(x))_i \geq (f(y))_i$.*

We wish to reformulate these conditions for the system (5.1). Our reformulation is as follows:

5.4.1 The P-K conditions

Definition 5.4.2. *We say (5.1) satisfies the P-K conditions (Piecewise-Kamke) if the following hold:*
(i): $x \in \Omega_A$ and $y \in \Omega_B$, $x \leq y$ and $x_i = y_i$ implies

$$(Ax)_i \leq (By)_i.$$

(ii) $x \in \Omega_A$ *and* $y \in \Omega_B$*,* $x \geq y$ *and* $x_i = y_i$ *implies*

$$(Ax)_i \geq (By)_i.$$

Note that as both A and B are Metzler, if x and y are either both in Ω_A (or Ω_B) then the Kamke conditions, as given in Definition 5.4.1, hold. Also note that if follows from the P-K conditions that $x \leq y$, $x_i = y_i$, $x, y \in \mathbb{R}^n_+$ implies

$$f_i(x) \leq f_i(y).$$

We next investigate what restrictions must be placed on the matrices A and B to ensure that the P-K conditions hold for the system (5.1). In order to do this, we define the sets I_0^c, I_+^c and I_-^c as follows.

Definition 5.4.3. *Given a vector* $c \in \mathbb{R}^n$*, form the following sets:*

$I_0^c := \{i \in \{1, ...n\} : \exists j_1, j_2 \text{ s.th. } j_1 \neq i, j_2 \neq i, j_1 \neq j_2, c_{j_1} c_{j_2} < 0\}$,

$I_+^c \ := \{i \in \{1, ...n\} : c_j \geq 0, j \neq i\}$,

$I_-^c \ := \{i \in \{1, ...n\} : c_j \leq 0, j \neq i\}$

Let us clarify this definition with the following examples:

Example 5.4.1. *(i)* $c = (3, -5, 2, -9)$*. Then*

$$I_0^c = \{1, 2, 3, 4\}, \tag{5.6}$$

$$I_+^c = I_-^c = \emptyset. \tag{5.7}$$

(ii) $c = (4, 5, 2, -4)$*. Then*

$$I_0^c = \{1, 2, 3\}, \tag{5.8}$$

$$I_+^c = \{4\}. \tag{5.9}$$

We now state and prove the following proposition which tells us precisely when the P-K conditions hold for (5.1). We assume that there exists $x \gg 0$ such that $c^T x = 0$. This means that for $i \in I_+^c$, there exists $j \neq i$ with $c_j > 0$. Also, for $i \in I_-^c$, there exists $j \neq i$ with $c_j < 0$.

Proposition 5.4.1. *The system (5.1) above satisfies the P-K conditions if and only if the following three conditions are satisfied:*

(i) $\forall x \in \Omega$, $i \in I_+^c$ *we have* $(Ax)_i \leq (Bx)_i$*;*

(ii) $\forall x \in \Omega$, $i \in I_-^c$ *we have* $(Ax)_i \geq (Bx)_i$*;*

(iii) $\forall x \in \Omega$, $i \in I_0^c$ *we have* $(Ax)_i = (Bx)_i$*.*

Proof. Assume that (i), (ii), and (iii) hold. We shall show the the P-K conditions in Definition 5.4.2 hold. There are two cases to consider.

Case (i) Suppose we have $x \in \Omega_A$, $y \in \Omega_B$ with $x \leq y$ and $x_i = y_i$. In this case we have

$$\begin{aligned} c^T x &< 0 \\ c^T y &> 0 \end{aligned}$$

which implies

$$c^T(y - x) = c_1(y_1 - x_1) + ... + c_n(y_n - x_n) > 0.$$

This implies that $i \in I_+^c \cup I_0^c$. To see this, note that if $i \in I_-^c$ this would mean that

$$c^T(y - x) = c_1(y_1 - x_1) + ... + \underbrace{c_i(y_i - x_i)}_{=0} + ... + c_n(y_n - x_n) \leq 0.$$

Since $i \in I_+^c \cup I_0^c$, it follows that

$$(Az)_i \leq (Bz)_i \tag{5.10}$$

for any $z \in \Omega$.

Write

$$\begin{aligned} x^{(\alpha)} &= (1 - \alpha)x + \alpha y && (5.11) \\ &= x + \alpha(y - x) && (5.12) \end{aligned}$$

where $\alpha \in [0,1]$. By continuity, there exists some $\alpha_0 \in (0,1)$ such that $x^{(\alpha_0)} \in \Omega$. By (5.10) we have

$$(Ax^{(\alpha_0)})_i \leq (Bx^{(\alpha_0)})_i.$$

Claim 1

$$(Ax)_i \leq (Ax^{(\alpha_0)})_i.$$

Proof of Claim 1: By (5.12), it follows that

$$(Ax^{(\alpha_0)})_i = (Ax)_i + \alpha(A(y-x))_i.$$

We also have $(y-x)_i = 0$ and $(y-x)_j \geq 0$ for $j \neq i$. Since A is Metzler, this implies that

$$\alpha(A(y-x))_i \geq 0.$$

Hence

$$(Ax)_i \leq (Ax^{(\alpha_0)})_i,$$

which proves the claim.

Claim 2

$$(Bx^{(\alpha_0)})_i \leq (By)_i.$$

Proof of Claim 2: Assume $(Bx^{(\alpha_0)})_i > (By)_i$.

Then by (5.11) it follows that

$$\alpha_0(By)_i + (1-\alpha_0)(Bx)_i > (By)_i.$$

Hence

$$(1-\alpha_0)(Bx)_i > (1-\alpha_0)(By)_i.$$

Dividing across by $1-\alpha_0$ and noting that $1-\alpha_0 > 0$ we get

$$0 > (B(y-x))_i$$

which is a contradiction since $(y-x)_i = 0$, $(y-x)_j \geq 0$ for $j \neq i$ and B is Metzler. So

$$(Bx^{(\alpha_0)})_i \leq (By)_i,$$

which proves the claim.

Using Claim 1 and Claim 2 it follows that

$$(Ax)_i \leq (Ax^{(\alpha_0)})_i \leq (Bx^{(\alpha_0)})_i \leq (By)_i,$$

i.e. the P-K conditions hold.

Case (ii) Suppose we have $x \in \Omega_A$, $y \in \Omega_B$ with $x \geq y$ and $x_i = y_i$. Again we have

$$\begin{aligned} c^T x &< 0 \\ c^T y &> 0 \end{aligned}$$

which implies

$$c^T(y-x) = c_1(y_1 - x_1) + ... + c_n(y_n - x_n) > 0.$$

Hence $i \in I_-^c \cup I_0^c$. For if we had $i \in I_+^c$ this would mean that

$$c^T(y-x) = c_1(y_1 - x_1) + ... + \underbrace{c_i(y_i - x_i)}_{=0} + ... + c_n(y_n - x_n) \leq 0,$$

since $y_j - x_j \leq 0$ for all j.

Since $i \in I_-^c \cup I_0^c$, it follows that

$$(Az)_i \geq (Bz)_i \tag{5.13}$$

for any $z \in \Omega$.

Write

$$x^{(\alpha)} = (1-\alpha)x + \alpha y \tag{5.14}$$

$$= x + \alpha(y - x) \tag{5.15}$$

where $\alpha \in [0,1]$. By continuity, there exists some $\alpha_0 \in (0,1)$ such that $x^{(\alpha_0)} \in \Omega$. By (5.10) we have

$$(Ax^{(\alpha_0)})_i \geq (Bx^{(\alpha_0)})_i.$$

Claim 3

$$(Ax)_i \geq (Ax^{(\alpha_0)})_i.$$

Proof of Claim 3: The argument is identical to the argument for Claim 1.

Claim 4

$$(By)_i \leq (Bx^{(\alpha_0)})_i.$$

Proof of Claim 4: The argument is identical to the argument for Claim 2.

By Claim 3 and Claim 4 it follows that

$$(Ax)_i \geq (By)_i,$$

i.e. the P-K conditions hold.

Conversely, let us assume that the P-K conditions hold. We shall now consider each case in turn.

Case (i) $i \in I_+^c$.

Let $v = (v_1, v_2, ..., v_{i-1}, 0, v_{i+1}, ..., v_n) \in \mathbb{R}^n_+$ where $v_j > 0$ if $i \neq j$. Suppose $z \in \Omega$. For $\delta > 0$, let us define x_δ and y_δ as follows.

$$\begin{aligned} x_\delta &= z - \delta v \\ y_\delta &= z + \delta v. \end{aligned} \tag{5.16}$$

Then, for all $\delta > 0$ we have

$$c^T x_\delta = c^T(z - \delta v) = -\delta c^T v < 0$$

$$c^T y_\delta = c^T(z + \delta v) = +\delta c^T v > 0$$

So $x_\delta \in \Omega_A$, $y_\delta \in \Omega_B$, $x_\delta \leq y_\delta$ and $(x_\delta)_i = (y_\delta)_i$.

Since the P-K conditions hold, we conclude that $(Ax_\delta)_i \leq (By_\delta)_i$. Letting $\delta \to 0$ we get $(Az)_i \leq (Bz)_i$.

Case (ii) $i \in I_-^c$.

Let $v = (v_1, v_2, ..., v_{i-1}, 0, v_{i+1}, ..., v_n) \in \mathbb{R}^n_+$. Suppose $z \in \Omega$. For $\delta > 0$, let us define x_δ and y_δ as follows.

$$\begin{aligned} x_\delta &= z - \delta v \\ y_\delta &= z + \delta v. \end{aligned} \tag{5.17}$$

Then, for all $\delta > 0$ we have

$$c^T x_\delta = c^T(z - \delta v) = -\delta c^T v > 0$$

$$c^T y_\delta = c^T(z + \delta v) = +\delta c^T v < 0$$

So $y_\delta \in \Omega_A$, $x_\delta \in \Omega_B$, $x_\delta \leq y_\delta$ and $(x_\delta)_i = (y_\delta)_i$.

Since the P-K conditions hold, we conclude that $(Ay_\delta)_i \geq (Bx_\delta)_i$. Letting $\delta \to 0$ we get $(Az)_i \geq (Bz)_i$.

Case (iii) $i \in I_0^c$.

Suppose $z \in \Omega$. For $\delta > 0$, let us define x_δ and y_δ as follows.

$$\begin{aligned} x_\delta &= z - \delta v \\ y_\delta &= z + \delta v. \end{aligned} \tag{5.18}$$

Furthermore, choose $v \geq 0$ such that $v_i = 0$, $v_j = 0$ whenever $c_j < 0$, and $v_j > 0$ otherwise.

Then

$$c^T x_\delta = -\delta c^T v < 0$$
$$c^T y_\delta = +\delta c^T v > 0$$

.

So we have $x_\delta \in \Omega_A$ and $y_\delta \in \Omega_B$ with $(x_\delta)_i = (y_\delta)_i$. Therefore $(Ax_\delta)_i \leq (By_\delta)_i$ since the Kamke conditions hold. Letting $\delta \to 0$ we get $(Az)_i \leq (Bz)_i$. (*)

Finally, suppose $v_i = 0$ and $v_j = 0$ whenever $c_j > 0$.

Then

$$c^T x_\delta = -\delta c^T v > 0$$
$$c^T y_\delta = +\delta c^T v < 0$$

.

So we have $x_\delta \in \Omega_B$ and $y_\delta \in \Omega_A$ with $(x_\delta)_i = (y_\delta)_i$. Therefore $(Ax_\delta)_i \geq (By_\delta)_i$ since the Kamke conditions hold. Letting $\delta \to 0$ we get $(Az)_i \geq (Bz)_i$. (**)

By (*) and (**), we have $(Az)_i = (Bz)_i$.

$\square$

Example. We will now use Proposition 5.4.1 to find an example of a simple three-dimensional bimodal system which satisfies the P-K conditions.

Let

$$c = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix}.$$

Let us consider the region Ω defined by

$$\Omega = \{x \in \mathbb{R}^n_+ : c^T x = 0\} \tag{5.19}$$

which divides $\mathbb{R}^n_+$ into two regions, Ω_A and Ω_B.

Let $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{pmatrix}$.

$I_c^+ = \{2\}$ and $I_c^0 = \{1, 3\}$.

For the PK-conditions to be satisfied, the matrix B must satisfy

$$(Ax)_1 = (Bx)_1 \tag{5.20}$$

$$(Ax)_2 \leq (Bx)_2 \tag{5.21}$$

$$(Ax)_3 = (Bx)_3 \tag{5.22}$$

for $x \in \Omega$.

Write $B^{(i)}$ for the i^{th} row of the matrix B, $i \in \{1, 2, 3\}$. Define $B^{(1)}$ and $B^{(3)}$ as

$$B^{(1)} = A^{(1)} + tc^T \tag{5.23}$$

$$B^{(3)} = A^{(3)} + tc^T, \tag{5.24}$$

where $t \in \mathbb{R}$.

Hence (5.20) and (5.22) above will be satisfied.

Finally, write $B^{(2)} = tc^T + v$ where $v > 0$. This will ensure that (5.21) above is satisfied.

So if we choose $t = 1/2$ and $p = \begin{pmatrix} 1 & 1/2 & 1/8 \end{pmatrix}$ we get $B = \begin{pmatrix} -1/2 & 7/4 & 11/8 \\ 5/2 & -15/4 & 5/2 \\ 5/2 & 3/4 & -3/4 \end{pmatrix}$ and using our results we conclude that the system defined as in (1) above, with the matrices A and B as in this example, satisfies the P-K conditions.

5.5 Monotonicity of Piecewise Smooth Systems

In this section we investigate the relationship between the P-K conditions and monotonicity for systems such as (5.1). Our first result shows that the P-K conditions are necessary for (5.1) to be monotone.

5.5.1 Necessity of P-K conditions

Proposition 5.5.1. *Suppose the dynamical system given by (5.1) is monotone. Then the P-K conditions hold.*

Proof. We will now show that if (5.1) is monotone then the P-K conditions must hold. We will do this by assuming that the P-K conditions do not hold and this will mean that monotonicity is violated.

Assume the P-K conditions do not hold.

Case (i): Suppose there exist $x \in \Omega_A, y \in \Omega_B$ with

$$\begin{aligned} x &\leq y \\ x_i &= y_i \end{aligned}$$

such that

$$(Ax)_i > (By)_i.$$

Next write

$$d(t) := x_i(t,x) - x_i(t,y).$$

d is smooth ($\mathcal{C}^1$) in $(0,\delta)$ for sufficiently small $\delta > 0$. We also have

$$d(0) = 0,$$

as $x_i = y_i$, and

$$\dot{d}(0) = (Ax)_i - (By)_i > 0.$$

But

$$\dot{d}(0) = \lim_{t\to 0} \frac{d(t) - d(0)}{t} = \lim_{t\to 0} \frac{d(t)}{t} > 0$$

which means that

$$d(t) > 0$$

for $t \in (0, \delta_1)$ for some $\delta_1 > 0$. i.e.

$$x_i(t,x) > x_i(t,y)$$

for $t \in (0, \delta_1)$. This means that (5.1) is not monotone. Thus, if (5.1) is monotone, part (i) of the PK-conditions must hold. The argument to show that (ii) is necessary is identical. □

5.5.2 Sufficient conditions for monotonicity of (5.1) in certain cases.

Proposition 5.4.1 implies that if

$$I_c^0 = \{1, 2, ..., n\}$$

the P-K conditions are equivalent to $Ax = Bx$ on Ω. Note, this can only happen if $n \geq 4$.

As n increases, the number of vectors c for which this happens increases. Furthermore, Proposition 5.4.1 tells us that the P-K conditions are automatically satisfied when we have equality on the bounding hyperplane. We shall focus our attention now on the case where

$$Ax = Bx$$

for all x such that

$$c^T x = 0.$$

As this makes f continuous, the concept of solution becomes much simpler.

5.6 Monotonicity and Continuity

5.6.1 Lipschitz Continuity of (5.1)

We shall show that under the assumption

$$Ax = Bx$$

for $x \in \Omega$, f in (5.1) is Globally Lipschitz.

Consider (5.1) and suppose that $x, y \in \mathbb{R}^n$. Let us examine $\|f(x) - f(y)\|$ where $\| - \|$ is the l^2 norm (we could choose any norm as we are in a finite dimensional linear space).

Case 1: $x, y \in \Omega_A$.

$$\|f(x) - f(y)\| = \|Ax - Ay\| \leq \|A\|\|x - y\|,$$

where $\|A\|$ is the induced matrix norm.

Case 2: $x, y \in \Omega_B$.

$$\|f(x) - f(y)\| \leq \|B\|\|x - y\|.$$

Case 3: $x \in \Omega_A, y \in \overline{\Omega}_B$ (or vice-versa).

There exists $\alpha \in [0, 1]$ such that

$$z := x + \alpha(y - x) \in \Omega.$$

We also have

$$\begin{aligned} x &= z + \alpha(x - y) \\ y &= z + (1 - \alpha)(y - x) \end{aligned}$$

so that x, y, z are collinear.

Also

$$\begin{aligned} x - z &= \alpha(x - y) \\ y - z &= (1 - \alpha)(y - x). \end{aligned}$$

Consider $\|f(x) - f(y)\|$.

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|Ax - Az\| + \|Bz - By\| \\
&\leq \|A\|\|x - z\| + \|B\|\|z - y\| \\
&\leq K(\|x - z\| + \|y - z\|) \\
&= K\|x - y\|,
\end{aligned}$$

where

$$K = max(\|A\|, \|B\|).$$

This shows that (5.1) is globally Lipschitz. This guarantees the existence and uniqueness of solutions of (5.1) defined on $[0, \infty)$ [8].

5.6.2 Solutions of (5.1) - A closer look.

Let us now look more closely at the form the solution of (5.1) takes under the assumption that

$$Ax = Bx$$

for $x \in \Omega$.

Suppose we have an initial condition $x_0 \in \Omega_A$. There exists $T > 0$ such that dynamics are given by $\dot{x}(t) = Ax(t)$ for $t \in [0, T]$ and the corresponding solution is given by $x(t, x_0) = e^{At}x_0$. This will remain the case until $x(t, x_0)$ hits the hyperplane, i.e. until such time as

$$c^T x(t, x_0) = 0.$$

Similarly, if we have an initial condition $x_0 \in \Omega_B$ then dynamics will evolve according to $\dot{x}(t) = Bx(t)$ with corresponding solution $x(t) = e^{Bt}x_0$ for as long as the trajectory remains in Ω_B.

By assuming that $Ax = Bx$ on Ω, we considerably simplify the situation for initial conditions on the boundary. First note that a conventional sliding mode such as in Figure 4.2 cannot occur. For, consider $x_0 \in \Omega$. According to the conventional definition for sliding modes, for a sliding mode to occur, we would need to have

$$c^T A x_0 \neq 0 \quad c^T B x_0 \neq 0$$
$$c^T A x_0 > 0 \quad c^T B x_0 < 0.$$

However this cannot happen since we have assumed that $Ax = Bx$ for $x \in \Omega$.

We next present some technical results to highlight solutions of (5.1) when $x_0 \in \Omega$.

Proposition 5.6.1. *Suppose* $x_0 \in \Omega$.

Define

$$k + 1 = min\{p \geq 0 : c^T A^p x_0 \neq 0\}.$$

Then (i) if $k < \infty$,

$$A^p x_0 = B^p x_0$$

for $0 \leq p \leq k + 1$.

(ii) if $k = \infty$ *then*

$$\begin{aligned} e^{At}x_0 &= e^{Bt}x_0 \;\forall\; t \in \mathbb{R} \\ e^{At}x_0 &\in \Omega \;\forall t \;\in \mathbb{R} \end{aligned}$$

Proof. (i) If $k < \infty$, then

$$c^T x_0 = c^T A x_0 = c^T A^2 x_0 = ... = c^T A^k x_0 = 0,$$

i.e.

$$A^p x_0 \in \Omega$$

for $0 \leq p \leq k$.

Since $x_0 \in \Omega$, we have

$$Ax_0 = Bx_0.$$

But we also have $Ax_0, Bx_0 \in \Omega$. This implies

$$A^2x_0 = B^2x_0,$$

and it follows inductively that

$$A^px_0 = B^px_0$$

for $0 \leq p \leq k+1$.

(ii) If $k = \infty$, then

$$c^TA^px_0 = 0 \tag{5.25}$$

for all p. It follows from the same argument as in (i) that

$$A^px_0 = B^px_0,$$

for $0 \leq p \leq \infty$, and hence

$$x_0 + Atx_0 + \frac{(At)^2}{2!}x_0 + ... = x_0 + Btx_0 + \frac{(Bt)^2}{2!}x_0 + ...,$$

i.e.

$$e^{At}x_0 = e^{Bt}x_0$$

for all $t \in \mathbb{R}$.

Furthermore, it follows from (5.25) and by looking at the power series expansions of $e^{At}x_0$ and $e^{Bt}x_0$ that

$$c^Te^{At}x_0 = c^Te^{Bt}x_0 = 0$$

for all $t \in \mathbb{R}$. Hence

$$e^{At}x_0 \in \Omega$$

for all $t \in \mathbb{R}$.

$\square$

Comment: From our previous result, if $k = \infty$, it follows that

$$e^{At}x_0 = e^{Bt}x_0$$

for all t and

$$e^{At}x_0 \in \Omega,$$

where $x_0 \in \Omega$. In the next result, we consider the case where $k < \infty$.

Proposition 5.6.2. *Let k be defined as in Propositon 5.6.1. Suppose $x_0 \in \Omega$ and $k < \infty$. Assume*

$$c^T A^{k+1} x_0 > 0.$$

Then

$$c^T e^{At} x_0 > 0$$

and

$$c^T e^{Bt} x_0 > 0$$

for $t \in (0, \delta)$, for some $\delta > 0$.

Proof. Taking a power series expansion of $c^T e^{At} x_0$ we obtain

$$\begin{aligned} c^T e^{At} x_0 &= c^T x_0 + c^T A t x_0 + c^T \frac{(At)^2}{2!} x_0 + ... + c^T \frac{(At)^k}{k!} x_0 + c^T \frac{(At)^{k+1}}{(k+1)!} x_0 + ... \\ &= c^T \frac{(At)^{k+1}}{(k+1)!} x_0 + c^T \frac{(At)^{k+2}}{(k+2)!} x_0 ... \\ &= t^{k+1} (c^T \frac{(A)^{k+1}}{(k+1)!} x_0 + t(c^T \frac{(A)^{k+2}}{(k+2)!} x_0 + ...)). \end{aligned} \tag{5.26}$$

We can choose $\delta_1 > 0$ so that

$$c^T \frac{(A)^{k+1}}{(k+1)!} x_0 > t(c^T \frac{(A)^{k+2}}{(k+2)!} x_0 + ...),$$

for $t \in (0, \delta_1)$, and so

$$c^T e^{At} x_0 > 0.$$

Using Proposition 5.6.1 the same argument shows that there exists $\delta_2 > 0$ such that for $t \in (0, \delta_2)$,

$$c^T e^{Bt} x_0 > 0.$$

Choose $\delta = min(\delta_1, \delta_2)$. Then

$$\begin{aligned} c^T e^{At} x_0 &> 0, \\ c^T e^{Bt} x_0 &> 0 \end{aligned}$$

for all $t \in (0\delta)$.

□

The form of solutions of (5.1) for intitial conditions on Ω is now clear.

(i) If $k = \infty$, then the trajectory remains on Ω for all time t.

(ii) If $k < \infty$ and $c^T A^{k+1} x_0 < 0$, then the trajectory moves into Ω_A.

(iii) If $c^T A^{k+1} x_0 > 0$, the the trajectory moves into Ω_B.

5.6.3 Monotonicity of (5.1)

We now prove that if $A, B \in \mathbb{R}^{n \times n}$ are Metzler, then under the assumption

$$Ax = Bx$$

on Ω, (5.1) is monotone. First recall that

$$\begin{aligned} x &\leq y \\ x_i &= y_i \end{aligned}$$

implies

$$f_i(x) \leq f_i(y),$$

and that

$$\dot{x}(t, x_0) = f(x(t, x_0) \tag{5.27}$$

for all $t \geq 0$.

Proposition 5.6.3. *Let $A, B \in \mathbb{R}^{n \times n}$ be Metzler. Suppose the P-K conditions hold for (5.1) and $Ax = Bx$ for $x \in \Omega$. Then (5.1) is monotone.*

Proof. Suppose we have $x_0, y_0 \in \mathbb{R}^n_+$ with

$$x_0 \leq y_0.$$

For $\delta > 0$, define g_δ as

$$g_\delta(x) = f(x) + \delta v,$$

where $v \gg 0$. Let y_δ be the solution of

$$\begin{aligned} \dot{y}_\delta(t) &= g_\delta(y_\delta(t)), \\ y_\delta(0) &= y_0 + \delta v. \end{aligned}$$

Let $x(t, x_0)$ be the solution of

$$\begin{aligned} \dot{x}(t, x_0) &= f(x(t, x_0)), \\ x(0) &= x_0. \end{aligned}$$

Clearly

$$x(0, x_0) \ll y_\delta(0).$$

We claim that

$$x(t, x_0) \ll y_\delta(t)$$

for all $t > 0$.

Suppose our claim is false. Then there exists $t_0 > 0$ such that

$$\begin{aligned} x_i(t_0, x_0) &= (y_\delta)_i(t_0), \text{ for some } i, \\ x(t_0, x_0) &\leq y_\delta(t_0), \\ x(s, x_0) &\ll y_\delta(s) \end{aligned}$$

for $0 \leq s \leq t_0$.

Since

$$x_i(s, x_0) - (y_\delta)_i(s) < 0$$

for $0 \leq s < t_0$, it follows that

$$\frac{d}{dt}(x_i(t_0, x_0)) \geq \frac{d}{dt}((y_\delta)_i(t_0)),$$

i.e.

$$f_i(x(t_0, x_0)) \geq f_i(y_\delta(t_0)) + \delta v_i.$$

Since the P-K conditions hold we must have

$$\begin{aligned} f_i(x(t, x_0)) &\leq f_i(y_\delta(t_0)) && (5.28) \\ &< f_i(y_\delta(t_0)) + \delta v_i && (5.29) \end{aligned}$$

which is a contradiction.

Hence,

$$x(t, x_0) \ll y_\delta(t)$$

for all $t > 0$.

Now, let $\delta \to 0$. As f and g_δ are Lipschitz for any $\delta > 0$, it follows from the continuous dependence of solutions on parameters and initial conditions (see Theorem 55, Appendix C of [50]) that

$$x(t, x_0) \leq x(t, y_0).$$

This completes the proof. $\square$

Finally for this section, we consider some implications of Proposition 5.6.3.

Lemma 5.6.1. *If $Ax = Bx$ on Ω, then A and B differ by a rank 1 matrix.*

Proof. Suppose

$$Ax = Bx$$

for all $x \in \mathbb{R}^n_+$ such that $c^T x = 0$. Then

$$(A - B)x = 0$$

for all x satisfying $c^T x = 0$. The kernel of the matrix $A - B$ is the hyperplane defined by $c^T x = 0$ and is of dimension $n - 1$. Therefore the rank of $A - B$ is 1 by the rank-nullity theorem. Any rank 1 matrix can be written as

$$xy^T$$

for vectors $x, y \in \mathbb{R}^n$. Since the kernel of $A - B$ is given by

$$\{x : c^T x = 0\},$$

it follows that

$$A - B = bc^T$$

for some vector $b \in \mathbb{R}^n$. □

Note also, that if A and B differ by a rank 1 matrix of the form bc^T for some $b \in \mathbb{R}^n$, i.e.

$$A = B + bc^T,$$

then it follows that

$$Ax = (B + bc^T)x = Bx + bc^T x = Bx \tag{5.30}$$

for x such that $c^T x = 0$.

Based on this discussion, we can now state the following proposition:

Proposition 5.6.4. *Let $A, B \in \mathbb{R}^{n\times n}$ be Metzler. If $B = A + bc^T$, then the piecewise system (5.1) is monotone.*

Furthermore, in this case, we can add to Proposition 5.6.4, and state the following result:

Proposition 5.6.5. *Let $A, B \in \mathbb{R}^{n\times n}$ be Metzler. If $I_0^c = \{1, ..., n\}$, (5.1) is monotone if and only if there exists a vector b such that $B = A + bc^T$.*

Proof. If $B = A + bc^T$, (5.1) is monotone by Proposition 5.6.5.

Conversely, if (5.1) is monotone then Proposition 5.5.1 implies that the P-K conditions hold. It follows from Proposition 5.4.1 that $I_0^c = \{1, ..., n\}$ and that

$$Ax = Bx,$$

for $x \in \Omega$. Lemma 5.6.1 now gives the result.

□

Finally, we construct a system which according to Proposition 5.6.5 is monotone.

Example 5.6.1. *A piecewise linear system in $\mathbb{R}^4$ which is monotone.*

Let $c^T = \begin{pmatrix} 1 & -1 & 2 & -2 \end{pmatrix}$ so that $I_0^c = \{1, 2, 3, 4\}$.

Define A as

$$A = \begin{pmatrix} -1 & 1 & 2 & 3 \\ 1 & -2 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 2 & 3 & -3 \end{pmatrix}.$$

Let

$$b = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}.$$

Hence, if

$$B = A + bc^T = \begin{pmatrix} \frac{-3}{2} & \frac{3}{2} & 3 & 2 \\ \frac{3}{4} & \frac{-7}{4} & \frac{3}{2} & \frac{3}{2} \\ \frac{2}{3} & \frac{7}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{5}{2} & 4 & -4 \end{pmatrix},$$

then the system (5.1) is monotone.

Chapter 6

Conclusion

In this Chapter, we summarise and briefly discuss the work done in the thesis. We then outline various open problems which provide scope for future research in this area.

6.1 Concluding Remarks

Non-smooth systems occur ubiquitously in applications ranging from engineering problems to biological models. The motivation for this thesis was to investigate the monotonicity properties of piecewise smooth systems. In particular, we considered the following question. Given a state-dependent switched linear system in which each of the component systems is monotone, what can we say about the monotonicity of the whole system?

To address this, in Chapters 2-4 we discussed the mathematics behind positive LTI systems and switched systems, outlining some of the major results in these areas and drawing from some more recent work, such as [30], [37]. In Chapter 3, we also discussed monotone systems and indicated how monotonicity can be used in deriving stability results for such systems. We also discussed the Kamke conditions and quoted an important result from [49] which states that the Kamke conditions are equivalent to monotonicity for smooth systems. In Chapter 4, we highlighted some of the issues which can arise in the study of state-dependent switched linear systems. In particular, we described how solutions for such systems are defined.

In Chapter 5, our aim was to introduce new results pertaining to the monotonicity of state-dependent switched linear systems. For simplicity, we considered a bimodal system in which the state-space was partioned via hyperplane through the origin. Dynamics in this setting were given by (5.1). We extended the Kamke conditions to take into account this new setting, and in doing so, formulated the P-K conditions (Piecewise-Kamke) for (5.1). We then showed in Proposition 5.5.1 that the P-K conditions were necessary for (5.1) to be monotone. Our next goal was to provide a simple algebraic characterisation of the P-K conditions which we succeeded in doing with Propostion 5.4.1. We next showed in Proposition 5.6.3, (5.1) is monotone provided that the vector fields agree on the separating hyperplane. This implied that for certain hyperplanes, continuity of (5.1) is equivalent to monotonicity. The work described in Chapter 5 gives rise to many interesting questions which could become the focus for future research. We discuss some of these in the next section.

6.2 Directions for Future Work

The first natural extension to the work here is to determine whether or not the P-K conditions are equivalent to monotonicity in general. We have shown this to be true when the hyperplane is defined by a vector c with $I_0^c = \{1, ..., n\}$. This ruled out sliding modes, which made things considerably simpler. A more general approach would have to take into account sliding modes, as described in Chapter 4. This introduces technical difficulties arising from the form of solutions to (5.1). In particular, results from the study of Differential Inclusions guarantee the existence of solutions satisfying the inclusion almost everywhere. The possibility that a solution doesn't satisfy (5.1) at all times t creates potential problems for the argument used in the proof of Proposition 5.6.3.

Another extension would be to generalise the form of the bounding hyperplane. General hyperplanes, of the form

$$c^T x + b = 0,$$

where $b \neq 0$, which do not go through the origin could be considered. Indeed, one could consider more general surfaces for partitioning $\mathbb{R}^n_+$, and not necessarily hyperplanes, and investigate the monotonicity properties of systems whose operating regions are defined by these surfaces.

Thus far, we have only been considering bimodal systems. However, many systems which occur in practical applications are multimodal in nature, such as Examples 2.2.2, 2.2.3. Furthermore, the bounding hyperplanes in these examples do not go through the origin so it is important that results such as Proposition 5.6.3 are extended to take such situations into account.

Finally, one could investigate whether or not the results presented in Chapter 5 could be extended to nonlinear state-dependent switched systems.

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