Gravitating BPS Monopoles in all $d = 4p$ Spacetime Dimensions

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Abstract

We have constructed, numerically, both regular and black hole static solutions to the simplest possible gravitating Yang-Mills–Higgs (YMH) in $4p$ spacetime dimensions. The YMH systems consist of $2p-$th power curvature fields without a Higgs potential. The gravitational systems consist of the 'Ricci scalar' of the $p-$th power of the Riemann curvature. In 4 spacetime dimensions this is the usual Einstein-YMH (EYMH) studied in [1, 2], whose qualitative results we emulate exactly.

1 Introduction

Gravitating monopoles were studied intensively in [1, 2, 3], subsequent to the discovery of the finite energy regular solutions [4] to the gravitating Yang–Mills (EYM) system. These regular EYM solutions were soon extended to black holes [5]. EYM solutions in $3 + 1$ dimensional spacetime are reviewed in [6] and an exhaustive analysis is given in [7]. Along with the introduction of the (dimensionful) Higgs field that leads to the construction of regular and black monopoles [1, 2, 3] in the gravitating EYM-Higgs (EYMH) system, the (dimensionful) cosmological constant was introduced. The case of positive cosmological constant is reviewed quite adequately in [6] and that of negative is in the review

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[8]. The latter led to very interesting new features of the solutions and is, in some sense, an alternative to the Higgs field.

EYM solutions in (higher) dimensions $D + 1$ ($D \geq 4$) were considered only relatively recently. Here there are two possibilities of identifying the D spacelike dimensions: Either as asymptotically Minkowski with S^{D-1} boundary, or, with boundary $S^{D-N-1} \times R^N$ in which case the N codimensions are frozen as in the case of the $z-$coordinate of the Abrikosov-Nielsen-Olesen vortex. In the case of S^3 boundary for $D = 4$ it was found in [9] that the energy of the regular solution is infinite, and in [10] (with negative cosmological constant) it turned out that the energy of the black hole is also infinite. In addition, in [9] solutions with boundary $S^2 \times R^1$ $D = 4$ with finite energy *per unit* length along R^1 were constructed, the total energy remaining infinite. This is not surprising because the usual EYM system in $D \geq 4$ does not have the requisite scaling properties for there to exist finite energy solutions. This obstacle is circumvented by extending the definition of the EYM system to feature higher order YM curvature terms with the appropriate scaling.

Suitably defined EYM systems featuring higher order YM curvature[1] terms in all $d = D + 1$ spacetime dimensions were studied in [11, 12] without cosmological constant, and in [13, 14] with cosmological constant. These latter differ from their $d = 3+1$ dimensional EYM analogues [4, 6], which are unstable (sphaleronic) and have no gravity decoupling limits, in that some of them do have gravity decoupling limits which are stabilised by Chern–Pontryagin (instantonic) charges. In this respect these are more akin to the gravitating monopoles, and like them these systems exhibit one or more dimensionful constants that cannot be scaled away and hence parametrise the ensuing solutions. As a result they feature certain bifurcation properties like the monopoles but some of these exhibit in addition what were named *conical singularities* in [15], where this analysis was carried out. To date no gravitating monopoles in dimensions higher than $d = 3 + 1$ are constructed and it is the aim of the present work to do that, in the simplest class of YMH models defined in $4p$ spacetime dimensions. Before going into the details of this choice, we point out that this is a very interesting restriction since in our previous study of higher dimensional EYM solutions in [15], it was found that the qualitative features of these repeated *modulo* every $4-$dimensions. Like in all the work on gravitating gauge fields quoted above, our solutions take into account the backreaction of gravity on the YMH fields.

Before considering the possible choices of suitable higher dimensional YMH systems, it is in order to consider the particular interest in gravitating mono-poles in higher dimensions, **a)** on a technical level, and **b)** from the viewpoint of

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[1]Higher order Riemann curvature terms can also be introduced, but since all studies are in practice carried out for systems subject to symmetries, such terms of high enough order Riemann curvature vanish due to the symmetry imposed. Their inclusion is thus unnecessary. Any nonvanishing such terms included do not play an essential role, but rather result only in a quantitative difference.

applications.

a) On a technical level, it is the only way which enables the construction of static solutions of EYM systems in spacetime dimensions higher than *four*, which describe a nonvanishing 'non Abelian electric' connection A_0. In this respect EYM solutions in the presence of a negative cosmological constant differ in spacetime dimension $d = 4$, from those in $d \geq 5$. In $d = 4$ the presence of the negative cosmological constant results in such asymptotic properties, which in contrast to the asymptotically flat case, allow for nonvanishing electric field. This is a consequence of the fact that the 'magnetic' connection of these solutions can [8], among other possibilities, behave as a *half pure gauge* just like a monopole. We have verified that the corresponding EYM systems (with negative cosmological constant) in $d \geq 5$ support 'magnetic' connections that behave asymptotically as *pure gauge*, like their asymptotically flat counterparts. Hence they do not support a nonvanishing electric field.

When a Higgs field is introduced however (irrespective of the presence of a cosmological constant), the situation changes. Already in flat space, EYMH systems automatically support dyonic solutions, *e.g.*, in all even dimensional spacetimes for models employed in [16]. Clearly, when the gravitational force is switched on, the 'electric' field of the resulting EYMH solution will persist.

b) From the viewpoint of applications, like their higher dimensional EYM analogues, they are expected to be relevant to various aspects of the study of $D-$branes. Some examples in the literature concern monopoles in string theory [17, 18], and selfgravitating supersymmetric solitons in [19]. In the former examples [17, 18] only the Yang-Mills system in the absence of Higgs fields appears, so that their higher dimensional extensions will feature the gauge field configurations described by the Yang-Mills hierarchy.[2] The work of [19] on the other hand concerns monopoles of the usual YMH system in the *spacelike* subspace. Thus its higher dimensional extensions will involve field configurations of the YM-Higgs hierarchies (see below), in other words higher dimensional monopoles. What is more, is that the monopoles in [19] satisfy the selfduality constraint in the spacelike dimensions, and that the higher dimensional extensions of the YMH systems employed here are chosen precisely such that the corresponding extended Bogomol'nyi constraints are likewise in force.

When choosing a YMH system in arbitrary (Euclidean) dimension D supporting a monopole solution we are faced with a plethora models. The most efficient way of constructing these YMH models is *via* dimensional descent [21] from a YM system [20] on $D + N$ (even) dimension. Integrating out the coordinates on the (compact) N codimensions results in the $D-$dimensional residual YMH theory supporting monopoles, whose topological (monopole) charge is the descendent of

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[2]In [17, 18] instanton fields of the usual YM system are exploited, while their higher dimensional extensions would employ the corresponding instantons of the YM hierarchy introduced in [20].

the $\frac{1}{2}(D+N)$−th Chern–Pontryagin charge. We shall not dwell on the detailed properties of these D−dimensional monopoles here, save to emphasise their most relevant feature pertinent to the present work. Because of the high degree of nonlinearity of the extended selfuality equations [20] in $D+N$ dimensions, the descendent Bogomol'nyi equations in D dimensions are *overdetermined* [22] and in general cannot be saturated. It turns out that these Bogomol'nyi equations can be saturated only when the descent is over $N = 1$ and $N = D - 2$ codimensions, the latter case being irrelevant here. Since the Euclidean dimensions $D + N$ are only even, then the residual Euclidean space is $D - 1$ dimensional, restricting us to models in even spacetime dimensions $d = (D - 1) + 1$ only.

In the present work we have restricted our attention to $d = 4p$ dimensional spacetimes because the YM hierarchy in $D + 1 = 4p$ dimensions is scale invariant and does not feature an additional dimensionful constant.[3] This results in the simplest possible residual YMH models, keeping a tight analogy with the 't Hooft–Polyakov monopole (in the BPS limit). Like the latter, these models feature only one dimensional constant, namely the Higgs VEV with inverse dimension of a length. A more direct construction, with the sole criterion of achieving a topological lower bound and not applying dimensional descent, was employed in [24, 25, 16], but these models always feature an extra dimensional constant in addition to the Higgs VEV, considerably complicating the numerical analysis in the gravitating case. Thus we have eschewed such models here.

Finally, we state the action densities to be employed. It is convenient to express the connection and the Higgs field in terms of the chiral $so(4p)$ representation matrices

$$\Sigma_{\mu\nu} = (\Sigma_{ij}, \Sigma_{i,D+1}) , \qquad \mu = 0, i , \quad i = 1, 2, \ldots, D , \tag{1}$$

such that the connection A_i takes its values in the algebra of $SO(4p-1)$, namely Σ_{ij}, and the Higgs field is

$$\Phi = \phi^i \, \Sigma_{i,D+1} . \tag{2}$$

In terms of the curvature $F \equiv F(2)$ of A and the covariant derivative $D\Phi$, the action densities of this hierarchy of YMH models in $4p$ spacetime are defined as

$$\mathcal{S}_{\text{matter}}^{(4p)} = \text{Tr} \left[\frac{1}{2\,(2p)!} F(2p)^2 - \frac{1}{2\,(2p-1)!}(F(2p-2) \wedge D\Phi)^2 \right] , \tag{3}$$

the $2k$−form $F(2k)$ being the k−fold totally antisymmetrised product of $F = F(2)$. The flat space static dyons of these systems were constructed in [16]. It is the monopole solutions in [16] that are gravitated in the present work, in the spirit of [1, 2].

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[3]YM systems on R^{2p+2} supporting instantons involve at least one additional dimensionful constant (see *e.g.*, [23]) whose dimensional descendent therefore will feature one more dimensional constant in addition to the Higgs VEV.

A pertinent comment at this point is that had we chosen the Higgs kinetic term in (3) to be $(F(2k-2) \wedge D\Phi)^2$, with $k \neq p$ in $2(p+k)$ dimensions, the solutions would still be topologically stable [16] but now a new dimensional constant will appear in (3). In this respect such systems would be more akin to the higher dimensional models studied in [15] and their gravitating solutions could then exhibit *conical fixed points*. The fixed point analysis for such selfgravitating YMH has not been carried out to date and it promises to be appreciably more involved than the corresponding analysis in [15], basically because now there is a Higgs field function in addition to the gauge field function. We have eschewed this choice here keeping strictly the analogy with the gravitating monopoles in $3+1$ dimensions, and encounter no solutions featuring *conical fixed points*. We will return to this question elsewhere.

To complete the definition of the gravitating YMH systems, the gravitational Lagrangian must be specified. Restricting to Levi–Civita connections, the hierarchy of Einstein systems in $d = 2(p+q)$ dimenional spacetimes are

$$e\, R_{(p,q)} = \langle E(2p), R(2p) \rangle \,, \tag{4}$$

where $E(2p) = {}^\star e(2q)^\star$ is the double–Hodge dual of $e(2q)$, the $2q-$fold anti-symmetrised product of the *Vielbein* fields. e in (4) is the determinant of the *Vielbein* and $R(2p)$ is the $2p-$form Riemann curvature. The $p = 1$ member of the hierarchy (4) is the Einstein–Hilbert Lagrangian in $d = 2(q+1)$, $p = 2$ the Gauss–Bonnet system in $d = 2(q+2)$, *etc.* Our choice of Lagrangian employed for gravitating the YMH system (3) for given p is the $p = q = \frac{d}{4}$ member of (4), like in the $p = 1$ case studied in [1, 2]. One can of course choose the $p = 1$ member of (4) with all YMH actions (3) but in that case the relative dimensionality of the YMH and gravitational Lagrangians would be $[L^{2(2p-1)}]$ instead of $[L^{2p}]$, which then would match with the $p = q = 1$ case studied in [1, 2], only for $p = 1$. This would result in the dilution of the analogy with the latter at least on the quantitative level. This choice was made previously in the study of the EYM systems in $4p$ dimensional spacetimes in [26], where it resulted in the tight similarity in the qualitative properties of the solutions of the $p-$th YM system gravitated with the $p(=q)-$th Einstein systems in all $4p$ dimensions.

The hierarchy of monopoles to be studied here is that of static and spherically symmetric (in $4p - 1$ space dimensions) solutions to the equations arising from the action densities

$$\mathcal{S}^{(4p)} = e\left(R_{(p,p)} + \mathcal{S}^{(4p)}_{\text{matter}}\right) \,. \tag{5}$$

2   Ansatz, Action, and Differential Equations

Using the spherically symmetric metric Ansatz in 'Schwarzschild' coordinates

$$ds^2 = A^2(r)\mu(r)dt^2 - \frac{dr^2}{\mu(r)} - r^2 d\Omega^2_{(d-2)} \,, \tag{6}$$

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together with a generalised 't Hooft–Polyakov Ansatz for the magnetic monopole

$$A_i = \Sigma_{ij}\frac{x^j}{r^2}\Big(1 - w(r)\Big) , \qquad \phi^i = \frac{x^i}{r}h(r) . \tag{7}$$

we obtain the reduced action

$$S = \frac{p(d-2)!}{(d-2p-1)!}(\tilde{\kappa}_p S_G + \tilde{\tau}_p S_M) , \tag{8}$$

with $\tilde{\kappa}_p = \kappa_p/2^{2(p-1)}$, $\tilde{\tau}_p = \tau_p/(2p)!$, and

$$S_G = -\frac{1}{2p}\int dr\, A\frac{d}{dr}\Big(r^{d-2p-1}(1-\mu)^p\Big) , \tag{9a}$$

$$S_M = \int dr\, A\, r^{d-4p}\bigg[W^{p-1}\mu\Big(\frac{dw}{dr}\Big)^2 + \frac{d-2p-1}{2p}\frac{W^p}{r^2}$$
$$+ \frac{\mu}{2p}\Big(r\frac{dH_p}{dr}\Big)^2 + (d-2p-1)w^2H_p^2\bigg] , \tag{9b}$$

where $W = (w^2 - 1)^2$ and $H_p = (w^2 - 1)^{p-1}h$.

A Derrick-type scaling argument shows that static finite energy solutions of the field eqs. derived form the action (8) can only exist in spacetime dimensions d with

$$2p + 1 < d < 4p + 1 , \tag{10}$$

and indeed in [26] the EYM system has been studied for all values in this range. The presence of the Higgs field, however, requires $d = 4p$.

The action (8) depends on three dimensionful parameters, $\tilde{\kappa}_p$, $\tilde{\tau}_p$, and the vacuum expectation value $\eta = \lim_{r\to\infty} h(r)$ of the Higgs field, with the dimensionless ratio

$$\alpha = \eta\left(\frac{\tilde{\tau}_p}{\tilde{\kappa}_p}\right)^{\frac{1}{2p}} . \tag{11}$$

Since an overall factor in front of the action has no effect on the eqs. of motion, we can rescale r such that either

$$S = \text{const.} \cdot (S_G + \alpha^{2p}S_M) , \qquad \eta = 1 , \tag{12}$$

resulting in α-dependent eqs. of motion with α-independent boundary conditions, or

$$S = \text{const.} \cdot (S_G + S_M) , \qquad \eta = \alpha , \tag{13}$$

resulting in α-independent eqs. of motion with an α-dependent boundary condition.

These two formulations are clearly equivalent as long as $\alpha \neq 0$. In the limit $\alpha \to 0$ they are, however, quite different. The first alternative describes

a YMH system in a fixed gravitational background, either flat (no gravity) or a Schwarzschild type black hole. The second alternative describes the (self gravitating) EYM system of [26].

In the following we will mostly use the second formulation, but will occasionally refer to the first formulation as 'unscaled variables'.

The dependence of the action (8) on $A(r)$ suggests to define the 'mass function'

$$M(r) = \frac{4p + 1 - d}{2p} r^{d-2p-1}(1 - \mu)^p , \qquad (14)$$

that has a finite limit $M = \lim_{r \to \infty} M(r)$ for asymptotically flat solutions with finite energy.

For the Reissner-Nordström solution with $w \equiv 0$ and $h \equiv \alpha$ the mass function satisfies

$$\frac{dM(r)}{dr} = \frac{d - 2p - 1}{2p}(4p + 1 - d)r^{-(4p+2-d)} , \qquad (15a)$$

$$M(r) = M - \frac{d - 2p - 1}{2p} r^{-(4p+1-d)} . \qquad (15b)$$

The normalization of $M(r)$ in Equ. (14) is chosen such that $M = 1$ for the extremal Reissner-Nordström solution. The limit $\alpha \to 0$ (in the original, unscaled variables) yields the generalised Schwarzschild solutions with constant $M(r) = M$, and thus

$$\mu(r) = 1 - \left(\frac{2pM}{4p + 1 - d}\right)^{1/p} r^{-(d-2p-1)/p} . \qquad (16)$$

2.1 New Variables – General

The metric Ansatz (6) in terms of Schwarzschild coordinates used so far can not describe situations where the radius r of $D - 1$-spheres first increases and then decreases, as in the solutions studied in [27, 28, 29] with spaces of spherical topology. Even for asymptotically flat spaces with monotonically increasing r, the equations of motion resulting from the action (8) are singular when $\mu(r) = 0$, i.e., at a horizon as well as for certain 'critical' solutions with a double zero of $\mu(r)$. Following [2], we avoid this coordinate singularity by the most general static, spherically symmetric metric Ansatz

$$ds^2 = e^{2\nu(\tau)}dt^2 - e^{2\lambda(\tau)}d\tau^2 - r^2(\tau)d\Omega_{(d-2)}^2 , \qquad (17)$$

with r now a function of τ, and substituting

$$A \, dr = e^{\nu + \lambda} \, d\tau , \qquad \mu = (e^{-\lambda}\dot{r})^2 , \qquad (18)$$

with the notation $\dot{x} = dx/d\tau$.

In order to obtain a system of first order differential equations (dynamical system), we introduce additional variables n, κ, u, and V_p

$$e^{-\lambda}\dot{r} = n \ , \tag{19a}$$

$$re^{-\lambda}\dot{\nu} = \kappa - n \ , \tag{19b}$$

$$e^{-\lambda}\dot{w} = u \ , \tag{19c}$$

$$re^{-\lambda}\dot{H}_p = V_p \ , \tag{19d}$$

following the procedure in, $e.g.$, [2, 30].

First we observe that the metric Ansatz (17) is explicitly invariant under reparametrisation of the new 'radial' variable τ. Consequently varying the action cannot result in a differential equation for λ, and thus all derivatives of $e^{-\lambda}\dot{r}$ in the action can be absorbed into a surface term that can be discarded. Then we introduce the new variables n, κ, u, and V_p as Lagrange multipliers such that the resulting action contains at most one τ-derivative and variation w.r.t. to the new variables yields Eqs. (19) as field equations. Finally we discard yet another surface term to obtain a compact expression.

Each of these three steps is straightforward for $p = 1$, but requires some new techniques for $p > 1$ (see Appendix A for the details). As a result Eqs. (9) are expressed as

$$S_G = \int d\tau \Big[e^{\nu+\lambda} r^{d-2p-2}(1-n^2)^{p-1}\Big((\kappa-n)n - \frac{d-2p-1}{2p}(1-n^2) $$
$$+ (re^{-\lambda}\dot{n}) - (\kappa-n)(e^{-\lambda}\dot{r})\Big)\Big] \ , \tag{20a}$$

$$S_M = \int d\tau\, e^{\nu+\lambda} r^{d-4p}\Big[(d-2p-1)\Big(\frac{1}{2p}\frac{W^p}{r^2} + w^2 H_p^2\Big) - W^{p-1}u^2 - \frac{V_p^2}{2p} $$
$$+ 2\Big(W^{p-1}u(e^{-\lambda}\dot{w}) + \frac{V_p}{2p}(re^{-\lambda}\dot{H}_p)\Big)\Big] \ . \tag{20b}$$

Varying the Action (20) w.r.t. λ yields the 'reparametrization constraint' $C_1 = 0$ with

$$C_1 = (\kappa-n)n - \frac{d-2p-1}{2p}(1-n^2) + \frac{1}{(r^2(1-n^2))^{p-1}} \cdot $$
$$\cdot \Big[(d-2p-1)\Big(\frac{1}{2p}\frac{W^p}{r^2} + w^2 H_p^2\Big) - W^{p-1}u^2 - \frac{V_p^2}{2p}\Big] \ . \tag{21}$$

Varying w.r.t. n, κ, u, and V_p by construction yields Eqs. (19) as field equations, and we will use them in the remainig variations. Varying w.r.t. λ for fixed $\nu + \lambda$ yields

$$re^{-\lambda}\dot{n} = (\kappa-n)n - \frac{2}{(r^2(1-n^2))^{p-1}}\Big(W^{p-1}u^2 + \frac{V_p^2}{2p}\Big) \ , \tag{22a}$$

varying w.r.t. r for fixed $e^{\nu+\lambda} r^{d-4p}$ yields

$$re^{-\lambda}\dot{\kappa} = \Big(\frac{4p+1-d}{p}n - \kappa\Big)(\kappa - n)$$
$$- 2(p-1)\frac{1-\kappa n}{1-n^2}\Big((re^{-\lambda}\dot{n}) - \frac{d-2p-1}{2p}(1-n^2)\Big)$$
$$+ \frac{2}{(r^2(1-n^2))^{p-1}}\Big(\frac{W^p}{r^2} - \frac{V_p^2}{2p}\Big)\,, \tag{22b}$$

and varying w.r.t. w and H_p yields the remaining field equations

$$re^{-\lambda}\dot{u} = (d-2p-1)\Big(\frac{w^2-1}{r} + r\frac{H_p^2}{W^{p-1}}\Big)w$$
$$- \Big(2(p-1)w\frac{ru}{w^2-1} + \kappa + (d-4p-1)n\Big)u\,, \tag{22c}$$
$$re^{-\lambda}\dot{V}_p = 2p(d-2p-1)w^2 H_p - \Big(\kappa + (d-4p)n\Big)V_p\,, \tag{22d}$$

Next, we fix the freedom to reparametrise the radial variable τ by the gauge choice $e^\lambda = r$ and define $V_p = (w^2 - 1)^{p-1}v$.

2.2 New Variables – Specific

The equations of motion resulting from the action (20) are singular when $w^2 = 1$, due to the term (3) with $p > 1$, or when $n^2 = 1$, due to the term (4) with $p > 1$. In order to avoid these singularities, we have to introduce new variables specifically adapted to the form of the action (20), whereas the procedure used to obtain this action is essentially the same as used for other gravitating matter systems in [2, 7, 30, 31, 15]. The *conical fixed point* observed in [15] is caused by the new variables required to remove the singularity at $w^2 = 1$ from the equations of motion for that particular model.

Here, we define new variables $\bar{u} = u/t$, $\bar{h} = h/t$, $\bar{v} = v/t$ where $t = (w^2 - 1)/r$, and introduce an additional, redundant variable $y = t^2/(1-n^2)$, resulting in the system of differential equations with polynomial r.h.s.

$$\dot{r} = rn\,, \qquad \text{or} \quad \dot{s} = -sn\,, \qquad \text{where} \quad s = r^{-1}\,, \tag{23a}$$
$$\dot{\nu} = \kappa - n\,, \tag{23b}$$
$$\dot{n} = (\kappa - n)n - 2y^p(1-n^2)\Big(\bar{u}^2 + \frac{\bar{v}^2}{2p}\Big)\,, \tag{23c}$$
$$\dot{\kappa} = \Big(\frac{4p+1-d}{p}n - \kappa\Big)(\kappa - n) + 2y^p\Big[(1-n^2)\Big(\frac{d-2p-1}{2p} - \frac{\bar{v}^2}{2p}\Big)$$
$$+ (p-1)(1-\kappa n)\Big((d-2p-1)(\frac{1}{2p} + w^2\bar{h}^2) + \bar{u}^2 + \frac{\bar{v}^2}{2p}\Big)\Big]\,, \tag{23d}$$
$$\dot{y} = \Big\{\Big(4w\bar{u} + \frac{d-4p-1}{p}n\Big)$$

$$-2y^p\, n\left[(d-2p-1)\left(\frac{1}{2p}+w^2\bar{h}^2\right)+\left(\bar{u}^2+\frac{\bar{v}^2}{2p}\right)\right]\Big\}y\;,\tag{23e}$$

$$\dot{w}=(w^2-1)\bar{u}\;,\tag{23f}$$

$$\dot{u}=(d-2p-1)\left(1+(w^2-1)\bar{h}^2\right)w-\left(2pw\bar{u}+\kappa+(d-4p-2)n\right)\bar{u}\;,\tag{23g}$$

$$\dot{h}=\bar{v}-(2pw\bar{u}-n)\bar{h}\;,\tag{23h}$$

$$\dot{v}=2p(d-2p-1)w^2\bar{h}-\left(2pw\bar{u}+\kappa+(d-4p-1)n\right)\bar{v}\;,\tag{23i}$$

subject to the two constraints

$$C_1=0\;,\tag{24a}$$

$$C_2=0\;,\tag{24b}$$

where the expressions

$$C_1=(\kappa-n)n-(1-n^2)\Big[\frac{d-2p-1}{2p}$$

$$-y^p\Big((d-2p-1)(\frac{1}{2p}+w^2\bar{h}^2)-\bar{u}^2-\frac{\bar{v}^2}{2p}\Big)\Big]\;,\tag{25a}$$

$$C_2=r^2y(1-n^2)-(w^2-1)^2\;,\tag{25b}$$

obey

$$\dot{C}_1=-2y^p\,n\Big(\frac{d-2p-1}{2}(1+2pw^2\bar{h}^2)+(p-2)(\bar{u}^2+\frac{\bar{v}^2}{2p})\Big)C_1\;,\tag{26a}$$

$$\dot{C}_2=-r^2y\,nC_1+4w\bar{u}C_2\;,\tag{26b}$$

and thus the two constraints are preserved by, and therefore compatible with the differential Eqs. (23).

Note that Eqs. (23b–i) are independent of r (or s) and thus remain regular for $r\to0$ as well as for $r\to\infty$, i.e., for $s\to0$.

2.3   Singular Points of the Differential Equations

The differential Eqs. (23) determine the derivatives of the dependent variables $Y=(r,Z)$ (or $Y=(s=r^{-1},Z)$) where $Z=(n,\kappa,w,\bar{u},\bar{h},\bar{v},y)$ w.r.t. τ,

$$\dot{Y}=f(Y)\;.\tag{27}$$

They are singular when one of the dependent variables diverges and at the fixed points (f.p.s) of the Dynamical System (23). For each such f.p. Y_0 with $f(Y_0)=0$ we can introduce new variables $\tilde{Y}=Y-Y_0$ and linearise the equations

$$\dot{\tilde{Y}}=M\tilde{Y}+O(\tilde{Y}^2)\;.\tag{28}$$

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Excluding the possibility that the matrix M has eigenvalues with vanishing real part, we can rewrite Eq. (28) in terms of a suitable basis $\tilde{Y} = (\tilde{Y}_-, \tilde{Y}_+)$ as

$$\frac{d}{d\tau} \begin{pmatrix} \tilde{Y}_- \\ \tilde{Y}_+ \end{pmatrix} = \begin{pmatrix} -M_- & 0 \\ 0 & M_+ \end{pmatrix} \begin{pmatrix} \tilde{Y}_- \\ \tilde{Y}_+ \end{pmatrix} + O(\tilde{Y}^2) , \tag{29}$$

with positive definite matrices M_- and M_+. Due to the theory of dynamical systems there exists a 'stable manifold' $\tilde{Y} = (\tilde{Y}_-, \tilde{Y}_+ = O(\tilde{Y}_-^2))$ of initial data such that $\tilde{Y} \to 0$ as $\tau \to +\infty$ as well as an 'unstable manifold' $\tilde{Y} = (\tilde{Y}_- = O(\tilde{Y}_+^2), \tilde{Y}_+)$ such that $\tilde{Y} \to 0$ as $\tau \to -\infty$. Prop. 1 of [7] states conditions such that these solutions can be characterised by and depend analytically on parameters determined at the f.p.; this however is not possible in general.

First, there is the case $\bar{u} \to \infty$, occuring when $w \to \pm 1$ while $\dot{w} = ru$ remains finite. Such solutions are of no interest since Eqs. (19c) and (22c) exclude maxima of $|w|$ with $|w| > 1$.

Another type of singularity with $n \to 0$ while $\kappa n \neq 0$ occurs at regular horizons and will be discussed below.

2.3.1 Regular Origin

A regular origin with $r \to 0$ as $\tau \to -\infty$ is described by the f.p. with $n = \kappa = w = 1$. Eq. (23g) then yields either $\bar{u} = -(d - 2p - 1)/2p$ or $\bar{u} = 1$, but only the second of these solutions is compatible with a regular origin.

2.3.1.1 Regular Origin without Higgs Field

With $h \equiv 0$ and $2p + 1 < d < 4p + 1$ Eq. (23e) implies

$$y^p = 1 , \tag{30}$$

and suggests the Ansatz

$$w(r) = 1 - b\, r^2 + O(r^4) , \tag{31a}$$
$$u(r) = -2b\, r + O(r^3) , \tag{31b}$$
$$n(r) = 1 - c_n\, r^2 + O(r^4) , \tag{31c}$$
$$\kappa(r) = 1 + c_\kappa\, r^2 + O(r^4) , \tag{31d}$$

where b is a free parameter, whereas

$$c_n = c_\kappa = 2b^2 , \tag{32}$$

(compare [26]). Expressing Eqs. (23c–g) in terms of the dependent variables $\tilde{n} = (1 - n)/r^2$, $\tilde{\kappa} = (\kappa - 1)/r^2$, $\tilde{w} = (w - 1)/r^2$, $\tilde{u} = (\bar{u} - 1)/r$, and $\tilde{y} = (y - 1)/r$

as functions of r yields the equations

$$r\frac{d\tilde{n}}{dr} = \tilde{n} - \tilde{\kappa} + rf_{\tilde{n}} ,\tag{33a}$$

$$r\frac{d\tilde{\kappa}}{dr} = (d-2)(\tilde{n} - \tilde{\kappa}) + rf_{\tilde{\kappa}} ,\tag{33b}$$

$$r\frac{d\tilde{w}}{dr} = rf_{\tilde{w}} ,\tag{33c}$$

$$r\frac{d\tilde{u}}{dr} = -d\,\tilde{u} + rf_{\tilde{u}} ,\tag{33d}$$

$$r\frac{d\tilde{y}}{dr} = -d\,\tilde{y} + rf_{\tilde{y}} ,\tag{33e}$$

or, using the linear combinations $\tilde{n}_+ = (d-2)\tilde{n} - \tilde{\kappa}$ and $\tilde{n}_- = \tilde{n} - \tilde{\kappa}$,

$$r\frac{d\tilde{n}_+}{dr} = r\Big((d-2)f_{\tilde{n}} - f_{\tilde{\kappa}}\Big) ,\tag{33f}$$

$$r\frac{d\tilde{n}_-}{dr} = -(d-3)\tilde{n}_- + r\Big(f_{\tilde{n}} - f_{\tilde{\kappa}}\Big) .\tag{33g}$$

The nonlinear terms $f_{\tilde{n}}$ etc. are analytic functions of r and $\tilde{n}$ etc., and thus Eqs. (33c–g) fulfill the assumptions of Prop. 1 of [7]. Furthermore Eqs. (25) can be written as

$$\frac{C_1}{r^2} = \tilde{\kappa} - \tilde{n} + rf_{C_1} , \qquad \frac{C_2}{r^4} = 2\tilde{n} - 4\tilde{w}^2 + rf_{C_2} ,\tag{34}$$

and allow us to obtain the relation $\tilde{n}(0) = \tilde{\kappa}(0) = 2\tilde{w}^2(0)$. This guarantees the local existence of a one parameter family of solutions with the boundary Conditions (31, 32), analytic in r and b, defined for all b and $|r| < \xi(b)$ with some $\xi(b) > 0$.

2.3.1.2 Regular Origin with Higgs Field

We supplement Eqs. (31) with the Ansatz for the Higgs field,

$$h(r) = a\,r + O(r^3) , \qquad i.e., \quad \bar{h}(r) = -\frac{a}{2b} + O(r^2) ,\tag{35}$$

such that $\Phi(\vec{x}, t)$ is regular near $\vec{x} = 0$. In view of Eqs (23h,i) this requires $d = 4p$ and consequently

$$v(r) = (2p-1)a\,r + O(r^3) , \qquad i.e., \quad \bar{v}(r) = -(2p-1)\frac{a}{2b} + O(r^2) .\tag{36}$$

Evaluating Eq. (23e) at the f.p. yields

$$y^p = \Big(1 + (2p-1)\bar{h}^2\Big)^{-1} .\tag{37}$$

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We now have two free parameters, a and b, whereas Eq. (32) is replaced by

$$c_n = 2b^2(1+\gamma)^{1/p} , \qquad c_\kappa = \zeta(\gamma)c_n , \qquad \text{with} \quad \zeta(\gamma) = 1 - \frac{2}{p}\frac{\gamma}{1+\gamma} , \qquad (38)$$

and $\gamma = (2p-1)a^2/4b^2$. Expressing Eqs. (23c–i) in terms of the dependent variables $\tilde{n}$, $\tilde{\kappa}$, $\tilde{w}$, and $\tilde{u}$ as above together with $\tilde{h}_+ = (2p\bar{h}+\bar{v})/(4p-1)$, $\tilde{h}_- = ((2p-1)\bar{h}-\bar{v})/((4p-1)r)$, and $\tilde{y} = (y^p(1+\tilde{\gamma})-1)/r$ with $\tilde{\gamma} = (2p-1)\tilde{h}_+^2$ as functions of r yields the equations

$$r\frac{d\tilde{n}}{dr} = \zeta(\tilde{\gamma})\tilde{n} - \tilde{\kappa} + rf_{\tilde{n}} , \qquad (39a)$$

$$r\frac{d\tilde{\kappa}}{dr} = (4p-2)\left(\zeta(\tilde{\gamma})\tilde{n} - \tilde{\kappa}\right) + rf_{\tilde{\kappa}} , \qquad (39b)$$

$$r\frac{d\tilde{w}}{dr} = rf_{\tilde{w}} , \qquad (39c)$$

$$r\frac{d\tilde{u}}{dr} = -4p\,\tilde{u} + rf_{\tilde{u}} , \qquad (39d)$$

$$r\frac{d\tilde{h}_+}{dr} = rf_{\tilde{h}_+} , \qquad (39e)$$

$$r\frac{d\tilde{h}_-}{dr} = -4p\,\tilde{h}_- + rf_{\tilde{h}_-} , \qquad (39f)$$

$$r\frac{d\tilde{y}}{dr} = -4p\,\tilde{y} + rf_{\tilde{y}} , \qquad (39g)$$

or, rewriting Eqs (39a–b) in terms of the new variables $\tilde{n}_+ = (4p-2)\tilde{n} - \tilde{\kappa}$ and $\tilde{n}_- = \zeta(\tilde{\gamma})\tilde{n} - \tilde{\kappa}$,

$$r\frac{d\tilde{n}_+}{dr} = r\left((4p-2)f_{\tilde{n}} - f_{\tilde{\kappa}}\right) , \qquad (39h)$$

$$r\frac{d\tilde{n}_-}{dr} = -\left(4p - 2 - \zeta(\tilde{\gamma})\right)\tilde{n}_- + r\left(\zeta(\tilde{\gamma})f_{\tilde{n}} - f_{\tilde{\kappa}} - \frac{4}{p}\frac{2p-1}{(1+\tilde{\gamma})^2}\tilde{n}\tilde{h}_+ f_{\tilde{h}_+}\right) . \qquad (39i)$$

Furthermore Eqs. (25) can be written as

$$\frac{C_1}{r^2} = \tilde{\kappa} - \zeta(\tilde{\gamma})\tilde{n} + rf_{C_1} , \qquad \frac{C_2}{r^4} = 2\tilde{n}(1+\tilde{\gamma})^{-1/p} - 4\tilde{w}^2 + rf_{C_2} . \qquad (40)$$

We use the Constraints (24) to eliminate $\tilde{n}_-$ as well as Eq. (39i) with the $\tilde{h}_+$ dependent 'eigenvalue'. In addition we obtain two relations between $\tilde{n}(0)$, $\tilde{\kappa}(0)$, $\tilde{w}(0)$, and $\tilde{h}_+(0)$. Eqs. (39c–h) then satisfy the assumptions of Prop. 1 of [7]. This guarantees the local existence of a two parameter family of solutions with the boundary Conditions (31, 35, 36, 38), analytic in r, a, and b, and defined for all a, $b \neq 0$, and $|r| < \xi(a,b)$ with some $\xi(a,b) > 0$.

2.3.2 Regular Horizons

For regular, *i.e.*, non-degenerate horizons we may use $z = 1/\kappa$ as the independent variable tending to zero (compare [31]), and thus replace Eqs. (23) by

$$z\frac{d}{dz}\tau = -\frac{\kappa}{\dot\kappa} = z(1 + zf_\tau) , \tag{41a}$$

$$z\frac{d}{dz}y = z(1 + zf_\tau)\dot y , \tag{41b}$$

$$z\frac{d}{dz}r = z(1 + zf_\tau)\dot r , \tag{41c}$$

$$z\frac{d}{dz}w = z(1 + zf_\tau)\dot w , \tag{41d}$$

$$z\frac{d}{dz}\bar u = -\bar u + zf_{\bar u} , \tag{41e}$$

$$z\frac{d}{dz}\bar h = z(1 + zf_\tau)\dot{\bar h} , \tag{41f}$$

$$z\frac{d}{dz}\bar v = -\bar v + zf_{\bar v} , \tag{41g}$$

with expressions f_τ etc. determined from Eqs. (23) that are regular at $z = 0$, while $n = O(z)$ can be computed from the Constraint (24a). These equations satisfy the assumptions of Prop. 1 of [7] and thus guarantee the existence of a family of solutions with $\bar u \to 0$ and $\bar v \to 0$ as $z \to 0$, and with finite limits τ_h, y_h, r_h, w_h, and $\bar h_h$ for the remainig variables τ, y, r, w, and $\bar h$. Finally, the Constraint (24b) yields the relation $y_h = (w_h^2 - 1)^2/r_h^2$.

In terms of the coordinate τ the behaviour near the horizon is (performing a shift in τ such that $\tau_h = 0$)

$$r(\tau) = r_h\left(1 + \frac{n_1\tau^2}{2}\right) + O(\tau^4) , \tag{42a}$$

$$n(\tau) = n_1\tau + O(\tau^3) , \tag{42b}$$

$$\kappa(\tau) = \tau^{-1} + O(\tau) , \tag{42c}$$

$$w(\tau) = w_h + \frac{r_h w_1\tau^2}{2} + O(\tau^4) , \tag{42d}$$

$$u(\tau) = w_1\tau + O(\tau^3) , \tag{42e}$$

$$h(\tau) = h_h\left(1 - (p - 1)\frac{r_h w_h w_1\tau^2}{w_h^2 - 1}\right) + \frac{h_1\tau^2}{2} + O(\tau^4) , \tag{42f}$$

$$v(\tau) = h_1\tau + O(\tau^3) , \tag{42g}$$

with expressions n_1, w_1, and h_1 determined by the free parameters r_h, w_h, and h_h.

We, thus, have a three parameter family of solutions that satisfy the black hole boundary Conditions (42) and are (except for the pole in κ) analytic in

τ, r_h, w_h, and h_h, defined for all r_h, w_h, h_h, and $|\tau| < \xi(r_h, w_h, h_h)$ with some $\xi(r_h, w_h, h_h) > 0$.

Solutions with a regular origin can be seen as the limits $r_h \to 0$ of those with a regular horizon. Consider a sequence of black hole solutions with finite limits $h_h/r_h \to a_h$ and $(1 - w_h)/r_h^2 \to b_h$ as $r_h \to 0$. Taken as functions of r they converge to a solution with regular origin and parameters a and b. Starting from the differential equations for $w(r)$ and $h(r)$, derived form the action (9b), we can linearise around $w = 1$ and $h = 0$, use a rescaled radial variable $\rho = r/r_h$, and neglect all terms that vanish as $r_h \to 0$, resulting in two hypergeometric equations. We thus obtain the relations

$$b^p = \left(\frac{\frac{(p-1)d+p+1}{d-2p-1}}{p-1} \right) b_h^p , \qquad \text{and} \quad a\, b^{p-1} = \left(\frac{\frac{4p^2-3p+1}{2p-1}}{p} \right) a_h b_h^{p-1} , \tag{43}$$

valid for $d = 4p$ or, without Higgs field, for $2p + 1 < d < 4p + 1$. Defining the ratios $\zeta_b = b/b_h$ and $\zeta_a = a/a_h$, and evaluating the binomial coefficients in Eqs. (43) for $p \le 4$ and $d = 4p$ yields the values shown in Tab. 1, with the well known ratios $\zeta_b = 1$ and $\zeta_a = 2$ for $p = 1$ (compare Fig. 8b in [1]).

Table 1: Ratios between boundary conditions for the limit $r_h \to 0$ of black monopoles and the corresponding regular monopoles for $p = 1$, 2, 3, and 4 in $d = 4p$ spacetime dimensions.

p	ζ_b	ζ_a
1	1	2
2	1.91485422	2.55313895
3	2.34407744	2.81289292
4	2.60713329	2.97958090

2.3.3 Reissner-Nordström Fixed Point

The Reissner-Nordström (RN) f.p. is characterised by $n = w = u = v = 0$, $h = \alpha$, $r = 1$, and $\kappa = \kappa_0$ where $\kappa_0^2 = d - 2p - 1$, as for the degenerate horizon of an extremal RN black hole. Introducing $\tilde{r} = r - 1$, $\tilde{u} = ru$, $\tilde{h} = h - \alpha$, and $\tilde{\kappa} = \kappa - \kappa_0$, we obtain

$$\dot{\tilde{r}} = n + \tilde{r}\, n , \tag{44a}$$

$$\dot{w} = \tilde{u} , \tag{44b}$$

$$\dot{\tilde{u}} = \kappa_0^2(\alpha^2 - 1)\, w - \kappa_0\, \tilde{u} + f_u , \tag{44c}$$

$$\dot{\tilde{h}} = v + f_h , \tag{44d}$$

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$$\dot{v} = -\kappa_0\, v + f_v \,, \tag{44e}$$

$$\dot{n} = \kappa_0\, n + f_n \,, \tag{44f}$$

$$\dot{\tilde{\kappa}} = -2\kappa_0\, \tilde{\kappa} + f_\kappa \,, \tag{44g}$$

with expressions $f_i = O((\tilde{r}, w, \tilde{u}, \tilde{h}, v, n, \tilde{\kappa})^2)$. Since $\tilde{r}$ can be eliminated by the Constraint (24a), and assuming $\kappa_0 > 0$, the linearised Eqs. (44) have one unstable mode n with eigenvalue κ_0, one stable mode $\tilde{\kappa}$ with eigenvalue $-2\kappa_0$. The $(\tilde{h}, v)$ subsystem contributes the stable mode v with eigenvalue $-\kappa_0$ and the zero mode $\tilde{h} + v/\kappa_0$ (that has to be considered as unstable due to the requirement $\tilde{h} \to 0$). The $(w, \tilde{u})$ subsystem contributes two modes with eigenvalues

$$\lambda = -\frac{\kappa_0}{2}\left(1 \pm \sqrt{4\alpha^2 - 3}\right). \tag{45}$$

For $\alpha^2 < 3/4$ there are two stable oscillating modes with complex conjugate eigenvalues with negative real part, for $3/4 < \alpha^2 < 1$ there are two stable modes with negative eigenvalues, whereas for $\alpha^2 > 1$ there is one stable and one unstable mode, *i.e.*, one positive and one negative eigenvalue.

Thus, the dimension of the stable manifold is 4 for $\alpha^2 < 1$ and 3 for $\alpha^2 > 1$, corresponding to a 3- resp. 2-parameter family of solutions converging to the f.p. as $\tau \to +\infty$. However, it is possible to extend the 3-dimensional stable manifold for $\alpha^2 > 1$ into the region $3/4 < \alpha^2 \leq 1$ and define a 3-dimensional submanifold of the 4-dimensional stable manifold by the requirement that the variables in Eqs. (44) decrease faster than $e^{-k_0\tau/2}$. When a family of asymptotically flat solution reaches a critical limit with a double zero of $\mu(r)$, the interior part with $r < 1$ will be a member of the corresponding 2-parameter family, while the exterior part with $r > 1$ will be the exterior of the extremal RN solution with $w \equiv 0$ and $h \equiv \alpha > \sqrt{3/4}$.

Note, however, that these solutions do not approach a degenerate horizon as $\tau \to \infty$ and $r \to 1$, because e^ν and hence $A = e^\nu/n$ diverge. They describe geodesically complete spacetimes, asymptotically like $AdS_2 \times S^{d-2}$,

$$ds^2 \to e^{2\kappa_0\tau}dt^2 - d\tau^2 - d\Omega^2_{(d-2)}\,. \tag{46}$$

2.3.4 Asymptotically Flat Infinity without Higgs Field

Asymptotically flat infinity with $s \to 0$ as $\tau \to +\infty$ is described by a f.p. with $n = \kappa = 1$. In the absence of a Higgs field this requires $w = \pm 1$ and Eq. (23g) again yields either $\bar{u} = -(d - 2p - 1)/2p$ or $\bar{u} = 1$, but this time only the first of these solutions is compatible with asymptotically flat infinity.

Together with Eq. (14), this suggests the Ansatz

$$n(s) = 1 - \frac{m}{2}z^2 + O(z^4)\,, \tag{47a}$$

$$\kappa(s) = 1 + c_\kappa z^2 + O(z^4)\,, \tag{47b}$$

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$$w(s) = 1 - c\,z^2 + O(z^4) \,, \tag{47c}$$

$$u(s) = \frac{d - 2p - 1}{p} c\,s\,z^2 + O(sz^4) \,, \tag{47d}$$

with $z = s^{(d-2p-1)/2p}$, where the total mass $M = \frac{4p+1-d}{2p} m^p$ and c are free parameters, while

$$c_\kappa = \frac{(d - 3p - 1)m}{2p} \,. \tag{48}$$

Expressing Eqs. (23c–g) in terms of the dependent variables $\tilde{n} = (1 - n)/z^2$, $\tilde{\kappa} = (\kappa - 1)/z^2$, $\tilde{w} = (w - 1)/z^2$, $\tilde{u} = (\bar{u} + (d - 2p - 1)/2p)/z$, and $\tilde{y} = y/(s\,z)^2$ as functions of s yields the equations

$$s\frac{d\tilde{n}}{ds} = \frac{3p + 1 - d}{p}\tilde{n} + \tilde{\kappa} + z^2 f_{\tilde{n}} \,, \tag{49a}$$

$$s\frac{d\tilde{\kappa}}{ds} = \frac{d - 3p - 1}{p}\tilde{n} - \tilde{\kappa} + z^2 f_{\tilde{\kappa}} \,, \tag{49b}$$

$$s\frac{d\tilde{w}}{ds} = z\,f_{\tilde{w}} \,, \tag{49c}$$

$$s\frac{d\tilde{u}}{ds} = -\left(d - 2 + \frac{d - 1}{2p}\right)\tilde{u} + z f_{\tilde{u}} \,, \tag{49d}$$

$$s\frac{d\tilde{y}}{ds} = z^2 f_{\tilde{y}} \,, \tag{49e}$$

or, using the linear combinations $\tilde{n}_+ = \tilde{n} + \tilde{\kappa}$ and $\tilde{n}_- = \tilde{n}(3p + 1 - d)/p + \tilde{\kappa}$,

$$s\frac{d\tilde{n}_+}{ds} = z^2\left(f_{\tilde{n}} + f_{\tilde{\kappa}}\right) \,, \tag{49f}$$

$$s\frac{d\tilde{n}_-}{ds} = -\frac{d - 2p - 1}{p}\tilde{n}_- + z^2\left(\frac{3p + 1 - d}{p}f_{\tilde{n}} + f_{\tilde{\kappa}}\right) \,. \tag{49g}$$

The nonlinear terms $f_{\tilde{n}}$ etc. are analytic functions of s, z, and $\tilde{n}$ etc., and thus, Eqs. (49c–g) fulfill the assumptions of Prop. 1 of [7] with z as independent variable. Furthermore Eqs. (25) can be written as

$$\frac{C_1}{z^2} = \tilde{n}_- + z^2 f_{C_1} \,, \qquad \frac{C_2}{z^4} = 2\tilde{n}\tilde{y} - 4\tilde{w}^2 + z^2 f_{C_2} \,, \tag{50}$$

and allow us to obtain the relations $\tilde{\kappa}(0) = \tilde{n}(0)(d - 3p - 1)/p$ and $\tilde{n}(0)\tilde{y}(0) = 2\tilde{w}^2(0)$. This guarantees the local existence of a two parameter family of solutions with boundary Conditions (47), analytic in M, c, and z, defined for all M, c, and $|s| < \xi(M, c)$ with some $\xi(M, c) > 0$.

2.3.5 Asymptotically Flat Infinity with Higgs Field

Asymptotically flat infinity with Higgs field is described by the f.p. with $n = \kappa = 1$, $w = u = v = 0$, and $h = \alpha > 0$. The difficulty here is, that w and u decrease

exponentially, approximately as $e^{-\sqrt{2p-1}\alpha r}$, while n, κ, h, and v converge much more slowly, as some power of $s \equiv 1/r$.

Assuming solutions with finite energy, we can use the Definition (14) and the Constraints (25) to obtain expressions for $\tilde{n} = r(n-1)$ and $\tilde{\kappa} = r(\kappa-1)$, that are bounded as long as w, u, h, v, and the mass function $M(r)$ are bounded. This suggests to derive differential equations for w, u, $h - \alpha$, v, and M with r as independent variable

$$\frac{dw}{dr} = u + f_w \,, \tag{51a}$$

$$\frac{du}{dr} = (2p-1)\alpha^2 w + f_u \,, \tag{51b}$$

with nonlinear terms f_w and f_u built from w, u, $h - \alpha$, v, $\tilde{n}$, $\tilde{\kappa}$, and s, while $d(h-\alpha)/dr$, dv/dr, and

$$\frac{dM}{dr} = (w^2 - 1)^{2(p-1)} \Big(\frac{2p-1}{2p} \frac{(w^2-1)^2}{r^2} + (2p-1)w^2 h^2 + u^2 + \frac{v^2}{2p} \Big) \,, \tag{51c}$$

consist entirely of such nonlinear terms. This yields one dimensional stable and unstable manifolds together with a 'center' manifold. In order to further analyze the orbits in the center manifold, in this particular case simply $w = u = 0$, we intoduce $\tilde{h} = r(h - \alpha + v)$, $\tilde{v} = rv$ and obtain

$$s\frac{dM}{ds} = s\, f_M \,, \tag{52a}$$

$$s\frac{d\tilde{h}}{ds} = -\tilde{h} + s\, f_{\tilde{h}} \,, \tag{52b}$$

$$s\frac{d\tilde{v}}{ds} = s\, f_{\tilde{v}} \,, \tag{52c}$$

with nonlinear terms f_M, $f_{\tilde{h}}$, and $f_{\tilde{v}}$ that are bounded expressions in terms of M, $\tilde{h}$, and $\tilde{v}$. Consequently there exists a three parameter family of solutions, partially characterised by the total mass and the asymptotic value of $\tilde{v}$.

3   Numerical Results

3.1   Numerical Procedure

We are mainly interested in regular monopole solutions, $i.e.$, solutions of Eqs. (23) connecting a regular origin, $r = 0$, with asymptotically flat infinity, $r \to \infty$. In addition there are 'black monopoles' starting from a non-degenerate horizon at $r = r_h$. To better understand some limiting cases of these two types of solutions, we also need solutions ending at the RN fixed point $w = 0$, $h = \alpha$ at $r = 1$ (joined with the exterior extremal RN solution $w \equiv 0$, $h \equiv \alpha$ for $r > 1$).

We use a procedure that could be called 'shooting and matching'. For each of the four cases mentioned, we use a suitable independent variable $0 \leq \xi < \infty$. First we integrate Eqs. (23) from $\xi = 0$ to some $\xi = \xi_{\max}$ with a slightly modified Runge-Kutta algorithm, using the shooting parameters a and b or h_h and w_h to replace expressions that become undetermined at the singular starting point $\xi = 0$. Next we express the solution at $\xi = \xi_{\max}$ in terms of stable and unstable modes suitable for the chosen endpoint $\xi \to \infty$. A solution will only converge to that endpoint for data on the 'stable manifold' where (the values of) the unstable modes are functions of the stable modes given by the solution of a system of integral equations. Finally, the shooting parameters must be adjusted to satisfy the matching condition that the solution at $\xi = \xi_{\max}$ lies on the stable manifold.

As usual for nonlinear systems, the procedure outlined above has a limited domain of convergence and requires good approximate initial data. Since we will study families of solutions depending smoothly on one or two parameters, such approximate initial data are given by varying these parameters in small steps.

In order to emphasise the similarities as well as the differences between the members of the hierarchy of p-EBPS solutions, we present results (figures and numbers) for $p = 1$, 2, 3, and 4, repeating results for $p = 1$ from [1, 2] (for $\beta = 0$, *i.e.*, without Higgs potential). In addition we will use results obtained in [16] for the p-BPS hierarchy in flat space and in [26] for the p-Bartnik-McKinnon (BK) hierarchy.

3.2   Regular Monopoles

As noted above, the limit $\alpha \to 0$ can be taken in two different ways. The 'unscaled' variables $\tilde{r} = \alpha r$ and $\tilde{h} = h/\alpha$ satisfy α-dependent equations together with the α-independent boundary condition $\tilde{h} \to 1$ as $\tilde{r} \to \infty$. For $\alpha = 0$ this yields the equations for the BPS monopole in flat space. Near $\tilde{r} = 0$ we have $w = 1 - \tilde{b}\tilde{r}^2 + O(\tilde{r}^4)$ and $\tilde{h} = \tilde{a}\tilde{r} + O(\tilde{r}^3)$ with $\tilde{a} = \tilde{b}/\sqrt{2p-1}$ dictated by the Bogomol'nyi equations, and $\tilde{b} = 1/6$, 0.1623, 0.13740, and 0.11775 for $p = 1$, 2, 3, and 4 respectively (as found in [16] but normalised differntly). Starting from these values we could increase α and adjust the parameters $\tilde{a}(\alpha)$ and $\tilde{b}(\alpha)$ such that the boundary conditions $\tilde{h} \to 1$ and $w \to 0$ as $\tilde{r} \to \infty$ remain satisfied.

Alternatively, we can use the 'rescaled' variables r and h with the α-dependent boundary condition $h \to \alpha$ as $r \to \infty$. Since there is no Higgs potential, the resulting equations are α-independent. For $\alpha = 0$ and thus $h \equiv 0$ they describe the EYM system [26]. Near $r = 0$ we have $w = 1 - br^2 + O(r^4)$ with $b = \alpha^2 \tilde{b}$ and $h = ar + O(r^3)$ with $a = \alpha^2 \tilde{a}$, and thus $a \approx b/\sqrt{2p-1}$ when $\alpha \ll 1$. In the absence of a Higgs potential it suffices to adjust one of the parameters a or b such that $w \to 0$ as $r \to \infty$, whereas α is determined by the solution as $\lim_{r \to \infty} h(r)$.

In addition to these fundamental monopole solutions with $w(r) \geq 0$, there exist excited solutions with $N = 1$, 2, $\ldots$ zeros of $w(r)$. Starting from the Nth generalised BK solutions with $a = 0$ and $b = b_N^{(p)}$ as given in Tab. 2, we can

increase a and adjust $b(a)$ such that $w(r) \to 0$ as $r \to \infty$.

Table 2: Parameters of generalised Bartnik-McKinnon solutions for $p = 1$, 2, 3, and 4 in $d = 4p$ spacetime dimensions.

p	$b_1^{(p)}$	$b_2^{(p)}$	$b_3^{(p)}$	$b_4^{(p)}$	$b_\infty^{(p)}$
1	0.453716	0.651726	0.697040	0.704878	0.706420
2	0.415609	0.532402	0.574678	0.588191	0.593799
3	0.410745	0.502893	0.539835	0.554239	0.562700
4	0.410127	0.489756	0.523142	0.537311	0.547598

For $N \to \infty$ the BK solutions converge to a non-trivial limiting solution for $r < 1$, and to the exterior of an extremal RN black hole for $r > 1$. Starting from these limiting solutions with $a = 0$ and $b = b_\infty^{(p)}$ we obtain limiting monopole solutions.

The properties of the excited and limiting solutions are dominated by the behaviour near the RN fixed point, where w decreases exponentially (in τ) and oscillates with a frequency proportional to $\sqrt{\alpha^2 - 0.75}$. Consequently all excited solutions coverge to the limiting solution as $\alpha^2 \to 0.75$ and cease to exist beyond that value.

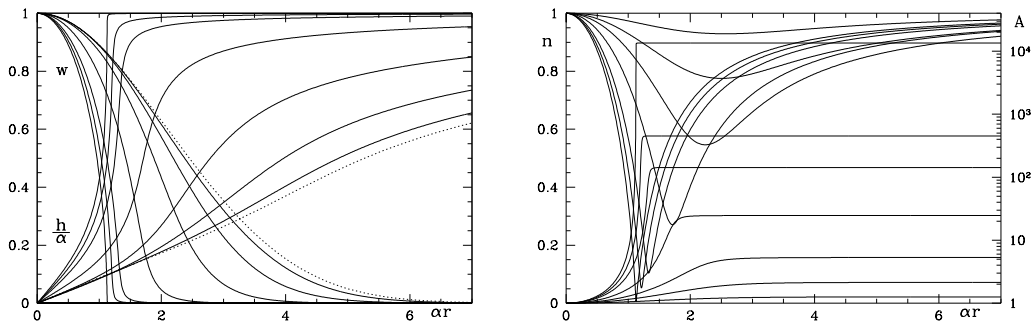Figure 1: Fundamental solutions for $p = 4$ with $\alpha = 1.0$, 1.6, α_{max}, 1.6, 1.3, 1.2, and 1.125; the dotted curves are for the monopole in flat space.

We thus obtain smooth one parameter families of solutions. Some fundamental solutions for $p = 4$ are shown in Fig. 1 and $N = 2$ excited ones for $p = 2$ are shown in Fig. 2. These families start with the (rescaled) flat monopole for $N = 0$ or the BK solutions for $N > 0$ and end with critical solutions where $\mu(r)$ has a double zero at $r = 1$. These critical solutions consist of two parts describing two geodesically complete ($t =$const) spaces: a non-trivial interior part for $r < 1$ and a trivial part with $w \equiv 0$, $h \equiv \alpha$, and $A =$const for $r > 1$, $i.e.$, the exterior of an extremal RN black hole. The extremal RN black hole has a degenerate horizon at $r = 1$ and is a geodesically incomplete spacetime. The non-trivial interior part
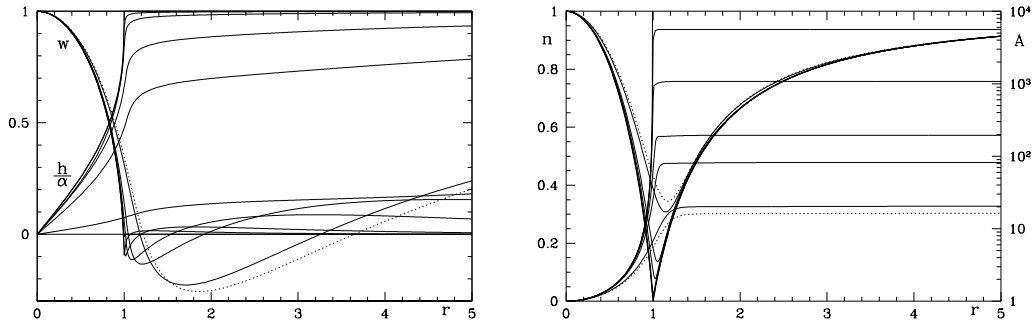
Figure 2: Second excited solutions for $p = 2$ with $\alpha = 0.1$, 0.15, 0.2, 0.35, 0.5 and 0.8; the dotted curves are for the corresponding BK solution.

describes a geodesically complete spacetime with $A \to \infty$ as $r \to 1$. The initial data for these families are quite similar for $p = 1, 2, 3,$ and 4 (see Fig. 3).
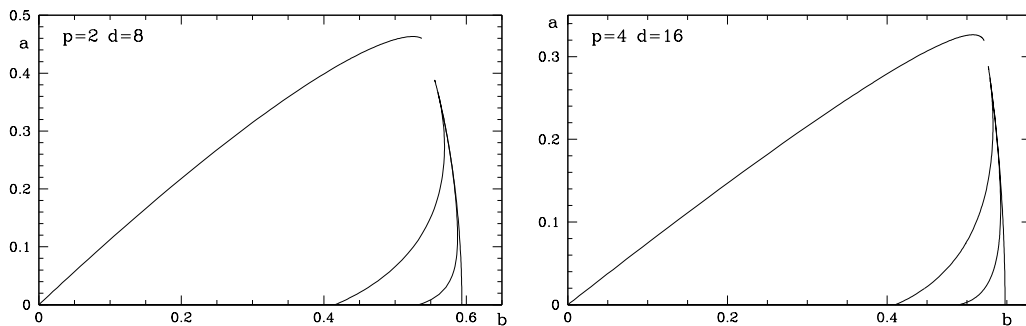

Figure 3: Initial data for the regular monopole solutions for $p = 2$ and 4.

An interesting structure emerges when we include the values of α obtained from these solutions. For $N = 0$ and $1 \leq p \leq 4$, α first increases from 0 to some $\alpha_{\max}$ and subsequently decreases to some α_{cr}. Simultaneously the mass M starts as α times the mass $M_{\mathrm{flat}} = \sqrt{2p-1}/p$ of the flat space monopole, increases to a maximum $M_{\max} > 1$ and subsequently decreases to $M_{\mathrm{cr}} = 1$ (see Figs. 4 and 5). In Tab. 3 we have collected some relevant numbers. The difference $\alpha_{\max} - \alpha_{\mathrm{cr}}$ is very small for $p = 1$, but quite large for $p > 1$.

Table 3: Maximal and critical values of α and masses of the fundamental monopole solutions for $p = 1, 2, 3,$ and 4.

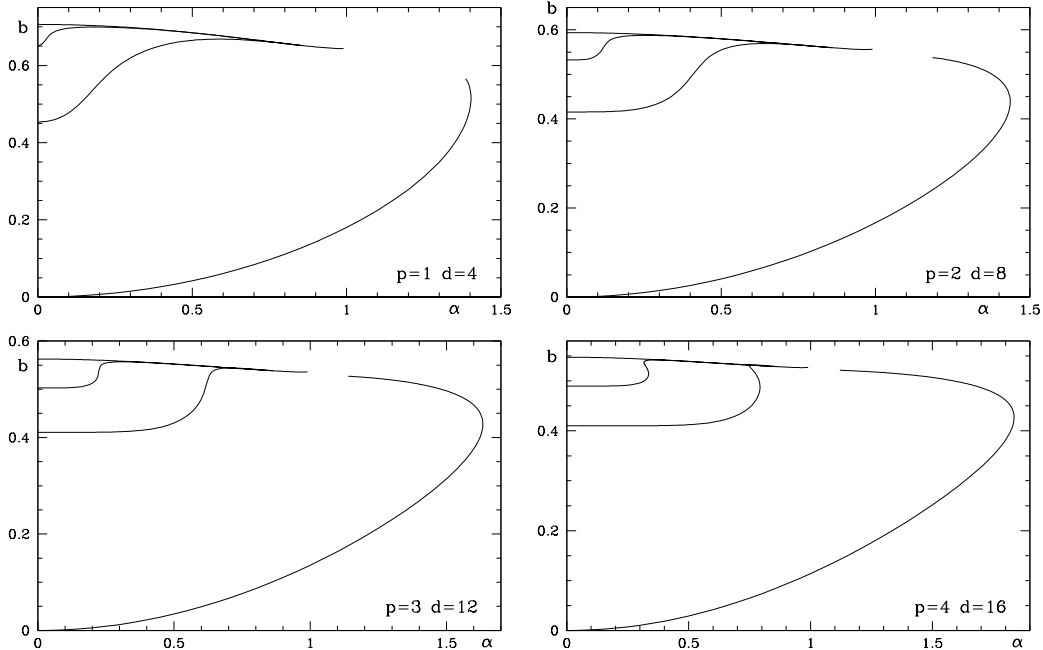| p	$\alpha_{\max}$	$M_{\max}$	α_{cr}
1	1.40303	1.00022	1.38585
2	1.43629	1.01212	1.18516
3	1.63363	1.02921	1.14018
4	1.83513	1.04432	1.12034

21

Figure 4: b *vs.* α for the regular monopole solutions with $p = 1, 2, 3$, and 4.

This may be a good point to comment on the numerical procedure described above. The possible values of $\xi_{\max}$ are limited severely by the necessity to suppress the unstable YM mode growing exponentially with $\xi \equiv r$, whereas $h(r)$ converges only slowly to its asymptotic value $h(\infty) = \alpha$. For the $p = 4$ solution with $\alpha_{\max}$ as shown in Tab. 3, $h(r)$ differs by more than 10% for the largest possible values of $\xi_{\max}$. Requiring that the unstable Higgs mode, $\tilde{h}$ in Eq.(52b) vanishes at $\xi_{\max}$ yields excellent results for the BPS monopole in flat space. But in the gravitating case considered here, $f_{\tilde{h}}$ has slowly decreasing contributions proportional to $\tilde{n}$ or $\tilde{\kappa}$, preventing to estimate $h(\infty)$ with an error much better than 10^{-3}. The results obtained with 'shooting and matching' as outlined, however, are accurate up to rounding and discretization errors, typically around 10^{-10} and are with that accuracy independent of the choice of $\xi_{\max}$.

The (first three) excited solutions for $p = 1, 2$, and 3 show a monotonic increase of α from 0 to $\alpha_{\mathrm{cr}} = \sqrt{0.75}$, with the mass increasing from its BK value to 1. For $p = 4$, however, the values of α first increase to some $\alpha_{\max}$ with $M_{\max} > 1$, then decrease to some $\alpha_{\min}$ with $M_{\min} < 1$, and finally increase to α_{cr} with $M = 1$ (see Figs. 4 and 5). One might speculate that higher excited solutions, with $N > 3$, exhibit such maxima and minima also for $p < 4$.

The maxima of α (and M) observed for the fundamental monopoles give rise to two 'branches' of solutions. Furthermore, for $\alpha = \alpha_{\max}$ there exists a 'zero mode', indicating a change in the number of instabilities against small time-dependent perturbations. Whereas the 'lower' branch from 0 to $\alpha_{\max}$ inherits the stability of the flat space BPS monopole, the 'upper' branch from $\alpha_{\max}$ to α_{cr} has
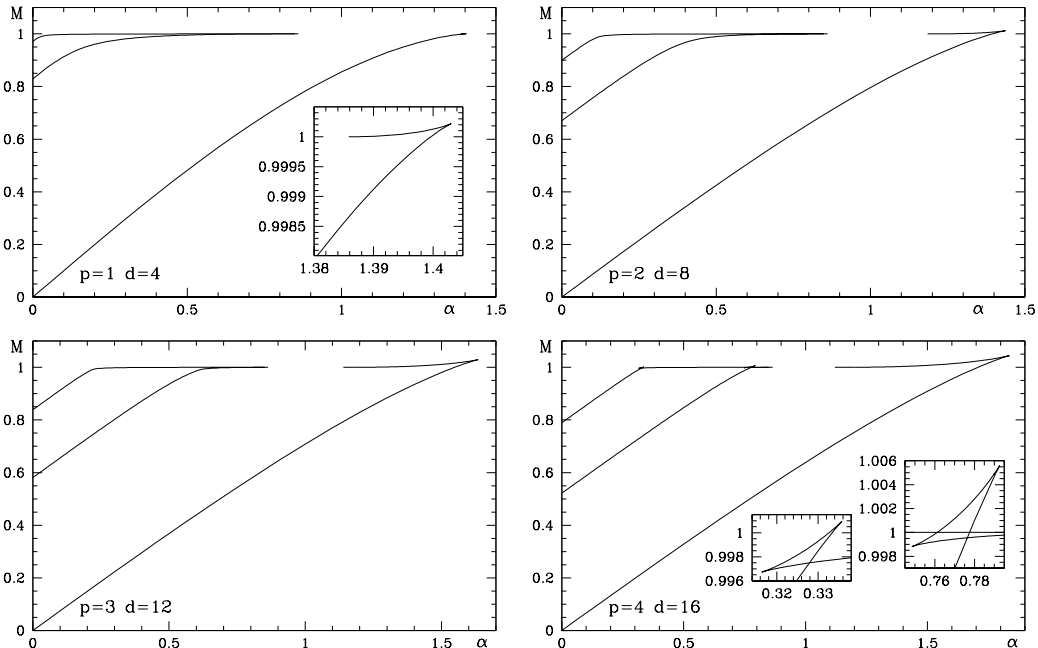
22

Figure 5: Mass of the regular monopole solutions with $p = 1$, 2, 3, and 4.

one such instability.

For $\alpha \ll 1$ the excited monopoles with mass $\approx M_{\text{BK}} + \alpha M_{\text{flat}}$ may be seen as 'superposition' of a BK solution for $\alpha r \ll 1$ and a flat space monopole for $r \gg 1$, and inherit the instabilities of the BK solution. The Nth BK solution for $p = 1$ has $2N$ such instabilities (N radial and N sphaleronic ones) and this might equally be true for $p > 1$. The maxima and minima of α (and M) observed for the excited $p = 4$ monopoles give rise to three branches of solutions, where the 'middle' branch from α_{max} to α_{min} has one additional instability.

3.3 Black Monopoles

As noted in Subsect. 2.3.2 the limits $r_h \to 0$ of black monopole solutions yields regular monopoles. Conversely the (fundamental, excited, or limiting) regular monopoles can be taken as starting points for corresponding black monopoles with $r_h \ll 1$.

Using 'rescaled' variables one can, *e.g.*, choose a value r_h and adjust one of the parameters h_h or w_h such that $w \to 0$ as $r \to \infty$, whereas α is again determined by the solution as $\lim_{r \to \infty} h(r)$. We thus obtain smooth two parameter families of solutions.

The initial data are quite similar for different values of p and are shown in Fig. 6 for $p = 3$ with $r_h = 0.5$ and for $p = 4$ with $r_h = 1.5$. For $h_h \ll 1$, and thus $\alpha \ll 1$ they start with $w_h \approx 1$ for the fundamental, $N = 0$ solution (tiny Schwarzschild black hole inside a flat space BPS monopole), or with w_h as for
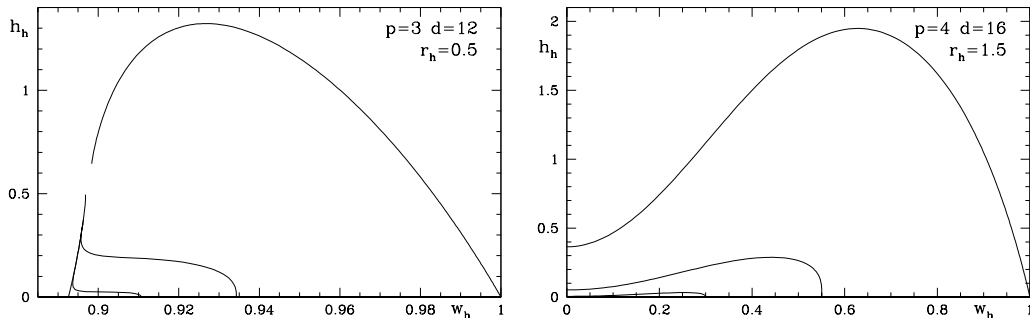
Figure 6: Initial data for the black monopole solutions with $r_h = 0.5$ for $p = 3$ and $r_h = 1.5$ for $p = 4$.
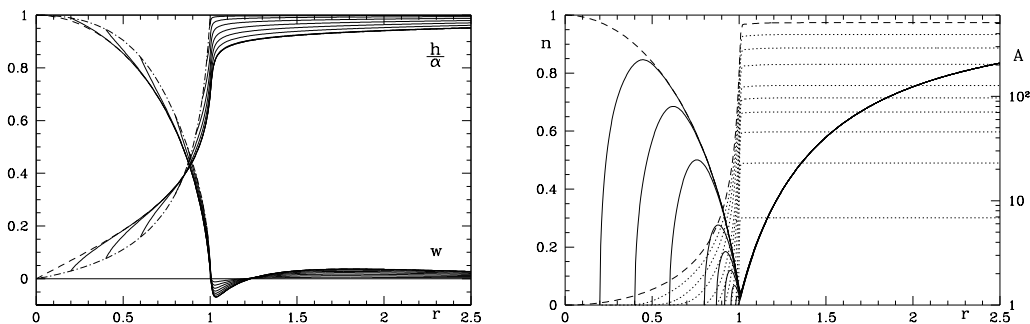

Figure 7: Excited, $N = 2$ black monopole solutions for $p = 4$ and $\alpha = 0.4$, with $r_h = 0.2$, 0.4, 0.6, 0.8, and $w_h = 0.5$, 0.4, 0.3, 0.2, and 0.1. The dashed lines are for the corresponding regular monopole. The dashed-dotted lines show the r_h-dependence of h_h and w_h.

the corresponding EYM black hole, $N = 1, 2, \ldots$, or $N \to \infty$ if $r_h < 1$ (compare Fig. 3 of [7]).

For $r_h < 1$ the families end with critical solutions as for the regular monopoles. The critical fundamental solutions occur at a value $\alpha_{\mathrm{cr}}(r_h)$, starting for $r_h = 0$ from α_{cr} as given in Tab. 3 for the regular monopoles and reaching the value $\sqrt{0.75}$ for some r_h close to but less than 1. For larger r_h the fundamental solutions converge to the limiting one as $\alpha^2 \to 0.75$ and cease to exist beyond that value, as do the excited solutions for all $r_h < 1$.

The situation is quite different for $r_h > 1$. All solutions end with $w_h = 0$ where they bifurcate with a regular, $i.e.$, non-extremal RN black hole. Black monopole solutions with various values of r_h are shown for $p = 4$, $N = 2$, $\alpha = 0.4$ in Fig. 7.

The relatively simple domain of existence of the families of black monopoles described above gets rather more complicated by the existence of maxima and minima (of α for fixed r_h or vice versa), giving rise to various branches. Fig. 8 shows these domains as αr_h $vs.$ α for $p = 1, 2, 3$, and 4. Regular RN black holes only exist for $r_h > 1$ (above the dashed diagonal). Fundamental and excited
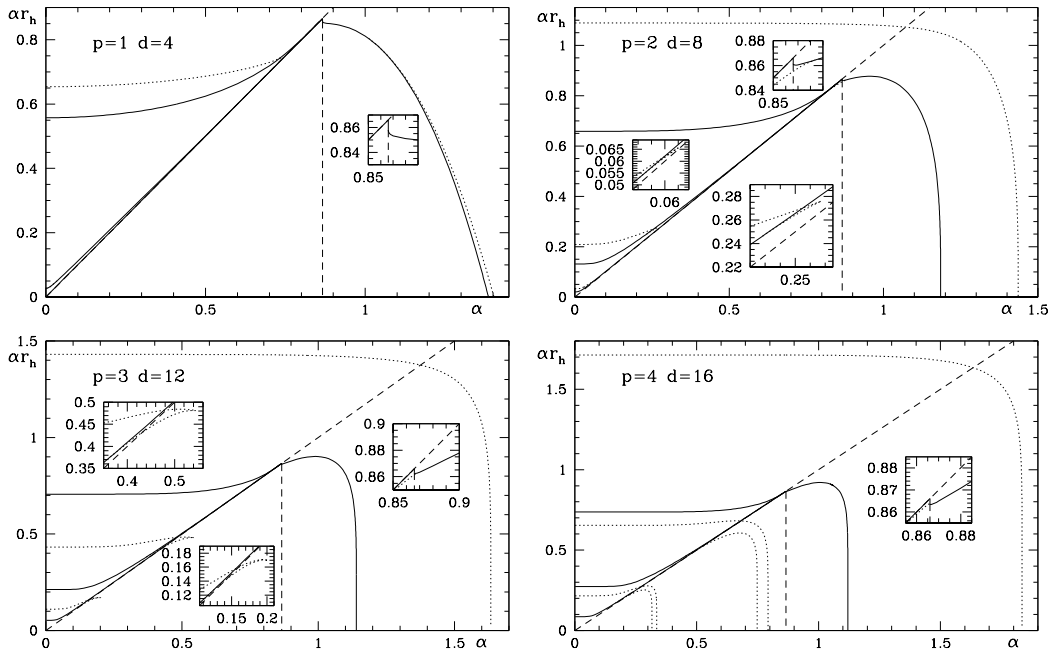
Figure 8: Domains of existence for (fundamental and excited) black monopole solutions with $p = 1$, 2, 3, and 4.

black monopoles always exist for $r_h < 1$ (below the diagonal) and $\alpha^2 < 0.75$ (left of the dashed vertical line), but extend beyond that region. Exhibiting maxima and minima (dotted lines) they either end as critical solutions (solid lines for $\alpha^2 \geq 0.75$, including a short vertical line near $r_h = 1$) or bifurcate with a regular RN black hole (solid lines for $\alpha^2 < 0.75$).

The fundamental solutions with $r_h \ll 1$ always have two branches, a 'lower' one from $\alpha = 0$ to some $\alpha_{\max}(r_h)$ and an 'upper' one from $\alpha_{\max}(r_h)$ to $\alpha_{\mathrm{cr}}(r_h)$. Likewise, the fundamental and excited solutions with $\alpha \ll 1$ always have two branches, a lower one from $r_h = 0$ to some $r_{\max}(\alpha)$ and an upper one from from $r_{\max}(\alpha)$ to $r_{\mathrm{bif}}(\alpha)$.

For $p = 1$ these upper branches of the fundamental solution cease to exist near $r_h = 1$. For $p > 1$ they exist for all r_h and are connected. In these cases continuity requires the existence of yet another branch near r_h and $\alpha = \sqrt{0.75}$, shown as detail in Fig. 8.

For $p = 1$ the upper branch of the excited solutions ceases to exist for larger values of α. For $p > 1$ the situation is somewhat more complicated. Increasing α, there first appears a minimum of r_h near the bifurcation, giving rise to a third branch. For $p = 2$ and 3 the maximum and minimum eventually meet, *i.e.*, the upper branch disappears and the lower and third branch are joined (compare Fig. 9). For $p = 4$, however, all three branches continue as $r_h \to 0$, with the 'upper' branch of the black monopoles corresponding to the 'middle' one of the regular monopoles. This correspondence suggests that the third branch observed
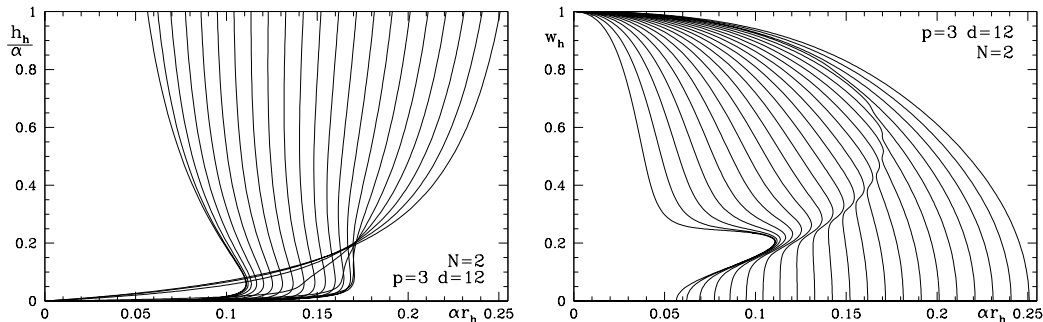
25

Figure 9: Initial data for the $N = 2$ excited black monopole solutions for $p = 3$ with $\alpha = 0.04, 0.05, \ldots, 0.25$.

for $p = 2$ and 3 is related to the steep (almost vertical) increase of b vs. α shown in Fig. 4.

4   Summary and Discussion

When it comes to choosing a model for a gravitating monopole in higher dimensions, the number of options proliferate. Our choices are guided by the twin criteria of having a Bogomol'nyi lower bound that can be saturated in the flat space limit of the Yang-Mills–Higgs (YMH) subsystem, and that aside from the gravitational coupling and the Higgs VEV there feature no other dimensionful constants in the model. This restriction is made to simplify both the analytic and the numerical analyses. The YMH systems employed, in $d = 4p$ spacetime, are those resulting from dimensional descent from $R^{4p-1} \times S^1$ which in flat space support selfdual solutions [16]. The gravitational systems are then chosen to be those members of the gravitational hierarchy (4) with $p = q$, such that the relation between the dimensions of the gravitational term in the Lagrangian (5) has exactly the same relation to the dimensions of the YMH term, for all p.

We have carried out a precise quantitative analysis of gravitating static monopoles, both regular and black, in $d = 4p$ spacetime dimensions. The models we have employed are in a sense very special, but the properties of their solutions are generic. Our template has been the gravitating Georgi-Glashow model in $d = 4$, in the BPS limit, *i.e.*, in the absence of the Higgs self interaction potential. This was the part of the subject of study in [1, 2] in which the Higgs potential was absent. We have emulated the results of [1, 2] exactly here, and have thus achieved our aim of showing that the generic properties of gravitating monopoles are fully understood at least within the restricted choice of models that we have exercised. As such, the present study is a general preliminary investigation into the nature of gravitating monopoles in higher dimensions, both regular and black.

In Section 2, we have presented an exhausive analytic analysis of the residual one dimensional action of the static system subject to spherical symmetry in

the spacelike dimensions. Especially prominent is the analysis of the Reissner–Nordström fixed point, and due to our restricted choice of systems, no *conical fixed points* [15] feature here. (Some numerical analysis of less restricted models was carried out which indicated the existence of conical fixed points, but we do not report on these here.) A preliminary step in the analysis for the gravitating Yang–Mills $p-$hierarchy was carried out before proceeding to the case of YMH in $d = 4p$. This underpins the numerical results of [26]. That was followed in Section 3 by the numerical analysis of the $4p-$dimensional gravitating monopoles, where the qualitative features discovered in [1, 2] for the $p = 1$ case were reproduced quantitatively for $p = 1, 2, 3$ with high accuracy. It is clear that the regular pattern observed *modulo* $4p$ up to $p = 3$ will repeat for all p.

Having achieved our aim of exhibiting the regularity of features *modulo* $4p$ of the simplest class of gravitating monopoles, it is perhaps in order to point to natural succeeding investigations. In this context we would exclude the otherwise obvious choice of employinging the usual ($p = 1$) Einstein–Hilbert system in models in $d \geq 5$, since this would result in the additional complication of encountering conical fixed points. As for introducing the usual Yang–Mills system in $d \geq 5$, this is excluded on the grounds that already in the flat space limit it would cause the mass/energy to diverge because of the *half–pure-gauge* asymptotic decay of the YM connection in all dimensions. The following options are open.

a) Extend the static spherically symmetric Ansatz (7) to allow for the electric YM connection $A_0 = u(r) \frac{x_j}{r} \Sigma_{j\,D-1}$. These would yield the gravitating versions of the dyons constructed in [16].

b) Construct the gravitating monopoles in *odd* spacetime dimensions. The main difference of the YMH models to be employed in this task, and the ones used in the present work, is that unlike the latter the former arise from the dimensional descent from $R^D \times K^N$ where N is now even and hence $N \geq 2$. As such, the gravity decoupling limits of these solutions do not saturate the Bogomol'nyi lower bound [22], but this is not important since in any case the gravitating monopoles are not selfdual. In this respect gravitating monopoles in *odd* spacetimes are similar to those in *even* spacetimes for $N \geq 3$, with odd N, so that the two tasks should be performed in parallel. Another difference between $N = 1$ and $N > 1$ models is that in the $N > 1$ models expressions of the Lagrangians for increasing p become progressively more cumbersome, in contrast with the $N = 1$ models for which the $p-$hierarchy is quite uniform. For this reason in the $N > 1$ case it is reasonable to restrict to $p = 2$ to get a glimpse of the qualitative features. Were one to restrict attention to the $p = 2$ YMH systems, then presumably the most aesthetic (if not necessary) choice for the gravitational system would be the $p = 2$ member of the gravitational hierarchy. Restricting to $p = 2$ YMH systems, the pertinent examples are $d = 6+1$, $d = 5+1$ and $d = 4+1$ spacetimes. The $p = 2$, $d = 3 + 1$ case is also of some interest, as it presents very different properties from the usual $p = 1$, $d = 3 + 1$ case, notably supporting mutually attracting like-charged monopoles [32].

Acknowledgements

A   The New Variables

With the substitutions Eqs. (18) the Action (9) takes the form

$$S_G = -\frac{1}{2p} \int d\tau \, e^{\nu+\lambda} \frac{1}{\dot r} \frac{d}{d\tau} \left(r^{d-2p-1}(1 - e^{-2\lambda} \dot r^2)^p \right) , \tag{53a}$$

$$S_M = \int d\tau \, e^{\nu+\lambda} \, r^{d-4p} \left[W^{p-1} \left(e^{-\lambda} \frac{dw}{d\tau} \right)^2 + \frac{d - 2p - 1}{2p} \frac{W^p}{r^2} \right.$$

$$\left. + \frac{1}{2p} \left(r e^{-\lambda} \frac{dH_p}{d\tau} \right)^2 + (d - 2p - 1) w^2 H_p^2 \right] . \tag{53b}$$

As a first step we want to absorb all τ-derivatives of λ in Eq. (53a) into a surface term. To write this in compact form we introduce a polynomial F_p uniquely determined by the property $F_p(z) + 2z F_p'(z) = (1 - z)^p$. We thus obtain

$$S_G = \int d\Phi(e^{-\lambda} \dot r) - \int d\tau \, e^{\nu+\lambda} \, r^{d-2p-2} \left[(r e^{-\lambda} \dot\nu)(e^{-\lambda} \dot r) F_p(e^{-2\lambda} \dot r^2) \right.$$

$$\left. + (d - 2p - 1) \left[\frac{1}{2p} (1 - e^{-2\lambda} \dot r^2)^p + (e^{-\lambda} \dot r)^2 F_p(e^{-2\lambda} \dot r^2) \right] \right] , \tag{54}$$

with

$$\Phi(x) = e^\nu r^{d-2p-1} x \, F_p(x^2) . \tag{55}$$

Next, we consider an action of the form

$$S = \int d\tau \, e^\lambda \, L(\varphi, e^{-\lambda} \dot\varphi) , \tag{56}$$

and introduce ψ_i as *abbreviation* for $e^{-\lambda} \dot\varphi_i$. Varying this action w.r.t. φ_i yields the second order equations

$$\sum_j M_{ij} \, e^{-\lambda} \dot\psi_j = \frac{\partial L}{\partial \varphi_i} - \sum_j \frac{\partial^2 L}{\partial \psi_i \partial \varphi_j} \psi_j , \qquad \text{where} \quad M_{ij} = \frac{\partial^2 L}{\partial \psi_i \partial \psi_j} , \tag{57}$$

and we assume that the determinant $|M_{ij}|$ does not vanish identically. Varying the action w.r.t. λ yields the first order equation

$$L = \sum_i \psi_i \frac{\partial L}{\partial \psi_i} \psi_i , \tag{58}$$

i.e., the reparametisation constraint.

In order to obtain a system of first order differential equations we now want to introduce *new variables* ψ_i as Lagrange multipliers. For a Lagrangian of the form

$$L(\varphi, \psi) = P(\varphi) + \sum_{ij} \psi_i \psi_j \, K_{ij}(\varphi) \,, \tag{59}$$

i.e., qudratic in the derivatives, this is achieved by the replacement

$$L(\varphi, e^{-\lambda}\dot{\varphi}) \Rightarrow \tilde{L} = L - \sum_{ij}(e^{-\lambda}\dot{\varphi}_i - \psi_i)(e^{-\lambda}\dot{\varphi}_j - \psi_j)\, K_{ij}$$
$$= P + \sum_{ij}(e^{-\lambda}\dot{\varphi}_i\psi_j + \psi_i e^{-\lambda}\dot{\varphi}_j - \psi_i\psi_j)K_{ij} \,. \tag{60}$$

We can generalise this procedure to the case with higher powers of derivatives by the replacement

$$L(\varphi, e^{-\lambda}\dot{\varphi}) \Rightarrow \tilde{L} = L(\varphi, \psi) + \sum_{i}(e^{-\lambda}\dot{\varphi}_i - \psi_i)\frac{\partial L(\varphi, \psi)}{\partial \psi_i} \,. \tag{61}$$

Varying the new action $\tilde{S} = \int d\tau \, e^{\lambda} \, \tilde{L}$ w.r.t. the new variables ψ_i yields

$$\sum_{j} M_{ij}(e^{-\lambda}\dot{\varphi}_j - \psi_j) = 0 \,, \tag{62}$$

i.e., what used to be abbreviations are now field equations. Varying w.r.t. φ_i or λ and substituting these new field equations reproduces the previous results Eqs. (57) and (58).

Applying this procedure to $(S_G - \int d\Phi)$ in Eq. (54) and replacing $\Phi(e^{-\lambda}\dot{r})$ by $\Phi(n)$ yields Eq. (20a). Applying the procedure to Eq. (53b) yields Eq. (20b).

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