# Topological Degeneracy and Vortex Manipulation in Kitaev's Honeycomb Model

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The classification of loop symmetries in Kitaev's honeycomb lattice model provides a natural framework to study the Abelian topological degeneracy. We derive a perturbative low-energy effective Hamiltonian that is valid to all orders of the expansion and for all possible toroidal configurations. Using this form we demonstrate at what order the system's topological degeneracy is lifted by finite size effects and note that in the thermodynamic limit it is robust to all orders. Further, we demonstrate that the loop symmetries themselves correspond to the creation, propagation, and annihilation of fermions. We note that these fermions, made from pairs of vortices, can be moved with no additional energy cost.

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Recently, Kitaev introduced a spin-$1/2$ quantum lattice model with Abelian and non-Abelian topological phases [1]. This model is relevant to ongoing research into topologically fault-tolerant quantum information processing [2–4]. The system is comprised of two-body interactions and is exactly solvable, which makes it attractive both theoretically [5–16] and experimentally [17–20].

Here, by classifying the loop symmetries of the system according to their homology, we derive a convenient form of the effective Hamiltonian on the torus. The result is valid for all orders of the Brillouin-Wigner perturbative expansion around the fully dimerized point as well as for all toroidal configurations. This allows the system's topological degeneracy to be addressed and shows at what order in the expansion the degeneracy is lifted. In the thermodynamic limit the system's topological degeneracy remains to all orders. In a separate analysis, valid for the full parameter space, we examine the paired-vortex excitations created by applying certain open string operations to the ground state. These vortex pairs are fermions and can be freely transported in a way that keeps additional unwanted excitations to a minimum.

The Hamiltonian for the system can be written as

$$H = -\sum_{\alpha\in\{x,y,z\}}\sum_{i,j} J_\alpha K_{ij}^{\alpha,\alpha}, \tag{1}$$

where $K_{ij}^{\alpha,\beta} \equiv \sigma_i^\alpha \otimes \sigma_j^\alpha$ denotes the exchange interaction occurring between the sites $i$ and $j$ connected by a $\beta$ link; see Fig. 1. In what follows, we will use $K_{ij}^{\alpha} \equiv K_{ij}^{\alpha,\alpha}$ whenever $\alpha = \beta$. Following Ref. [1], we consider loops of $n$ nonrepeating $K$ operators, $K_{ij}^{\alpha^{(1)}} K_{jk}^{\alpha^{(2)}}, \ldots, K_{li}^{\alpha^{(n)}}$, where $\alpha^{(m)} \in x, y, z$. Any loop constructed in this way commutes with the Hamiltonian and with all other loops. When the model is mapped to free Majorana fermions coupled to a $\mathbb{Z}_2$ gauge field, these loop operators become Wilson loops [1]. The plaquette operators

$$W_p = \sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z, \tag{2}$$

where the numbers 1–6 label lattice sites on single hexagonal plaquette $p$ (see Fig. 1), are the closed loop operators around each of the hexagons of the lattice. The commutation relations imply that we may choose energy eigenvectors $|n\rangle$ such that $w_p = \langle n|W_p|n\rangle = \pm 1$. If $w_p = -1$, we say that the state $|n\rangle$ carries a vortex at $p$.

For a finite system of $N$ spins on a torus, there are $N/2$ plaquettes. The product of all plaquette operators is the identity, and this is the single nontrivial relation between them. Hence there are only $N/2 - 1$ independent quantum numbers: $\{w_1, \ldots, w_{N/2-1}\}$. All homologically trivial loops are products of plaquettes. The relevant homology is $\mathbb{Z}_2$, since loop operators square to the identity. To describe the full symmetry group generated by loop operators, we introduce generators for the nontrivial $\mathbb{Z}_2$ homology classes of the surface that the lattice lives on. At most one generator per homology class is necessary, since all elements of any homology class can be generated from an arbitrary element of that class using the plaquettes. The $\mathbb{Z}_2$ homology group of the torus is $\mathbb{Z}_2 \times \mathbb{Z}_2$, so it is enough to add two homologically nontrivial loops as generators. The third nontrivial class is generated from the product of these two. The full loop symmetry group of the torus is the Abelian

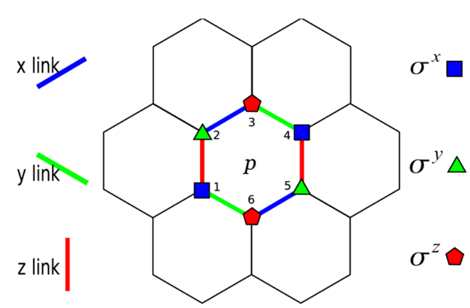


FIG. 1 (color online). The honeycomb lattice and plaquette $W_p$.

 

group with $N/2+1$ independent generators of order 2, that is, $\mathbb{Z}_2^{N/2+1}$. All closed loop symmetries can be written as

$$C_{(k,l)} = G_k F_l(W_1, W_2, \ldots, W_{N-1}). \tag{3}$$

Here $k \in \{0, 1, 2, 3\}$, $G_0 = I$, and $G_1$, $G_2$, and $G_3$ are arbitrarily chosen symmetries from the three nontrivial homology classes. The $F_l$, with $l \in \{1, \ldots, 2^{N/2-1}\}$, run through all monomials in the $W_p$.

The loop symmetries play an important role in the perturbation theory of the Abelian phase of the model. Following Kitaev, we take $J_z \gg J_x, J_y$ and write the Hamiltonian as $H = H_0 + U$, where $H_0 = -J_z \sum_{ij} K_{ij}^z$ is the unperturbed Hamiltonian and $U = -\sum_{\alpha\in\{x,y\}} J_\alpha \sum_{ij} K_{ij}^\alpha$ is the perturbative contribution. $H_0$ has a $2^{N/2}$-fold degenerate ground state space spanned by ferromagnetic configurations of the dimers on $z$ links. To understand how this degeneracy behaves under perturbation, we analyze the Brillouin-Wigner expansion [21,22]. The method returns the systems energies $E$ as an implicit nonlinear eigenvalue problem and thus, for the actual calculation of coefficients to high orders, can be difficult to apply [23]. However, we will take advantage of the infinite but exact nature of the series by recognizing that on the torus the form of the Hamiltonian is restricted, allowing one term for each element of the group of loop symmetries. This will facilitate a general discussion on the system's topological degeneracy.

Define $\mathcal{P}$ to be the projector onto the ferromagnetic subspace, and note that, for any exact eigenstate of the full Hamiltonian $|\psi\rangle$, the projection $|\psi_0\rangle = \mathcal{P}|\psi\rangle$ satisfies

$$\Big[E_0 + \sum_{n=1}^{\infty} H^{(n)}\Big]|\psi_0\rangle = E|\psi_0\rangle = H_{\mathrm{eff}}|\psi_0\rangle, \tag{4}$$

where $H^{(n)} = \mathcal{P}U\mathcal{G}^{n-1}\mathcal{P}$, $\mathcal{G} = [1/(E-H_0)](1-\mathcal{P})U$, and $E_0$ is the ground state energy of $H_0$. The eigenstates, with eigenvalue $E$, of the effective system and full system are related by the expression $|\psi\rangle = (1-\mathcal{G})^{-1}|\psi_0\rangle = \sum_{n=0}^{\infty}\mathcal{G}^n|\psi_0\rangle$.

Calculating the $n$th order correction is equivalent to finding the nonzero elements of the matrix $H^{(n)}$. Contributions to $H^{(n)}$ come from the length $n$ products $K_{ij}^{\alpha^{(1)}}, \ldots, K_{kl}^{\alpha^{(n)}}$, with $\alpha^{(m)} \in x, y$, that preserve the low-energy subspace. Hence any such contribution comes from an element of the group of loop symmetries from which all factors $K_{ij}^z$ have been removed.

The resulting low-energy effective Hamiltonian can be written in terms of operators acting on the spins of the dimers using the following transformation rules:

$$\begin{aligned} &\mathcal{P}[\sigma^x \otimes \sigma^y] \to +\sigma_e^y, \qquad \mathcal{P}[\sigma^x \otimes \sigma^x] \to +\sigma_e^x,\\ &\mathcal{P}[\sigma^y \otimes \sigma^y] \to -\sigma_e^x, \qquad \mathcal{P}[\sigma^z \otimes I] \to +\sigma_e^z,\\ &\qquad\qquad \mathcal{P}[\sigma^z \otimes \sigma^z] \to +I_e, \end{aligned} \tag{5}$$

where the subscript $e$ indicates the effective spin operation and the arrow $\to$ can be read as "is represented by." Importantly, this transformation can be applied directly to the loop symmetries themselves, without removing the $z$ links first, and does not change the resulting operator on the low-energy subspace. The lowest order nonconstant contributions therefore come from the plaquette operators $\mathcal{P}[W_p] \to Q_p = \sigma_{e(l)}^y \sigma_{e(r)}^y \sigma_{e(u)}^z \sigma_{e(d)}^z$, where $l$, $r$, $u$, and $d$ denote the positions (left, right, up, and down, respectively) of the effective spins, relative to the plaquette $p$ [1]. Expanding to all orders, we have contributions from all loop symmetries, both homologically trivial and nontrivial. To come to an explicit expression for the effective Hamiltonian, we now introduce a particular generating set for the loop symmetry group, constructed from $N/2-1$ plaquettes and the operators $Z \equiv \prod_i \sigma_i^z$, where $i$ represents lattice sites in the horizontal direction of alternating $x$ and $y$ links, and $V \equiv \prod K_{jk}^{x,y} \prod K_{lm}^{y,x}$, where the products take place over vertically arranged $x$ and $y$ links. The projections $\mathcal{P}(Z) \to z$ and $\mathcal{P}(V) \to y$ act by $\sigma_e^z$ and $\sigma_e^y$ on the relevant effective spins; see Fig. 2. In analogy to (3), we can now write the full effective Hamiltonian as

$$H_{\mathrm{eff}} = \sum_{k=0}^{3} \sum_{l=1}^{2^{N/2-2}} d_{k,l} G_k(z, y) F_l(Q_1, Q_2, \ldots, Q_{N/2-2}), \tag{6}$$

where $G_0 = I$, $G_1 = z$, $G_2 = y$, $G_3 = zy$, and the $d_{k,l}$ are constants which depend on $J_x$, $J_y$, and $J_z$. This form is strictly valid for when the effective square toroidal lattice has an even number of plaquettes $Q_p$ along both directions. The inside sum only runs to $2^{N/2-2}$ because, as a result of the projection, we now have two nontrivial relations $\prod Q_b = \prod Q_w = 1$; see Fig. 2. These arguments apply to even-even toric code configurations but can be generalized to the configurations examined in [24].

In general, $d_{k,l} \sim O(J_x^{n_x} J_y^{n_y})$, where $n_x$ and $n_y$ are the respective number of $x$ links and $y$ links used to make $G_k(z, y)F_l(Q_1, Q_2, \ldots, Q_{N/2-2})$. In the thermodynamic

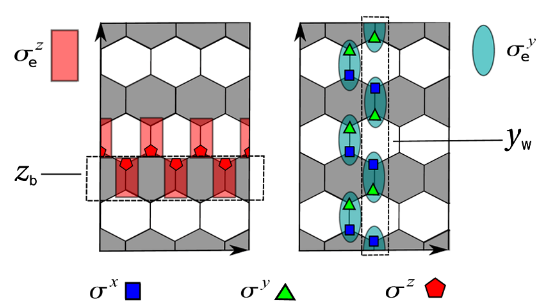


FIG. 2 (color online). The $Z$ and $V$ chains with their projections onto the dimerized subspace. The projections may be factorized into products $\mathcal{P}[Z] \to z_b z_w$ and $\mathcal{P}[V] \to y_b y_w$. Each of the individual factors $z_b$, $z_w$, $y_b$, and $y_w$ also commute with the homologically trivial components of the effective Hamiltonian but obey the relation $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$. In the text, dimers are referred to as black (white) if they are shared by a black (white) plaquette.

limit, and for homologically nontrivial loops ($k > 0$), the values of $n_x$ and $n_y$ go to infinity and the limiting form of (6) is similar to the form addressed in Ref. [12] but with additional topological degrees of freedom.

We can now analyze the topological degeneracy of the Abelian phase. The general argument for topological ground state degeneracy depends on the existence of operators $T_1$ and $T_2$ that both create particle-antiparticle pairs from the vacuum, bring the particle (or antiparticle) around the torus, and then annihilate the pair [2,25]. These operators should commute with the Hamiltonian but not with each other. Hence $T_1$ and $T_2$ operators for the honeycomb system cannot be contained within the group of commuting loop symmetries. However, the low-energy effective representations of the homologically nontrivial loops' generators have the factorizations $z = z_b z_w$ and $y = y_b y_w$, where $z_b$ and $y_b$ act with effective $\sigma^z$'s and $\sigma^y$'s, respectively, on the spins of the "black" dimers involved in $z$ and $y$, while $z_w$ and $y_w$ do the same for the "white" dimers (see Fig. 2).

These black and white operators correspond to the nontrivial loop operators on the square lattice and dual square lattice of the toric code (cf. [2]) and thus obey the commutation relations $z_j^{-1} y_k^{-1} z_j y_k = e^{i\pi(1-\delta_{jk})} I$. Since these operators commute with the effective plaquette operators $Q_p$, they also commute with all homologically trivial components of $H_{\text{eff}}$. However, they do not commute with all of the homologically nontrivial components. If we define $C'$ as the homologically nontrivial loop with the least number of $x$ and $y$ links, then the topological degeneracy is first broken at the order $M$, where $M$ is the number of $x$ and $y$ links in $C'$.

At any size, the plaquette and homologically nontrivial operators together generate all conserved quantities and, in particular, determine the energy. For the typical system sizes that can be handled by numerical diagonalization and other numerical methods, the homologically nontrivial terms in the effective Hamiltonian are appreciable and must be taken into account to produce a good fit to exact numerical results. In larger tori these homologically nontrivial terms become less relevant to the energy, and the topological degeneracy of the system can be robust beyond the 4th order toric code approximation. Indeed, in the thermodynamic limit, the fourfold topological degeneracy exists to all orders of the perturbation theory and the eigenstates of the effective Hamiltonian are exactly those of the toric code. One should note, however, that even in this limit, and unlike the toric code, the energy of a particular eigenstate is also determined by the relative positions of the vortex excitations [12].

We now concentrate on the full Hamiltonian and consider the physical properties associated with open-ended strings of overlapping $K^\alpha$ operators. We first note that $\{\sigma_i^\alpha, W_p\} = 0$ when the site $i$ belongs to an $\alpha$ link at plaquette $p$. Hence, the operator $\sigma_j^\alpha$ changes the vorticity of the two plaquettes sharing this $\alpha$ link by either creating or annihilating a pair of vortices or moving a vortex from one plaquette to the other. It follows that the $K$ operators satisfy $[K_{ij}^\alpha, W_p] = 0$ ($\forall\ i, j$), $[K_{ij}^{\alpha,\beta}, W_p] = 0$ ($i, j \notin p$), and $\{K_{ij}^{\alpha,\beta}, W_p\} = 0$ ($i \vee j \in p$).

Now define a path $s$ on the lattice as some ordered set of $|s|$ neighboring sites connecting the end points $i$ and $j$. A string operator, denoted as $S_{ij}^s$, of overlapping $K^\alpha$ operators along this path $s$ can be represented as a site ordered product of $\sigma^\alpha$ and $K^{\alpha,\beta}$ operators. We use the $K^{\alpha,\beta}$ notation in what follows when we wish to explicitly indicate the simultaneous operation of the constituent $\sigma^\alpha$ operators. If we assume that a $K^{\alpha,\beta}$ always acts first, we see that the total operator can be interpreted as creation of two vortex pairs and subsequent movement of one of the pairs along the path $s$. Importantly, we see that $\sigma^\alpha$ correspond to a rotation of one vortex pair, whereas $K^{\alpha,\beta}$ moves the pair without a rotation (see Fig. 3). If $i$ and $j$ are neighboring sites and $s$ is a homologically trivial loop, then by definition $C_{(k,l)} = S_{ij}^s = \prod_p W_p$, where the product is over all plaquettes enclosed by $s$ [see (3)]. If we treat a vortex pair as a composite particle, then the simplest loop operator $C_{(k,l)} = W_p$ (constructed from single $\sigma^\alpha$ operators) rotates the composite particle by $2\pi$. The resulting overall phase of $e^{i\pi}$ suggests that the vortex pairs are fermions for all values of $J_\alpha$.

Suppose now that the first and last links along the path $s$ are $\alpha$ and $\mu$ links, respectively, and that the ends of the string $S_{ij}^s$ are given by the operators $\sigma^\beta$ or $K^{\beta,\alpha}$ and $\sigma^\nu$ or $K^{\nu,\mu}$. Then

$$S_{ij}^s H S_{ij}^s = H + 2J_\gamma K_{ia}^\gamma + 2J_\tau K_{jb}^\tau, \tag{7}$$

where $a$ and $b$ are the sites connected to $i$ and $j$ by the respective $\gamma$ and $\tau$ links $\gamma \neq \alpha \neq \beta$ and $\tau \neq \mu \neq \nu$. Taking the expectation value of both sides with respect to any translationally invariant state $|\psi\rangle$, which includes the

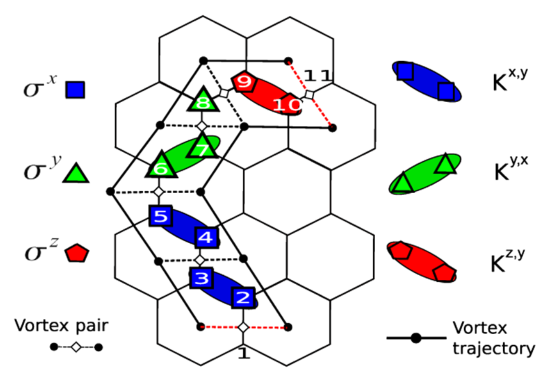


FIG. 3 (color online). The operator $K_{2,3}^{x,y}$ is used to create two vortex pairs from the vacuum with an energy cost of $2J_z\langle K_{1,2}^z + K_{3,4}^z\rangle$. The subsequent operators $K_{4,5}^{x,y}$ and $K_{6,7}^{y,x}$ move one of the pairs in the direction shown. The Pauli operator $\sigma_8^y$ rotates this pair, and the energy of the system at this time is $E_0 + 2J_z\langle K_{1,2}^z\rangle + 2J_x\langle K_{8,9}^x\rangle$. This new pair is then moved horizontally with no additional energy cost by $K_{9,10}^{z,y}$.

vortex-free ground state [1,26], we see that the expectation energy of the state $S^s_{ij}|\psi\rangle$ depends only on the ends of the string and this energy contribution is the same for links of the same type. This implies, even when $J_x \neq J_y \neq J_z$, that vortex pairs created from the ground state can be propagated freely provided the relative orientation of each pair remains constant. The expectation energy of the states created in this way can be calculated explicitly with respect to the ground state [1,9].

These fermionic vortex pairs are distinct from the fermions introduced as redundant degrees of freedom in Ref. [1], those obtained by Jordan-Wigner transformation [7,8,11], and the vorticity-preserving free-fermionic excitations of Ref. [12]. In the gapped phase, however, the low-energy vortex-pair configurations are related to certain fermionic $e$-$m$ composites of the toric code [1,2]. This last point is potentially relevant to the connection between the Abelian and the non-Abelian phases [16].

The movement of vortex pairs is in contrast to the situation encountered when one wants to separate individual vortices. Crucially, this cannot be done using overlapping $K^\alpha$ terms and indeed can be achieved only if we use single $\sigma^\alpha$'s that do not, in general, act on neighboring sites. To this end we define $D^s_{ij} \equiv \sigma^\alpha_i \sigma^\beta_k \ldots \sigma^\gamma_j$, where it is understood that if $k$ and $l$ are neighboring sites along some link, then $\alpha \neq \beta$. These operators satisfy

$$D^s_{ij} H D^s_{ij} = E_0 + aJ_x K^x + bJ_y K^y + cJ_z K^z, \qquad (8)$$

where $a + b + c = |s|$ for some integers $a$, $b$, and $c$ depending on the path $s$. As before, suppose we take the expectation value of (8) with respect to a translationally invariant state $|\psi\rangle$. In this case we see that the energy expectation value of the state $D^s_{ij}|\psi\rangle$ scales with $|s|$ and implies a string tension for states created in this way [27].

The above results, valid for all values of the parameters $J_\alpha$, are in agreement with the perturbative analysis of the gapped Abelian phase [15]. There it was shown that, while the repeated application of single $\sigma^\alpha$ excites $e$ or $m$ toric code semions in the low-energy dimerized subspace, it also introduces contributions to the wave function from higher-energy eigenstates. These high-energy eigenstate contributions also occur when low-energy vortex pairs are excited, but in this case two effective toric code $e$-$m$ pairs are created in the effective system. An easy way to see this is to compare the expectation energy $\langle K^{y,z} H K^{y,z}\rangle = E_0 + 4J\langle K^x\rangle \approx E_0 + 2J^2/J_z$ (see [15]) to that of the ground state energy of the 4-vortex configuration $E \approx E_0 + J^4/2J_z^3$, where $J = J_x = J_y$. However, since (7) implies that the vortex pairs can be moved freely, there can be no increase in the contribution from these high-energy states as these pairs are propagated. This may be useful for the experimental detection of anyons because, in the toric code, the $e$-$m$ pairs and single $e$ (or $m$) excitations have mutually anyonic statistics.

In conclusion, we have associated each of the loop symmetries of the full toroidal system with a term in the perturbative expansion. We then demonstrated the order at which the topological degeneracy is broken and noted that, in the thermodynamic limit, it remains to all orders. In a further analysis, we showed that the symmetries correspond to propagation of vortex pairs along closed loops. When treated as composite particles, the vortex pairs are fermions. We showed that these pairs can be propagated with no additional energy cost but that, in general, single vortices cannot. In relation to the Abelian phase, we have included a discussion that shows how vortex pairs maybe transported in way that keeps the fermionic population to a minimum. This argument relies on a detailed understanding of the supporting spectrum. It may be possible to make similar arguments in the non-Abelian phase; however, the necessary understanding of the supporting eigenspectrum is currently lacking.

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[27] This result does not imply vortex confinement because the state $D^s_{ij}|\psi\rangle$ is not necessarily the ground state of the relevant vortex configuration sector. Note that the perturbative results of Ref. [12] suggest that the vortex configuration ground state energies, as a function of the distance between vortices, are bounded.