NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

Stability Analysis of Positive Systems

with Applications to Epidemiology

A dissertation submitted for the degree of
Doctor of Philosophy

By:

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To all those who offered me their help and support,
in my ongoing journey,
to become who I want to be.

Contents

Declaration

I herewith declare that I have produced this manuscript without the prohibited assistance of third parties and without making use of aids other than those specified.

The thesis work was conducted from December 2008 to January 2011 under the supervision of Dr. Oliver Mason in Hamilton Institute, National University of Ireland Maynooth. This work was supported by Science Foundation Ireland (SFI) grant 08/RFP/ENE1417.

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Acknowledgement

Looking back at my life, I can see that more than anything else, my life has been influenced by a chain of accidents and events over which I had little or no influence. That of course has never stopped me in trying to fully utilise the consequences of those events which were in my favour, and mitigate the effects of those that were against me. In doing so, nothing has been more valuable to me than the support and assistance of those people who decided to help me, out of love, compassion, kindness or even a sense of responsibility. Thus, at the beginning of my Ph.D. dissertation, I like to acknowledge those individuals who made my Ph.D study one of the most productive and interesting stages of my life.

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While still thy body's breath is warm and sweet,
Follow thy pleasures with determined feet,
Ere death, the coldest lover in the world,
Catches thee up with footsteps still more fleet.

Omar Khayyam (1048-1131), Persian poet and mathematician.

The Richard Le Gallienne's translation.

Abstract

In this thesis, we deal with stability of uncertain positive systems. Although in recent years much attention has been paid to positive systems in general, there are still many areas that are left untouched. One of these areas, is the stability analysis of positive systems under any form of uncertainty. In this manuscript we study three broad classes of positive systems subject to different forms of uncertainty: nonlinear, switched and time-delay positive systems. Our focus is on positive systems which are monotone. Naturally, monotonicity methods play a key role in obtaining our results.

We start with presenting stability conditions for uncertain nonlinear positive systems. We consider the nonlinear system to have a certain kind of parametric uncertainty, which is motivated by the well-known notion of D-stability in positive linear time-invariant systems. We extend the concept of D-stability to nonlinear systems and present conditions for D-stability of different classes of positive nonlinear systems. We also consider the case where a class of positive nonlinear systems is forced by a positive constant input. We study the effects of adding such an input on the properties of the equilibrium of the system.

We then present conditions for stability of positive time-delay systems, when the value of delay is fixed, but unknown. These types of results are known in the literature as delay-independent stability results. Based on some recent results on delay-independent stability of linear positive time-delay systems, we present conditions for delay-independent stability of classes of positive nonlinear time-delay systems.

After that, we present conditions for stability of different classes of positive linear and nonlinear switched systems subject to a special form of structured uncertainty. These results can also be considered as the extensions of the notion of D-stability to positive switched systems.

And finally, as an application of our theoretical work on positive systems, we study a class of epidemiological systems with time-varying parameters. Most of the work done so far in epidemiology has been focused on models with time-independent parameters. Based on some of the recent results in this area, we describe the epidemiological model as a switched system and present some results on stability properties of the disease-free state of the epidemiological model.

We conclude this manuscript with some suggestions on how to extend and develop the presented results.

Notations

Symbol	**Meaning**
$\mathbb{R}$	The field of real numbers
$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathbb{R}^n$	The space of column vectors of size n of real numbers
$\mathbb{R}^{n\times n}$	The space of $n \times n$ matrices of real numbers
$\mathbb{C}$	The field of complex numbers
$\Re(c)$	The real part of the complex number c
$\Im(c)$	The imaginary part of the complex number c
x_i	The ith entry of the vector x in $\mathbb{R}^n$, for $i \in \{1, \cdots, n\}$
x_0	A vector in $\mathbb{R}^n$ that usually represents initial condition
$y = \lvert x\rvert$	A vector with the same dimension as x with $y_i = \lvert x_i\rvert$
$\lVert x\rVert$	Any p-norm of the vector x
a_{ij}	The (i, j) entry of the matrix A
$D = \text{diag }(x)$	An $n \times n$ diagonal matrix in which $d_{ii} = x_i$ for all i
A^T	The transpose of the matrix A
A^{-1}	The inverse of the matrix A
$\lvert A\rvert$	The determinant of the matrix A
$\lVert A\rVert$	Any p-norm of the matrix A
$A \gg B$	$a_{ij} > b_{ij}$, for all $i, j \in \{1, \cdots, n\}$
$A > B$	$a_{ij} \geq b_{ij}$, for all $i, j \in \{1, \cdots, n\}$ and $A \neq B$
$A \geq B$	$a_{ij} \geq b_{ij}$, for all $i, j \in \{1, \cdots, n\}$
$\mathbb{R}^n_+$	The positive orthant of $\mathbb{R}^n$, given by $\{x \in \mathbb{R}^n : x \geq 0\}$
I	The identity matrix of proper dimensions
0	The zero matrix of proper dimensions
$\sigma(A)$	The set of all eigenvalues (spectrum) of the matrix A
$\rho(A)$	The spectral radius of the matrix A
$\mu(A)$	The spectral abscissa of the matrix A

CHAPTER 1

Introduction

In this chapter, we discuss the motivations behind the work of this thesis and provide an overview of the material presented in the following chapters.

1.1 Introductory Remarks

Positive systems are those systems whose states remain nonnegative for all future times, if started from nonnegative initial conditions. Positivity is not an inherent property of a system; we might be able to turn a positive system into a non-positive system with a simple change of variable. However, in many cases the variables which are normally chosen as state variables can only take nonnegative values, thus, leading to models which are in fact positive systems. Such systems can be found in different areas of science and technology. We recall a few examples here.

Biology and physiology. The theory of positive systems has been used repeatedly and successfully to model biological and physiological systems. That is mainly motivated by the fact that an important characteristic of biochemical models is that most variables take only nonnegative values, since they usually represent chemical concentrations. [Son05] and [HC05] are among the many references that describe the links between biological systems and positive systems theory.

Ecology and population dynamics. Positive systems are natural choices for modelling ecological systems. The reason is that in ecological systems, usually the states of the system are the populations of different species, which are of course nonnegative valued. That is why since the early works of Lotka [Lot25] and Volterra [Vol26] on ecological systems, positive systems have played an important role in this field. Look for example at [HS98b] for a review of some of the more common ecological models.

Economics. Perhaps the most significant, but certainly not the only, area in economics in which the theory of positive systems plays an important role is *input-output analysis*. Input-output analysis is introduced by Leontief in [Leo36]. The input-output analysis deals with the specific question of what should be the production level of different industries in an economy, in order to satisfy the demand for a certain product in a particular economic situation. Another important application of positive systems in economics is in the study of the von Neumann model, which is developed by von Neumann in [vN45]. This model introduces the concept of equilibrium growth and provides the first proof that a solution to a general equilibrium model exists. [BP94, Chapter 9] provides more details on the applications of the theory of positive systems in the input-output model.

Communications. Positive systems have been used successfully in the modelling, analysis and control of different phenomena in communications. For example in congestion control in TCP networks [SWL06, SKWL07], and in stability analysis of distributed power control algorithms for cellular communications [PAMS04].

Compartmental systems. A compartment is an amount of some material that is kinetically homogeneous. Kinetically homogeneous means that the material of a compartment is at all times homogeneous; any material entering it is instantaneously mixed with the material of the compartment [JS93]. A compartment can, for example, be a physical model of a part of the body of a living organism with a more or less homogeneous concentration of a chemical compound or of a material. A compartmental system is compromised of a number of compartments.

Compartmental systems were first used in physiology, but are natural models for pharmacokinetics, epidemiology, ecology, and other systems whose models are derived from mass balance considerations. The variables of interest in a compartmental system are usually the mass of each compartment and the flows into and out of it. Since we deal with mass of different substances in these cases, therefore, we can usually describe the compartmental systems as a positive system. In other words, compartmental systems can be described as a subset of the class of positive systems. Look at [JS93] and [BF02] and references therein for more information on compartmental systems.

Genetic regulatory network. A genetic regulatory network is a collection of DNA segments in a cell which interact with each other indirectly (through their RNA and protein expression products) and with other substances in the cell, thereby governing the rates at which genes in the network are transcribed into mRNA. In general, each mRNA molecule goes on to make a specific protein (or set of proteins). Genes can be viewed as nodes in a network, with inputs being proteins such as transcription factors, and outputs being the level of gene expression. It is common to model such a network with a set of coupled (possibly stochastic) ordinary differential equations describing the reaction kinetics of the constituent parts. In such cases, the states of the systems are the concentrations of the corresponding substances, which are in fact nonnegative variables. [dJ02] provides a review of different approaches in studying genetic regulatory networks.

Markov chains. A Markov chain is a stochastic process with finite or countable states. It is a random process whose next state depends only on the current state and not on the sequence of events that preceded it. If the state space is finite, the transition probability distribution of the Markov chain at each step (or instant of time) can be represented by a matrix, called the transition matrix. The transition matrix is a stochastic matrix, meaning each of its rows sums to one and all elements are nonnegative. Markov chains have found numerous applications in different areas of science. Look for example at [BP94, Chapter 8] and [Sen06] for a better understanding of Markov chains in the context of positive systems.

Chemostats and bioreactors. A bioreactor is a vessel in which a chemical process is carried out which involves organisms or biochemically active substances derived from such organisms. A chemostat is a bioreactor to which fresh medium is continuously added, while culture liquid is continuously removed to keep the culture volume constant. Chemostats are used for investigations in cell biology, as a source for large volumes of uniform cells or protein. The chemostat is often used to gather steady state data about an organism in order to generate a mathematical model relating to its metabolic processes. Chemostats are also used as microcosms in ecology and evolutionary biology. The parameters of interest in a chemostat (for example culture volume, dissolved oxygen concentration, nutrient and product concentrations, pH, cell density, etc.) are nonnegative variables. Thus a chemostat can be modelled as a positive system. Look at [SW95] for more information on chemostats and on the applications of dynamical systems to their analysis.

Epidemiology. Usually in epidemiological models, the population that is under study is divided into different classes and the numbers of individuals in each of these classes are considered as state variables. That is why sometimes epidemiological models are studied as a special case of ecological models. We have dedicated a whole chapter in this manuscript to epidemiological models. Chapter 6 discusses the stability analysis of a class of epidemiological models described as a positive nonlinear switched system.

Of course the positivity of systems would not have been of much significance, if it was not because of the theoretical properties of positive systems that make their analysis more straightforward. That is ,in particular, true for positive linear systems that have a rich literature and been investigated thoroughly. The theory of positive linear systems is intertwined with that of nonnegative matrices, i.e., matrices with nonnegative entries. In fact, in many applications of the theory of the positive systems (including some of the above mentioned examples), only properties of nonnegative matrices have been utilised without any explicit references to positive systems. In the next chapter, we see that for a discrete-time linear system to be positive, its representative matrix should be a nonnegative matrix but that is not necessary for positivity of a continuous-time linear system. Although, with a simple transformation, some of those powerful properties of nonnegative matrices can be used in the analysis of continuous-time positive linear systems.

Even though stability properties of positive linear time-invariant systems are now well understood, there are still a lot of unanswered questions in the study of stability properties of other classes of positive systems, such as positive nonlinear, time-delay and switched systems. That is in particular true if we consider positive systems which are subject to uncertainty. Although there is a rich literature on properties of uncertain systems, they have been rarely studied in the context of positive systems. To fill this void, in this thesis work, we deal with stability properties of nonlinear, time-delay and switched positive systems under different forms of uncertainty. The results presented for stability of uncertain nonlinear and switched positive systems are inspired by the concept of D-stability, which is one of the well-known properties of positive linear systems (as discussed in the next chapter). We also present conditions for stability of positive time-delay systems, when the value of delay is fixed, but unknown. Such results are known as delay-independent stability results.

1.2 Overview

We begin by setting the context and providing the background material for much of the later work in Chapter 2. Basic concepts of matrix analysis and dynamical systems as well as the relevant properties of positive linear time-invariant and nonlinear systems are all reviewed in this chapter. We also introduce homogeneous and subhomogeneous systems and recall some of the relevant properties of monotone systems. We also state sufficient conditions for positivity of monotone systems which are used in the following chapters. The KKM lemma, which is a well-known result in fixed-point theory is also discussed in Chapter 2.

In Chapter 3, we introduce the concept of D-stability for positive nonlinear systems and present results on D-stability for homogeneous and subhomogeneous monotone systems. We also present a local D-stability result for general cooperative systems. Moreover, we study the case where a monotone subhomogeneous system is forced by a constant input and study how such an input affects positivity and stability properties of its equilibria.

In Chapter 4, we study positive nonlinear systems which are subject to time-delay. Inspired by the recent results on delay-independent stability of positive linear time-delay systems, we present delay-independent stability results for homogeneous and subhomogeneous cooperative time-delay systems. We also state a local delay-independent result for general cooperative systems.

In Chapter 5, we deal with positive switched systems. In this chapter, we extend the concept of D-stability to positive switched systems. Most of the work done on positive switched systems so far has been focused on positive linear switched systems. In line with those efforts, we present novel necessary and sufficient stability conditions for positive linear switched systems. Then we add an irreducibility assumption to state a single necessary and sufficient condition for positive linear switched systems. We also state D-stability conditions for homogeneous cooperative systems and positive switched systems with commuting vector fields.

In Chapter 6, we study a class of epidemiological models with time-varying parameters. We consider the SIS model, which is used to model diseases that do not confer immunity. SIS models have been used in modelling diseases like tuberculosis and gonorrhoea and also in studying propagation of computer viruses. Most of the research done on different classes of epidemiological models in general and SIS models in particular have been focused on networks with time-independent parameters. That is clearly a simplistic assumption, because apart from spatial movements of individuals, their health conditions are also subject to change, for example via vaccination or malnutrition. As a step in extending the current results to networks with time-varying parameters, we describe the SIS model as a positive switched system and then state different stability conditions for the disease-free equilibrium of the switched SIS model.

In Chapter 7, we summarize the stated results and conclude the thesis by outlining possible directions for extending those results.

1.3 Publications

The following journal publications were prepared and published in the course of this doctorate:

V. S. Bokharaie, O. Mason, and M. Verwoerd. D-stability and delay-independent stability of homogeneous cooperative systems. *IEEE Transactions on Automatic Control*, 55(12):1996-2001, December 2010.

V. S. Bokharaie, O. Mason, and M. Verwoerd. Correction to D-stability and delay-independent stability of homogeneous cooperative systems. *IEEE Transactions on Automatic Control*, 56(6):1489, June 2011.

V. S. Bokharaie, O. Mason, and F. Wirth. Stability and positivity of equilibria for subhomogeneous cooperative systems. *Nonlinear Analysis*, 74:6416-6426, 2011.

The following journal publication (with preliminary title) is in the final stages of preparation:

V. S. Bokharaie, O. Mason, F. Wirth and M. Ait Rami. Stability Criteria for SIS Epidemiological Models under Switching Policy. To be submitted to *Discrete and Continuous Dynamical Systems - Series S, (Special Issue on Positive Dynamical Systems)*.

The following conference publications were also prepared and presented in the course of this doctorate:

O. Mason, V. S. Bokharaie, and R. Shorten. Stability and D-stability for switched positive systems. In *Proceeding of the third Multidisciplinary Symposium on Positive Systems (POSTA09)*, pages 101-109, 2009.

V. S. Bokharaie, O. Mason, and F. Wirth. On the D-stability of linear and nonlinear positive switched systems. In *Proceedings of International Symposium on Mathematical Theory of Networks and Systems (MTNS2010)*, pages 1717-1719, 2010.

V. S. Bokharaie, O. Mason, and F. Wirth. Spread of epidemics in time-dependent networks. In *Proceedings of International Symposium on Mathematical Theory of Networks and Systems (MTNS2010)*, pages 795-798, 2010.

CHAPTER 2

Background

This chapter provides the mathematical basis required to understand the results presented in the following chapters. We recall some definitions in matrix and nonlinear analysis and state some basic definitions and results related to dynamical systems.

2.1 Introduction

In this chapter, we define various concepts and results that will be used in the following chapters. Apart from the results concerning subhomogeneous vector fields and their properties, the rest of the results presented in this chapter are well-known results in the literature. We first recall some basic results in linear algebra and then discuss some known properties and definitions concerning dynamical systems. Then we focus on linear and nonlinear positive systems and their properties. Classes of homogeneous and subhomogeneous systems, which will be repeatedly used in the following chapters, are discussed in the context of positive systems. Switched systems will be discussed thoroughly in Chapter 5, therefore, we leave the discussion on basic properties of switched systems to that chapter.

2.2 Basic Concepts in Matrix Analysis

Throughout this manuscript, $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers, respectively. The real part of $c \in \mathbb{C}$ is represented by $\Re(c)$ and its imaginary part by $\Im(c)$. $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries. $\mathbb{R}^n$ is the space of column vectors of size n with real entries. For $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, x_i denotes the i^{th} coordinate of x. Note that, we usually use x_0 for the initial condition of a system or just a general vector in $\mathbb{R}^n$. For $A \in \mathbb{R}^{n\times n}$, a_{ij} denotes the $(i,j)^{th}$ entry of A. A^T represents the transpose matrix of A and A^{-1} is the inverse matrix of A. Also, for $x \in \mathbb{R}^n$, $D = \text{diag}\,(x)$ is the $n \times n$ diagonal matrix in which $d_{ii} = x_i$. We use 0 to refer to a vector or matrix of appropriate dimensions with all entries equal to zero. I refers to identity matrix of proper dimensions.

We define:

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$$

and

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$$

$\mathbb{R}^n_+$ is called the *positive orthant* of $\mathbb{R}^n$.

Given a point $x_0 \in \mathbb{R}^n$ and a real number $r > 0$, $B(x_0; r)$ is called an *open ball* around x_0 if we have:

$$B(x_0; r) = \{x \in \mathbb{R}^n : d(x, x_0) < r\}$$

where $d(\cdot,\cdot)$ represents any metric in $\mathbb{R}^n$. x_0 is called the centre and r the radius of the ball. For any subset $\mathcal{U}$ of $\mathbb{R}^n$, a point x_0 is called an *interior point* of $\mathcal{U}$ if there is an open ball around x_0 which is wholly contained in $\mathcal{U}$. The set of all interior points of $\mathcal{U}$ is called the interior of $\mathcal{U}$ and is denoted by $\text{int}\,(\mathcal{U})$. In the special case where $\mathcal{U} = \mathbb{R}^n_+$, we have:

$$\text{int}\,(\mathbb{R}^n_+) := \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}$$

For a closed subset $\mathcal{U}$ of $\mathbb{R}^n$ the boundary of $\mathcal{U}$ is defined as:

$$\text{bd}\,(\mathcal{U}) := \mathcal{U} \setminus \text{int}\,(\mathcal{U})$$

Let x_0 be a point in $\mathbb{R}^n$. An open ball $B(x_0; \epsilon)$ of radius $\epsilon > 0$ is often called an *ϵ-neighbourhood* of x_0. By a *neighbourhood* of x_0, we mean any subset of $\mathbb{R}^n$ which contains an ϵ-neighbourhood x_0 [Kre78, p. 19].

Let $\mathcal{U}$ and $\mathcal{W}$ be subsets of $\mathbb{R}^n$. The set $\mathcal{W}$ is a *neighbourhood* of the set $\mathcal{U}$, if it contains a neighbourhood of every point of $\mathcal{U}$.

For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$; $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ if $x_i > y_i$ for $1 \leq i \leq n$. Similarly, for matrices $A, B \in \mathbb{R}^{n\times n}$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$; $A > B$ if $A \geq B$ and $A \neq B$; and $A \gg B$ if $a_{ij} > b_{ij}$ for $1 \leq i, j \leq n$. Note that in this manuscript, we will *never* use $A > 0$ ($A < 0$) symbol to refer to positive-definite (negative definite) matrices.

For $A \in \mathbb{R}^{n\times n}$, we denote the *spectrum* of A by $\sigma(A)$. The *spectral radius* of A is denoted by $\rho(A)$ and is defined as:

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$$

Also, the notation $\mu(A)$ denotes the *spectral abscissa* of A which is defined as follows:

$$\mu(A) := \max\{\Re(\lambda) : \lambda \in \sigma(A)\}$$

A matrix A is called *Hurwitz*, if $\mu(A) < 0$.

A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if its off-diagonal entries are nonnegative.

The matrix A is *irreducible* if and only if for every non-empty proper subset K of $N := \{1, \cdots, n\}$, there exists an $i \in K$, $j \in N \setminus K$ such that $a_{ij} \neq 0$. When A is not irreducible, it is *reducible*.

2.3 KKM Lemma

The classical theorem of Knaster-Kuratowski-Mazurkiewicz (often called the KKM theorem, or the KKM lemma or KKM principle) has numerous applications in various fields of pure and applied mathematics. Extensions and applications of the core idea of theorem are now known as the KKM theory [Yua99]. The KKM lemma was first published in 1929, by three Polish mathematicians, Bronislaw Knaster (1893 - 1990), Kazimierz Kuratowski (1896 - 1980) and Stefan Mazurkiewicz (1888 - 1945) [KKM29].

In Chapter 3, we will make considerable use of the KKM Lemma. The KKM lemma as originally stated was concerned with coverings of a simplex by closed sets, but it is a later version of the result concerning open coverings that we make use of here [Par00]. A more detailed description of KKM lemma and its variations can be found in [Yua99]. Before stating the lemma, we first need to recall some definitions which are adopted from [Yua99].

A set of vectors in $\mathbb{R}^n$ represented by $\{a_0, a_1, \cdots, a_r\}$ is *affinely independent* if the system of vectors

$$(a_1 - a_0), \cdots, (a_r - a_0)$$

is linearly independent.

Given a set of affinely independent vectors, $a_0, a_1, \cdots, a_r$, the set of all vectors of the form

$$x = \lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_r a_r \tag{2.1}$$

where $\lambda_i \geq 0$, $0 \leq i \leq r$, $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$ is called an r-dimensional *simplex*, or briefly an r-simplex. The points $a_0, a_1, \cdots, a_r$ are the *vertices* of the simplex. For simplicity, we denote the simplex by $S(a_0, \ldots, a_r)$.

For any $c \in \mathbb{R}$, $cS(a_0, \ldots, a_r)$ is the set of all vectors cx where x is of the form (2.1). For any $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$, $p + cS(a_0, \ldots, a_r)$ is the set of all vectors $p + cx$ where x is of the form (2.1).

The simplex whose vertices are the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$ is referred to as the *standard simplex* and denoted by Δ_n.

Given a simplex $S(a_0, \ldots, a_r)$ and indices $0 \leq i_0 < i_1 < \cdots < i_p \leq r$, the simplex $S(a_{i_0}, \ldots, a_{i_p})$ is a *face* of $S(a_0, \ldots, a_r)$.

In the sequel, we shall use the following open version of the KKM Lemma [Par00].

Theorem 2.3.1 (KKM Lemma)**.** *Let $\Delta := S(a_0, a_1, \ldots, a_r)$ be an r-simplex and let $F_0, F_1, \ldots, F_r$ be open subsets of Δ. If*

$$S(a_{i_0}, \ldots, a_{i_p}) \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_p}$$

holds for all faces $S(a_{i_0}, \ldots, a_{i_p})$, $1 \leq p \leq r$, $0 \leq i_0 < i_1 < \cdots < i_p \leq r$, then

$$F_0 \cap F_1 \cdots \cap F_r \neq \emptyset.$$

The geometric meaning of the KKM lemma is illustrated by Figure 2.1.

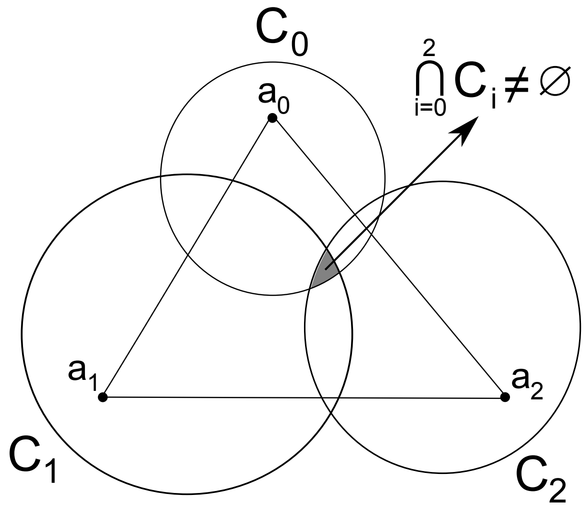

Figure 2.1: The geometric representation of the KKM Lemma.

2.4 Dynamical Systems

In this section, we recall some basic definitions and results concerning dynamical systems. The systems we deal with in this chapter are autonomous continuous-time nonlinear systems of the form:

$$\dot{x}(t) = f(x), \quad x(0) = x_0 \tag{2.2}$$

where $f : \mathcal{D} \mapsto \mathbb{R}^n$ is a nonlinear vector field on a subset $\mathcal{D}$ of $\mathbb{R}^n$ and $x_0 \in \mathcal{D}$ is called the initial condition. The *forward solution* (sometimes referred to as *solution*) or *trajectory* of (2.2) with initial condition x_0 at $t = 0$ is denoted by $x(t, x_0)$ and is defined on the maximal forward interval of existence $\mathcal{I}_{x_0} := [0, T_{max}(x_0))$ [Kha02]. Hereafter, if we do not explicitly specify the maximal forward interval of existence for an initial condition x_0, we always assume $\mathcal{I}_{x_0} = [0, +\infty)$.

For some $x_0 \in \mathcal{D}$, $x(t, x_0)$ is said to be *decreasing*, if for all $t \geq 0$ and $s > 0$ with $t + s \in \mathcal{I}_{x_0}$, we have:

$$x(t + s, x_0) < x(t, x_0)$$

Also, $x(t, x_0)$ is said to be *non-increasing*, if

$$x(t + s, x_0) \leq x(t, x_0)$$

increasing and *non-decreasing* trajectories are defined in the obvious manner.

A set $\mathcal{U} \subset \mathcal{D}$ is *forward invariant* or *positive invariant* for system (2.2) if and only if $\forall x_0 \in \mathcal{U}, x(t, x_0) \in \mathcal{U}$ for all $t \in \mathcal{I}_{x_0}$.

A point p is an *omega limit point* of x_0 if there exists an increasing sequence of time instances $\{t_k\}$, with $t_k \to +\infty$ when $k \to +\infty$, such that $\lim\limits_{t_k \to +\infty} x(t_k, x_0) = p$. The set of all the omega limit points of x_0 is called its *omega limit set* and is represented by $\omega(x_0)$. Note that omega limit set of x_0 can be empty, for example if the solution starting from x_0 diverges.

If $T_{max}(x_0) = +\infty$, then the set $\mathcal{O}(x_0) := \{x(t, x_0) | t \in \mathbb{R}_+\}$ is the *forward orbit* of the forward solution $x(t, x_0)$. $\mathcal{O}(x)$ is said to be a T-periodic orbit for some $T > 0$ if $x(T, x_0) = x_0$. In that case, $x(t + T, x_0) = x(t, x_0)$ for all $t \geq 0$, so $\mathcal{O}(x) = \{x(t, x_0) : 0 \leq t \leq T\}$ [Smi95].

If the model (2.2) is to be a useful mathematical representation of a dynamical system, then it should have two important properties. The solution for every initial condition of interest should exist, and it should be unique. To state the condition for existence and uniqueness of the solution of (2.2), we need to define the Lipschitz condition.

Definition 2.4.1 (Lipschitz Condition)**.** Let $D \subset \mathbb{R}^n$ and let $f : D \mapsto \mathbb{R}^n$ be a nonlinear vector field. We say f is locally Lipschitz in a closed subset $\mathcal{U}$ of $\mathcal{D}$, if there exists a positive real L such that

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for all $x, y \in \mathcal{U}$ where $\| \cdot \|$ represents any p-norm.

The Lipschitz property is weaker than continuous differentiability, as stated in the next lemma which is Lemma 3.2 in [Kha02].

Lemma 2.4.1. *If $f(a)$ and $\frac{\partial f}{\partial x}(a)$ are continuous in a subset $\mathcal{U}$ of $\mathcal{D}$, then f is locally Lipschitz in $\mathcal{U}$.*

The following theorem, states condition for existence and uniqueness of the solutions of (2.2) [Kha02, Theorem 3.1].

Theorem 2.4.2 (Local Existence and Uniqueness)**.** *Let $D \subset \mathbb{R}^n$ and let $f : D \mapsto \mathbb{R}^n$ be a nonlinear vector field. Let f be continuous and Lipschitz in*

$B = \{x \in \mathcal{D} : \|x - x_0\| \leq r\}$ *for some real* r *with* $r > 0$. *Then there exists some* $\delta > 0$ *such that system* (2.2) *with* $x(0) = x_0$ *has a unique solution over* $[0, \delta]$.

We now extend the concept of irreducibility to nonlinear dynamical systems. Following [AdL02] System (2.2) is *irreducible* in $\mathbb{R}^n_+$, if

- For all $a \in \text{int}\,(\mathbb{R}^n_+)$, $\frac{\partial f}{\partial x}(a)$ is irreducible,
- For all $a \in \text{bd}\,(\mathbb{R}^n_+) \setminus \{0\}$, either $\frac{\partial f}{\partial x}(a)$ is irreducible or $f_i(a) > 0$ for all i such that $a_i = 0$.

2.4.1 Stability

We next recall various fundamental stability concepts but before formally stating definitions of stability, we should define the concept of equilibrium of a system.

Definition 2.4.2 (Equilibrium Point)**.** Let $\mathcal{D} \subset \mathbb{R}^n$ and let $f : \mathcal{D} \mapsto \mathbb{R}^n$ be a nonlinear vector field. Any point $\bar{x} \in \mathcal{D}$ that satisfies $f(\bar{x}) = 0$ is an equilibrium point of the system (2.2).

Now we are ready to define different concepts of stability.

Definition 2.4.3. Let $f : \mathcal{D} \mapsto \mathbb{R}^n$ be a vector field on an open subset $\mathcal{D}$ of $\mathbb{R}^n$. Let the system (2.2) have an equilibrium at p in a positive invariant and closed subset $\mathcal{U}$ of $\mathcal{D}$. We consider $\mathcal{U}$ to be the state space of the system (2.2). Then we say that the equilibrium point p is

- stable, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x_0 - p\| < \delta \Rightarrow \|x(t, x_0) - p\| < \epsilon, \quad \forall t > 0.$$

- unstable, if it is not stable;
- asymptotically stable if it is stable and there exists a neighbourhood N of p such that

$$x_0 \in N \Rightarrow \lim_{t \to \infty} x(t, x_0) = p.$$

The set

$$A(p) := \{x_0 \in \mathcal{U} : x(t, x_0) \to p, \text{ as } t \to \infty\}$$

is the *domain of attraction* or *region of attraction* of p. If $A(p) = \mathcal{U}$, then we say p is globally asymptotically stable (GAS for short).

2.4.2 Cooperative Systems

Of special interest for us in this manuscript are cooperative systems and monotonicity. Monotone methods have been applied since at least the 1920s [Mül26, Kam32], but not until the work of M. W. Hirsch in the 1980s was the potential of monotonicity widely appreciated in dynamical systems theory (look at [Hir88b] and references therein). We start with the definition of monotone systems.

Definition 2.4.4 (Monotonicity)**.** Suppose $\mathcal{D} \subset \mathbb{R}^n$ is a forward invariant set for system (2.2). The system (2.2) is *monotone* in $\mathcal{D}$ if and only if $\forall x_0, y_0 \in \mathcal{D}$ with $x_0 \leq y_0$, it holds that $x(t, x_0) \leq x(t, y_0)$, $\forall t \in (\mathcal{I}_{x_0} \bigcap \mathcal{I}_{y_0})$.

There is another property which is closely related to monotonicity.

Definition 2.4.5 (Strong Monotonicity)**.** Let $\mathcal{D} \subset \mathbb{R}^n$ be a forward invariant set for system (2.2). System (2.2) is *strongly monotone* in $\mathcal{D}$ if and only if $\forall x_0, y_0 \in \mathcal{D}$ with $x_0 < y_0$, it holds that $x(t, x_0) \ll x(t, y_0)$, $\forall t \in (\mathcal{I}_{x_0} \bigcap \mathcal{I}_{y_0})$.

There is an easy way to check monotonicity of a system which is due to [Kam32]. It is called the Kamke Condition.

Definition 2.4.6 (Kamke Condition)**.** The vector field $f : \mathcal{D} \mapsto \mathbb{R}^n$ on an open subset $\mathcal{D}$ of $\mathbb{R}^n$ is said to be of *type K* or to satisfy *Kamke Condition*, if for each i, $f_i(a) \leq f_i(b)$ for any two points a and b in $\mathcal{D}$ satisfying $a \leq b$ and $a_i = b_i$.

The following Proposition, which is a restatement of Proposition 3.1.1 in [Smi95], links Kamke condition with monotonicity.

Proposition 2.4.3. *Let f be type K in an open subset $\mathcal{D}$ of $\mathbb{R}^n$. Then system* (2.2) *is monotone.*

Another concept that we will use repeatedly in this manuscript, and is closely tied with monotonicity, is the concept of cooperativity. A cooperative system is defined as follows.

Definition 2.4.7 (Cooperativity)**.** We say that the C^1 vector field $f : \mathcal{D} \to \mathbb{R}^n$ is *cooperative* on a closed subset $\mathcal{U}$ of $\mathcal{D}$ if the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is Metzler for all $a \in \mathcal{U}$. Also, system (2.2) is said to be cooperative, if f is cooperative.

Example 2.4.1. As an example of a cooperative system, consider the following system:

$$\dot{x} = f(x) = \text{diag }(x)(Ax + b) \tag{2.3}$$

where A is Metzler. System (2.3) represents a Mutualistic Lotka-Volterra system [HS98b, Chapter 15] and is a model for interactions between two or more species, all benefiting from the interaction. For example, in ant-plant mutualisms, ants defend plants from herbivores or perform other services in exchange for nest sites and nutrition provided by the plants [Roc06, Chapter 8]. Since x_i represents the population of the ith species, therefore, it is logical to assume $x > 0$.

Calculating the Jacobian of $f(\cdot)$, we have:

$$J^i(x) = (a_{i1}x_i, a_{i2}x_i, \cdots, 2a_{ii}x_i + b_i + \sum_{j \neq i} a_{ij}x_j, \cdots, a_{i(n-1)}x_i, a_{in}x_i)$$

where J^i represents the ith row of the Jacobian matrix J. It is easy to see that J is Metzler, which means the system (2.3) is cooperative in $\mathbb{R}^n_+$. □

It can be proved that every cooperative system defined on a suitable set satisfies Kamke condition, hence, is monotone. The following remark, which is Remark 3.1.1 in [Smi95], describes this relation.

Remark 2.4.1. A subset $\mathcal{D}$ of $\mathbb{R}^n$ is said to be *p-convex* if $\alpha x + (1 - \alpha)y \in \mathcal{D}$ for all $\alpha \in [0, 1]$ whenever $x, y \in \mathcal{D}$ and $x \leq y$. Obviously, if $\mathcal{D}$ is convex, then it is also p-convex. Let $\mathcal{D}$ be a p-convex subset of $\mathbb{R}^n$ and let $f : \mathcal{D} \mapsto \mathbb{R}^n$ be cooperative, which means we have

$$\frac{\partial f_i}{\partial x_j}(a) \geq 0 \quad i \neq j \quad \forall a \in \mathcal{D} \tag{2.4}$$

Then the fundamental theorem of calculus, implies that f satisfies the Kamke condition in $\mathcal{D}$. In fact, if $a \leq b$ and $a_i = b_i$, then

$$f_i(b) - f_i(a) = \int_0^1 \sum_{i \neq j} \frac{\partial f_i}{\partial x_j}(a + r(b-a))(b_j - a_j)dr \geq 0$$

by (2.4).

Monotonicity is a powerful property and provides a range of different mathematical tools that will help us in the following chapters. One of the properties of monotone systems that we will repeatedly use is the following lemma which is a restatement of Proposition 3.2.1. in [Smi95].

Lemma 2.4.4. *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathcal{D} \to \mathbb{R}^n$ be a cooperative vector field. Assume there exists a vector w such that $f(w) \ll 0$ ($f(w) \gg 0$). Then the trajectory $x(t, w)$ of system (2.2) is decreasing (increasing) for $t \geq 0$. In the case of $f(w) \leq 0$ ($f(w) \geq 0$), the trajectory will be non-increasing (non-decreasing).*

2.4.3 Positive Systems

In this manuscript, we shall be almost exclusively concerned with *positive systems*. A system is called positive, if starting from any initial condition in the positive orthant ($\mathbb{R}^n_+$), the trajectory of the system will remain in the positive orthant. The formal definition of a positive system is as follows.

Definition 2.4.8. System (2.2) is called positive, if

$$x(t, x_0) \geq 0 \text{ for all } t \geq 0, x_0 \geq 0 \tag{2.5}$$

In other words, if $\mathbb{R}^n_+$ is an invariant set for the system (2.2), then the system is positive.

Due to the fact that we deal with positive systems in this manuscript, hereafter, when we say f is cooperative without specifying the set $\mathcal{U}$, we understand that it is cooperative in $\mathbb{R}^n_+$. To avoid complications that may arise from calculating the Jacobian matrix on the $\operatorname{bd}(\mathbb{R}^n_+)$, hereafter, unless stated otherwise, we always assume any vector field is well-defined in $\mathcal{W}$, where $\mathcal{W}$ is a neighbourhood of $\mathbb{R}^n_+$.

The following lemma is an immediate consequence of Lemma 2.4.4.

Lemma 2.4.5. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be a vector field such that system* (2.2) *is cooperative and positive. Suppose that the system (2.2) has an equilibrium point at $p \in \mathbb{R}^n_+$. The following statements hold.*

(i) If there exists $w > p$ with $f(w) \geq 0$, then $x(t, w) \geq w$ for all $t \geq 0$;

(ii) If there exists $0 \leq w < p$ with $f(w) \leq 0$, then $x(t, w) \leq w$ for all $t \geq 0$.

In particular, in both case (i) and case (ii), w cannot lie in the domain of attraction $A(p)$ of p.

Proof: (i) Based on Lemma 2.4.4, if $f(w) \geq 0$, then $x(t, w)$ is non-decreasing, which means $x(t, w) \geq w$ for all $t \geq 0$. If w is in the domain of attraction of p, then $x(t, w) \to p$ as $t \to \infty$. Therefore, since $p < w$, then for some $t_1 > 0$, $x(t_1, w) < w$ which contradicts the fact that $x(t, w)$ is non-decreasing. Hence w cannot be in the domain of attraction of p.

(ii) Similar to the proof of (i) with proper changes in inequalities. □

2.5 Positive LTI Systems

In this section, we recall some basic properties of linear time-invariant (LTI) systems with an emphasis on positive LTI systems. Although linear systems can be rarely found in nature and even in industry, control design methods based on linear models have proven quite successful in practice. Furthermore, a lot of concepts and methods related to linear systems can be extended to nonlinear systems and that is what we will do in this chapter. We present some properties of LTI systems in this section and in the next section we will see how some of those properties can be extended to nonlinear systems. A continuous-time LTI system with zero input can be described as follows:

$$\dot{x}(t) = Ax(t), \ \ \forall t \geq 0 \tag{2.6}$$

$$x(0) = x_0 \tag{2.7}$$

where $A \in \mathbb{R}^{n \times n}$. It is easy to check that the origin is an equilibrium of the system (2.6). It is a well-known fact that the origin is a GAS equilibrium of system (2.6) if and only if the matrix A is Hurwitz. Look for example at [Rug96] for a comprehensive study of linear systems and their properties. Another condition for stability of LTI systems is the Lyapunov condition that in its simplest form can be stated as follows [Che99, Theorem 5.5].

Theorem 2.5.1. *The origin is a GAS equilibrium of the LTI system* (2.6) *if and only if there exists a symmetric positive-definite matrix P such that the matrix $A^T P + PA$ is negative-definite.*

Of special interest to us, are positive LTI systems. The following well-known theorem, provides a straightforward and easy to check condition for positivity of LTI systems [FR00, Theorem 2].

Theorem 2.5.2. *The continuous-time LTI system* (2.6) *is positive if and only if A is Metzler.*

Comparing the condition stated in Theorem 2.5.2 with the definition of cooperative vector fields, it is clear that a continuous-time LTI system is positive, if and only if it is cooperative and hence monotone. The following lemma formally states this property.

Lemma 2.5.3. *The LTI system* (2.6) *is monotone, if and only if it is positive.*

Proof: System (2.6) is cooperative if and only if A is Metzler. Therefore, the proof follows directly from Theorem 2.5.2 and the fact that cooperative systems are monotone. □

It must come as no surprise that in the following chapters we will be mainly dealing with nonlinear systems which are cooperative. We will use cooperativity as a tool to extend the properties of positive LTI systems to the realm of positive nonlinear systems.

A famous and very useful property of positive LTI systems is stated in the *Perron-Frobenius Theorem*. Perron-Frobenius Theorem is evolved from the contributions of German mathematicians Oskar Perron (1880-1975) and Ferdinand Georg Frobenius (1849-1917). A short and interesting biography of these two mathematicians can be found in [Mey00, Chapter 11]. Perron published his treatment of positive matrices in 1907. In [Per07], he stated some results on matrices with positive entries which are the basis of what we know today as the Perron-Frobenius Theorem. Frobenius, was intrigued by Oskar Perron's discovery. In Frobenius' words from the introduction to [Fro08], the "characteristic determinant and its sub-determinant" of matrices with positive real-entries "have some remarkable properties" [Cur99]. He published three papers on the subject himself, from 1908 to 1912 [Fro08, Fro09, Fro12], containing simplified proofs and significant extensions of the Perron's results to matrices with nonnegative entries.

The Perron-Frobenius Theorem, as originally stated by Perron, can be stated as follows [HJ85, Theorem 8.2.11].

Theorem 2.5.4 (Perron-Frobenius Theorem for strictly positive matrices). *Let $A \in \mathbb{R}^{n\times n}$ and $A \gg 0$, then*

(i) $\rho(A) > 0$;

(ii) $\rho(A)$ *is an eigenvalue of* A.

(iii) $\rho(A)$ *is an algebraically and hence geometrically simple eigenvalue of* A;

(iv) If x *is the eigenvector corresponding to* $\rho(A)$, *then* $x \gg 0$;

(v) For every eigenvalue $\lambda \neq \rho(A)$, $|\lambda| < \rho(A)$.

Frobenius, extended the above theorem to positive irreducible matrices.

Theorem 2.5.5 (Perron-Frobenius Theorem for positive irreducible matrices). *Let $A \in \mathbb{R}^{n\times n}$ be irreducible and $A > 0$, then*

(i) $\rho(A) > 0$;

(ii) $\rho(A)$ *is an eigenvalue of* A.

(iii) $\rho(A)$ *is an algebraically and hence geometrically simple eigenvalue of* A;

(iv) If x *is the eigenvector corresponding to* $\rho(A)$, *then* $x \gg 0$.

Extensions of the Perron-Frobenius Theorem to Metzler matrices have also been developed. Since we mostly deal with Metzler matrices, we are more interested in the Perron-Frobenius Theorem for Metzler matrices. Before stating the theorem, we need to define some terminology.

Definition 2.5.1 (Dominant Eigenvalue and Eigenvector)**.** λ_F is dominant or Frobenius eigenvalue of A if and only $\Re(\lambda_F) = \mu(A)$. Eigenvectors corresponding to dominant eigenvalues are called the dominant or Frobenius eigenvectors of A.

Remembering the conditions of stability for LTI systems, it is clear that a continuous-time LTI system is globally asymptotically stable if and only if $\Re(\lambda_F) < 0$.

Now we are ready to state the Perron-Frobenius Theorem for Metzler matrices [FR00, Theorem 11].

Theorem 2.5.6 (Perron-Frobenius Theorem for Metzler Matrices)**.** *Let $A \in \mathbb{R}^{n\times n}$ be Metzler. Then*

(i) $\mu(A) \in \sigma(A)$;

(ii) If $Ax = \mu(A)x$, then $x > 0$;

(iii) If x is an eigenvector of A with $x \gg 0$, then $Ax = \mu(A)x$.

If we add the extra assumption of irreducibility, then even more can be said about dominant eigenvalues and eigenvectors. Note that when A is irreducible, then we say the system (2.6) is irreducible. The Perron-Frobenius Theorem for irreducible Metzler matrices can be stated as follows [FR00, Theorem 17].

Theorem 2.5.7 (Perron-Frobenius Theorem for Irreducible Metzler Matrices)**.** *Let $A \in \mathbb{R}^{n\times n}$ be Metzler and irreducible. Then*

(i) $\mu(A)$ is an algebraically simple eigenvalue of A;

(ii) $x_F \gg 0$ and is unique (up to a scalar multiplication);

(iii) If x is an eigenvector of A with $x > 0$, then $Ax = \mu(A)x$.

Many necessary and sufficient conditions for stability of LTI systems become simpler for positive LTI systems. For example, the Lyapunov condition for stability of positive LTI systems has a simpler form [FR00, Theorem 15].

Theorem 2.5.8. *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. The origin is a GAS equilibrium of LTI system* (2.6) *if and only if there exists a diagonal matrix P with diagonal positive entries such that the matrix $A^TP + PA$ is negative-definite.*

Another important property of positive LTI systems, that is of utmost importance in this manuscript and is the inspiration for most of the results presented in Chapters 3 and 5, is the *D-stability* of positive LTI systems. D-stability means that the stability of a positive LTI systems is maintained while the matrix A is perturbed under certain rules. More specifically, it means if we substitute a Hurwitz matrix A in (2.6) with DA, where D is a diagonal matrix with positive diagonal entries, then the origin is a GAS equilibrium of the corresponding system. The following theorem, which is Theorem 16 in [FR00] states this property in a formal way.

Theorem 2.5.9. *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and Metzler. Then the origin is a GAS equilibrium of the system:*

$$\dot{x}(t) = DAx(t) \tag{2.8}$$

for all diagonal D with positive diagonal entries.

The following proposition summarizes the relevant and well-known stability properties of continuous-time positive LTI systems. The proof for each part of the following proposition can be found in numerous references including [Rug96, HJ85, BP94, FR00]

Proposition 2.5.10. *Let A be a Metzler matrix. Then the following statements are equivalent:*

(a) The origin is a GAS equilibrium of the LTI system (2.6);

(b) A is Hurwitz;

(c) There exists a symmetric positive-definite $P \in \mathbb{R}^{n\times n}$ such that A^TP+PA is negative-definite;

(d) There exists a diagonal $D \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that $A^TD + DA$ is negative-definite;

(e) *There exists a vector $v \gg 0$ such that $Av \ll 0$;*

(f) *The origin is a GAS equilibrium of system* (2.8) *for every diagonal D with positive diagonal entries.*

(e) *There exists a vector $v \gg 0$ such that $A^T v \ll 0$;*

Of special interest for us in the remainder of this manuscript is the equivalence between items (a) and (f) in the above proposition. In fact our aim in Chapters 3 and 5 will be to extend this equivalence to different classes of nonlinear and switched positive systems. This interest is motivated by the fact that uncertainty is an inherent property of most physical systems. That is why we are interested in methods that enable us to analyse and control uncertain systems. More on that will be discussed in the following chapters.

We finish this section with a motivational example of D-stability for positive LTI systems.

Example 2.5.1. Consider we have a market composed of n commodities and let $y_i(t) \in \mathbb{R}_+$ be the price of commodity i at time t. Also, let the expected price of the commodity i be $\bar{y}_i$. We assume that $\bar{y}_i$ is constant and that $\bar{y}_i > 0$. The assumption that the expected prices for all future times are constant cannot be always justified, but in this simple example we do not consider more general cases (for information on more general models, you can look for example at [EA56]).

Let $x_i(t)$ be the difference between the price of commodity i at time t and the expected price of the commodity i, i.e., $x_i(t) = y_i(t) - \bar{y}_i$. We assume that the demand and the supply of commodity i at time t depend only on $x(t)$ and also the changes in the $x(t)$ are proportional to the surplus of demand. Therefore, we have:

$$\dot{x}_i(t) = d_i[f_i^D(x(t)) - f_i^S(x(t))] \quad i = 1, \cdots, n \tag{2.9}$$

where $d \gg 0$ is the vector of proportionality factors and $f^D(\cdot)$ and $f^S(\cdot)$ represent the dependence of demand and supply on the prices, respectively. When analysing complex economic systems it is unrealistic to imagine that we know the proportionality factors. Therefore we are interested to know if there are cases that the system is insensitive to changes in the proportionality factor.

A simple answer to this question can be given by assuming that:

- Demand and supply are linear functions of $x(t)$;
- x_i has a negative effect on $f_i^D(x)$ and a nonnegative effect on $f_j^D(x)$ for $j \neq i$;
- x_i has a positive effect on $f_i^S(x)$ and a nonpositive effect on $f_j^S(x)$ for $j \neq i$.

Considering these assumptions, the model (2.9) can be simplified to the following linear system:

$$\dot{x}(t) = DA(x(t)) \tag{2.10}$$

where A is a Metzler matrix and $D = \text{diag}\ (d)$. Based on Theorem 2.5.9, we can conclude that if A is Hurwitz, then the system (2.10) has a globally asymptotically stable equilibrium at the origin for every diagonal D with positive diagonal entries. In other words, if the prices of the n commodities tend toward a set of expected prices for some vector of proportionality factors, then they do so for any other positive values of proportionality factors. □

2.6 Positive Nonlinear Systems

In this section, we review some properties and basic definitions related to positive nonlinear systems. The concepts and results presented in this section will be used in Chapter 3.

Unlike LTI systems, checking the positivity of nonlinear systems is not always straightforward. The following lemma provides a general condition for positivity of nonlinear systems [dL00].

Lemma 2.6.1. *System* (2.2) *is positive, if and only if the following condition is satisfied:*

$$\forall x \in \text{bd}\,(\mathbb{R}_+^n) : x_i = 0 \Rightarrow f_i(x) \geq 0 \tag{2.11}$$

Since in the following chapters we will almost exclusively deal with cooperative systems, the following lemma provides an easy to check condition for positivity of cooperative systems.

Lemma 2.6.2. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative. If the system* (2.2) *has an equilibrium at the origin then it is positive.*

Proof: Since f is cooperative, then the system (2.2) is monotone, which means for every initial conditions y_0 and z_0 we have:

$$y_0 \leq z_0 \Rightarrow x(t, y_0) \leq x(t, z_0) \text{ for all } t \geq 0$$

Since $x(t, 0) = 0$ for all $t \geq 0$, then for all initial conditions $x_0 \geq 0$ we have

$$x_0 \geq 0 \Rightarrow x(t, x_0) \geq x(t, 0) = 0 \text{ for all } t \geq 0$$

This means the positive orthant is an invariant set for the system (2.2) and this concludes the proof. □

It should be noted that the condition stated in Lemma 2.6.2 is a sufficient but not a necessary condition for positivity of cooperative systems. It means, it is possible to have a positive cooperative system that does not have an equilibrium in the origin. The following simple example clarifies this notion.

Example 2.6.1. Consider the system (2.2), where f is defined as follows:

$$f(x) = \begin{pmatrix} 1 + x_1^2 + x_2^2 \\ 1 + x_1^2 + x_2^2 \end{pmatrix}$$

It is easy to check that condition (2.11) is satisfied which means the system is positive.

Computing the Jacobian of $f(\cdot)$ at a point a, we have:

$$\frac{\partial f}{\partial x}(a) = \begin{pmatrix} 2a_1 & 2a_2 \\ 2a_1 & 2a_2 \end{pmatrix}$$

which is a Metzler matrix for all $a \in \mathbb{R}^n_+$, therefore, system (2.2) is cooperative. On the other hand

$$f(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence with the above choice of $f(\cdot)$, system (2.2) is a positive and cooperative system that does not have an equilibrium at the origin. □

Among different classes of cooperative systems, we will be mainly dealing with *homogeneous cooperative* and *subhomogeneous cooperative* systems. In the remainder of this chapter, we define both classes and present their relevant properties.

2.6.1 Homogeneous Cooperative Systems

Homogeneous vector-fields are vector-fields possessing a symmetry with respect to a family of dilations [MAPS02]. Homogeneity has proven to be a fruitful concept for solving stabilisation problems [Kaw90, Her91, MM, MPS99b]. Apart from that, due to their special structure, they provide a natural framework to extend some of the stability properties of linear systems to nonlinear systems.

A homogeneous system can be defined as follows.

Definition 2.6.1. Given an n-tuple $r = (r_1, \ldots, r_n)^T$ of positive real numbers and $\lambda > 0$, the *dilation map* $\delta^r_\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\delta^r_\lambda(x) = (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n)^T$$

If $r = (1, \cdots, 1)^T$, then the dilation map is called the *standard dilation map.*

For an $\alpha \geq 0$, the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *homogeneous* of degree α with respect to $\delta^r_\lambda(x)$ if

$$\forall x \in \mathbb{R}^n, \lambda \geq 0, \quad f(\delta^r_\lambda(x)) = \lambda^\alpha \delta^r_\lambda(f(x)) \tag{2.12}$$

When f is homogeneous, then we say the system (2.2) is homogeneous.

Let $f(x) = Ax$ where $A \in \mathbb{R}^{n\times n}$. Then we have

$$f(\lambda x) = \lambda f(x), \text{ for all } \lambda > 0, x \in \mathbb{R}^n$$

which means that an LTI system is homogeneous of degree 0 with respect to the standard dilation map.

Example 2.6.2. Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \mapsto \mathbb{R}^n$ be defined as follows:

$$f(x_1, x_2, x_3) = \begin{pmatrix} -2x_1^2 + x_3^2 \\ x_1^2 - 2x_2^2 + x_3 x_1 \\ x_1 x_2 + x_2^2 - 5x_3^2 \end{pmatrix}$$

$f(\cdot)$ is homogeneous of degree $\alpha = 1$ with respect to the standard dilation map, meaning the dilation map δ_λ^r with $r = (1,1,1)^T$. To see this, note:

$$\begin{aligned} f(\delta_\lambda^r(x)) &= f(\lambda x_1, \lambda x_2, \lambda x_3) \\ &= \begin{pmatrix} -2\lambda^2 x_1^2 + \lambda^2 x_3^2 \\ \lambda^2 x_1^2 - 2\lambda^2 x_2^2 + \lambda^2 x_3 x_1 \\ \lambda^2 x_1 x_2 + \lambda^2 x_2^2 - 5\lambda^2 x_3^2 \end{pmatrix} \\ &= \left(\lambda^2 f_1(x), \lambda^2 f_2(x), \lambda^2 f_3(x)\right)^T \\ &= \lambda \left(\lambda f_1(x), \lambda f_2(x), \lambda f_3(x)\right)^T \\ &= \lambda^\alpha \delta_\lambda^r(f(x)) \end{aligned}$$

□

Example 2.6.3. Now we consider a more general homogeneous vector field. Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}_+^n$ and let $g : \mathcal{W} \mapsto \mathbb{R}^n$ be defined as

$$g(x_1, x_2, x_3) = \begin{pmatrix} -2x_1^{5/3} + x_3 \\ x_1^2 - 2x_2^{3/2} + x_3 x_1^{1/3} \\ x_1 x_2 + x_2^{7/4} - 5x_3^{7/5} \end{pmatrix}$$

$g(\cdot)$ is homogeneous of degree $\alpha = 2$ with respect to the dilation map δ_λ^r, with $r = (3,4,5)$. In fact:

$$\begin{aligned} g(\delta_\lambda^r(x)) &= g(\lambda^3 x_1, \lambda^4 x_2, \lambda^5 x_3) \\ &= \begin{pmatrix} -2\lambda^5 x_1^{5/3} + \lambda^5 x_3 \\ \lambda^6 x_1^2 - 2\lambda^6 x_2^{3/2} + \lambda^6 x_3 x_1^{1/3} \\ \lambda^7 x_1 x_2 + \lambda^7 x_2^{7/4} - 5\lambda^7 x_3^{7/5} \end{pmatrix} \\ &= \left(\lambda^5 g_1(x), \lambda^6 g_2(x), \lambda^7 g_3(x)\right)^T \\ &= \lambda^2 \left(\lambda^3 g_1(x), \lambda^4 g_2(x), \lambda^5 g_3(x)\right)^T \\ &= \lambda^\alpha \delta_\lambda^r(g(x)) \end{aligned}$$

□

The following lemma, states a well-known property of homogeneous vector fields that will be useful for us. The statement of the lemma is adopted from [AdL02].

Lemma 2.6.3 (Euler's Formula)**.** *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathcal{D} \mapsto \mathbb{R}^n$ be a C^1 and homogeneous vector field of degree α with respect to the dilation map $\delta_\lambda^r(x)$. Euler's formula for homogeneous vector fields is as follows:*

$$\frac{\partial f}{\partial x}(a)\text{diag }(r)a = \text{diag }(r + \alpha^*)f(a) \quad \forall a \in \mathcal{D} \tag{2.13}$$

where $\alpha^* := (\alpha, \cdots, \alpha)^T$.

Proof: Euler's formula can be easily proved by first taking the derivative with respect to λ on both sides of (2.12) and then evaluating the resulting equation for $\lambda = 1$. □

As mentioned before, the Perron-Frobenius Theorem has been extended to different classes of systems, including homogeneous systems [NL99, GG00, Kra01, Sin90]. We state the Perron-Frobenius theorem for homogeneous systems based on the version discussed in [AdL02].

Before stating this result, we need to first present some definitions and a preliminary result, which are all adopted from [AdL02].

Definition 2.6.2. For $x \in \mathbb{R}^n \setminus \{0\}$ and a fixed but arbitrary dilation map $\delta_\lambda^r(x)$, $R_x := \{\delta_\lambda^r(x) | \lambda \in \mathbb{R}_+^n\}$ is the *ray* through x.

Lemma 2.6.4. *If system* (2.2) *is homogeneous and if there exists a point $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ such that*

$$f(\bar{x}) = \gamma_{\bar{x}}\text{diag }(r)\bar{x} \tag{2.14}$$

for some $\gamma_{\bar{x}} \in \mathbb{R}$, then the vector field $f(x)$ is tangent to $R_{\bar{x}}$ at each point of $R_{\bar{x}}$.

Lemma 2.6.4 implies that if there exists a point $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ such that (2.14) holds, then the forward solution of the system (2.2) starting from an arbitrary point in $R_{\bar{x}}$ stays in this ray for all future times for which this solution is defined. Such a ray is called an *invariant ray* for system (2.2).

An invariant ray R_x is said to be *asymptotically stable*, *stable* or *unstable* for some $x \in R_x$ if and only if $\gamma_x < 0$, $\gamma_x \leq 0$ or $\gamma_x > 0$, respectively.

The following theorem, which is Theorem 5.2 in [AdL02], can be considered as the extension of Perron-Frobenius Theorem to irreducible homogeneous cooperative systems.

Theorem 2.6.5. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \mapsto \mathbb{R}^n$ be homogeneous of degree α with respect to the dilation map $\delta^r_\lambda(x)$, cooperative and irreducible such that system* (2.2) *has a unique equilibrium at the origin. Then there exists $x^* \in \mathrm{int}\,(\mathbb{R}^n_+)$ such that*

$$f(x^*) = \gamma_{x^*}\mathrm{diag}\ (r)x^*$$

for some $\gamma_{x^*} \in \mathbb{R}$. The equilibrium of system (2.2) *is asymptotically stable, if and only if $\gamma_{x^*} < 0$, and is unstable if and only if $\gamma_{x^*} > 0$.*

Theorem 2.6.5 shows that the invariant rays can be interpreted as the equivalents of dominant eigenvectors for homogeneous systems.

2.6.2 Subhomogeneous Cooperative Systems

Another class of nonlinear systems that we will deal with in the following chapters is the class of subhomogeneous systems, that can be defined as follows [BMW11].

Definition 2.6.1. Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$. A vector field $f : \mathcal{W} \to \mathbb{R}^n$ is *subhomogeneous* of degree $\alpha > 0$ if $f(\lambda v) \leq \lambda^\alpha f(v)$, for all $v \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}$ with $\lambda \geq 1$. System (2.2) is subhomogeneous when f is subhomogeneous.

Example 2.6.4. Let $f : \mathcal{U} \mapsto \mathbb{R}^n$ with $\mathcal{U} = \{x : 0 \leq x \leq 1\}$ such that

$$\dot{x} = f(x) = [D + B - \mathrm{diag}\ (x)B]x \tag{2.15}$$

where D is a diagonal matrix and $B > 0$. We prove that f is subhomogeneous of degree $\alpha = 1$.

We have:

$$f(\lambda x) = \lambda(D + B - \lambda\mathrm{diag}\ (x)B)x \tag{2.16}$$

and

$$\lambda f(x) = \lambda(D + B - \mathrm{diag}\ (x)B)x \tag{2.17}$$

Considering the fact that $\lambda \geq 1$ and based on the assumptions imposed on D, B and x, we can conclude $f(\lambda x) \leq \lambda f(x)$, which means $f(\cdot)$ is subhomogeneous of degree 1.

The vector field introduced in this example arises in compartmental models in epidemiology and will be the basis of our discussions in Chapter 6. □

The class of subhomogeneous vector fields given above includes different classes of vector fields. For example, it is easy to check that linear and homogeneous vector fields with respect to the standard dilation map are subhomogeneous.

Remark 2.6.1. It should be noted that in the definition of subhomogeneity, unlike homogeneity, we do not introduce the concept of the dilation map. As mentioned earlier, a homogeneous vector field of degree $\alpha > 0$ with respect to standard dilation map will satisfy the equality $f(\lambda x) = \lambda^\alpha \lambda f(x)$ while a vector field which is subhomogeneous of degree $\alpha > 0$ satisfies the inequality $f(\lambda v) \leq \lambda^\alpha f(v)$. This means that a homogeneous vector field of degree 0 with respect to standard dilation map will be subhomogeneous of degree 1.

The following lemma, shows another important class of subhomogeneous vector fields.

Lemma 2.6.6. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a concave vector field such that $f(0) \geq 0$. Then f is subhomogeneous of degree 1.*

Proof: Based on the definition of concave vector fields [HJ85, p. 534], we have

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \tag{2.18}$$

for all $0 < \alpha < 1$ and for all $x, y \in \mathbb{R}^n$, $x \neq y$. Considering $x \neq 0$ and $y = 0$, we have:

$$\alpha f(x) + (1-\alpha)f(0) \leq f(\alpha x)$$

Since $f(0) \geq 0$ and $\alpha < 1$, we can conclude:

$$\alpha f(x) \leq f(\alpha x)$$

Changing the variable to $z = \alpha x$, we have:

$$\alpha f(\frac{1}{\alpha} z) \leq f(z)$$

$$\Rightarrow f(\frac{1}{\alpha}z) \leq \frac{1}{\alpha}f(z)$$

We define $\lambda = 1/\alpha$. Since $\alpha \in (0,1)$, we have $\lambda > 1$. Therefore:

$$f(\lambda z) \leq \lambda f(z) \quad \forall z \in \mathbb{R}^n, \lambda \geq 1$$

Note that the inequality is trivially true for $\lambda = 1$. This means $f(\cdot)$ is a subhomogeneous vector field of degree 1. $\square$

The following result establishes an inequality for subhomogeneous vector fields that is reminiscent of Euler's formula for homogeneous functions.

Lemma 2.6.7. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$. The vector field $f : \mathcal{W} \to \mathbb{R}^n$ is subhomogeneous of degree $\alpha > 0$ if and only if:*

$$\frac{\partial f}{\partial x}(a)a \leq \alpha f(a) \quad \forall a \geq 0. \tag{2.19}$$

Proof: We first show that f is subhomogeneous of degree α if and only if for any $a \geq 0$, the mapping

$$\lambda \to \lambda^{-\alpha} f(\lambda a)$$

is a non-increasing function for $\lambda > 0$.

Let $a \geq 0$ be given. If f is subhomogeneous, then for any $\mu \geq \lambda > 0$ we have

$$\begin{aligned} f(\mu a) = f\left(\frac{\mu}{\lambda}\lambda a\right) &\leq \left(\frac{\mu}{\lambda}\right)^{\alpha} f\left(\lambda a\right) \\ &\Rightarrow \mu^{-\alpha} f\left(\mu a\right) \leq \lambda^{-\alpha} f\left(\lambda a\right) \end{aligned}$$

Thus, we can conclude that $\lambda^{-\alpha} f\left(\lambda a\right)$ is a non-increasing function with respect to λ for all $\lambda > 0$. Conversely, if this function is non-increasing for $\lambda > 0$, then by choosing $\mu \geq \lambda = 1$, we see immediately that $f(\mu a) \leq \mu^{\alpha} f(a)$.

Differentiating with respect to λ, we see that f is subhomogeneous if and only if for all $\lambda > 0$

$$\frac{d}{d\lambda}\left(\lambda^{-\alpha} f\left(\lambda a\right)\right) \leq 0$$

$$\Leftrightarrow -\alpha\lambda^{-\alpha-1} f\left(\lambda a\right) + \lambda^{-\alpha}\frac{\partial f}{\partial x}\left(\lambda a\right) a \leq 0$$

Rearranging this inequality, we see that f is subhomogeneous if and only if

$$\frac{\partial f}{\partial x}(\lambda a)(\lambda a) \leq \alpha f(\lambda a) \ \ \forall a \geq 0; \forall \lambda > 0$$

Evaluating the last statement at $\lambda = 1$, we have

$$\frac{\partial f}{\partial x}(a)a \leq \alpha f(a) \quad \forall a \geq 0$$

This concludes the proof. □

In the following corollary, some of the basic properties of subhomogeneous systems are stated.

Corollary 2.6.8. *(i) The set of subhomogeneous vector fields of degree α on $\mathbb{R}^n_+$ is a convex cone.*

(ii) A non-negative constant vector field $f(x) \equiv c$ is subhomogeneous of any degree $\alpha > 0$.

(iii) Any affine map $f(x) = Ax + b$ where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n_+$ is subhomogeneous of degree 1.

Proof:

(i) The claim follows as the condition (2.19) has to be satisfied pointwise and is clearly convex in f and invariant under positive scaling of f.

(ii) Immediate from (2.19) as $f(x) \geq 0 = \partial f/\partial x(x)$ for all $x \geq 0$.

(iii) Let $f(x) = Ax + b$. Then $f(\lambda x) = \lambda Ax + b$ and $\lambda f(x) = \lambda Ax + \lambda b$. Since $b \geq 0$ and $\lambda \geq 1$, we have $b \leq \lambda b$, therefore, we can conclude $f(\lambda x) \leq \lambda f(x)$ for all $\lambda \geq 1$. □

□

And as the last result of this section, we show that subhomogeneous cooperative systems are positive.

Theorem 2.6.9. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree $\alpha > 0$ and cooperative. Then the system* (2.2) *is positive.*

Proof: It follows from Lemma 2.6.7 and the fact that the Jacobian matrix is Metzler for all $x \in \mathbb{R}^n_+$ that $f_i(x) \geq 0$ for all $x \in \mathbb{R}^n_+$ with $x_i = 0$. Therefore, condition (2.11) is satisfied and this implies that (2.2) is positive. □

CHAPTER 3

Stability Conditions for Positive Nonlinear Systems

In this chapter, we extend the notion of D-stability to nonlinear systems and present some results on D-stability of different classes of positive nonlinear systems. We exclusively deal with positive systems which are monotone, therefore, monotonicity plays an important role in the proofs. We also study the behaviour of a class of positive nonlinear systems forced by a constant positive input.

3.1 Introduction

In this chapter, we focus on stability analysis of positive nonlinear systems in the presence of uncertainty. Control systems are designed so that certain designated signals, such as tracking errors and actuator inputs, do not exceed pre-specified levels. One of the main issues that prevents us from achieving these goals is uncertainty in the system. Uncertainty can take different forms and can appear in the plant to be controlled or as errors in measurement signals (sensors can measure signals only to a certain accuracy) [DFT90]. Dealing with uncertainty in measurement signals is out of the scope of this manuscript. Our focus in this chapter is on uncertainties about the plant.

There are different sources of uncertainty in the plants. Uncertainty can be caused by the fact that mathematical models we use in representing real physical systems are idealisations. Or it can be caused by the changes in parameters of the system due to, for example, ageing of the system. Therefore, it is preferable, and usually necessary, to consider such uncertainties in analysing the stability properties of the systems and designing controllers. In this chapter, we are concerned solely with the stability analysis of uncertain systems. We focus on a special class of model uncertainties. More specifically, our main goal is to extend the concept of D-stability of positive LTI systems (as discussed in Chapter 2) to nonlinear systems.

In Section 3.2, we present preliminary results and definitions, including the formal definition of D-stability for nonlinear systems. In Section 3.3, we review the relevant results on stability of positive nonlinear systems, with more emphasis on monotone systems and their properties. Then, we present the D-stability results in four sections. In Section 3.4, we state a local D-stability condition for general cooperative systems. In Sections 3.5 and 3.6, we add extra assumptions of homogeneity and subhomogeneity to cooperative systems to obtain stronger D-stability results. In Section 3.7, we present an alternative result for D-stability of cooperative systems, in the planar case. In this chapter, we also study another class of perturbations applied to positive nonlinear systems. In section 3.8, we consider the effects of forcing cooperative subhomogeneous systems by a positive constant input and study its effects on the positivity and stability properties of the equilibria.

3.2 Background

As discussed in Chapter 2, one well-known fact about positive LTI systems is that they are *D-stable*. Formally, if the positive LTI system

$$\dot{x} = Ax, \;\; x(0) = x_0$$

has a globally asymptotically stable (GAS) equilibrium at the origin, then so does the system

$$\dot{x} = DAx, \;\; x(0) = x_0$$

for all diagonal matrices D with positive diagonal entries, as stated in Theorem 2.5.9.

Extending the notion of D-stability to nonlinear positive systems is the central theme of this chapter. In this section, we consider continuous-time autonomous nonlinear systems of the form:

$$\dot{x}(t) = f(x(t)), \;\; x(0) = x_0 \in \mathcal{D} \tag{3.1}$$

where $f : \mathcal{D} \mapsto \mathbb{R}^n$ is a C^1 vector field on $\mathcal{D} \subset \mathbb{R}^n$.

To define D-stability for system (3.1), we consider the system

$$\dot{x}(t) = \text{diag }(d(x(t))).f(x(t)), \;\; x(0) = x_0 \in \mathcal{D} \tag{3.2}$$

where $d \in \Delta$ and Δ is the set of all mappings $d : \mathcal{D} \mapsto \mathbb{R}^n_+$ that satisfy the following conditions:

(i) d is C^1 in $\mathcal{D}$;

(ii) $d_i(x) = d_i(x_i)$ for $1 \leq i \leq n$;

(iii) $d_i(x_i) > 0$, for $x_i > 0$, for $1 \leq i \leq n$.

Example 3.2.1. Here are some examples of mappings that satisfy the above mentioned conditions:

$$d^{(1)}(x) = \left(\frac{x_1^2}{x_1^2+1}, x_2, 1+\sin^2(x_3)\right)^T \text{ for all } x \in \mathbb{R}^3_+;$$

$$d^{(2)}(x) = (e^{x_1}, 1 + 0.5\cos(x_2))^T \text{ for all } x \in \mathbb{R}^2_+;$$

$$d^{(3)}(x) = (1, 2, 3)^T \text{ for all } x \in \mathbb{R}^3_+;$$

Definition 3.2.1 (D-stability)**.** Let the system (3.1) have an equilibrium p in $\mathcal{D}$ with a region of attraction $A_p \subset \mathcal{D}$. Then we say that p is a D-stable equilibrium of the system (3.1), if it is an asymptotically stable equilibrium of the system (3.2) (with possibly different region of attraction than A_p), for all $d \in \Delta$. If p is a GAS equilibrium of both systems (3.1) and (3.2), then we call it a *globally D-stable* equilibrium of the system (3.1).

It is easy to see that if a vector $d \in \mathbb{R}^n$ is a vector with all elements equal to 1, then $d \in \Delta$. In other words, when an equilibrium point p is a D-stable equilibrium of the system (3.1), then it is an asymptotically stable equilibrium of the system (3.1).

Remark 3.2.1. A vector with positive constant elements belongs to Δ. In other words, if $d_i(x) = c_i$ for $i = 1, \cdots, n$, where $c_i \in \mathbb{R}_+$ is a constant, then $d \in \Delta$. This means that the definition of D-stability for linear systems is a special case of the more general definition of D-stability for nonlinear vector fields presented above. In the remaining parts of this chapter, every time we use the term D-stability, we refer to Definition 3.2.1 for nonlinear systems. Also note that sometimes in the literature (and in this manuscript), in a slight abuse of notation, instead of its equilibrium, a positive LTI system itself is said to be D-stable. That should not cause any confusion.

We make the following simple observation that the properties of cooperativity and positivity are preserved under pre-multiplication by diag $(d(x))$ for all $d \in \Delta$. Just as a reminder, system (3.1) is positive if and only if it satisfies the following condition.

$$\forall x \in \mathrm{bd}\,(\mathbb{R}^n_+) : x_i = 0 \Rightarrow f_i(x) \geq 0 \tag{3.3}$$

Proposition 3.2.1. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative and satisfy condition* (3.3) *and $d \in \Delta$. Then the vector field $g : \mathcal{W} \to \mathbb{R}^n$ given by $g(x) =$ diag $(d(x))f(x)$ is cooperative and satisfies condition* (3.3)*.*

Proof: If $x_i = 0$, then $f_i(x) \geq 0$ as f satisfies condition (3.3). It is now immediate that $g_i(x) = d_i(x_i)f_i(x) \geq 0$ as $d_i(0) \geq 0$ by continuity. Hence g satisfies condition (3.3). Direct calculation shows that for $i \neq j$

$$\frac{\partial g_i}{\partial x_j}(a) = d_i(a_i)\frac{\partial f_i}{\partial x_j}(a) \geq 0$$

for all $a \in \mathbb{R}^n_+$. Therefore, since f is cooperative, then g is also cooperative as claimed. $\square$

Now we are ready to state the main results of this chapter, but before that, we review the relevant results in the literature.

3.3 Literature Review

The properties of positive matrices have been studied since at least the 1910s, with the seminal works of Perron and Frobenius, as discussed in Section 2.5. The formal study of positive dynamical systems however began years later. The first major textbook in dynamical systems that studies positive systems is probably [Lue79]. However, previously the powerful properties of positive matrices were utilised in different areas, most particularly, in economics. One of the first researchers to investigate the properties of positive economical systems is Lloyd Metzler, he of the Metzler matrix. Metzler in [Met45] used a linear positive system to model a market with n commodities. To justify his linear model, he argues that:

"I believe the case for linear systems is much stronger than is commonly supposed. Most of the statistical investigations of such important functions as the propensity to consume and the propensity to import fail to show any significant departure from linearity."

Among the earlier works on positive systems, is the *Peter's Principle*. Peter's principle can be explained in non-technical terms as follows: "In a hierarchical organisation, people rise until they reach their level of incompetence" [PH69]. Peter's principle was first stated in non-mathematical terms in [PH69] and modelled as a positive LTI system in [Kan70]. Another early work utilising properties of positive matrices is done by Lewis F. Richardson. He developed a series of linear dynamical models of war [Ric60]. His study, which utilises the theory of positive matrices is a classical example of the potential of dynamical system theory in the study of social phenomena [Lue79].

In this chapter, we exclusively deal with positive systems which are cooperative. Therefore, most of our proofs utilise monotonicity methods. Monotonicity methods in differential equations have been studied since at least the 1920s. Müller [Mül26] and Kamke [Kam32] laid the foundations of the theory of monotone ordinary differential equations. Müller and Kamke's works were extended, for example in [Cop65, Wal70], to more general cases, but not until the seminal work of M. W. Hirsch, were monotonicity methods fully integrated in the theory of dynamical systems. Hirsch published a series of papers in the 80s and early 90s which are the basis of what is known today as the theory of monotone dynamical systems [Hir82, Hir85, Hir88b, Hir89, Hir90, Hir91].

Also, in [Hir88a] he extended the monotonicity methods to infinite dimensional differential equations. Most of the basic stability and convergence properties of monotone systems are also due to [Hir88a].

Influenced by the works of Hirsch, other researchers started to develop his results in different directions. For example, Matano in [Mat84] introduced the concept of 'strongly order preserving' differential equations and developed the concept in [Mat86, Mat87]. Smith and Thieme in [ST90] showed that Hirsch's results, which are stated for strongly monotone systems, are also valid under the weaker condition of strongly order preserving. The monograph written by Smith [Smi95], is probably the first book dedicated solely to the theory of monotone systems. Most of the basic definitions and properties of monotone systems stated in this manuscript are adopted from [Smi95].

Studying the stability properties of systems subject to different forms of uncertainty is the subject of numerous manuscripts on robust and adaptive control, for example look at [Dor87, ZDG95, Gup86, AW08] and references therein. Studying uncertain positive systems has been mostly limited to positive LTI systems. A class of results on stability analysis of positive systems are studied under the subject of *comparative statics*. The term comparative statics refers to an analysis procedure that focuses on the equilibrium point of a dynamic system, and how that equilibrium point changes when various system parameters are changed [Lue79, Section 6.7]. This form of analysis is concerned with how the new equilibrium is related to the old one, although to make the result meaningful, the new equilibrium should be proved to be asymptotically stable. The results presented in Section 3.8 fall under this category.

Another class of results on stability of uncertain positive LTI systems is related to the notion of D-stability as stated in Section 2.5. The notion of D-stability for positive LTI systems comes from economics. Arrow and McManus in their work on price equilibrium in economic system, coined the term D-stability [AM58]. In the early references, the D-stability is defined as a property of a matrix not a property of the equilibrium of an LTI system. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be D-stable, if DA is Hurwitz for all diagonal D with positive diagonal entries. It is easy to see why these two definitions of D-stability has been often used interchangeably. Apart from D-stability, the notion of *S-Stability* is also introduced in [AM58]. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be

S-stable, if the matrix SA is Hurwitz for every symmetric positive definite S. It is evident that if a matrix is S-stable, then it is D-stable. As is shown in [AM58] via a simple numerical example, Hurwitz Metzler matrices are not necessarily S-stable, therefore, the concept of S-stability has not been of much significance in the study of positive LTI systems. [Joh74] and [Kim81] provide a review of some of the most well-known sufficient conditions for D-stability of real and complex matrices. In [Kaf02], the notion of *strong D-stability* is introduced. According to [Kaf02], a matrix $A \in \mathbb{R}^{n\times n}$ is called strongly D-stable if there exists an $\alpha > 0$ such that $A + G$ is D-stable, where $G \in \mathbb{R}^{n\times n}$ satisfies $\|G\| < \alpha$. The thirteen sufficient conditions for D-stability stated in [Joh74] are proved in [Kaf02] to be also sufficient conditions for strong D-stability.

There have also been some attempts in applying the robust control methodologies to positive linear systems. One approach toward robust stability of positive systems is based on the concept of *stability radii* developed by Hinrichsen and Pritchard in [HP86a, HP86b]. Stability radii are defined as follows.

Definition 3.3.1. Consider a positive LTI system $\dot{x}(t) = Ax(t)$ which is subject to the structured perturbations of the form:

$$A \to A + D\Delta E \tag{3.4}$$

where $D \in \mathbb{K}^{n\times l}$ and $E \in \mathbb{K}^{q\times n}$ are given positive matrices defining the structure of the perturbation and $\Delta \in \mathbb{K}^{l\times q}$ is the disturbance matrix with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. It should be noted that if $D = E = I$, then this perturbation represents unstructured perturbations. Let $\|\cdot\|$ represent any *monotonic* norm in $\mathbb{K}$, meaning:

$$|x| < |y| \Rightarrow \|x\| < \|y\| \quad \text{for all } x, y \in \mathbb{K}$$

Then the *stability radius* of such an uncertain continuous-time positive LTI system is defined as follows:

$$r_{\mathbb{K}} = r_{\mathbb{K}}(A; D, E) := \inf\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q}, \mu(A + D\Delta E) \geq 0\}$$

where $\|\Delta\|$ represents the induced norm on $\mathbb{K}^{l\times q}$. For a discrete-time positive LTI system of the form $x(k + 1) = Ax(k)$, the stability radius is defined as follows:

$$r_{\mathbb{K}} = r_{\mathbb{K}}(A; D, E) := \inf\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q}, \rho(A + D\Delta E) \geq 1\}$$

In [Son95] it is shown that for any positive linear discrete-time system $x(k+1) = Ax(k)$, the complex and real stability radii under unstructured uncertainty $A \to A+\Delta$ are equal and can be computed directly by a simple formula, provided that the spaces under consideration are endowed with vector norms $\|\cdot\|_\alpha$, for $\alpha = 1, 2, \infty$. In [HS94] the results of [Son95] are extended to discrete-time positive LTI systems under perturbations of the form (3.4) and in [HS96], to continuous-time positive LTI systems. Also, in [HS98a], a μ-analysis for nonnegative matrices is presented and results are applied to analyse robust stability of continuous-time positive linear systems under arbitrary affine parameter perturbations.

Our work is motivated by the need to obtain robust stability conditions for nonlinear positive systems [BMV10, BMV11, BMW11]. In [BMV10], the concept of D-stability in linear systems was extended to nonlinear systems. Also, D-stability conditions for subhomogeneous cooperative systems are presented in [BMW11] without explicitly referring to them as D-stability results. These two papers are the basis for some of the results presented in this chapter.

3.4 D-stability for General Cooperative Systems

In this section, we present D-stability conditions for cooperative nonlinear systems. We state the results for two distinct cases: the case where the system has a unique equilibrium at the origin, and the case where the system has a unique equilibrium in the interior of $\mathbb{R}^n_+$.

3.4.1 Unique Equilibrium at the Origin

In this section, we present D-stability conditions for system (3.1) when it has a GAS equilibrium at the origin. Before stating the main theorem of this section, we need to establish some preliminary results.

Lemma 3.4.1. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field. If the system* (3.1) *has a GAS equilibrium at the origin, then the system* (3.2) *has a unique equilibrium at the origin for all $d \in \Delta$.*

Proof: Clearly, (3.2) has an equilibrium at the origin. It remains to show that it is unique.

Since $d \in \Delta$, we know that (3.2) cannot have any equilibrium points in $\text{int}\,(\mathbb{R}^n_+)$. Now, by way of contradiction, suppose $\text{diag}\,(d(p))f(p) = 0$ for some $p \neq 0$ in $\text{bd}\,(\mathbb{R}^n_+)$.

We define $Z := \{i : p_i = 0\}$ and $NZ := \{i : p_i \neq 0\}$. As $d_i(p_i) > 0$ for all $i \in NZ$ by assumption, we must have $f_i(p) = 0$ for all $i \in NZ$, otherwise $\text{diag}\,(d(p))f(p) \neq 0$. As the origin is a GAS equilibrium of (3.1), it follows from Lemma 2.4.5 that we cannot have $f(p) \geq 0$. Hence, there must be some $i_0 \in Z$ such that $f_{i_0}(p) < 0$.

On the other hand

$$\frac{\partial f_{i_0}}{\partial x_j}(s) \geq 0$$

for all $j \neq i_0$ and for all $s \in \mathbb{R}^n_+$. Furthermore, $p_{i_0} = 0$ as $i_0 \in Z$. Thus, from the fundamental theorem of calculus, we have:

$$f_{i_0}(p) = f_{i_0}(0) + \int_0^1 \sum_{j=1}^n \frac{\partial f_{i_0}}{\partial x_j}(sp)p_j ds \geq 0.$$

This is a contradiction and we can conclude that the origin is the only equilibrium of (3.2). □

The following proposition plays a key role in proving later results. We utilise the KKM lemma (Theorem 2.3.1) in the proof of this proposition.

Proposition 3.4.2. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field. If the system* (3.1) *has a GAS equilibrium at the origin then there exists a vector $v \gg 0$ such that $f(v) \ll 0$.*

Proof: Lemma 2.6.2 implies that the system (3.1) is positive. Consider the standard simplex Δ_n. We define $C_i = \{x \in \Delta_n : f_i(x) < 0\}$ for $i = 1, \cdots, n$. As f is continuous, C_i is a relatively open set in Δ_n for $i = 1, \cdots, n$. On the other hand, since the system (3.1) has a GAS equilibrium at the origin, there is no $w > 0$ in the simplex, such that $f(w) \geq 0$ by Lemma 2.4.5. Therefore, $\cup_{i=1}^n C_i = \Delta_n$.

Let $S(e_{i_0}, e_{i_1}, \cdots, e_{i_s})$ be an arbitrary face of the simplex and let $x \in S(e_{i_0}, e_{i_1}, \cdots, e_{i_s})$. Then $x_j = 0$ for $j \notin \{i_0, \cdots, i_s\}$. Since the positive orthant is an invariant set for (3.1), it follows that $f_j(x) \geq 0$ for $j \notin \{i_0, \cdots, i_s\}$. Therefore as (3.1) has a GAS equilibrium at the origin, Lemma 2.4.5 implies that $f_k(x) < 0$ for some $k \in \{i_0, \cdots, i_s\}$. This means that

$$x \in C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_s}.$$

As x was arbitrary, we conclude that for any face of the simplex, we have

$$S(e_{i_0}, e_{i_1}, \cdots, e_{i_s}) \subset C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_s}.$$

Now it follows from Theorem 2.3.1 that $\cap_{i=1}^{n} C_i \neq \varnothing$. As f is continuous, this means there exists a $v \gg 0$ in Δ_n such that $f(v) \ll 0$. □

Remark 3.4.1. Note that since the sets C_i (as defined in the proof of Proposition 3.4.2), for $i = 1, \cdots, n$ are open, then $\cap_{i=1}^{n} C_i$ have infinitely many elements. In other words, there are infinitely many vectors v that lie on the standard simplex with $f(v) \ll 0$. Also, since system (3.1) is assumed to be globally asymptotically stable, then the same applies to any simplex $r\Delta_n$ for all real r with $r > 0$.

To prove the main result of this section, we also need the following theorem but before stating it, we need to define some terminology which are adopted from [Kre78].

The following theorem, which is commonly known as *Convergence Criterion* for monotone systems, is a simple adaptation of Theorem 1.2.1 in [Smi95]. Note that $\overline{\mathcal{O}(x_0)}$ represents the closure of the orbit of the system (3.1) corresponding to initial condition x_0.

Theorem 3.4.3 (Convergence Criterion)**.** *Let the system* (3.1) *be cooperative defined on a subset $\mathcal{D}$ of $\mathbb{R}^n$ such that $\overline{\mathcal{O}(x_0)}$ is closed and bounded for all $x_0 \in \mathcal{D}$. If $x(t, x_0) \leq x_0$ for t belonging to some non-empty subset of* $(0, \infty)$*, then $x(t, x_0) \to p \in E$ as $t \to \infty$ where E is the set of all equilibria of the system* (3.1)*. In particular, if the system* (3.1) *is strongly order preserving and $x(T, x_0) < x_0$ for some $T > 0$ then $x(t, x_0) \to p \in E$ as $t \to \infty$.*

Now we are ready to state and prove the main theorem of this section.

Theorem 3.4.4. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. If the origin is a GAS equilibrium of the system* (3.1)*, then it is a D-stable equilibrium of the system* (3.1)*.*

Proof: It follows from Proposition 3.2.1 that the system (3.2) is positive and monotone. As the origin is a GAS equilibrium of system (3.1), Proposition 3.4.2 implies that there exists a $v \gg 0$ such that $f(v) \ll 0$.

Lemma 2.4.4 implies that the trajectory $x(t, v)$ of (3.2) starting from $x(0) = v$ is decreasing. In addition $\mathbb{R}^n_+$ is invariant under (3.2). It now follows from Theorem 3.4.3 that $x(t, v)$ converges to an equilibrium of (3.2) as $t \to \infty$. Lemma 3.4.1 implies that the origin is the only equilibrium of (3.2). It follows immediately that $x(t, v) \to 0$ as $t \to \infty$.

As (3.2) is positive and monotone, for every $x_0 < v$, we have:

$$0 \leq x(t, x_0) \leq x(t, v)$$

for all $t \geq 0$. This implies that $x(t, x_0) \to 0$ as $t \to \infty$ for all $x_0 \in \{x \in \mathbb{R}^n_+ : x \leq v\}$. This concludes the proof. □

Remark 3.4.2. Note that as shown in the proof of the above theorem, the domain of attraction of the origin for system (3.2) includes the region defined as $\{x \in \mathbb{R}^n_+ : 0 \leq x \leq v\}$. As discussed in Remark 3.4.1, there are infinitely many vectors v with $f(v) \ll 0$ lying on every simplex $r\Delta_N$ with $r > 0$. It should be noted that this does not mean that for every $x_0 \in \mathbb{R}^n_+$, we can find a $v > x_0$ with $f(v) \ll 0$. To make this point more clear, imagine the case where for all $v \gg 0$ with $f(v) \ll 0$, there is a real number $b > 0$ such that $v_i < b$ for some $i \in \{1, \cdots, n\}$. Figure 3.1 illustrates this situation. In this case, we cannot say whether a point x with $x_i > a$ belongs to the region of attraction of the equilibrium of the system (3.2). As will be seen in Sections 3.5 and 3.6, we need to impose extra assumptions on the system (3.1) to resolve this issue.

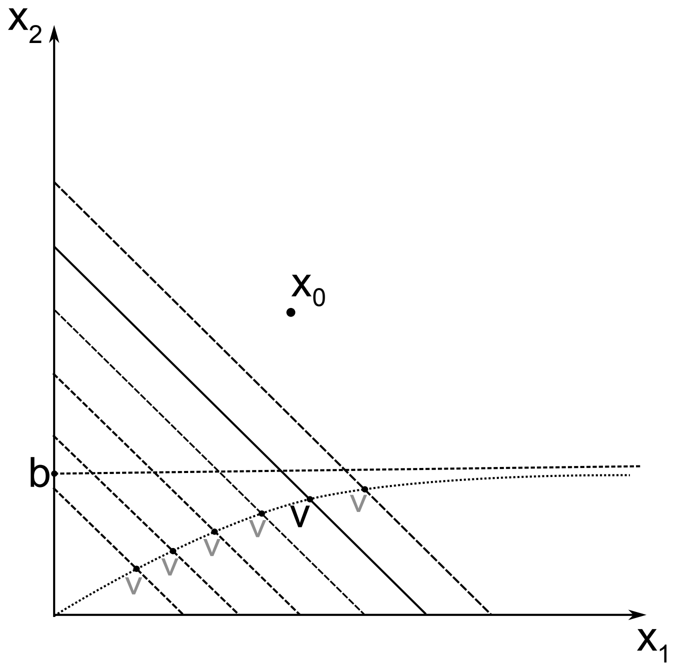

Figure 3.1: The case discussed in Remark 3.4.2 where for all initial condition x_0 with $x_{0_2} > b$ there exists no $v \gg x_0$ such that $f(v) \ll 0$.

3.4.2 Unique Equilibrium in the Interior of the Positive Orthant

We next state a D-stability result for the case where the system (3.1) has a unique equilibrium p in int $(\mathbb{R}^n_+)$. To understand why we only consider systems with unique equilibrium in int $(\mathbb{R}^n_+)$, not the whole $\mathbb{R}^n_+$, note that many systems have an equilibrium at the origin or on the bd $(\mathbb{R}^n_+)$, whose stability properties are not interesting for us. The following example clarifies this notion.

Example 3.4.1. Consider a Mutualistic Lotka-Volterra system [HS98b]:

$$\dot{x} = f(x) = \text{diag }(x)(Ax + b) \tag{3.5}$$

where A is Metzler. We also assume $b \gg 0$. Based on the discussions in Example 2.4.1, we know that system (3.5) is cooperative.

Lotka-Volterra systems are commonly used as models for interactions of different species in ecological systems. One of the equilibria of (3.5) is the origin which represents the undesirable situation where all the species have died out.

The other equilibrium is $\bar{x} = -A^{-1}b$. It is a known fact that if A is Metzler and Hurwitz, then A^{-1} is Hurwitz and $-A^{-1} > 0$ [Lue79, Theorem 6.5.3]. Also, $b \gg 0$, hence $-A^{-1}b \gg 0$, which means that system (3.5) has a unique equilibrium in $\operatorname{int}(\mathbb{R}^n_+)$.

In Lotka-Volterra model, $x_i = 0$ means the extinction of the ith species. It is easy to check that each axis is an invariant set for the system (3.5). Hence, the domain of attraction of $\bar{x} = -A^{-1}b$ cannot contain $\operatorname{bd}(\mathbb{R}^n_+)$. □

To prove the main result of this section, we need the following variant of Proposition 3.4.2.

Proposition 3.4.5. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. Assume that* (3.1) *has an asymptotically stable equilibrium at $p \gg 0$ and that the domain of attraction of p contains* $\operatorname{int}(\mathbb{R}^n_+)$. *Then there exists vectors $v, u \in \mathbb{R}^n_+$, such that $v \gg p$ with $f(v) \ll 0$ and $0 \ll u \ll p$ with $f(u) \gg 0$.*

Proof: We utilise KKM lemma (Theorem 2.3.1). Let $R_1(p)$ and $R_2(p)$ be defined as follows:

$$R_1(p) = \{x \in \operatorname{int}(\mathbb{R}^n_+) : x \gg p\}$$

$$R_2(p) = \{x \in \operatorname{int}(\mathbb{R}^n_+) : x \ll p\}$$

for all $p \in \mathbb{R}^n_+$. Figures 3.2 shows these two regions for a planar system.

Firstly, we prove there exists a $v \in R_1(p)$ such that $f(v) \ll 0$. Let Δ_n be the standard simplex. We consider $p + \Delta_n$, the standard simplex shifted to point p and define $C_i = \{x \in p + \Delta_N : f_i(x) < 0\}$ for $i = 1, \cdots, n$. Note the following facts.

(i) The set $\{x \in \mathbb{R}^n_+ : x \geq p\}$ is forward invariant under (3.1). This is because p is an equilibrium and (3.1) is monotone.

(ii) There is no $x > p$ with $f(x) \geq 0$. This follows from Lemma 2.4.5 as the domain of attraction of p contains $\operatorname{int}(\mathbb{R}^n_+)$.

Using (i) and (ii), we can apply the KKM lemma in the same way as in Proposition 3.4.2 to conclude that there exists $v \gg p$ such that $f(v) \ll 0$.

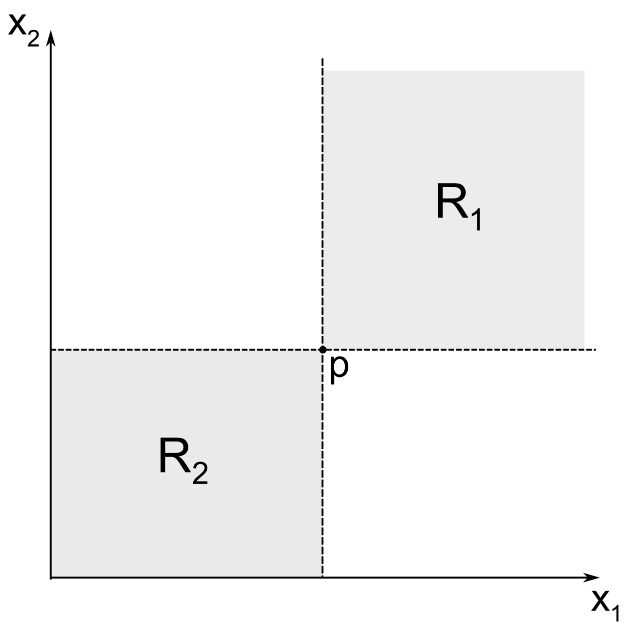

Figure 3.2: Regions R_1 and R_2 for a planar system with an equilibrium p in the int $(\mathbb{R}^2_+)$.

We next show that there exists a $u \in R_2(p)$ with $f(u) \gg 0$. First, choose $r > 0$ small enough to ensure that the shifted simplex $p - r\Delta_n$ is wholly contained in int $(\mathbb{R}^n_+)$. As above, it follows that $\{x \in \mathbb{R}^n_+ : x \leq p\}$ is forward invariant under (3.1) and that there can be no $x < p$ with $f(x) \leq 0$. Again applying the KKM Lemma, we conclude that there exists a $u \ll p$, such that $f(u) \gg 0$. $\square$

Now we are ready to state the main theorem of this section.

Theorem 3.4.6. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative in a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$ and let $d \in \Delta$. Assume that* (3.1) *has an asymptotically stable equilibrium at $p \gg 0$ and that the domain of attraction of p under* (3.1) *contains* int $(\mathbb{R}^n_+)$*. Then p is a D-stable equilibrium of the system* (3.1)*.*

Proof: Proposition 3.4.5 implies that there exists a $v \gg p$ such that $f(v) \ll 0$ and there exists a $0 \ll u \ll p$ such that $f(u) \gg 0$.

From the properties of d, it follows immediately that $d(v)f(v) \ll 0$ and $d(u)f(u) \gg 0$. Hence the trajectory $x(t, v)$ of (3.2) is decreasing while the trajectory $x(t, u)$ is increasing. Further, as p is an equilibrium of (3.2) and (3.2) is monotone, the trajectories $x(t, v)$, $x(t, u)$ of (3.2) satisfy $p \leq x(t, v) \leq v$, $u \leq x(t, u) \leq p$ for all $t \geq 0$.

Taken together this implies that the trajectories $x(t, v)$, $x(t, u)$ of (3.2) converge monotonically to p.

Let $x_0 \in \mathrm{int}\,(\mathbb{R}^n_+)$ be an initial condition such that $u \leq x_0 \leq v$. It now follows from the monotonicity of the system (3.2) that

$$x(t, u) \leq x(t, x_0) \leq x(t, v)$$

for all $t \geq 0$. Hence, $x(t, x_0)$ must also converge to p. This means p is an asymptotically stable equilibrium of (3.1) with $\{x \in \mathbb{R}^n_+ : u \leq x \leq v\}$ as domain of attraction. This concludes the proof. □

Remark 3.4.3. Similar to Theorem 3.4.4, Theorem 3.4.6 does not provide any information on whether or not the points outside $\{x \in \mathbb{R}^n_+ : u \leq x \leq v\}$ belong to the domain of attraction of the equilibrium of the system (3.2). Also, note that a cooperative system with a unique equilibrium in $\mathrm{int}\,(\mathbb{R}^n_+)$ may not even be positive. The following example illustrates this fact.

Example 3.4.2. Consider the system (3.1), with $f : \mathbb{R}^2_+ \mapsto \mathbb{R}^2$ defined as follows:

$$f(x) = \begin{pmatrix} x_1^2 - 3 \\ x_2^2 - 2 \end{pmatrix}$$

It is easy to see that f is cooperative and that the system (3.1) with the above choice of f has an a unique equilibrium in $\mathbb{R}^n_+$ at $(\sqrt{3}, \sqrt{2})^T$. On the other hand, for all x with $x_1 = 0$ we have $f_1(x) < 0$ and for all x with $x_2 = 0$ we have $f_2(x) < 0$ which means the system is not positive. Figure 3.3 shows the trajectory of the system starting from initial condition $x_0 = (1, 1)^T$ for $t \in [0, 6]$. □

In Section 3.6, we add the extra assumption of subhomogeneity to obtain a D-stability result where the domain of attraction of system (3.2) is also $\mathrm{int}\,(\mathbb{R}^n_+)$.

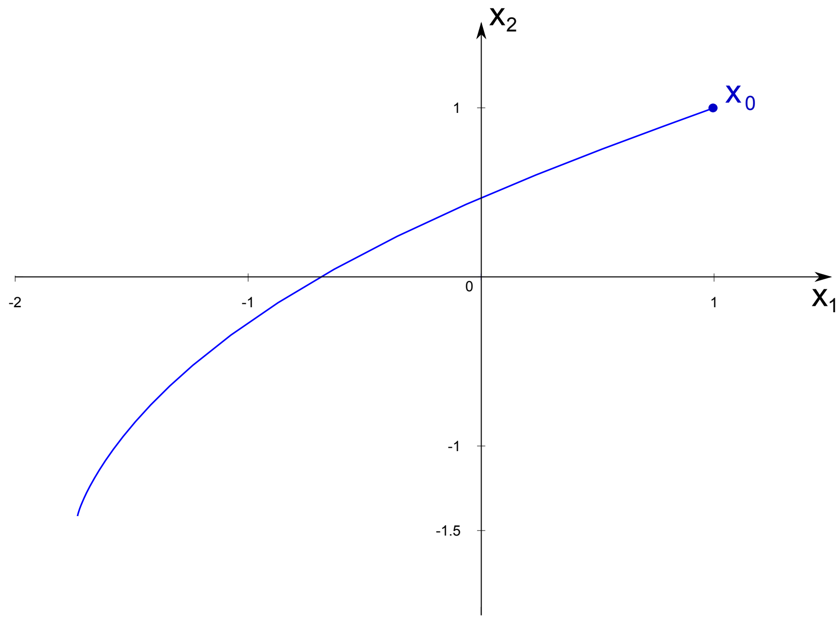

Figure 3.3: Trajectory of a cooperative system with an equilibrium in int $(\mathbb{R}^n_+)$ and without an equilibrium at the origin. Evidently, the system is not positive.

3.5 D-stability for Homogeneous Cooperative Systems

In this section, we add the extra assumption of homogeneity to the system (3.1) to obtain a D-stability result where the regions of attraction of systems (3.1) and (3.2) are the same.

Throughout this section, we assume $f : \mathcal{W} \to \mathbb{R}^n$ satisfies the following assumption on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$.

Assumption 3.5.1. (i) f is continuous and C^1 on $\mathcal{W}$;

(ii) f is homogeneous of degree α with respect to the dilation map δ^r_λ;

(iii) f is cooperative in $\mathbb{R}^n_+$.

The following proposition plays a key role in the proof of the main result of this section. It should be noted that the following proposition can be proved using the KKM lemma, but in the interest of completeness, we present an alternative proof which is based on the extension of the Perron-Frobenius theorem to homogeneous systems (Theorem 2.6.5).

Proposition 3.5.1. *Let f be cooperative and homogeneous of degree α with respect to dilation map δ_λ^r and assume system* (3.1) *has a GAS equilibrium at the origin. Then for any $x_0 \in \mathbb{R}_+^n$, there exists a $v \gg x_0$ with $f(v) \ll 0$.*

Proof: If f were irreducible, this result would be an immediate consequence of Theorem 2.6.5. The main step in the proof is to show that we can find an irreducible, homogeneous cooperative vector field f_1 such that $f_1(x) \geq f(x)$ for all $x \in \mathbb{R}_+^n$ and such that $\dot{x} = f_1(x)$ has a GAS equilibrium at the origin. Consider the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$g_i(x) = \left((x_1^2)^{\frac{M}{r_1}} + (x_2^2)^{\frac{M}{r_2}} + \ldots + (x_n^2)^{\frac{M}{r_n}}\right)^{\frac{(r_i+\alpha)}{M}} \tag{3.6}$$

for all $1 \leq i \leq n$ where M is a real number such that $M/r_i > 1$ for $i = 1, \ldots, n$. It can be easily checked that:

$$\frac{\partial g_i}{\partial x_j} = \frac{2(r_i+\alpha)}{r_j}\left(x_j^{\frac{2M}{r_j}-1}\right)\left(x_1^{\frac{2M}{r_1}} + \ldots + x_n^{\frac{2M}{r_n}}\right)^{\frac{r_i+\alpha}{M}-1} \tag{3.7}$$

It follows from (3.6) and (3.7) that:

- $g(a) \geq 0$ and $\frac{\partial g_i}{\partial x_j}(a) \geq 0$ for all $a \geq 0$ and $i \neq j$;
- g is continuous on $\mathbb{R}^n$ and C^1 on $\mathbb{R}^n \setminus \{0\}$;
- g is irreducible;
- g is homogeneous of degree α with respect to δ_λ^r.

We now claim that $f + \epsilon g$ has a GAS equilibrium at the origin for some $\epsilon > 0$. We prove this by contradiction. For all $\epsilon > 0$, we know that $(f + \epsilon g)$ is irreducible and satisfies Assumption 3.5.1. Further, $(f + \epsilon g)(v) \geq f(v)$ for all $v \geq 0$ because $g(v) \geq 0$ for all $v \geq 0$. If there is no $\epsilon' > 0$ such that the system $\dot{x} = (f + \epsilon' g)(x)$ has a GAS equilibrium at the origin, Theorem 2.6.5 implies that for every $\epsilon > 0$, there exists a non-zero $w \geq 0$ such that $(f + \epsilon g)(w) \geq 0$. We could then pick a sequence $\epsilon_n \to 0$, such that there exists a corresponding sequence $w^{(n)} \geq 0$, $w^{(n)} \neq 0$ with $(f + \epsilon_n g)(w^{(n)}) \geq 0$ for all

n. By homogeneity, we can normalize all $w^{(n)}$ such that $\|w^{(n)}\| = 1$. Choosing a subsequence, if necessary, we can assume that $w^{(n)} \to w'$ with $w' \geq 0$ and $\|w'\| = 1$. Since $\epsilon_n \to 0$, we know that

$$\lim_{n\to\infty} (f + \epsilon_n g)(w^{(n)}) = f(w') \geq 0$$

Since $\|w^{(n)}\| = 1$ and $w^{(n)} \geq 0$, it follows immediately from Lemma 2.4.4, that $x(t, w') \geq w' > 0$ for all $t \geq 0$ which contradicts the fact that (3.1) has a GAS equilibrium at the origin. Therefore there must exist an $\epsilon_1 > 0$, such that $f + \epsilon_1 g$ has a GAS equilibrium at the origin.

Theorem 2.6.5 implies that there is a vector $u \gg 0$ such that $(f + \epsilon_1 g)(u) = f(u) + \epsilon_1 g(u) \ll 0$ and since $g(u) \geq 0$, then we have $f(u) \ll 0$. To conclude the proof, simply choose $\lambda > 0$ such that $v := \delta_\lambda^r(u) \gg x_0$; the homogeneity of f implies that $f(v) \ll 0$. This completes the proof. □

We are now in a position to prove the following theorem, which is the main result of this section and appears in [BMV10]. The following theorem can be considered as an extension of Theorem 2.5.9 to homogeneous cooperative systems.

Theorem 3.5.2. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \mapsto \mathbb{R}^n$ be cooperative and homogeneous of degree α with respect to dilation map δ_λ^r. If the origin is a GAS equilibrium of the system* (3.1)*, then it is a globally D-stable equilibrium of the system* (3.1)*.*

Proof: Let $d \in \Delta$ and let $a \in \mathbb{R}^n_+$ be given. It follows from Proposition 3.5.1 that there exists $v \gg a$ with diag $(d(v))f(v) \ll 0$. Lemma 2.4.4 immediately implies that the trajectory $x(t, v)$ of the system (3.2) is non-increasing and bounded. Theorem 3.4.3 implies that it must converge to an equilibrium. Thus, the theorem can be proven provided we can show that the origin is the only equilibrium of the system (3.2).

Lemma 3.4.1 has already established the fact that the origin is the unique equilibrium of the system (3.2) under the stated conditions but we present an alternative proof in here using properties of homogeneous systems. To this end, note that since $f(0) = 0$ and $d_i(x_i) > 0$ for all $i = 1, \cdots, n$, we know that the origin is an equilibrium of (3.2). We shall show that it is the only equilibrium of the system by way of contradiction. Suppose that there is some $e := [e_1, e_2, ..., e_n]^T > 0$ satisfying diag $(d(e))f(e) = 0$. Let $x(t, a)$ denote the solution of (3.2) with initial condition a.

Choose some $v \gg 0$ with $f(v) \ll 0$. It is immediate that diag $(d(v))f(v) \ll 0$. Define $\kappa = \max\{(\frac{e_i}{v_i})^{(1/r_i)} : 1 \leq i \leq n\}$ and let $j \in \{1, ..., n\}$ be such that $(\frac{e_j}{v_j})^{1/r_j} = \kappa$. Note that as $e \neq 0$, $\kappa > 0$. It follows from the definition of κ that $e \leq \delta_\kappa^r(v)$ and that $e_j = (\delta_\kappa^r(v))_j$. As f is homogeneous, we have that $f(\delta_\kappa^r(v)) \ll 0$ and hence diag $(d(\delta_\kappa^r(v)))f(\delta_\kappa^r(v)) \ll 0$. Thus, we can pick t_1, such that for all $0 < t < t_1$,

$$x(t, \delta_\kappa^r(v)) \ll \delta_\kappa^r(v)$$

In particular, $x(t, \delta_\kappa^r(v))_j < \kappa^{r_j} v_j = e_j$. But as $e \leq \delta_\kappa^r(v)$ and the system (3.2) is monotone, we must have $x(t, \delta_\kappa^r(v))_j \geq e_j$ for all $t \geq 0$. This contradiction shows that the origin is the only equilibrium of (3.2) as claimed. This completes the proof. □

Remark 3.5.1. The proof of the previous result shows that under the stated assumptions, the origin is a GAS equilibrium of the cooperative homogeneous system (3.2) if for any initial condition $x_0 \in \mathbb{R}_+^n$, there exists a $v \gg x_0$ with $f(v) \ll 0$. In other words, any other assumption apart from homogeneity imposed on cooperative system (3.1) that guarantees existence of such a v for every $x_0 \in \mathbb{R}_+^n$, guarantees global D-stability of the origin for the system (3.1). In the next section, we see that subhomogeneity is one such assumption.

The following example illustrates Theorem 3.5.2.

Example 3.5.1. Consider the system (3.1) where

$$f(x) = \begin{pmatrix} -2x_1^{5/3} + x_3 \\ x_1^2 - 2x_2^{3/2} + x_3 x_1^{1/3} \\ x_1 x_2 + x_2^{7/4} - 5x_3^{7/5} \end{pmatrix}$$

The Jacobian of the system is

$$J = \frac{\partial f}{\partial x} = \begin{pmatrix} -\frac{10}{3}x_1 & 0 & 1 \\ 2x_1 + \frac{1}{3}x_3 x_1^{-2/3} & -3x_2^{1/2} & x_1^{1/3} \\ x_2 & x_1 + \frac{7}{4}x_2^{3/4} & -7x_3^{2/5} \end{pmatrix}$$

It is easy to see that J is a Metzler matrix for all $x \geq 0$, which makes system (3.1) a cooperative system. On the other hand, based on the Example 2.6.3

we know that f is homogeneous of degree $\alpha = 2$ with respect to dilation map δ_λ^r, with $r = (3, 4, 5)$

Also, for $v = (0.5, 0.5, 0.3)^T$, $f(v) = (-0.33, -0.22, -0.38)^T \ll 0$. Therefore, based on Theorem 3.5.2, the origin is a globally D-stable equilibrium of the system (3.1) with the above choice of f. For example, if we consider d given by:

$$d(x) = \left(\frac{x_1^2}{x_1^4 + 1}, x_2^2, 1 + \sin^4(x_3) \right)^T$$

then $d \in \Delta$ and it follows that with the above mentioned choices of d and f, system (3.2) has a GAS equilibrium at the origin. Note that system (3.2) is not homogeneous with these choices of d and f.

Figure 3.4 shows the trajectories for $\dot{x} = f(x)$ and $\dot{x} = \text{diag}\,(d(x)).f(x)$ for a set of initial conditions. □

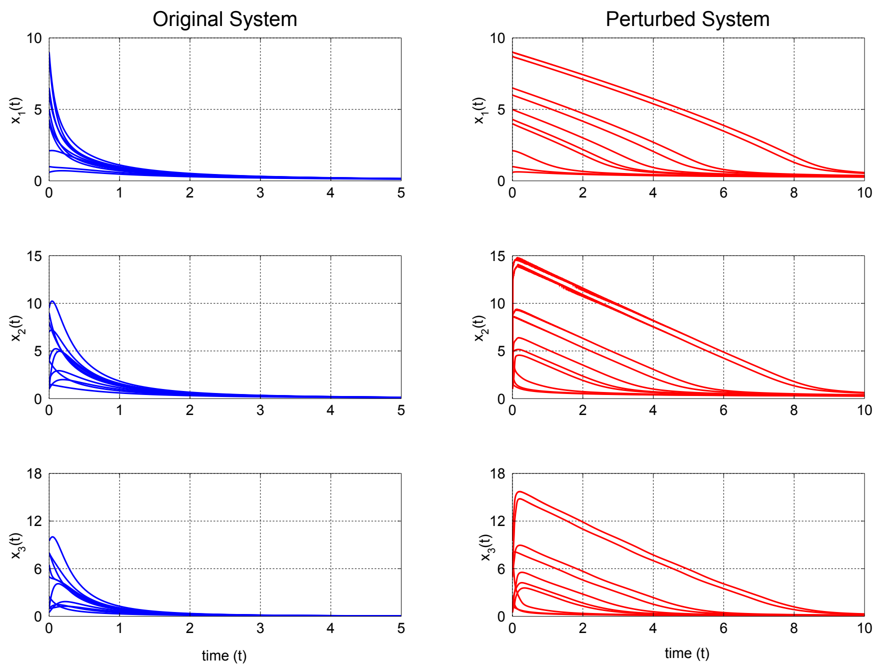

Figure 3.4: Evolution of states for the systems (3.1) and (3.2) corresponding to the Example 3.5.1

3.6 D-stability for Subhomogeneous Cooperative Systems

In this section, building on the results presented in Sections 3.4 and 3.5, we present conditions for D-stability for subhomogeneous cooperative systems. Since subhomogeneous systems and their basic properties were discussed in Section 2.6.2, we do not discuss them again in this section. We only recall the definition of a subhomogeneous system.

Definition 3.6.1. Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$. A vector field $f : \mathcal{W} \to \mathbb{R}^n$ is subhomogeneous of degree $\alpha > 0$ if $f(\lambda v) \leq \lambda^\alpha f(v)$, for all $v \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}$ with $\lambda \geq 1$. When f is subhomogeneous, then we say system (3.1) is subhomogeneous.

Throughout this section, we assume that the vector field f is:

- C^1 on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$;
- Subhomogeneous of degree α;
- Cooperative on $\mathbb{R}^n_+$.

As mentioned in Section 2.6.2, the class of subhomogeneous vector fields given above includes concave vector fields. Furthermore, it includes vector fields which are homogeneous with respect to the standard dilation map and consequently, linear vector fields.

We state the results for two distinct cases of cooperative subhomogeneous systems. First, when the system (3.1) has a unique equilibrium in the origin and then when the system (3.1) has a unique equilibrium in $\operatorname{int}(\mathbb{R}^n_+)$.

3.6.1 Equilibrium at the origin

In this section, we add the subhomogeneity assumption to Theorem 3.4.4 in order to attain more information on the domain of attraction of the equilibrium of (3.2). We use the preliminary results stated in Section 3.4.1 to prove the following theorem.

Theorem 3.6.1. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field that is subhomogeneous of degree $\alpha > 0$. Assume that* (3.1) *has a GAS equilibrium at the origin. Then the equilibrium is a globally D-stable equilibrium of the system* (3.1).

Proof: It follows from Proposition 3.2.1 that the system (3.2) is positive and monotone. As the origin is a GAS equilibrium of system (3.1), Proposition 3.4.2 implies that there exists a $v \gg 0$ such that $f(v) \ll 0$.

Let $x_0 \in \mathbb{R}^n_+$ be given. We can find a $\lambda \geq 1$ such that $w = \lambda v > x_0$. From subhomogeneity, it follows that

$$f(w) = f(\lambda v) \leq \lambda^\alpha f(v) \ll 0.$$

Further, since $d \in \Delta$, then

$$\text{diag }(d(w))f(w) \ll 0.$$

Lemma 2.4.4 implies that the trajectory $x(t, w)$ of (3.2) starting from $x(0) = w$ is decreasing. In addition $\mathbb{R}^n_+$ is invariant under (3.2). It now follows from Theorem 3.4.3 that $x(t, w)$ converges to an equilibrium of (3.2) as $t \to \infty$. Lemma 3.4.1 implies that the origin is the only equilibrium of (3.2). It follows immediately that $x(t, w) \to 0$ as $t \to \infty$.

As (3.2) is positive and monotone and as $x_0 < w$, it follows that

$$0 \leq x(t, x_0) \leq x(t, w)$$

for all $t \geq 0$. This implies that $x(t, x_0)$ is bounded for all $x_0 \in \mathbb{R}^n_+$ and $x(t, x_0) \to 0$ as $t \to \infty$. This concludes the proof. □

The following example shows a simple application of Theorem 3.6.1.

Example 3.6.1. Consider the system

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 + \dfrac{x_2}{m + x_2} \\ \\ -x_2 + \dfrac{x_1}{n + x_1} \end{pmatrix} \tag{3.8}$$

where $m > 1$, $n > 1$. It can be easily checked that f is C^1 on $\mathbb{R}^n \setminus \{(-n, -m)\}$, which means it is C^1 in $\mathbb{R}^n_+$. The Jacobian of this system is as follows:

$$J = \frac{\partial f}{\partial x} = \begin{pmatrix} -1 & \frac{m}{(m+x_2)^2} \\ \frac{n}{(n+x_1)^2} & -1 \end{pmatrix}$$

which is a Metzler matrix for all $x \in \mathbb{R}^n_+$. Also for $\lambda \geq 1$, we have:

$$\begin{aligned} f(\lambda x) &= \begin{pmatrix} -\lambda x_1 + \frac{\lambda x_1}{m+\lambda x_2} \\ -\lambda x_2 + \frac{\lambda x_1}{n+\lambda x_1} \end{pmatrix} \\ &\leq \begin{pmatrix} -\lambda x_1 + \frac{\lambda x_1}{m+x_2} \\ -\lambda x_2 + \frac{\lambda x_1}{n+x_1} \end{pmatrix} \\ &= \lambda f(x) \end{aligned}$$

which means f is subhomogeneous of degree 1. Also $f(x) = 0$ has two solutions, one is $x = 0$ and the other is

$$x = \left(\frac{1 - mn}{1 + m}, \frac{1 - mn}{1 + n} \right)$$

Since $m, n > 1$, the second solution is outside the positive orthant. Hence the origin is the unique equilibrium of the positive system (3.8) in $\mathbb{R}^n_+$. As $f(1, 1) \ll 0$, the argument in the previous proof can be readily adapted to show that the origin is a GAS equilibrium of (3.8).

If we define

$$d(x) = \begin{pmatrix} \dfrac{x_1^4}{x_1^3 + 1} \\ \\ 1 + \sin^{10}(x_2) \end{pmatrix}$$

then we have $d \in \Delta$. Now based on Theorem 3.6.1 we can say that the system $\dot{x} = \text{diag } (d(x))f(x)$ has a GAS equilibrium at the origin. Note that this new system is cooperative but not subhomogeneous.

Figure 3.5 shows trajectories of the systems (3.8) and (3.2) with the above choices of f and d with $a = 2$ and $b = 3$ for a variety of initial conditions. □

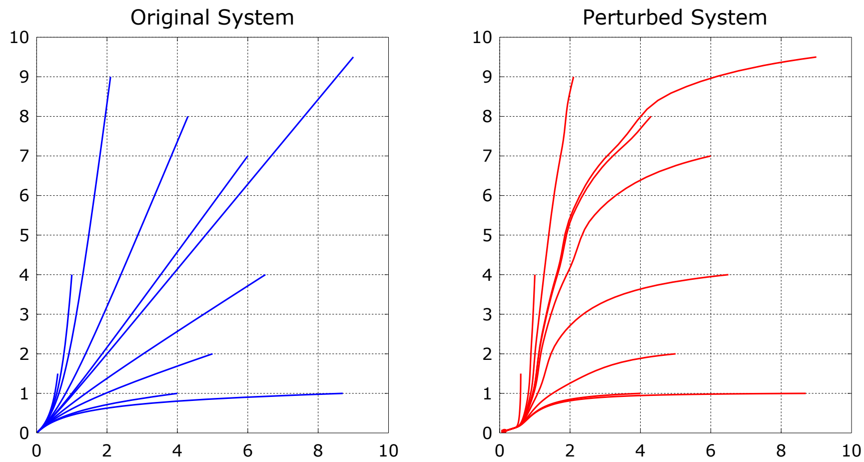

Figure 3.5: Trajectories of the systems (3.1) and (3.2) for Example 3.6.1

3.6.2 Equilibrium in the interior of the Positive Orthant

We next derive a version of Theorem 3.6.1 for the case where (3.1) has a unique equilibrium p in $\operatorname{int}(\mathbb{R}^n_+)$. For this scenario, apart from showing that asymptotic stability of p for (3.1) implies asymptotic stability of p for (3.2), we show that if the domain of attraction of p under (3.1) contains $\operatorname{int}(\mathbb{R}^n_+)$, then the domain of attraction of p under (3.2) also contains $\operatorname{int}(\mathbb{R}^n_+)$.

Theorem 3.6.2. *Let $f : \operatorname{int}(\mathbb{R}^n_+) \to \mathbb{R}^n$ be subhomogeneous of degree α and cooperative in a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. Assume that (3.1) has an asymptotically stable equilibrium at $p \gg 0$ and that the domain of attraction of p under (3.1) contains $\operatorname{int}(\mathbb{R}^n_+)$. Then p is a D-stable equilibrium of the system (3.1) and the domain of attraction of p under (3.2) contains $\operatorname{int}(\mathbb{R}^n_+)$.*

Proof: Proposition 3.4.5 implies that there exists a $v \gg p$ such that $f(v) \ll 0$ and there exists a $0 \ll u \ll p$ such that $f(u) \gg 0$. It follows from the subhomogeneity of f that for any $\lambda \geq 1$,

$$f(\lambda v) \leq \lambda^\alpha f(v) \ll 0$$

Similarly, for any $0 < \mu \leq 1$,

$$f(\mu u) \geq \mu^\alpha f(u) \gg 0$$

Let $x_0 \in \operatorname{int}(\mathbb{R}^n_+)$ be an arbitrary initial condition. Then we can choose $\lambda > 1$ and $\mu < 1$ such that

$$\mu u \leq x_0 \leq \lambda v \tag{3.9}$$

From the properties of d, it follows immediately that $d(\lambda v)f(\lambda v) \ll 0$ and $d(\mu u)f(\mu u) \gg 0$. Hence the trajectory $x(t, \lambda v)$ of (3.2) is decreasing while the trajectory $x(t, \mu u)$ is increasing. Further, as p is an equilibrium of (3.2) and (3.2) is monotone, the trajectories $x(t, \lambda v)$, $x(t, \mu u)$ of (3.2) satisfy $p \leq x(t, \lambda v) \leq \lambda v$, $\mu u \leq x(t, \mu u) \leq p$ for all $t \geq 0$.

Taken together this implies that the trajectories $x(t, \lambda v)$, $x(t, \mu u)$ of (3.2) converge monotonically to p. It now follows from the monotonicity of the system (3.2) and the inequality (3.9) that

$$x(t, \mu u) \leq x(t, x_0) \leq x(t, \lambda v)$$

for all $t \geq 0$. Hence, $x(t, x_0)$ must also converge to p. This completes the proof. □

3.7 D-stability for Planar Cooperative Systems

In Section 3.4.1, we presented a weak result concerning D-stability for general cooperative systems. While it applied to general systems, it only provided a local form of D-stability. It this section, we investigate whether or not the assumptions of homogeneity and subhomogeneity are necessary for the more general D-stability results presented in Sections 3.5 and 3.6. In other words, we check if the following conjecture is true.

Conjecture 3.1. Let $f : \mathcal{W} \mapsto \mathbb{R}^n$ be a cooperative vector field on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. If the system (3.1) has a GAS equilibrium at the origin, then the origin is a globally D-stable equilibrium of the system (3.2).

The conjecture was considered in [BMV10]. However, as the following example from [BMV11] shows, it fails even for planar systems.

Example 3.7.1. Consider the system on $\mathbb{R}^2_+$ given by

$$\dot{x} = f(x) = \begin{pmatrix} -\dfrac{x_1}{1 + x_1^3} + x_2 \\ \\ -x_2 \end{pmatrix} \tag{3.10}$$

It is easy to verify that f is cooperative and that the origin is the only equilibrium of this system. Also, for $v = (1, 0.25)^T$, $f(v) = (-0.25, -0.25)^T \ll 0$. This system is not homogeneous or subhomogeneous. We prove that the origin is a GAS equilibrium of (3.10) but is not a GAS equilibrium of (3.2) for the above choice of $f(\cdot)$.

First note that

$$\dot{x}_1 + \dot{x}_2 = -\frac{x_1}{1 + x_1^3} + x_2 - x_2 = -\frac{x_1}{1 + x_1^3} \leq 0$$

for all $x_1, x_2 \in \mathbb{R}_+$. This implies that for every $K > 0$, the bounded set

$$\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 \leq K\}$$

is invariant under (3.10). In particular, the trajectories of (3.10) have closed and bounded closure in $\mathbb{R}_+^2$. Using Theorem 3.2.2 in [Smi95], we can conclude that the single equilibrium of this system, which is the origin, is globally asymptotically stable.

Now choosing $d(x) = (1, x_2^3)$, (3.2) takes the form:

$$\dot{x} = \text{diag } (d(x)) f(x) = \begin{pmatrix} -\dfrac{x_1}{1 + x_1^3} + x_2 \\ \\ -x_2^4 \end{pmatrix} \tag{3.11}$$

As stated in Example 3.11 of [RKW10], the origin is *not* a GAS equilibrium of (3.11). In fact, for the initial condition $(x_1(0), x_2(0)) = (1, 1)$, the x_1 component of the associated solution grows without bound. This shows that the origin is not a GAS equilibrium of the system (3.2). This means the conjecture 3.1 is false even in planar case. □

Although the above example clearly shows that Conjecture 3.1 is not true, we can still ask what extra assumptions (apart from homogeneity or subhomogeneity) could be added to Conjecture 3.1, to make it true. In this section, we report on one of the possible answers to this questions, which appeared in [BMV11].

Throughout the section, $f : \mathcal{W} \to \mathbb{R}^2$ is assumed to satisfy the following assumptions in a neighbourhood $\mathcal{W}$ of $\mathbb{R}_+^2$:

Assumption 3.7.1. (i) f is C^1 on $\mathcal{W}$;

(ii) f is cooperative on $\mathbb{R}^2_+$;

(iii) For any $a_1 \geq 0$, there exists some δ_{a_1} and $\epsilon_{a_1} > 0$ such that

$$\frac{\partial f_2}{\partial x_2}(a_1, \delta) \leq -\epsilon_{a_1} \text{ for all } \delta \geq \delta_{a_1}; \tag{3.12}$$

(iv) For any $a_2 \geq 0$, there exists some δ_{a_2} and $\epsilon_{a_2} > 0$ such that

$$\frac{\partial f_1}{\partial x_1}(\delta, a_2) \leq -\epsilon_{a_2} \text{ for all } \delta \geq \delta_{a_2}. \tag{3.13}$$

Items (i) and (ii) of Assumption 3.7.1 are just the usual assumptions required for the existence and uniqueness of the solutions and monotonicity. The extra items (iii) and (iv) are useful to prove the next proposition which in turn is used to prove the main theorem of this section.

Proposition 3.7.1. *Assume that the system* (3.1) *has a GAS equilibrium at the origin and f satisfies Assumption 3.7.1. Then given any $a \in \mathbb{R}^2_+$, there exists $v \gg a$ with $f(v) \ll 0$.*

Proof: Let

$$\Omega_1 := \{x \gg a : f_1(x) < 0\}$$

and

$$\Omega_2 := \{x \gg a : f_2(x) < 0\}$$

where $a \in \mathbb{R}^2_+$. We prove that the items (iii) and (iv) of Assumption 3.7.1, guarantee that Ω_1 and Ω_2 are not empty. We show that there exists $u \geq a$ with $f_1(u) < 0$. (The proof that there is $w \geq a$ with $f_2(w) < 0$ is identical). By Assumption 3.7.1, there exist some constants $\delta_{a_2}, \epsilon_{a_2} > 0$ such that

$$\frac{\partial f_1}{\partial x_1}(\delta, a_2) < -\epsilon_{a_2}$$

for all $\delta \geq \delta_{a_2}$. Therefore, for $\delta \geq \delta_{a_2}$, we have

$$\begin{aligned} f_1(\delta, a_2) &= f_1(\delta_{a_2}, a_2) + \int_{\delta_{a_2}}^{\delta} \frac{\partial f_1}{\partial x_1}(s, a_2) ds \\ &\leq f_1(\delta_{a_2}, a_2) - \epsilon(\delta - \delta_{a_2}) \end{aligned}$$

This immediately implies that for every

$$\delta > \delta_{a_2} + \frac{f_1(\delta_{a_2}, a_2)}{\epsilon_{a_2}} \tag{3.14}$$

we must have $f_1(\delta, a_2) < 0$ which means Ω_1 is not an empty set. Using similar argument, we can prove that Ω_2 is also not an empty set. Therefore, we know for every $a \in \mathbb{R}^2_+$ there exists $u \geq a$ and $w \geq a$ with $f_1(u) < 0$ and $f_2(v) < 0$. Let

$$\Omega := \{x \in \mathbb{R}^2 : x \gg a\}$$

As f is cooperative and GAS, it follows from Lemma 2.4.4 that there cannot exist a non-zero vector $z \geq 0$ with $f(z) \geq 0$. Hence $\Omega = \Omega_1 \cup \Omega_2$. It is clear that Ω is a connected set and Ω_1, Ω_2 are open and non-empty subsets of Ω. Therefore, $\Omega_1 \cap \Omega_2$ is non-empty and this means that there exists some $v \gg a$ with $f(v) \ll 0$ as claimed. This completes the proof. □

Theorem 3.7.2. *Let f satisfy Assumption 3.7.1. If the origin is a GAS equilibrium of the system* (3.1)*, then it is a D-stable equilibrium of the system* (3.1)*.*

Proof: Let $d : \mathbb{R}^2 \to \mathbb{R}^2$ be in Δ. It follows from Lemma 3.4.1 that the origin is the only equilibrium of the system (3.2).

Theorem 3.7.1 implies that for any $x_0 \in \mathbb{R}^2_+$, there exists some $v \gg x_0$ with diag $(d(v))f(v) \ll 0$. It now follows from Lemma 2.4.4 that the trajectory $x(t, v)$ of (3.2) is non-increasing and bounded from below. Hence as the origin is the only equilibrium of the system (3.2), $x(t, v) \to 0$ as $t \to \infty$. It follows immediately from the monotonicity and positivity of (3.2) that $x(t, x_0) \leq x(t, v)$ also converges to the origin. This completes the proof. □

3.8 Positivity of Equilibria

In this section we study the case where the system (3.1) is forced by a constant input. Since we deal with positive systems, we are particularly interested in the case where the constant input is nonnegative. We are concerned with how the new equilibrium is related to the old one, although to make the result meaningful, the new equilibrium should be proved to be asymptotically stable. This form of analysis belongs to the domain of comparative statics, which we first mentioned in Section 3.3. For example, the result that relates the equilibrium of the positive LTI system $\dot{x} = Ax + b$ to the equilibrium of the positive LTI system $\dot{x} = Ax$ is a simple example of comparative statics. This simple result, can be stated as follows [FR00].

Theorem 3.8.1. *Consider the system*

$$\dot{x} = Ax + b\bar{u} \tag{3.15}$$

where $A \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz, $b \in \mathbb{R}^n$ with $b > 0$ and $\bar{u} \in \mathbb{R}$ with $\bar{u} > 0$. Let $\bar{x}$ be the equilibrium of the system (3.15)*. Then $\bar{x} > 0$.*

The above theorem only states a condition for positivity of the equilibrium of (3.15). It is shown in [MR91] that if the system (3.15) is excitable, then its equilibrium is globally asymptotically stable if and only if it is strictly positive (for the definition of excitability, look at [PR02]).

In [PR02], it is shown that the equivalence between asymptotic stability and strict positivity of equilibria for the system (3.15), does not hold for excitable nonlinear cooperative systems, although weaker results hold.

In [dLA01], results relating the stability properties of a homogeneous cooperative irreducible system $\dot{x} = f(x)$ to the existence of positive equilibria of the associated system $\dot{x} = f(x) + b$ are presented. The arguments of this paper rely on the extension of the Perron-Frobenius Theorem to homogeneous irreducible cooperative systems (as stated in the Theorem 2.6.5). In this section, we extend some of these results to subhomogeneous systems. Specifically, we consider the system (3.1), where f is assumed to be cooperative, subhomogeneous and irreducible, and relate it to the existence and stability of positive equilibria of the associated system

$$\dot{x} = f(x) + b, \quad b \in \mathbb{R}^n. \tag{3.16}$$

We do not specifically use the main theorem of [dLA01] to prove the results presented in this section. However, in the interest of completeness, we state it here.

Theorem 3.8.2. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \mapsto \mathbb{R}^n$ be cooperative, irreducible and homogeneous of degree α with respect to dilation map δ^r_λ. Then there exists a unique equilibrium point $\bar{x}$ in $\mathbb{R}^n_+$ for system* (3.16)*. This equilibrium is in* $\operatorname{int}(\mathbb{R}^n_+)$ *and is GAS for system* (3.16)*.*

The main theorem of this section, states a similar condition for the positivity and stability of the equilibrium of system (3.16) when f is subhomogeneous. We first state some preliminary results. The following proposition establishes a sufficient condition for the system (3.16) to be positive.

Proposition 3.8.3. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous and cooperative on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$ and let $b \geq 0$. Then the system* (3.16) *is positive.*

Proof: Let $g(\cdot) = f(\cdot) + b$. Since f is subhomogeneous and $\lambda \geq 1$, we have

$$\begin{aligned} g(\lambda x) &= f(\lambda x) + b \\ &\leq \lambda^\tau f(x) + b \\ &\leq \lambda^\tau f(x) + \lambda^\tau b \\ &= \lambda^\tau (f(x) + b) \\ &= \lambda^\tau g(x) \end{aligned}$$

which means $g(\cdot)$ is subhomogeneous. Also $\frac{\partial f}{\partial x}(a) = \frac{\partial g}{\partial x}(a)$ for all $a \in \mathbb{R}^n_+$ which means g is cooperative. The result now follows from Theorem 2.6.9. $\square$

The following proposition, provides a condition for positivity of equilibria for the systems of the form (3.16).

Proposition 3.8.4. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree α and cooperative and let $b > 0$ be given. Assume that the system* (3.1) *has a GAS equilibrium at the origin. Then the system* (3.16) *has at least one equilibrium in* $\operatorname{int}(\mathbb{R}^n_+)$.

Proof: From Proposition 3.4.2, we know there exists a $v \gg 0$ such that $f(v) \ll 0$. The subhomogeneity of f implies that $f(\lambda v) \leq \lambda^{\alpha} f(v)$ for all $\lambda \geq 1$. By choosing λ large enough we can ensure that $f(\lambda v) + b \ll 0$. Since $g(\cdot) = f(\cdot) + b$ is also cooperative, it follows from Lemma 2.4.4 that the trajectory $x(t, \lambda v)$ of (3.16) starting from λv is decreasing.

Given any $x_0 \in \mathbb{R}^n_+$, we can find $\lambda > 1$ with $\lambda v \geq x_0$ and $f(\lambda v) + b \ll 0$. Further, as (3.16) is positive, this implies that

$$0 \leq x(t, x_0) \leq x(t, \lambda v) \leq \lambda v$$

for all $t \geq 0$. Hence, the forward solution $\{x(t, x_0) : t \geq 0\}$ is bounded for any $x_0 \in \mathbb{R}^n_+$. It follows immediately from Theorem 3.4.3 that $x(t, \lambda v)$ converges to an equilibrium point $p \in \mathbb{R}^n_+$.

We have shown that there exists an equilibrium in $\mathbb{R}^n_+$. To complete the proof, we show that every equilibrium of (3.16) is in $\mathrm{int}\,(\mathbb{R}^n_+)$. Since $b > 0$ and $f(0) = 0$, the origin cannot be an equilibrium of the system (3.16). Next consider $z \in \mathrm{bd}\,(\mathbb{R}^n_+) \setminus \{0\}$ with $z_i = 0$. Since system (3.16) is positive, then based on (3.3), $f_i(z) > 0$. Also $b > 0$, hence $f(z) + b$ cannot be zero. This concludes the proof. □

To prove the main result of this section, we need the following proposition, which extends Proposition 4 of [dLA01] to subhomogeneous vector fields.

Note that in the remainder of this section, we impose the extra assumption of irreducibility on the vector field f.

Proposition 3.8.5. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree α, cooperative and irreducible and let $b > 0$ be given. Then the Jacobian matrix of $f(x) + b$ evaluated at an equilibrium point of the system* (3.16) *is a Hurwitz matrix.*

Proof: We prove the proposition by contradiction. Choose an arbitrary equilibrium point p. Based on Proposition 3.8.4 we know that $p \in \mathrm{int}\,(\mathbb{R}^n_+)$. As f is irreducible and cooperative and $p \in \mathrm{int}\,(\mathbb{R}^n_+)$, $\frac{\partial f}{\partial x}(p)$ is an irreducible Metzler matrix. By way of contradiction, suppose that $\frac{\partial f}{\partial x}(p)$ is not a Hurwitz matrix. Writing μ for the maximal real part of the eigenvalues of $\frac{\partial f}{\partial x}(p)$, we have $\mu \geq 0$. Perron-Frobenius Theorem for irreducible LTI systems (Theorem 2.5.7) then implies that there exists a vector $v \in \text{int}(\mathbb{R}^n_+)$ with

$$v^T \frac{\partial f}{\partial x}(p) = \mu v^T \tag{3.17}$$

On the other hand based on Lemma 2.6.7, we know that

$$\frac{\partial f}{\partial x}(p)p \leq \alpha f(p). \tag{3.18}$$

Multiplying (3.18) by v^T on the left and invoking (3.17) we have

$$\mu v^T p \leq \alpha v^T f(p) \tag{3.19}$$

We know that $f(p) = -b < 0$. Therefore there exists at least one j such that $f_j(p) < 0$. This implies that the right hand side of (3.19) is strictly negative, while the left hand side is nonnegative. We have therefore reached a contradiction and we can conclude that $\frac{\partial f}{\partial x}(p)$ is a Hurwitz matrix. □

Now we are ready to state and prove the main theorem of this section.

Theorem 3.8.6. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree α, cooperative and irreducible such that the system* (3.1) *has a GAS equilibrium at the origin. Then for any $b > 0$, the system* (3.16) *has a unique equilibrium in* $\text{int}(\mathbb{R}^n_+)$*, and this equilibrium is GAS.*

Proof: We know from Proposition 3.8.4 that (3.16) has an equilibrium in $\text{int}(\mathbb{R}^n_+)$. We first prove that this equilibrium is unique.

To this end, suppose that there are two distinct equilibria $p \gg 0, q \gg 0$. Proposition 3.8.5 implies that Jacobian of $g(\cdot) = f(\cdot) + b$ evaluated at each equilibrium point is Hurwitz. Further, as g is cooperative and irreducible, the Jacobian evaluated at each equilibrium point is irreducible and Metzler. Perron-Frobenius Theorem for irreducible LTI systems (Theorem 2.5.7) implies that there exist vectors x^p, x^q with $\|x^p\| = 1$, $\|x^q\| = 1$ such that

$$\frac{\partial g}{\partial x}(p)x^p \ll 0$$

and

$$\frac{\partial g}{\partial x}(q)x^q \ll 0$$

Without loss of generality, we can assume that

$$\max_i \frac{q_i}{p_i} > 1 \;\; \forall i = 1, \cdots, n$$

As g is C^1, it follows from Taylor's theorem that by choosing $t > 0$ sufficiently small, we can ensure that $g(p + tx^p) \ll 0$, $g(q - tx^q) \gg 0$. Define $v = p + tx^p$, $w = q - tx^q$. Then $g(v) \ll 0$, $g(w) \gg 0$. Also, choosing a smaller t if necessary, we can ensure that

$$\lambda := \max_i \frac{w_i}{v_i} = \frac{w_k}{v_k} > 1$$

Now note the following facts:

(i) $\lambda v \geq w$ and $\lambda v_k = w_k$;

(ii) $g(\lambda v) \leq \lambda^\alpha g(v)$ (as $b > 0$, g is also subhomogeneous).

As g is cooperative, it follows from (i) that

$$g_k(\lambda v) \geq g_k(w) > 0$$

On the other hand, it follows from (ii) that

$$g_k(\lambda v) \leq \lambda^\alpha g_k(v) < 0.$$

This is a contradiction, which shows that there can only be one equilibrium of (3.16) in $\text{int}\,(\mathbb{R}^n_+)$ as claimed.

To complete the proof, we show that this unique equilibrium point is GAS. Let $p \gg 0$ be the equilibrium point of (3.16). As the Jacobian of g evaluated at p is Hurwitz, Metzler and irreducible, it follows from Taylor's theorem (as in the previous paragraph) that there is some $v \geq p$ with $g(v) \ll 0$. Further, as $f(0) = 0$, we have $g(0) = b > 0$. Hence from Lemma 2.4.4 the trajectory $x(t, 0)$ of (3.16) is non-decreasing and satisfies

$$0 \leq x(t, 0) \leq p$$

for all $t \geq 0$. As p is the only equilibrium of (3.16) in $\text{int}\,(\mathbb{R}^n_+)$ it follows that $x(t, 0) \to p$ as $t \to \infty$.

Let $x_0 \in \mathbb{R}^n_+$ be given. As g is subhomogeneous, we can find a $\lambda > 1$ such that $w = \lambda v \gg x_0$, and $g(w) \ll 0$. Lemma 2.4.4 implies that the trajectory $x(t, w)$, starting from w is decreasing and satisfies

$$w \geq x(t, w) \geq p$$

for all $t \geq 0$. Thus $x(t, w) \to p$ as $t \to \infty$.

As $0 \leq x_0 \leq w$ and (3.16) is monotone, it follows that

$$x(t, 0) \leq x(t, x_0) \leq x(t, w)$$

for all $t \geq 0$. It is now immediate that $x(t, w) \to p$ as $t \to \infty$. This concludes the proof. □

Example 3.8.1. We again, consider the system defined in Example 3.6.1. In other words, we consider $f : \mathcal{W} \mapsto \mathbb{R}^n$ to be defined on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$ as follows:

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 + \dfrac{x_2}{m + x_2} \\ \\ -x_2 + \dfrac{x_1}{n + x_1} \end{pmatrix}$$

where $n > 1$, $m > 1$. In Example 3.6.1, it was proved that f is cooperative and subhomogeneous of degree 1. Also looking at the Jacobian of f:

$$J = \frac{\partial f}{\partial x} = \begin{pmatrix} -1 & \frac{m}{(m+x_2)^2} \\ \frac{n}{(n+x_1)^2} & -1 \end{pmatrix}$$

we can see that it is irreducible for all $x \in \mathbb{R}^n_+$. It was also shown that for $v = (1, 1)^T$, $f(v) \ll 0$ and the origin is the GAS equilibrium of the system (3.1) with the above choice of f. This means that the system (3.1) satisfies all the conditions stated in Theorem 3.8.6. Therefore, system (3.16) with the above mentioned choice of f has a unique GAS equilibrium in $\mathrm{int}\,(\mathbb{R}^n_+)$ for all vectors b with $b > 0$.

Figure 3.6 shows the trajectories of the systems (3.1) and (3.16) for a set of initial conditions with $m = 2$, $n = 3$ and $b = (1, 1)^T$.

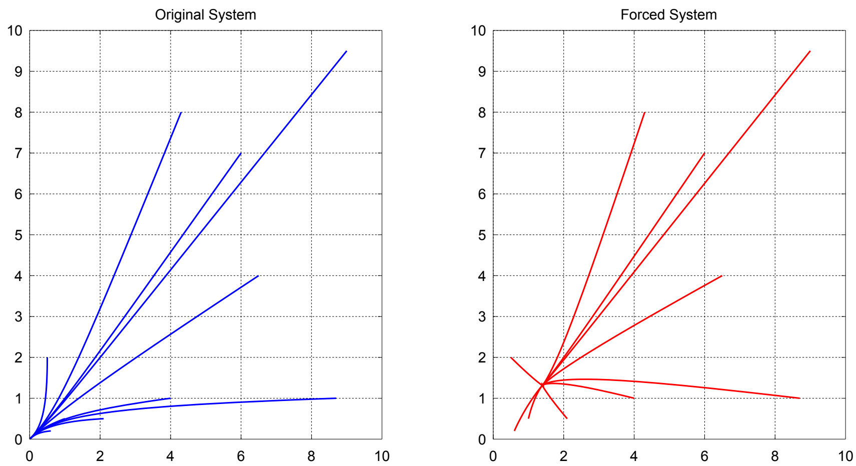

Figure 3.6: Trajectories of Systems (3.1) and (3.16) corresponding to Example 3.8.1

3.9 Concluding Remarks

The main theme of this chapter was extending the notion of D-stability to nonlinear positive systems. D-stability is a novel concept in nonlinear systems theory and therefore, much is left to study. Our focus in this chapter was on monotone positive systems. As mentioned before, for LTI systems, positivity and monotonicity properties are equivalent but that does not hold for nonlinear systems. Therefore, a natural extension to the results presented in this section is to study positive nonlinear systems which are not necessarily cooperative.

Even for monotone positive systems, there is still much left to do. The most immediate is to obtain a necessary and sufficient condition for the D-stability for the general cooperative system (3.1), such that the equilibria of the system (3.2) have the same domain of attraction as the equilibria of (3.1). Also, we only considered the cases where system (3.1) has a unique equilibrium in $\mathbb{R}^n_+$ or in int $(\mathbb{R}^n_+)$. D-stability of monotone positive systems with multiple equilibria is another interesting problem that can be studied.

Apart from the different types of positive systems we can consider, the definition of D-stability itself may also be extended. It is interesting to ask if similar D-stability conditions for a more general Δ could be derived, in particular, the case where the condition $d_i(x) = d_i(x_i)$, $i = 1, \cdots, n$, for $d \in \Delta$ is eased.

Also, there is a possible extension to the results presented in Section 3.8. In that section, we did not assume excitability in stating the results, instead we used the assumption of irreducibility. Therefore, studying the positivity and asymptotic stability of the equilibrium of the system (3.16) when system (3.1) is cooperative, excitable and homogeneous or subhomogeneous, is the next logical step in extending those results.

CHAPTER 4

Delay-Independent Stability of Positive Nonlinear Systems

In this chapter, we review the basic concepts and properties of time-delay systems. Then we present conditions for delay-independent stability of classes of positive time-delay systems.

4.1 Introduction

Dynamical systems with time-delay have attracted much interest [XFS09]. There are many reasons for this. One reason is that nature is full of systems subject to delay. Another reason is that time-delay systems are often used to model a large class of engineering systems, where propagation or transmission of information or material is involved. The presence of delays (especially long delays) can make system analysis and control design much more difficult [Zho06]. Systems with delay are often referred to as *time-delay* or *delayed systems*.

In this chapter, our attention is focused on positive nonlinear delayed systems, particularly, homogeneous and subhomogeneous cooperative delayed systems. We present conditions for stability of these systems that hold for fixed but unknown values of delay. We also report on an attempt to generalise these results to more general classes of cooperative delayed systems.

4.2 Background

A common hypothesis in the modelling of physical systems is to assume that the future behaviour of the system depends only on the present value of the states of the system. Such models, when the number of states is finite, fall into the category of ordinary differential equations, ODEs. Sometimes, we face situations in which the influence of the past states should also be considered. For example, in population dynamics, time-delay should be added to the model to account for hatching and maturation periods [Hut48]. A familiar example in control engineering is the delay in measuring the states of a plant due to technological or physical limitations.

As a theoretical example, consider the following system:

$$\dot{x}(t) = -x(t-\tau)$$

where τ is real with $\tau > 0$. In such cases, the state cannot be the vector $x(t)$ anymore. The above equation can have infinite number of solutions for the same value of $x(0)$ and in order to calculate the future values of $x(t)$, we need to know the history segment $\{x(s) : t-\tau \leq s \leq t\}$. In systems like the above example where a time-delay exists, the state should be defined over the time period $[-\tau, 0]$.

Mathematical models of time-delay systems, such as the above mentioned example, belong to the family of *functional differential equations*. Functional differential equations are generally infinite dimensional, which makes them harder to analyse than ordinary differential equations. If the past dependence appears only in the state variables and not the derivative of the state variables, then the functional differential equation is called *retarded functional differential equations* or *retarded differential equations* [Hal77].

There are also a number of applications in which the delayed argument occurs in the derivative of the state variable as well as in the independent variable. Such functional differential equations are called *neutral differential equations* [Hal77]. Neutral differential equations are less common compared to retarded differential equations and are out of the scope of this manuscript. Hereafter, every time we use the term functional differential equations, we only refer to retarded functional differential equations.

While modelling a time-delay system as a functional differential equation, based on the information we have on τ, the value of the delay, we are usually faced with three situations:

(1) τ is fixed and its exact value is known;

(2) τ is fixed but its exact value is unknown;

(3) τ is time-variable, i.e., $\tau = \tau(t)$ for $t \geq [-\tau, \infty)$.

In this chapter, we are interested in the second case, i.e., when we know τ is fixed, but we do not know its precise value. The results stated in this chapter hold for all positive (but fixed) values of time-delay. Such stability results are called *delay-independent* stability conditions.

The states of the functional differential equation we consider in this manuscript belong to $C([-\tau, 0], \mathbb{R}^n)$, where $C([a, b], \mathbb{R}^n)$ is the space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ for $a, b \in \mathbb{R}$ with $b > a$. For any $\phi \in C([a, b], \mathbb{R}^n)$, we use the norm defined as

$$\|\phi\| := \sup_{a \leq \theta \leq b} \|\phi(\theta)\|$$

where $\|\cdot\|$ in the right hand side can be any p-norm in $\mathbb{R}^n$, for example 2-norm. Even though $\|\cdot\|$ is used for norms in different spaces, no confusion should arise.

Let $t_f \in \mathbb{R}$ with $t_f \geq 0$. For $x \in C([-\tau, t_f], \mathbb{R}^n)$ and $t \in [0, t_f]$, we define $x^{(t)} \in C([-\tau, 0], \mathbb{R}^n)$ as follows:

$$x^{(t)}(\theta) = x(t + \theta), \;\; \theta \in [-\tau, 0]$$

Let Ω be a subset of $C([-\tau, 0], \mathbb{R}^n)$. Then a retarded functional differential equation can be represented by

$$\dot{x}(t) = h(x^{(t)}) \tag{4.1}$$

with $h : \Omega \mapsto \mathbb{R}^n$. A function x is said to be a *solution* or *trajectory* of (4.1) on $[-\tau, t_f)$, if there is $t_f > 0$ such that $x \in C([-\tau, t_f), \mathbb{R}^n)$, $x^{(t)} \in \Omega$ and $x(t)$ satisfies (4.1) for all $t \in [0, t_f)$ [Hal77].

For a given $\phi \in C([-\tau, 0], \mathbb{R}^n)$, we say $x(t, \phi)$ is a solution of (4.1) with initial condition ϕ, or simply a *solution through* ϕ, if there is a $t_f > 0$, such that $x(t, \phi)$ is a solution of (4.1) on $t \in [-\tau, t_f)$ and $x^{(0)}(\theta) = \phi(\theta)$ for all $\theta \in [-\tau, 0]$ [Hal77].

Similar to ordinary differential equations, we are only interested in those cases where the solution of a functional differential equation exists and is unique. The following two theorems are adopted from [Hal77].

Theorem 4.2.1 (Existence). *Let Ω be an open subset of $C([-\tau, 0], \mathbb{R}^n)$, and let $h : \Omega \mapsto \mathbb{R}^n$ be continuous in Ω. Then for every $\phi \in \Omega$, system* (4.1) *has a solution through ϕ.*

To state the condition for uniqueness of solutions of (4.1), we need to extend the Lipschitz condition (Definition 2.4.1) to vector fields defined on $C([-\tau, 0], \mathbb{R}^n)$.

Definition 4.2.1 (Lipschitz Condition). Let $h : K \mapsto \mathbb{R}^n$ where K is a closed and bounded subset of $C([-\tau, 0], \mathbb{R}^n)$. We say h is Lipschitz in K, if for any $\phi_1, \phi_2 \in K$, there exists a constant $k > 0$, such that

$$\|h(\phi_1) - h(\phi_2)\| \leq k\|\phi_1 - \phi_2\|$$

The condition for uniqueness of the solution of (4.1) can be stated as follows.

Theorem 4.2.2 (Uniqueness). *Let $h : \Omega \mapsto \mathbb{R}^n$ where Ω is an open subset of $C([-\tau, 0], \mathbb{R}^n)$. If h is Lipschitz on every closed and bounded subset of Ω, then system* (4.1) *has a unique solution for every initial condition $\phi \in \Omega$.*

In the remainder of this chapter, we always assume all vector fields satisfy the existence and uniqueness conditions, even if it is not explicitly mentioned.

In this chapter, we deal with positive time-delay systems. The definition of a positive time-delay system is as stated in the following and in principle, is similar to the concept of positivity for nonlinear systems stated in Chapter 2.

Definition 4.2.2 (Positivity). Let Ω be an open subset of $C([-\tau, 0], \mathbb{R}^n)$, and let $h : \Omega \mapsto \mathbb{R}^n$ be continuous in Ω. Also, let Ω_+ be defined as follows:

$$\Omega_+ := \{\phi \in \Omega : \phi(\theta) \geq 0 \text{ for all } \theta \in [-\tau, 0]\} \tag{4.2}$$

Then the system (4.1) is said to be *positive*, if for every $\phi \in \Omega_+$, we have $x(t, \phi) \geq 0$ for all $t \geq 0$.

In this chapter, we deal with monotone time-delay systems. Monotonicity in functional differential equations is defined as follows.

Definition 4.2.3 (Monotonicity)**.** Let Ω be a subset of $C([-\tau, 0], \mathbb{R}^n)$ and let $x(t, \phi)$ represent the trajectory of the system (4.1) with respect to initial condition $\phi \in \Omega$ at time t. Then the system (4.1) is said to be monotone if for every $\phi, \psi \in \Omega$, satisfying

$$\phi(\theta) \leq \psi(\theta), \quad \text{for all } -\tau \leq \theta \leq 0$$

we have:

$$x(t, \phi) \leq x(t, \psi)$$

It is known that the quasimonotone condition, provides a sufficient condition for monotonicity of the system (4.1) [Smi95, Section 5.1]. The quasimonotone condition can be stated as follows.

Definition 4.2.4 (Quasimonotone Condition)**.** Whenever $\phi(\theta) \leq \psi(\theta)$, for all $-\tau \leq \theta \leq 0$ and $\phi_i(0) = \psi_i(0)$ holds for some i, then $h_i(\phi) \leq h_i(\psi)$.

The next theorem, which is a restatement of Theorem 5.1.1 in [Smi95], formally states the relation between quasimonotone condition and monotonicity of the system (4.1).

Theorem 4.2.3. *Let Ω be a subset of $C([-\tau, 0], \mathbb{R}^n)$. The system* (4.1) *is monotone in Ω, if the quasimonotone condition is satisfied for every $\phi, \psi \in \Omega$.*

The quasimonotone condition is an extension of the Kamke condition (Definition 2.4.6) to vector fields in $C([-\tau, 0], \mathbb{R}^n)$. It can be easily seen that for the special case where $\tau = 0$, the quasimonotone and Kamke conditions are in fact the same.

From now on, for any $p \in \mathbb{R}^n$ we define $\hat{p} \in C([-\tau, 0], \mathbb{R}^n)$ to be:

$$\hat{p}(\theta) \equiv p, \quad \forall \theta \in [-\tau, 0] \tag{4.3}$$

Let Ω be an open subset of $C([-\tau, 0], \mathbb{R}^n)$. The equilibria of (4.1) in Ω are those $\phi \in \Omega$ such that

$$h(\phi) = 0 \tag{4.4}$$

This means that at $\theta = -\tau$ and for $t \in [0, \tau]$, we have:

$$x(t - \tau, \phi) = \phi(t - \tau) = \phi(-\tau)$$

This implies that $\phi(t - \tau) = \phi(-\tau)$ for all $t \in [0, \tau]$, which means ϕ should be a constant function. Thus the set of equilibria is given by

$$E = \{\hat{p} \in \Omega : p \in \mathbb{R}^n \text{ and } h(\hat{p}) = 0\} \tag{4.5}$$

If $\hat{p} \in E$ with $p = 0$, then we say the system (4.1) has an equilibrium at the origin.

The following lemma states that there is a one-to-one correspondence between the equilibria of the system (4.1) and the equilibria of the following system

$$\dot{y}(t) = H(y(t)) \tag{4.6}$$

where $H : \mathbb{R}^n \mapsto \mathbb{R}^n$ and

$$H(y) = h(\hat{y}) \tag{4.7}$$

with $y \in \mathbb{R}^n$ and $\hat{y}$ as defined in (4.3).

Lemma 4.2.4. *Let $h : \Omega \mapsto \mathbb{R}^n$, where Ω is an open subset of $C([-\tau, 0], \mathbb{R}^n)$. Let h satisfy quasimonotone condition and let p be an equilibrium of the system* (4.6)*. Then $\hat{p}$, as defined in* (4.3)*, is an equilibrium of the system* (4.1)*.*

Proof: The proof is based on discussions in Section 5.1 (pp. 77-78) of [Smi95]. Let Ω be an open subset of $C([-\tau, 0], \mathbb{R}^n)$. As already discussed, the set of equilibria of the system (4.1) in Ω is:

$$E = \{\hat{p} \in \Omega : p \in \mathbb{R}^n \text{ and } h(\hat{p}) = 0\}$$

Since $H(p) = h(\hat{p})$, $H(p) = 0$ if $h(\hat{p}) = 0$. Therefore, we can conclude that the equilibria of (4.1) consists of those $\hat{p}$ for which p is an equilibrium of (4.6). This concludes the proof. $\square$

The following result is the analogue for delayed systems of Lemma 2.4.4 and follows immediately from Corollary 5.2.2 of [Smi95].

Lemma 4.2.5. *Let $h : \Omega \mapsto \mathbb{R}^n$ satisfy the quasimonotone condition in Ω, where Ω is an open subset of $C([-\tau, 0], \mathbb{R}^n)$. If $v \in \mathbb{R}^n$ is such that $\hat{v} \in \Omega$ and $h(\hat{v}) \leq 0$, then $x(t, \hat{v})$, solution of* (4.1) *through $\hat{v}$, is non-increasing for all $t \geq 0$.*

Note that since $H(v) = h(\hat{v})$, as defined in (4.7), then $H(v) \leq 0$ implies $h(\hat{v}) \leq 0$ and vice versa. Therefore, Lemma 4.2.5 can be also stated as follows.

Lemma 4.2.6. *Let $h : \Omega \mapsto \mathbb{R}^n$ satisfy the quasimonotone condition in Ω, where Ω is an open subset of $C([-\tau, 0], \mathbb{R}^n)$ and let H be defined as* (4.7)*. Assume that there exists a vector $v \geq 0$ with $H(v) \leq 0$. Then the trajectory $x(t, \hat{v})$ of the system* (4.1) *is non-increasing for all $t \geq 0$.*

We use Lemma 4.2.6 instead of Lemma 4.2.5 in the following sections, because the statement of Lemma 4.2.6 is more suitable for the framework we have chosen for our stability conditions.

We also use a variation of the Convergence Criterion (Theorem 3.4.3) for delayed systems. The following theorem, is a simple adaptation of the Theorem 1.2.1 in [Smi95].

Theorem 4.2.7 (Convergence Criterion)**.** *Let $h : \Omega \mapsto \mathbb{R}^n$ satisfy the quasimonotone condition in an open subset Ω of $C([-\tau, 0], \mathbb{R}^n)$. Let $x(t, \hat{x}_0)$, the trajectory of the system* (4.1) *be bounded for all $t \geq 0$ and for all $\hat{x}_0 \in \Omega$. If $x(t, \hat{x}_0) \leq \hat{x}_0$ for t belonging to some non-empty subinterval of $(0, \infty)$, then $x(t, \hat{x}_0) \to \hat{p} \in E$ as $t \to \infty$, where E is the set of all equilibria of the system* (4.1)*.*

Note that the statement of the Convergence Criterion as stated in Theorem 4.2.7 is very similar to Theorem 3.4.3. In fact, Theorem 1.2.1 of [Smi95] is stated in a general framework and can be applied to both systems (3.1) and (4.1).

To close this section, we state a condition for positivity of monotone time-delay systems. The following result, is the equivalent of the Lemma 2.6.2 for time-delay systems.

Lemma 4.2.8. *Let Ω be an open subset of $C([-\tau,0],\mathbb{R}^n)$, and let $h:\Omega\mapsto\mathbb{R}^n$. If the system* (4.1) *is monotone and has an equilibrium at the origin, then it is positive.*

Proof: The proof is in principle, similar to the proof of Lemma 2.6.2. Since h is monotone, then based on the definition, we know that for two initial conditions $\phi,\psi\in\Omega$, if $\phi(\theta)\le\psi(\theta)$ for all $\theta\in[-\tau,0]$, then we have

$$x(t,\phi)\le x(t,\psi) \text{ for all } t\ge 0 \tag{4.8}$$

The origin is an equilibrium of the system (4.1), which means if $\psi(\theta)=0$ for all $\theta\in[-\tau,0]$, then $x(t,\psi)=0$ for all $t\ge 0$. Therefore, based on (4.8), we can conclude that for all $\phi\in\Omega_+$, with Ω_+ as defined in (4.2), we have $x(t,\phi)\ge 0$. This concludes the proof. □

In the remainder of this chapter, we deal with monotone time-delay systems that have an equilibrium at the origin. Therefore, based on Lemma 4.2.8, they are all positive systems, although we do not explicitly mention this fact in each of the following sections of this chapter.

4.3 Literature Review

Time delay systems have been studied since at least the 1920s. The great number of monographs written on the subject, particularly in recent years, is evidence for the continuing interest of mathematicians and engineers in delayed systems. For example, look at survey papers [Kha99, KNG99, Ric03], special issues [LR97, RK998, DDN01, NR002, FS003], monographs [Kua93, CL07, Zho06, Ern09, XFS09, LMNS09, LS10] and references therein. We note the following points concerning time-delay systems:

(i) With increasing expectations of dynamic performances, engineers need to use more and more realistic models. Many processes and systems include time-delay in their inner dynamics. Monographs [KM92, Nic01, CL07, Ern09] provide numerous examples of time-delay systems in biology, chemistry, economics, mechanics, physics, physiology, population dynamics, communication networks, queueing theory and other fields of science and engineering.

(ii) Time-delay systems are 'resistant' to many classical controllers [Ric03]. One apparent solution for this problem is to approximate the delay systems

with a finite-dimensional system, most commonly using Padé approximation. Although this method sometimes works, it can lead to complex controllers and in worst cases, can have disastrous results in terms of stability, as discussed below:

(a) Linear systems with a single constant delay remain the favourite domain of application for these methods. But, even in this case, designing controllers that stabilize Padé approximations may lead to unstable behaviours of the original system [SDB01].

(b) When there is uncertainty in the value of delay, the stability analysis results developed for certain constant delay may allow stability analysis for uncertain time-delay systems but cannot be applied straightforwardly to define a control law. And even if we are only interested in stability analysis, there exist other competitive methods, such as the pseudo-delay technique, which has proven to be preferable to any approximation method [HJZ84], [WM87], [MCH89, Chapter 7].

(c) Rational approximations are not an appropriate choice for time-varying delays [Ric03]. For instance, it is shown that a simple, first-order system with a variable delay can be unstable while each of the values taken by the delay (when constant) provides a stable model. For example, consider the following delay system in which the delay $s(t)$ corresponds to a sampling device with a unit period:

$$\dot{x}(t) = ax(t) + bx(t - s(t))$$

$$s(t) = t - k, \text{ for all } t \in (k, k+1],\ k \in \mathbb{N}$$

It is easy to see that $0 < s(t) \leq 1$. In [HS80] it is shown that the system has an unstable equilibrium for $a = -3.5$ and $b = -4$. It is also shown that for $a = -1$, $b = 1.5$, the equilibrium is asymptotically stable, whereas the system with $s(t) \equiv 0$ has an unstable equilibrium in this case. Thus, choosing a mean value of the delay and applying finite-dimensional rational approximations cannot be justified for the stability analysis of the time-delay systems with time-varying delays.

(d) Finite-dimensional rational approximations do not extend readily to the analysis of nonlinear or time-varying functional differential equations [Ric03].

(e) When dealing with positive systems, we are only interested in those approximations that preserve positivity of the system. Recently, it is been shown in [ZCKS12] that the Padé approximation does not preserve positivity of the system. Therefore, apart from the above mentioned disadvantages, applying Padé methods is a particularly poor technique for positive delayed systems, because we may lose the positivity of the system in the process.

(iii) Delay affects stability properties of the systems, sometimes drastically. That is one of the most important reasons that has led to the development of the theory of delayed systems. The effects of delay on stability can be considered from two different angles.

(a) Delay usually has a 'bad reputation' when it comes to its effects on stability properties of a system. Sometimes even small values of delay that seem harmless to ignore in modelling a system, can lead to instability. For example, consider the following LTI system, which has a GAS equilibrium at the origin:

$$\dot{x}(t) + 2\dot{x}(t) = -x(t)$$

On the other hand, the trivial solution of the following neutral functional differential equation

$$\dot{x}(t) + 2\dot{x}(t-\tau) = -x(t)$$

is unstable for any $\tau > 0$ (for a proof look at [Hal77, p. 28]).

(b) Delay can also have surprising effects on improving the stability of the systems. Several studies have shown that voluntary introduction of delays can also benefit the control. For example, in damping and stabilisation of ordinary differential equations [ADBRB93] [RGBTD98, Chapter 11], delayed resonators [JO98] and nonlinear limit cycle control [ARS00].

(iv) Although time-delay systems are infinite dimensional, the class of functional differential equations are relatively simple to analyse in the very complex area of partial differential equations (PDEs). That is why sometime time-delay systems are used to approximate PDEs. For instance, hyperbolic PDEs can

be locally understood as neutral delay systems [HVL93, KN86]. [Ric03, Table 1] provides a list of such approximations and their domain of application.

(v) Different approaches for solving functional differential equations have been suggested as early as the 1950s [Él'55]. Although, due to their more complex nature compared to ordinary differential equations, simulation of time-delay systems in many cases has not been possible until recently. Rapid advances in computational power have revived interest in time-delay systems in the past couple of decades. Apart from that, new areas of research have appeared that demand the use of theory of delayed systems. These two factors have led to numerous applications of time-delay systems in different areas. For example, time-delay has been considered in the analysis of congestion control in TCP-based communication networks [LAQ+08, EGOP10], in life science [Smi10], in consensus problems in networks [OSM04], in the analysis of traffic scheduling algorithms [SV98], in modelling blood production in patients with leukaemia [MG77, GM88], in modelling business cycles [SKT01, SK05], in modelling predator-prey systems [GE00], in studying different neural assemblies in body [SC00], in chemostat models [RW96, WX97, WXR97, WW06], in modelling HIV/AIDS [MGC07], to name a few. More examples of applications of time-delay systems in different areas of science and engineering can be found in [Ern09, CL07, BZ03] and references therein.

The above mentioned factors, have motivated the study of time-delay systems since its first appearance in mathematical literature, which is probably [Vol28]. In his work on predator-prey models and viscoelasticity [Vol28, VB30], the Italian mathematician Vito Volterra (1860-1940), formulated some relatively general differential equations incorporating past states of the system. Unfortunately, this aspect of his work was almost completely ignored and didn't have much immediate impact. In the early forties, Minorsky in his study of ship stabilization and automatic steering [Min42], showed the importance of considering delay in the feedback mechanisms. In the late forties and fifties, more attempts were made to develop the theory of systems with time-delay. Most notable among them is the work of Krasovskiĭ [Kra59, Kra63], which extended the Lyapunov methods to delayed systems. Approaches based on this line of work are commonly known today as the *Lyapunov-Krasovskiĭ functional method* [DV98, GKC03]. In the past four decades, an extensive number of textbooks and papers devoted to the theoretical properties of time-delay systems have appeared. The standard reference on properties of functional differential equations is [Hal77], from which most of the basic definitions and results stated in the previous section are adopted.

Time-delay in the context of monotone systems has been studied since the late 1970s and early 1980s. The first papers that introduced the quasimonotone condition and the basic properties of monotone functional differential equations are [KS79], [Mar81] and [Oht81]. Stability properties of monotone time-delay systems were studied by different researchers in the 1980s. Two of the most important of these manuscripts are [KN84] and [Smi87]. In [KN84] some results on stability for abstract linear delay differential equations are presented which are based on Perron-Frobenius theory. [Smi87], presented some fundamental results on stability of monotone functional differential equations. In [Smi87] it is proved that the stability of the equilibrium of a cooperative and irreducible time-delay system is determined by a real characteristic root and that this stability is the same as for an associated systems of cooperative ordinary differential equations. A good reference, that summarizes the fundamental properties of monotone time-delay systems is [Smi95, Chapter 5].

In the past two decades, different aspects of monotone and positive time-delay systems have been investigated and developed in different directions. In [HC04], a condition for delay-independent stability of linear positive time-delay systems is presented. This result is the basis of the results presented in this chapter. We discuss it in the next section. Similar results are also presented for neutral linear functional differential equations in [NNS07], for positive linear Volterra equations in [NNSM08] and for Linear Volterra-Stieltjes Differential Systems in [NMN^{+}09] and most recently, for a class of positive linear integro-differential equations with infinite delay in [Ngo11]. Also, systems with delay in inputs are considered in [DlS07]. The problem of designing a positive observer for positive time-delay systems is studied in [ARHT07, Hua08, LLS09] and excitability of linear positive time-delay system is addressed in [DlS09].

4.4 Time-delay Homogeneous Cooperative Systems

The main theorem of this section, can be considered as an extension of the main result of [HC04] to homogeneous cooperative delayed systems. In [HC04], a condition for asymptotic stability of the equilibrium of a class of linear functional differential equations is presented. That result can be stated as follows.

Theorem 4.4.1. *Consider a linear time-delay system of the form*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau), \;\; t \geq 0 \\ x(\theta) &= \phi(\theta), \;\; -\tau \leq \theta \leq 0 \end{aligned} \tag{4.9}$$

where $A \in \mathbb{R}^{n\times n}$ *is Metzler,* $A_d \geq 0 \in \mathbb{R}^{n\times n}$, $\tau \geq 0$ *and* $\phi(\cdot)$ *is the initial state. The origin is an asymptotically stable equilibrium for the system* (4.9) *for all* $\tau \in [0,\infty)$ *if and only if there exists vectors* $q \gg 0$ *and* $r \gg 0$ *such that*

$$(A + A_d)^T q + r = 0$$

In other words, Theorem 4.4.1 states that the linear time-delay system (4.9) is asymptotically stable for every $\tau > 0$ if and only if there exists a vector $v \gg 0$ such that $(A+A_d)^T v \ll 0$. Note that if $(A+A_d)^T v \ll 0$ for some $v \gg 0$, then based on Theorem 2.5.10, the undelayed system (system (4.9) with $\tau = 0$) has an asymptotically stable equilibrium at the origin. The similarity between this condition and the stability conditions stated in Chapter 3, inspired us to extend Theorem 4.4.1 to positive nonlinear time-delay systems.

The nonlinear time-delay system we consider in this chapter, is as follows:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g^{(i)}(x(t-\tau_i)) \qquad \tau_i \geq 0, \quad \forall i = 1,\cdots,m \tag{4.10}$$

where $f : \mathcal{W} \to \mathbb{R}^n$ and $g^{(i)} : \mathcal{W} \to \mathbb{R}^n$ are vector fields on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$ and $\tau_i > 0$ are delays for $i = 1,\cdots,m$. We define:

$$\tau := \max_{1\leq i\leq m} \tau_i$$

We also define $h : C([-\tau,0],\mathbb{R}^n) \mapsto \mathbb{R}^n$ to be

$$h(x^{(t)}) = f(x(t)) + \sum_{i=1}^{m} g^{(i)}(x(t-\tau_i)) \tag{4.11}$$

In the remainder of this chapter, f and $g^{(i)}$, for $i = 1, \cdots, m$, satisfy the following assumption.

Assumption 4.4.1. For $i = 1, \cdots, m$, we have:

- f and $g^{(i)}$ are C^1 on $\mathcal{W}$;
- f is cooperative in $\mathbb{R}^n_+$ and $g^{(i)}$ is non-decreasing in $\mathbb{R}^n_+$.

Since f and $g^{(i)}$, for $i = 1, \cdots, m$, are C^1, it can be easily seen that the conditions of the Theorems 4.2.1 and 4.2.2 are satisfied. Therefore, the first part of assumption 4.4.1, guarantees existence and uniqueness of the solutions of the system (4.10). The second part of the assumption 4.4.1, guarantees monotonicity of the system (4.10), as proved in the following lemma.

Lemma 4.4.2. *Let f and $g^{(i)}$, for $i = 1, \cdots, m$, satisfy Assumption 4.4.1. Then the system* (4.10) *satisfies the quasimonotone condition in $\mathbb{R}^n_+$.*

Proof: The proof follows directly from Theorem 4.2.3 as discussed in [Smi95, p. 79] for a system with single delay. In the interest of completeness, we repeat that argument in here. We define:

$$z(x, y^{(1)}, \cdots, y^{(m)}) = f(x(t)) + \sum_{i=1}^{m} g^{(i)}(y^{(i)}(t))$$

where $x, y^{(i)} \in \mathbb{R}^n$, for $i = 1, \cdots, m$. This means we can rewrite (4.10) as:

$$\dot{x}(t) = z(x(t), x(t-\tau_1), \cdots, x(t-\tau_m)) = f(x(t)) + \sum_{i=1}^{m} g^{(i)}(x(t-\tau_i)) \quad (4.12)$$

Now if $z(x, y^{(1)}, \cdots, y^{(m)})$ satisfies

$$\frac{\partial z_i}{\partial x_j} \geq 0, \quad \forall i \neq j \quad (4.13)$$

and

$$\frac{\partial z_i}{\partial y_l^{(k)}} \geq 0, \quad \forall i, l \quad (4.14)$$

then the system (4.12) satisfies the quasimonotone condition. To this end, note that for all $a \in \mathbb{R}^n_+$, we have:

$$\frac{\partial z}{\partial q}(a) = \frac{\partial f}{\partial q}(a) + \sum_{i=1}^{m} \frac{\partial g^{(i)}}{\partial q}(a), \quad \forall a \in \mathbb{R}^n_+$$

where q represents x or $y^{(k)}$ for some $k = 1, \cdots, m$. Since f is cooperative and $g^{(i)}$ is non-decreasing in $\mathbb{R}^n_+$, it can be easily checked that (4.13) and (4.14) hold. This concludes the proof. □

The aim of the following result, which is the main theorem of this section, is to relate the stability properties of the equilibrium of the system (4.10), to the following system:

$$\dot{x}(t) = H(x) = f(x(t)) + \sum_{i=1}^{m} g^{(i)}(x(t)) \quad i = 1, \cdots, m \tag{4.15}$$

It is easy to see that system (4.15) is generated from (4.10), with $\tau_i = 0$, for $i = 1, \cdots, m$. Note that based on Lemma 4.2.4, there is a one-to-one correspondence between the equilibria of (4.10) and (4.15).

Theorem 4.4.3. *Consider the system* (4.10) *where vector fields f and $g^{(i)}$, for $i = 1, \cdots, m$, are homogeneous of degree α with respect to the dilation map δ^r_λ and satisfy assumption 4.4.1. If the origin is a GAS equilibrium of the system* (4.15) *then the system* (4.10) *has a GAS equilibrium at the origin for all $\tau_i \geq 0$ with $i \in \{1, \cdots, m\}$.*

Proof: Note that if p is an equilibrium of (4.15), then $\hat{p}$, as defined in (4.3), is an equilibrium of (4.10) and vice versa. Since (4.15) has a unique equilibrium at the origin by assumption, therefore, the origin is the unique equilibrium of (4.10).

Let $h(\cdot)$ be defined as in (4.11). Based on Proposition 3.5.1, we know that there exists some $v \gg 0$ such that $H(v) \ll 0$. Based on homogeneity of $H(\cdot)$, we can conclude that $H(\delta^\lambda_r(v)) \ll 0$ which based on (4.7) implies $h(\delta^\lambda_r(\hat{v})) < 0$, where $\delta^\lambda_r(\hat{v}) \in C([-\tau, 0], \mathbb{R}^n_+)$ is defined as follows:

$$\delta^\lambda_r(\hat{v})(\theta) := \delta^\lambda_r(v) \quad \forall\theta \in [-\tau, 0]$$

It now follows from Lemma 4.2.6 that the solution $x(t, \delta^\lambda_r(\hat{v}))$ of (4.10) is non-increasing and bounded. Hence, the convergence criterion (Theorem 4.2.7) implies that it converges to an equilibrium of the system (4.10). Since the origin is the only equilibrium of this system, therefore, we can conclude $x(t, \delta^\lambda_r(\hat{v})) \to 0$ as $t \to \infty$.

Since $v \gg 0$, for any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n_+)$ there exists some $\lambda > 0$ such that

$$\phi \ll \delta_r^\lambda(\hat{v}) \quad \forall \theta \in [-\tau, 0]$$

Hence, based on positivity and monotonicity of the system (4.10), we can conclude that for all $\phi \in C([-\tau, 0], \mathbb{R}^n_+)$, $x(t, \phi) \to 0$ as $t \to \infty$. □

The following example, illustrates the result.

Example 4.4.1. Consider the system (4.10), with $m = 1$, where

$$f(x_1, x_2, x_3) = \begin{pmatrix} -5x_1^3 + x_1x_2 \\ x_1x_3 - 7x_2^2 \\ x_1^3x_2 + 0.5x_1^2x_3 - 10x_3^{5/3} \end{pmatrix}$$

$$g^{(1)}(x_1, x_2, x_3) = \begin{pmatrix} x_1^3 + 2x_3 \\ 0.5x_1^2x_2 + x_1x_3 + 1.5x_2^2 + 2x_3^{4/3} \\ x_1x_2^2 + x_3x_2 + 2x_3^{5/3} + x_1^5 \end{pmatrix}$$

It can be easily checked that both f and $g^{(1)}$ are homogeneous of degree 2 with respect to the dilation map δ_λ^r with $r = (1, 2, 3)$. Moreover, f is cooperative and $g^{(1)}$ is non-decreasing for $x \in \mathbb{R}^n_+$. Note that $(f + g^{(1)})(1, 1, 1) \ll 0$.

It can also be easily checked that the origin is the only equilibrium of the system (4.15) with these choices of f and $g^{(1)}$. Therefore, we can conclude that the origin is the unique equilibrium of the system (4.10) and it is globally asymptotically stable for every non-negative and fixed value of delay.

Figure 4.1 shows the evolution of the states for the above mentioned system, for $\tau = 2$ and initial condition ϕ which is defined as $\phi(\theta) = (3, 1, 5)^T$ for all $\theta \in [-2, 0]$.

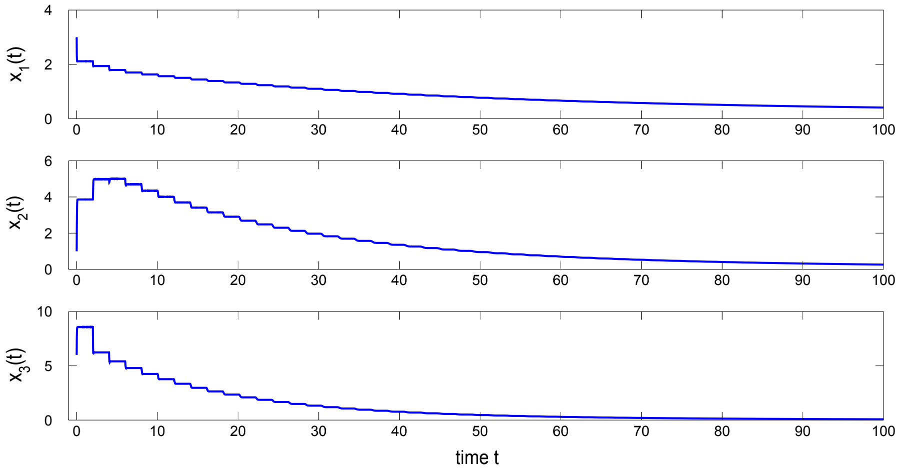

Figure 4.1: Evolution of states for the system (4.10) corresponding to Example 4.4.1.

4.5 Time-delay Subhomogeneous Cooperative Systems

In this section, we extend the result stated in the previous section to subhomogeneous cooperative systems. To prove the main result, we need the following proposition which can be considered as the equivalent of the Proposition 3.5.1 for subhomogeneous systems.

Proposition 4.5.1. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^n_+$ and let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree α and cooperative. If* (3.1) *has a GAS equilibrium at the origin then for every $x_0 \in \mathbb{R}^n_+$ there exists $v \gg x_0$ such that $f(v) \ll 0$.*

Proof: Since f is cooperative, then based on Proposition 3.4.2, we know that there exists a $w \gg 0$ such that $f(w) \ll 0$. Therefore, for every $x_0 \in \mathbb{R}^n_+$ there exists a $\beta > 1$ such that $v = \beta w \gg x_0$. On the other hand, since f is subhomogeneous of degree α, we know that $f(v) = f(\beta w) < \beta^\alpha f(w) \ll 0$ and this concludes the proof. $\square$

Now we are ready to prove the main result of this section.

Theorem 4.5.2. *Consider the system* (4.10) *where vector fields f and $g^{(i)}$, for $i = 1, \cdots, m$, are subhomogeneous of degree α and satisfy assumption 4.4.1. If the system* (4.15) *has a GAS equilibrium at the origin then the system* (4.10) *has a GAS equilibrium at the origin for all $\tau_i \geq 0$, for $i = 1, \cdots, m$.*

Proof: The proof is similar to the proof of Theorem 4.4.3. Since the origin is the only equilibrium of system (4.15), then system (4.10) has also a unique equilibrium at the origin.

Based on Proposition 4.5.1 and with the similar argument as presented in the proof of Theorem 4.4.3, we can conclude that for any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n_+)$, we can find a $v \gg 0$ such that $\phi \ll \hat{v}$ and $h(\hat{v}) \ll 0$ where $h(\cdot)$ is defined as in (4.11).

It now follows from Lemma 4.2.6 that the solution $x(t, \hat{v})$ of (4.10) is non-increasing and bounded. Hence, Convergence Criterion (Theorem 4.2.7) implies that it converges to an equilibrium which is the origin. Finally, the monotonicity and positivity of (4.10) imply that the solution $x(t, \phi)$ of (4.10) also converges to the origin as $t \to \infty$. $\square$

Remark 4.5.1. Note that in both Theorems 4.4.3 and 4.5.2, we used homogeneity and subhomogeneity only to show that for all $x_0 \in \mathbb{R}^n_+$, there exists a $v \gg x_0$ such that $H(v) \ll 0$. The rest of the proofs, follows from monotonicity of the systems. Therefore, Theorems 4.4.3 and 4.5.2 can be extended to any class of monotone systems for which a vector v with $H(v) \ll 0$ can be found for any initial condition $x_0 \in \mathbb{R}^n_+$.

4.6 Time-delay General Cooperative Systems

In this section, we generalise the results stated in the previous two sections to general cooperative delayed systems. Similar to the result stated in Section 3.4, the delay-independent stability result for general cooperative systems holds only locally.

In the proof of the following theorem, which is the main result of this section, we utilise the KKM lemma (Theorem 2.3.1).

Theorem 4.6.1. *Consider the system* (4.10) *where vector fields f and $g^{(i)}$, for $i = 1, \cdots, m$, satisfy assumption 4.4.1. If the system* (4.15) *has an asymptotically stable equilibrium at the origin then the system* (4.10) *has an asymptotically stable equilibrium at the origin for all $\tau_i \geq 0$, for $i = 1, \cdots, m$.*

Proof: Note that since the system (4.10) is monotone and (4.15) has a unique equilibrium at the origin, then system (4.10) also has a unique equilibrium at the origin. Let $h(\cdot)$ be defined as in (4.11) and $H(\cdot)$ be defined as in (4.15). Based on the Proposition 3.4.2, there exists a vector v with $v \gg 0$, such that $H(v) \ll 0$. Since $H(v) = h(\hat{v})$, we can conclude that $h(\hat{v}) \ll 0$ with $\hat{v}$ defined as in (4.3).

It now follows from monotonicity of the system (4.10) and Lemma 4.2.6 that the solution $x(t, \hat{v})$ of (4.10) is non-increasing and bounded. Hence, Convergence Criterion (Theorem 4.2.7) implies that it converges to an equilibrium which is the origin. Finally, the monotonicity of (4.10) implies that the solution $x(t, \phi)$ of (4.10) also converges to the origin as $t \to \infty$ for every $\phi \in C([-\tau, 0]$ that satisfies $\phi \leq \hat{v}$. □

Note that Theorem 4.6.1, unlike Theorems 4.4.3 and 4.5.2, states only a local condition for delay-independent stability of the system (4.10).

4.7 Time-delay Planar Cooperative Systems

In this section, we try to provide a more general delay-independent stability result for cooperative systems compared to the result presented in the previous section while removing the homogeneity and subhomogeneity assumptions. The main result of this section, which appeared in [BMV11], is based on the results presented in the Section 3.7 for planar monotone nonlinear systems.

Throughout this section, $f : \mathcal{W} \to \mathbb{R}^2$ is assumed to satisfy Assumption 3.7.1 and f and $g^{(i)} : \mathcal{W} \to \mathbb{R}^2$, for $i = 1, \cdots, m$, satisfy Assumption 4.4.1 on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^2_+$.

In the proof of the following theorem, which is the main and only result of this section, we use Proposition 3.7.1 and Lemma 4.2.6.

Theorem 4.7.1. *Let $\mathcal{W}$ be a neighbourhood of $\mathbb{R}^2_+$ and let $f : \mathcal{W} \mapsto \mathbb{R}^n$ and $g^{(i)} : \mathcal{W} \to \mathbb{R}^2$, for $i = 1, \cdots, m$, satisfy Assumption 4.4.1 and let $f + g^{(1)} + \cdots + g^{(m)}(\cdot)$ satisfies assumption 3.7.1. Then the origin is a GAS equilibrium of the system* (4.10)*, if system* (4.15) *has a GAS equilibrium at the origin.*

Proof: We know that (4.10) has the same equilibria as (4.15). Proposition 3.7.1 implies that for any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^2_+)$, there exists some $v \in \mathbb{R}^2_+$ with $\phi(s) \ll v$ for all $s \in [-\tau, 0]$ and $(f + \sum_{i=1}^{m} g^{(i)})(v) \ll 0$. Further as the equilibria of (4.15) coincide with those of (4.10), it follows that (4.10) has a unique equilibrium at the origin. These facts combined with Lemma 4.2.6 imply that $x(t, \hat{v}) \to 0$ as $t \to \infty$. Based on monotonicity, we can conclude $x(t, \phi) \to 0$ as $t \to \infty$ and this concludes the proof. □

4.8 Concluding Remarks

In this chapter, we extended some delay-independent stability results on linear time-delay systems to different classes of positive nonlinear time-delay systems. The basic properties of time-delay systems were introduced with emphasis on monotone systems. We first stated the condition for stability of homogeneous cooperative time-delay systems and then extended that result to subhomogeneous cooperative time-delay systems. Similar to Chapter 3, we also stated a local stability result for general cooperative systems without imposing homogeneity or subhomogeneity assumptions. We also presented a global stability result for planar cooperative systems while removing homogeneity and subhomogeneity assumptions.

The results presented in this chapter can be extended in a number of ways. One possible direction is to try to find a more general assumption to add to Theorem 4.6.1 to obtain global results. As stated in Remark 4.5.1, every assumption on the vector fields of the system (4.10) that guarantees for all $x_0 \in \mathbb{R}^n_+$, there exists a $v \gg x_0$ such that $H(v) \ll 0$ can be used to obtain global stability results similar to Theorems 4.4.3 and 4.5.2. But it may be possible to find alternative assumptions that may lead to such global results.

Another possible direction for extending the presented results is to consider delayed systems with time-varying delays. In this chapter, we only dealt with systems with constant delays, although the value of delay was considered to be uncertain. It should be noted that stability analysis of systems with time-varying delays requires the theory of non-autonomous systems which is generally more complicated than the theory of autonomous systems.

CHAPTER 5

D-stability in Positive Switched Systems

In this chapter, the concept of D-stability for switched systems is defined and conditions for D-stability of different classes of positive linear and nonlinear switched systems are presented.

5.1 Introduction

Recently, switched systems have attracted a lot of attention in the literature (look at [SWM+07, Lib03, SG11] and references therein). This has been primarily motivated by the fact that many man-made systems and some physical systems can be modelled within this framework. While major advances have been made in this area, many important questions that relate to the behaviour of switched systems still remain unanswered, even for linear switched systems. One of the most important of these relate to the stability of such systems.

In this chapter, we are concerned with the stability properties of positive switched systems. We introduce the concept of D-stability of switched systems which is a stability condition for switched systems subject to a particular form of uncertainty. This concept, is inspired by the notion of D-stability in positive linear time-invariant systems although different from the concept of D-stability for nonlinear systems presented in Chapter 3. In this chapter, we present conditions for D-stability for different classes of linear and nonlinear positive switched systems. Looking at the different resources in the literature, it can be seen that most of the research in stability analysis of positive switched systems has focused on positive linear switched systems. Therefore, there is a clear need to extend these results to positive nonlinear switched systems. In this chapter, we first state conditions for D-stability for different classes of positive linear switched systems and then extend those results to classes of positive nonlinear switched systems. It should be noted that the methods we use in this chapter in proving the results differ from most of the previous works in this area, in the sense that we do not use Lyapunov-based methods in the proofs. Apart from the difference in the proofs, the results themselves are also novel stability conditions that can be used as alternative conditions for the known results on positive linear switched systems.

The structure of this chapter is as follows. In the next section, we present the basic definitions concerning switched systems and introduce the concept of D-stability. In Section 5.3, we review the relevant results in the literature, with more emphasis on stability properties of positive switched systems. In Section 5.4, we present D-stability conditions for linear positive switched systems and show that by adding irreducibility assumptions, we can obtain a necessary and sufficient condition for D-stability of such systems. In Section 5.5 we are concerned with D-stability in nonlinear positive switched systems. We study three different classes of nonlinear positive switched systems and extend the D-stability conditions to these systems. Finally, in Section 5.6, we present the conclusions and outline some possible extensions of the results presented in this chapter.

5.2 Background

A switched system is a system whose behaviour is governed by two sets of states. In one level, a set of continuous or discrete-time states, which are described by differential or difference equations and in another level, a logic-based switching. In this chapter, we only deal with the case where switching happens between a set of continuous-time systems. Each of these continuous-time systems is called a *subsystem* or *constituent system* or *constituent subsystem* of the switched system. In the models we deal with in this chapter and the next, the logic based switching is represented by a piecewise continuous function $\sigma : \mathbb{R}_+ \mapsto \{1, \cdots, m\}$ called the *switching signal.* m is the number of constituent subsystems of the switched system. The points of discontinuity of $\sigma(\cdot)$ are called *switching instances* or *switching times* and are represented by $t_0 = 0, t_1, t_2, \cdots$. To avoid the complications caused by infinitely fast switching, hereafter, we assume there exists a *dwell time* $\tau > 0$. Formally, this means:

$$\exists \tau > 0 \text{ such that } t_i - t_{i-1} > \tau \text{ for all } i = 1, 2, 3, \cdots$$

Assuming the switched system has a dwell time is a reasonable assumption in most applications, specifically in epidemiology, which is the subject of the next chapter.

The most general switched model we consider in this chapter, is as follows:

$$\dot{x}(t) = f^{(\sigma(t))}(x(t)); \quad x(0) = x_0 \in \mathcal{D} \tag{5.1}$$

where $f^{(i)} : \mathcal{D} \mapsto \mathbb{R}^n$ for $i = 1, \cdots, m$ is a nonlinear continuous-time vector field in an open subset $\mathcal{D}$ of $\mathbb{R}^n$. Since we only deal with positive switched systems in this chapter, we always assume $f^{(i)}$, for $i = 1, \cdots, m$, is defined over a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$, unless explicitly stated otherwise. As already mentioned, each of the systems $\dot{x}(t) = f^{(i)}(x(t))$, for $i = 1, \cdots, m$ represents a constituent subsystem of the switched system.

We always assume $\sigma \in \mathcal{S}$, where $\mathcal{S}$ is a subset of the set of all the admissible switching signals for the switched system (5.1). The set of admissible switching signals is usually defined based on the context. For example, if the only limitation imposed on the switching signal is the existence of a dwell time, then every possible switching signal with a dwell time is an admissible switching signal. If there is no constraint on the admissible switching signals and if $\mathcal{S}$ includes *all* the admissible switching signals, then the switched system (5.1) is being considered under *arbitrary switching.* Note that in this chapter, in a slight abuse of notation, we use the term arbitrary switching to refer to the case where the switched signal has a dwell time. The reason for that is discussed in the next section.

A switching signal $\sigma(\cdot)$ can be a function of time, states and initial conditions. If $\sigma(\cdot)$ is only a function of time, then we have *time-dependent switching.* When $\sigma(\cdot)$ is a function of the states of the system, we have *state-dependent switching.*

The *(forward) solution* or *trajectory* of the switched system with respect to initial condition x_0 and switched signal σ is represented by $x(t, x_0, \sigma)$ and is defined for $t \in [0, +\infty)$. Also, $x^{(i)}(t, x_0)$ represents the trajectory of the ith subsystem.

Switched system (5.1) is said to be *well-defined* if for all $\sigma \in \mathcal{S}$ and for all $x_0 \in \mathbb{R}^n_+$, there exists a unique absolutely continuous solution of system (5.1) over $[0, +\infty)$. The switched system (5.1) is well-defined if each subsystem satisfies an appropriate Lipschitz condition [SG11]. Hereafter, we assume that the Lipschitz condition holds for all the subsystems, therefore, the switched system (5.1) is well-defined.

Remark 5.2.1. Note that in this manuscript, we do not deal with *impulse effects*, which are the instantaneous jumps of the trajectory of the switched system at switching instances. In other words, we assume that the trajectory of the switched system is continuous everywhere, including the switching instances, although in general it loses differentiability at the switching instances.

In this chapter, we exclusively deal with positive switched systems. A positive switched system can be defined as follows.

Definition 5.2.1 (Positive Switched System)**.** Switched system (5.1) is *positive*, if

$$x(t, x_0, \sigma) \in \mathbb{R}^n_+ \text{ for all } x_0 \in \mathbb{R}^n_+,\ t > 0,\ \sigma \in \mathcal{S}$$

The following lemma, provides a necessary and sufficient condition for positivity of switched systems.

Lemma 5.2.1. *The switched system* (5.1) *is positive, if all of its constituent subsystems are positive.*

Proof: The proof is almost trivial. Let $x_0 \geq 0$ be the initial condition for the switched system and without loss of generality, let $f^{(i)}$ be the vector field representing the system for $t \in [0, t_1]$ where $i \in \{1, \cdots, m\}$. Since system $\dot{x} = f^{(i)}(x)$ is positive, then $x^{(i)}(t_1, x_0) \geq 0$. Let $f^{(k)}$ be the vector field representing the switched system in $t \in [t_1, t_2)$ for some $k = \{1, \cdots, m\}$. $x^{(i)}(t_1^-, x_0)$ is the initial condition for the new subsystem, and since $\dot{x} = f^{(k)}(x)$ is positive, we have $x^{(k)}(t, x_0) \geq 0$ for $t \in [t_1, t_2)$. Following this procedure, we can prove that $x(t, x_0, \sigma) \geq 0$ for all $x_0 \geq 0$ and $\sigma \in \mathcal{S}$. This concludes the proof. $\square$

In the following sections of this chapter, we do not explicitly discuss positivity of each class of switched system we deal with. The reason is that the constituent subsystems of all the switched systems discussed in the sequel are in fact different classes of linear and nonlinear positive systems discussed in Chapters 2. Therefore, Lemma 5.2.1 guarantees the positivity of the switched systems discussed in this chapter and the next. Also, we always assume $x_0 \in \mathbb{R}^n_+$.

5.2.1 Stability and D-stability

We next recall various fundamental stability concepts, but first, we should define an equilibrium of a switched system.

Definition 5.2.2 (Equilibrium of a Switched System)**.** The point $\bar{x}$ is called an equilibrium of the switched system (5.1), if

$$f^{(\sigma(t))}(\bar{x}) = 0 \text{ for all } t \geq 0, \sigma \in \mathcal{S}$$

Comment 5.2.2. Note that, based on the Definition 5.2.2, a point $\bar{x}$ is an equilibrium of the switched system (5.1) under arbitrary switching, if and only if, it is an equilibrium point of each subsystem of the switched system (5.1). Also, since we deal with positive switched systems in this chapter, we are only interested in the equilibria of the switched system (5.1) in $\mathbb{R}^n_+$.

Although we defined different stability concepts in Chapter 2 for a single autonomous system, it is necessary for the rest of our discussion to extend those concepts for switched systems.

Definition 5.2.3. Let $\bar{x} \geq 0$ be an equilibrium of the system (5.1). Then, we say the equilibrium point $\bar{x}$ is

- *stable* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that:

$$\|x_0 - \bar{x}\| < \delta \Rightarrow \|x(t, x_0, \sigma) - \bar{x}\| < \epsilon, \quad \forall t > 0, \sigma(\cdot) \in \mathcal{S}$$

- *unstable*, if it is not stable;

- *uniformly asymptotically stable* if it is stable and there exists a neighbourhood N of $\bar{x}$ such that

$$x_0 \in N \Rightarrow \lim_{t\to\infty} x(t, x_0, \sigma) = \bar{x}, \quad \forall\sigma(\cdot) \in \mathcal{S}$$

The set

$$A(\bar{x}) := \{x_0 \in \mathbb{R}^n_+ : x(t, x_0, \sigma) \to \bar{x}, \text{ as } t \to \infty\}$$

is the *domain of attraction* of $\bar{x}$. Since we are exclusively dealing with positive switched systems, if $A(\bar{x}) = \mathbb{R}^n_+$, then we say $\bar{x}$ is globally uniformly asymptotically stable, GUAS for short.

Note that the term 'uniformly' in the definition refers to uniformity with respect to switching signals.

Comment 5.2.3. When an equilibrium of the switched system (5.1) is not uniformly asymptotically stable under arbitrary switching, it may be possible to find a switching signal for which the equilibrium is asymptotically stable. Finding such switching signals is referred to as the *stabilisation* problem [Lib03].

In Chapter 2, we talked about the concept of D-stability for positive LTI systems and in Chapter 3, we saw how can we extend it to classes of positive nonlinear systems. Our main aim in this chapter, is to extend D-stability property of positive LTI systems to positive switched systems.

D-stability of switched systems is defined as follows.

Definition 5.2.4 (D-stability)**.** Let $\bar{x}$ be an equilibrium of the system (5.1). Then $\bar{x}$ is said to be *D-stable*, if it is a uniformly asymptotically stable equilibrium of the corresponding switched system

$$\dot{x}(t) = D_{\sigma(t)} f^{(\sigma(t))}(x(t)); \quad x(0) = x_0 \in \mathbb{R}^n_+ \tag{5.2}$$

for any set $\{D_1, \cdots, D_m\}$ of diagonal matrices with positive diagonal entries and all $\sigma(\cdot) \in \mathcal{S}$. If $\bar{x}$ is a GUAS equilibrium of the system (5.2), then it is said to be a *globally D-stable* equilibrium of the system (5.1).

Comment 5.2.4. Note that in the definition of D-stability, nothing is said about the domain of attraction of the equilibrium. In other words, if $\bar{x}$ is an asymptotically stable equilibrium of both systems (5.1) and (5.2) with different domains of attraction, it is still said to be a D-stable equilibrium of the system (5.1).

As can be seen, the definition of D-stability is different from the definition of D-stability for nonlinear systems presented in Chapter 3 in the sense that the vector fields are pre-multiplied by constant matrices rather than nonlinear vector fields.

5.3 Literature Review

Switched systems belong to the more general class of *hybrid systems*. Loosely speaking, hybrid systems consist of continuous-time and/or discrete-time processes interfaced with some logical or decision-making process. The continuous/discrete time component might consist of differential/difference equations or continuous/discrete time state models [DBPL00]. In the case of continuous-time switched systems that we study in this manuscript, the switching signal has the role of the logical or decision-making process.

One of the earliest models of a hybrid system to appear in the literature is [Wit66], which studies "a class of continuous-time systems with part continuous, part discrete state". Since then, there has been a great interest in the subject and numerous papers and books are published that study hybrid and switched systems from different angles. Several monographs, review papers and special issues are dedicated to the subject. For example, look at books [SS00, MS00, SB99, SE02, SG11, SG05, Lib03], review paper [SWM+07] and special issues [PS995, AN998, MPS99a, ES999, Ben01, HL001]. This is motivated by the wide range of applications of hybrid systems in different areas of science and engineering. You can look at any of the above mentioned monographs for different examples of the applications of switched systems but as a simple example, consider a car with a manual gearbox. The motion of the car can be characterised by two continuous variables, the position of the car and its velocity. These two variables are controlled by the throttle angle and the engaged gear. In each mode (engaged gear), the dynamics of the system evolves in a continuous manner. Transitions between modes are abrupt and depend on the decision of the driver in changing the gear. In other words, we have a set of differential equations and a decision-making process that governs the transition between those differential equations.

It is a well-known fact that even if all the subsystems of a switched system have asymptotically stable equilibria, the equilibria of the switched system may be unstable. You can look at [Lib03, p. 19], [Mas04, Section 1.1] and [Kno11, Section 2.2.1] for some examples. That is one of the main reasons that has motivated the study of the stability properties of switched systems. Different stability conditions have been stated and proved for switched systems. Probably the most well-known of these conditions is the existence of a common Lyapunov function for all the subsystems of a switched system. This condition, can be stated as follows, but before stating the theorem, we need to clarify what we mean by a common Lyapunov function for a switched system.

Definition 5.3.1 (Common Lyapunov Function)**.** Given a positive definite C^1 function $V : \mathbb{R}^n \mapsto \mathbb{R}$, we say that it is a common Lyapunov function for the system (5.1) if there exists a positive definite continuous function $W : \mathbb{R}^n \mapsto \mathbb{R}$ such that we have

$$\frac{\partial V}{\partial x}(a) f^{(i)}(a) \leq -W(a), \quad \forall\, a \in \mathcal{D},\ i \in \{1, \cdots, m\} \tag{5.3}$$

The Lyapunov condition for stability in switched systems can be stated as follows [Lib03, Theorem 2.1]

Theorem 5.3.1. *If all the subsystems of the system* (5.1) *share a radially unbounded common Lyapunov function and a unique equilibrium at the origin, then the switched system* (5.1) *has a GUAS equilibrium at the origin.*

Theorem 5.3.1 can be proved using the same method that is used in proving the Lyapunov theorem for nonlinear systems [Lib03, Section A.3].

A few points regarding the Theorem 5.3.1 should be noted:

- Theorem 5.3.1 holds even if the switched system has an infinite number of subsystems. If the switched system has a finite number of subsystems (which is the case we consider in this chapter), then condition (5.3) can be replaced with the following condition and the Theorem (5.3.1) still holds:
$$\frac{\partial V}{\partial x}(a) f^{(i)}(a) < 0, \quad \forall\, a \in \mathcal{D} \setminus \{0\},\ i \in \{1, \cdots, m\}$$

- When the common Lyapunov function is quadratic, i.e. $V(x) = x^T P x$ where P is a positive definite matrix, then the switched system is said to be *quadratically stable*.
- Theorem 5.3.1 states a stability condition under arbitrary switching. That includes the case where the switching happens arbitrarily fast.

We could restrict ourselves to switching signals that have a specified lower bound for the dwell time. Such switching signals are usually studied as a special class of constrained switching called *slow switching*. It is a well-known fact that if all the subsystems of a switched system have asymptotically stable equilibria, then the equilibria of the switched system are asymptotically stable if the dwell time is large enough, so to allow the transient effects of switching to dissipate [Lib03]. In the results stated in the following sections of this chapter, we always consider that the switching signal has a dwell time, however, we only need the dwell time to take any positive value and do not set any positive lower bound on the value of dwell time. That is the reason that the results stated in the remainder of this chapter are said to hold under arbitrary switching. Although it is out of the scope of this manuscript, switched systems with a specified lower bound on the dwell time have been studied extensively in the literature. As a more recent example, in [AB10] a switched system with multiple equilibria is studied and a lower bound on the dwell time guaranteeing the attractivity of the equilibria is given.

Many papers which have dealt with stability analysis of positive switched systems have utilised copositive Lyapunov functions and in particular, linear copositive Lyapunov functions. Copositive Lyapunov functions are only required to satisfy the requirements of a Lyapunov function within the positive orthant and may lead to less conservative stability conditions for positive linear switched systems than can be obtained using traditional Lyapunov functions [SWM+07]. A linear copositive Lyapunov function has the form $V(x) = v^T x$, where $v \in \mathbb{R}^n$ and $v \gg 0$. Sufficient and necessary conditions for the existence of common copositive Lyapunov functions were initially studied in [MS07a] and [MS07b]. Those results were further extended to switched systems with arbitrary many subsystems in [KMS09]. [FV10a, FV10b, LYW10] are among the most recent papers to have utilised (linear) copositive Lyapunov functions. In [FV12], the results on linear copositive Lyapunov functions for continuous-time systems are extended to discrete-time case. It is also shown in [FV12] that for discrete-time positive linear systems, if a linear copositive common Lyapunov function can be found, then it ensures the existence of a quadratic copositive common function.

Efforts have been made to study stability properties of switched systems subject to uncertainty, in particular, parametric uncertainty [Lib03, Chapter 6]. To date, relatively little has been written on positive switched systems subject to uncertainty. The works of [MBS09], [BMW10a] and [BMW10a] are of this nature and are the basis for some of the results presented in this chapter and the next.

Besides stability, other properties of switched systems and positive switched systems are also studied from different angles in the literature. These include input to output (or input to state) properties [HM99b, HM99a, Lib99, Hes02, VCL07, EPL08, XWL01], controllability and observability [LAH87a, LAH87b, CvS04, BFTM00, SGL02, VCSS03], reachability [Val09b, XW03, FV09, VS08, Val09a, SV08, SV06a, SV06b, SV06c], excitability [Val07b, Val07a] and passivity [PJS98, ZBS01, ZH06, ZH08, GCB12].

5.4 D-stability for Linear Positive Switched Systems

In this section, we present different results on stability and D-stability of positive linear switched systems. A linear switched system is a switched system which consists of two or more linear subsystems. Linear switched systems provide a framework which bridges linear systems and complex and/or uncertain systems. On one hand, switching among linear systems may produce complex system behaviour such as chaos and multiple limit cycles [SG05]. On the other hand, linear switched systems are relatively easy to handle as many powerful tools from linear and multi-linear analysis are available to cope with these systems. Moreover, the study of linear switched systems provides additional insights into some complex problems, such as intelligent control, adaptive control, and robust analysis and control.

Switched linear systems have been investigated for a long time in the control literature and have attracted increasingly more attention since the 1990s (look at [SWM+07] and references therein). Quite a number of fundamental concepts and powerful tools have been developed since then. Despite the rapid progress made so far, many fundamental problems are still either unexplored or less well understood. One of these areas is the D-stability of linear positive switched systems which is the subject of this section.

The systems we consider in this section can be modelled as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t); \quad x(0) = x_0 \geq 0 \tag{5.4}$$

where $\{A_1, \cdots, A_m\}$ is a set of Metzler matrices in $\mathbb{R}^{n\times n}$ representing each of the m constituent subsystems of the switched system and $\sigma : \mathbb{R}_+ \mapsto \{1, \cdots, m\}$ is the switching signal. In this section, we always assume A_i is Hurwitz for $i = 1, \cdots, m$.

Following Definition 5.2.4, the origin is a D-stable equilibrium of the linear switched system (5.4), if it is a GUAS equilibrium of the following system:

$$\dot{x}(t) = D_{\sigma(t)}A_{\sigma(t)}x(t); \quad x(0) = x_0 \geq 0 \tag{5.5}$$

for all diagonal D_i, for $i = 1, \cdots, m$, with positive diagonal entries.

Remark 5.4.1. Note that when A_i, for $i = 1, \cdots, m$, is Hurwitz, then the origin is the only equilibrium of the switched systems (5.4) and (5.5). Therefore, if the origin is a D-stable equilibrium of the system (5.4), then it is a globally D-stable equilibrium.

Before stating the main results of this section, we recall some preliminary results on linear switched systems. The following is a well-known necessary stability condition for linear switched systems [SWM+07]. Note that this results holds for linear switched systems which are not necessarily positive.

Lemma 5.4.1. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Hurwitz. Suppose that the associated linear switched system* (5.4) *has a GUAS equilibrium at the origin. Then for any real $\gamma \geq 0$, $A_1 + \gamma A_2$ is Hurwitz.*

In [GSM07, Theorem 3.2], it is shown that Lemma 5.4.1 can be stated as a necessary and sufficient condition for positive linear switched systems defined on $\mathbb{R}^2$ as stated below.

Lemma 5.4.2. *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Metzler and Hurwitz. Then the switched system* (5.4) *has a GUAS equilibrium at the origin, if and only if $A_1 + \gamma A_2$ is Hurwitz for all real $\gamma \geq 0$.*

Note that Lemma 5.4.2 is not true for arbitrary dimensions, even in $\mathbb{R}^{3\times 3}$, as shown in [FMC09] via a counter-example.

Now we can state and prove the following theorem.

Theorem 5.4.3. *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Metzler and Hurwitz. The positive linear switched system (5.4) has a globally D-stable equilibrium at the origin if and only if $A_1 + DA_2$ is Hurwitz for all diagonal matrices D with positive diagonal entries.*

Proof:

Necessity:

Since the origin is a globally D-stable equilibrium of the system (5.4), we know that D_1A_1 and D_2A_2 are Hurwitz for all diagonal D_1, D_2 with positive diagonal entries. Therefore, based on Lemma 5.4.1, $D_1A_1 + \gamma D_2A_2$ is Hurwitz for all $\gamma > 0$ including $\gamma = 1$. We have:

$$D_1A_1 + D_2A_2 = D_1(A_1 + D_1^{-1}D_2A_2) = D_1(A_1 + DA_2)$$

where $D = D_1^{-1}D_2$ is an arbitrary diagonal matrix with positive diagonal entries.

Since $D_1(A_1 + DA_2)$ is Metzler and Hurwitz, then based on D-stability of positive LTI systems (Theorem 2.5.9), we can conclude $A_1 + DA_2$ is Hurwitz where D can be any diagonal matrix with positive diagonal entries.

Sufficiency:

To prove sufficiency, let $D_1 > 0$, $D_2 > 0$ be diagonal matrices and let $\gamma \geq 0$ be any nonnegative real number. By hypothesis, $A_1 + \gamma D_1^{-1}D_2A_2$ is Hurwitz for $\gamma > 0$ and it is trivially true for $\gamma = 0$. However, this matrix is also Metzler and hence based on D-stability of positive LTI systems (Theorem 2.5.9), $D_1A_1 + \gamma D_2A_2 = D_1(A_1 + \gamma D_1^{-1}D_2A_2)$ is also Hurwitz. It now follows immediately from Lemma 5.4.2 that the switched system associated with D_1A_1, D_2A_2 is stable. As this is true for any diagonal $D_1 > 0$, $D_2 > 0$, then the origin is a globally D-stable equilibrium of the system (5.4). $\square$

In Chapter 2, we mentioned that for positive LTI systems, stability and D-stability are equivalent (Theorem 2.5.9). The following example clearly shows that unlike positive linear systems, D-stability is not an intrinsic property of positive linear switched systems, even in the planar case.

Example 5.4.1. Consider the Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} -2 & 0 \\ 1 & -4 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}$$

It is straightforward to verify that eigenvalues of $A_1 + \gamma A_2$ are $\lambda_1 = -(3+\gamma) + \sqrt{11\gamma + 9}$ and $\lambda_2 = -(3+\gamma) - \sqrt{11\gamma + 9}$ which means $A_1 + \gamma A_2$ is Hurwitz for all $\gamma \geq 0$. Hence by Lemma 5.4.2, the associated switched system is stable. On the other hand, choosing

$$D = \begin{pmatrix} 20 & 0 \\ 0 & 0.5 \end{pmatrix}$$

it is easy to check that eigenvalues of $A_1 + DA_2$ are $\lambda_1 \simeq -26.54$ and $\lambda_2 \simeq +0.04$ which means $A_1 + DA_2$ is not Hurwitz. Hence by Theorem 5.4.3 the origin is not a D-stable equilibrium of the associated switched system (5.4). $\square$

5.4.1 General Linear Positive Switched Systems

In this section, we state new conditions for D-stability for different classes of positive linear switched systems. We state the necessary and sufficient conditions in two separate theorems. The next result states a sufficient conditions for D-stability in linear positive switched systems, which first appeared in [MBS09].

Theorem 5.4.4. *Let $A_i \in \mathbb{R}^{n\times n}$, for $i = 1, \cdots, m$, be Metzler and Hurwitz. If there is some $v \gg 0$ with $A_i v \ll 0$, for $i = 1, \cdots, m$, then the origin is a globally D-stable equilibrium of the system* (5.4).

Proof: The first step in the proof is to show that the existence of such a v is sufficient for the stability of the switched system (5.4). With this in mind, suppose that there exists some $v \gg 0$ satisfying $A_i v \ll 0$, for $i = 1, \cdots, m$, and let a switching signal $\sigma(\cdot)$ be given. Furthermore, let $0 = t_0, t_1, t_2, \ldots, t_k, \ldots,$ be the switching times. Let $x(., x_0, \sigma)$ denote the unique, piecewise C^1 solution of (5.4) corresponding to the initial condition x_0 and the switching signal σ. Also, for $i = 1, \cdots, m$, let $x^{(i)}(., x_0)$ denote the unique solution of the stable positive LTI system $\dot{x} = A_i x$ corresponding to the initial state x_0. Note the following readily verifiable facts concerning the solutions of the positive LTI system with system matrix A_i.

(a) For $i = 1, \cdots, m$, if $x_0 > 0, x_1 > 0$ satisfy $x_0 < x_1$, then $x^{(i)}(t, x_0) < x^{(i)}(t, x_1)$ for all $t \geq 0$. This simply records the well-known fact that positive LTI systems are monotone (Lemma 2.5.3);

(b) For $i = 1, \cdots, m$, as $\frac{d}{dt}x^{(i)}(0, v) = A_i v \ll 0$, it follows that there is some $\delta > 0$ such that $x^{(i)}(t, v) \ll v$ for $0 \leq t \leq \delta$.

Combining (a) and (b) we see immediately that for $0 < t \leq \delta$, and $i = 1, \cdots, m$,

$$x^{(i)}(t + \delta, v) = x^{(i)}(t, x^{(i)}(\delta, v)) < x^{(i)}(t, v) < v.$$

Simply iterating this process, it is easy to see that for $i = 1, \cdots, m$, $x^{(i)}(t, v) < v$ for all $t \geq 0$.

Now consider the solution $x(t, v, \sigma)$ of (5.4) corresponding to the initial condition v and the switching signal σ. The argument in the previous paragraph guarantees that for $0 \leq t \leq t_1$, $x(t, v, \sigma) < v$ (as the dynamics in this interval are given by one of the constituent positive LTI systems). But in the second interval $[t_1, t_2)$, the system dynamics are again given by a positive LTI system with $x(t_1, v, \sigma) < v$ as initial condition. Hence from the previous argument combined with point (a) above, we can conclude that for $t_1 \leq t \leq t_2$, $x(t, v, \sigma) < v$. Continuing in this way, we can easily see that for all $t \geq 0$, we have $x(t, v, \sigma) < v$. As the switching signal σ was arbitrary, we can conclude that $x(t, v, \sigma) < v$ holds for all switching signals.

It is now straightforward to show that the solutions of (5.4) are uniformly bounded. Firstly, note that for any $\alpha > 0$, we have:

$$x^{(i)}(t, \alpha v) = \alpha x^{(i)}(t, v)$$

for $i = 1, \cdots, m$. Therefore, we can use the same method as we used above to show that

$$x(t, \alpha v, \sigma) < \alpha v$$

Let $x_0 > 0$ be an initial condition such that $\|x_0\|_\infty \leq K_1$. Then $x_0 < (K_1/v_{min})v$ where $v_{min} = \min\{v_1, \ldots, v_n\}$. It now follows that for all $t \geq 0$,

$$x(t, x_0, \sigma) < (K_1/v_{min})v$$

and hence

$$\|x(t, x_0, \sigma)\|_\infty \leq K_1(v_{max}/v_{min})$$

for all $t \geq 0$ where $v_{max} = \max\{v_1, \ldots, v_n\}$.

So far, we have shown the trajectory of the switched system starting from any initial condition $x_0 > 0$ is bounded. Now we show that the trajectory is not only bounded, but tends to the origin. Now if there is some $v \gg 0$ with $A_i v \ll 0$ for $i = 1, \cdots, m$, then $(A_i + \epsilon I)v \ll 0$ for sufficiently small positive $\epsilon > 0$. It is easy to see that $(A_i + \epsilon I)$ is a Metzler matrix and also, based on Ostrowski's Theorem on continuity of eigenvalues [Wil65, Section 2.2], we can select a small enough ϵ such that $(A_i + \epsilon I)$ is Hurwitz for $i = 1, \cdots, m$. Therefore, based on the discussions so far, for some $\epsilon > 0$, the trajectories of the switched system

$$\dot{y}(t) = (A_i + \epsilon I)y(t) \tag{5.6}$$

are also uniformly bounded for all $x_0 \in \mathbb{R}^n_+$. If $y(\cdot, x_0, \sigma)$ is the solution of the system (5.6) corresponding to the initial condition x_0 and switched signal σ, then we have:

$$y(t, x_0, \sigma) = e^{(A_{\sigma(t)} + \epsilon I)t} x_0 = e^{\epsilon I t} e^{A_{\sigma(t)} t} x_0 = e^{\epsilon I t} x(t, x_0, \sigma)$$

Hence, based on the boundedness of $y(t)$ and the fact that $e^{\epsilon I t} \to \infty$ as $t \to \infty$, we can immediately conclude that the switched system (5.4) is GUAS.

To complete the proof, note that for all diagonal matrices D_i with positive diagonal entries, the matrices $D_i A_i$ are Metzler and Hurwitz for all $i = 1, \cdots, m$. Moreover, if $A_i v \ll 0$ for $i = 1, \cdots, m$, then $D_i A_i v \ll 0$. The above argument now immediately implies that the origin is a GUAS equilibrium of the system (5.5) and hence the origin is a globally D-stable equilibrium of the system (5.4) as claimed. □

Remark 5.4.2. It should be noted that the sufficient condition stated in the Theorem 5.4.4 is only a sufficient and *not* a necessary condition for D-stability in linear switched systems, as demonstrated by the following example.

Example 5.4.2. Consider the Metzler Hurwitz matrices A_1 and A_2 given by:

$$A_1 = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$$

Using Theorem 4.1 of [MS07a] it is straightforward to show that there is no vector $v \gg 0$ with $A_1 v \ll 0$ and $A_2 v \ll 0$. On the other hand, it can be verified algebraically that for any diagonal $D \in \mathbb{R}^{2\times 2}$ with positive diagonal entries, $A_1 + DA_2$ is Hurwitz and hence based on the Theorem 5.4.3, the origin is a globally D-stable equilibrium of the switched system (5.4). □

The next theorem, provides a necessary condition for D-stability in linear positive switched systems. To prove it, we need the following two lemmas. The first one, is a restatement of Theorem 3.1 in [MS07a].

Lemma 5.4.5. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Suppose that there is no vector $v \in \mathbb{R}^n$ with $v > 0$ such that $A_i v \leq 0$ for $i = 1, 2$. Then there is some diagonal D with positive diagonal entries such that $A_1 + DA_2$ is singular.*

The following lemma, follows from Lemma 5.4.1.

Lemma 5.4.6. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Suppose that the origin is a globally D-stable equilibrium of the associated linear positive switched system* (5.4)*. Then for any diagonal matrix D with positive diagonal entries, $A_1 + DA_2$ is Hurwitz.*

Proof: The origin is a globally D-stable equilibrium of the system (5.4), which means it is a GUAS equilibrium of the system (5.5) for all choices of diagonal D_1, D_2 with positive diagonal entries. Therefore, based on Lemma 5.4.1, $D_1 A_1 + \gamma D_2 A_2$ is Hurwitz. We have:

$$D_1 A_1 + \gamma D_2 A_2 = D_1 (A_1 + \gamma D_1^{-1} D_2 A_2)$$

Since D_1 is an arbitrary diagonal matrix with positive diagonal entries, based on Theorem 2.5.9 we can conclude that $A_1 + \gamma D_1^{-1} D_2 A_2$ is Hurwitz. Since D_2 is also an diagonal matrix with positive diagonal entries, then so is $D = D_1^{-1}D_2$. Therefore, $A_1 + DA_2$ is Hurwitz for all diagonal D with positive entries. This concludes the proof. □

Now we are ready to state the following theorem.

Theorem 5.4.7. *Let $A_i \in \mathbb{R}^{n\times n}$, for $i = 1, 2$, be Metzler and Hurwitz. If the origin is a globally D-stable equilibrium of the system* (5.4) *then there exists some $v > 0$ with $A_i v \leq 0$ for $i = 1, 2$.*

Proof: The proof follows immediately from Lemmas 5.4.5 and 5.4.6. Since the origin is a globally D-stable equilibrium of the system (5.4), then based on Lemma 5.4.5, $A_1 + DA_2$ is Hurwitz for all diagonal D with positive entries. Therefore, based on Lemma 5.4.6, there should exists a $v > 0$ such that $A_i v \leq 0$ for $i = 1, 2$. This concludes the proof. □

Remark 5.4.3. It is important to note the difference between the D-stability condition stated in this section and the existence of common linear copositive Lyapunov function. The following counter example shows that the existence of a common linear copositive Lyapunov function does not imply D-stability of the linear switched system (5.4).

Example 5.4.3. Consider the linear switched system (5.4) with $m = 2$, where:

$$A_1 = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -6 & 2 \\ 6 & -5 \end{pmatrix}$$

Clearly, A_1 and A_2 are Metzler and it can be easily checked that they are both Hurwitz. For $w = (2.4 \quad 1)^T$, we have:

$$A_1^T w = \begin{pmatrix} -0.4 \\ -0.6 \end{pmatrix} < 0, \quad A_2^T w = \begin{pmatrix} -8.4 \\ -0.2 \end{pmatrix} < 0$$

This means that there exists a common copositive Lyapunov function for the positive linear switched system (5.4) with the above choices of A_1 and A_2, therefore, the origin is a GUAS equilibrium of the system (5.4). On the other hand, it can be proved that the origin is not a D-stable equilibrium of the above mentioned system. To this end, note that based on the Theorem (5.4.7), if the origin is a D-stable equilibrium of the system (5.4), then there should exists a vector $v = (v_1 \quad v_2)^T > 0$, such that

$$A_1^T v = \begin{pmatrix} -v_1 + v_2 \\ 2v_1 - 3v_2 \end{pmatrix} \leq 0 \quad \text{and} \quad A_2^T w = \begin{pmatrix} -6v_1 + 2v_2 \\ 6v_1 - 5v_2 \end{pmatrix} \leq 0 \tag{5.7}$$

For (5.7) to hold, we should have $v_1 \geq v_2$ and $5v_2 \geq 6v_1$. These two inequalities hold only if $v_1 = v_2 = 0$. Therefore, there is no $v > 0$ such that $A_1 v \leq 0$ and $A_2 v \leq 0$ and that means the origin is not a D-stable equilibrium of the system (5.4) with the above choices of A_1 and A_2. $\square$

5.4.2 Irreducible Linear Positive Switched Systems

Looking at Theorems 5.4.4 and 5.4.7, it can be easily seen that there is a gap between the sufficient and necessary D-stability conditions given in these theorems. In the next result, which appears in [BMW10a], we show that under the extra assumption of irreducibility, it is possible to give a single necessary and sufficient condition for D-stability in linear switched systems.

Theorem 5.4.8. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler, irreducible and Hurwitz. Then the origin is a globally D-stable equilibrium of the switched system* (5.4) *if and only if there exists a vector $v \gg 0$ such that $A_1 v < 0$ and $A_2 v < 0$.*

Proof:

Proof of Necessity:

From Theorem 5.4.7, we already know that if the origin is a globally D-stable equilibrium of the switched system (5.4), then there exists a $v > 0$ such that $A_1 v < 0$ and $A_2 v < 0$. We shall show that if A_1 and A_2 are irreducible, then any such v must be strictly positive.

To this end, assume that $v > 0$, $A_i v < 0$ for $i = 1, 2$ and v is not strictly positive. Without loss of generality, we assume that precisely the first k elements of v are non-zero, so $v_i > 0$ for $i = 1, \cdots, k$ and $v_i = 0$ for $i = k+1, \cdots, n$. Now we partition A_1 and A_2 as follows:

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}$$

In which A_{11} and A'_{11} are $k \times k$, A_{22} and A'_{22} are $(n-k)\times(n-k)$ and A_{21} and A'_{21} are $(n-k)\times k$ sub-matrices. Note that A_{11}, A'_{11}, A_{22} and A'_{22} are Metzler and A_{12}, A'_{12}, A_{21} and A'_{21} are element-wise nonnegative.

We know that $A_1 v < 0$ and since the last $n-k$ elements of v are zero, then we should have $A_{21}v' < 0$, in which $v' = [v_1, \cdots, v_k]^T$. Since we know A_{21} is a nonnegative matrix and $v' \gg 0$, then the only way that this inequality can hold is that A_{21} is a zero matrix. Using the same method, we can easily conclude that A'_{21} should also be a zero matrix. This implies that both A_1 and A_2 are reducible, which is a contradiction. Therefore, v cannot have zero entries and we must have $v \gg 0$ as claimed.

Proof of Sufficiency:

Let σ be a given switching signal with switching instances $t_0, t_1, t_2, \ldots$ with a dwell time $\tau > 0$. We shall also write $i_j = \sigma(t_j)$ for $j = 0, 1, \ldots$, so the following formulae will be easier to read. We denote by $x(t, x_0, \sigma)$ the solution of switched system (5.4) corresponding to a given switched signal σ and initial condition $x_0 \in \mathbb{R}^n_+$.

Now note that for an irreducible Metzler matrix A, $e^{At} \gg 0$ for all $t > 0$ [BP94, Theorem 6.3.12]. Consider for any such A, the system $\dot{x}(t) = Ax(t)$. Then for any solution $x(t)$ of this system, $y(t) = Ax(t)$ also satisfies $\dot{y}(t) = Ay(t)$. As $e^{At} \gg 0$ for all $t > 0$, it immediately follows that if $y(0) < 0$ then we must have $y(t) \ll 0$ for all $t > 0$. In terms of the original system, this means that $Ax(0) < 0$ implies that $Ax(t) \ll 0$ for all $t > 0$. This argument guarantees that there is some $\alpha < 1$ such that for $i = 1, 2$:

$$e^{A_i \tau} v \leq \alpha v. \tag{5.8}$$

Further, as $t_{j+1} - t_j \geq \tau$ for all j, we can also conclude that for $i = 1, 2$ and $j = 0, 1, 2, 3, \ldots$

$$e^{A_i (t_{j+1} - t_j)} v \leq e^{A_i (\tau)} v. \tag{5.9}$$

Now consider any time $t > 0$ and assume that t_K is the final switching instant before t. Then

$$x(t, v, \sigma) = e^{A_{i_K}(t - t_K)} e^{A_{i_{K-1}}(t_K - t_{K-1})} \cdots e^{A_{i_0}(t_1 - t_0)} v. \tag{5.10}$$

It follows from (5.9) and (5.10) that $x(t, v, \sigma) \leq \alpha^K v$. A little thought (we can "lose" at most one power of α per switch) shows that if we define N_t to be the largest integer less than or equal to $\frac{t}{2\tau}$, then for any switching signal (whether there are finitely many switches or infinitely many switches) we must have

$$x(t, v, \sigma) \leq \alpha^{N_t} v$$

implying that $x(t, v, \sigma) \to 0$ as $t \to \infty$.

Now let $x_0 \in \mathbb{R}^n_+$ be given. Choose $\lambda > 0$ with $x_0 \leq \lambda v$. It follows from $e^{A_i t} \gg 0$ for all $t > 0$ and $i = 1, 2$ that

$$x(t, x_0, \sigma) \leq x(t, \lambda v, \sigma) = \lambda x(t, v, \sigma)$$

and hence $x(t, x_0, \sigma) \to 0$ as well. The result now follows as it is immediate that $D_i A_i v < 0$ for $i = 1, 2$, for any diagonal matrices D_i with positive diagonal entries. □

Remark 5.4.4. Note that in Theorem 5.4.4, the sufficient condition for D-stability of the equilibrium of positive linear switched system (5.4) is the existence of a vector $v \gg 0$ such that $A_i v \ll 0$, for $i = 1, \cdots, m$. That condition is relaxed to $A_i v < 0$ in Theorem 5.4.8, which is a consequence of adding the irreducibility assumption.

5.4.3 Commuting Linear Positive Switched Systems

We close this section with a result on commuting positive linear switched systems. A commuting linear switched system is a linear switched system with commutative system matrices.

Definition 5.4.1. Two Matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$ are commutative (or commute) if and only if

$$A_1 A_2 = A_2 A_1 \tag{5.11}$$

The above definition is a special case of commuting vector fields, which will be considered in Section 5.5.2.

It has been previously shown [NB94] that linear switched systems with commuting system matrices have a GUAS equilibrium. In the following result, we show that for positive linear switched systems, commutativity implies the stronger property of D-stability [BMW10a].

Theorem 5.4.9. *Let $A_1 \in \mathbb{R}^{n \times n}$ and $A_2 \in \mathbb{R}^{n \times n}$ be Metzler and Hurwitz. Further, assume that $A_1 A_2 = A_2 A_1$. Then the origin is a globally D-stable equilibrium of the switched system* (5.4) *under arbitrary switching.*

Proof: Recall that for Metzler, Hurwitz matrices A_1 and A_2, $A_1^{-1} < 0$ and $A_2^{-1} < 0$ [Lue79, Theorem 6.5.3]. Now let $w \gg 0$ in $\mathbb{R}^n$ be given. Then

$v = A_1^{-1}A_2^{-1}w \gg 0$, and therefore:

$$A_1 v = A_1 A_1^{-1} A_2^{-1} w = A_2^{-1} w \ll 0$$

and

$$A_2 v = A_2 A_1^{-1} A_2^{-1} w = A_2 A_2^{-1} A_1^{-1} w = A_1^{-1} w \ll 0$$

Thus, we have $v \gg 0$ such that $A_1 v \ll 0$ and $A_2 v \ll 0$, and it follows from Theorem 5.4.4 that the switched system (5.4) has a D-stable equilibrium at the origin. $\square$

5.5 D-Stability for Positive Nonlinear Switched Systems

Nonlinear switched systems are those switched systems whose subsystems are nonlinear. Due to the nature of nonlinear systems, usually the methods applied in attaining stability results for linear switched systems cannot be applied to nonlinear switched systems. Nevertheless, there have been different attempts made to obtain stability results for nonlinear switched systems [SG11]. In line with those efforts, in this section we present some results on stability and D-stability for nonlinear switched systems.

5.5.1 Homogeneous Cooperative Switched Systems

As already discussed in Chapters 2 and 3, homogeneous cooperative systems have some characteristics that makes them natural candidates when we want to extend some properties of positive linear systems to nonlinear systems. Therefore, as a first step in extending the results on D-stability in positive linear switched systems, we consider homogeneous cooperative switched systems.

Throughout this section, all vector fields are assumed to be cooperative and homogeneous of degree 0 with respect to a fixed dilation map δ_λ^r. Further, we shall assume that all vector fields are C^1 on a neighbourhood $\mathcal{W}$ of $\mathbb{R}_+^n$. As noted in [AdL02], this ensures existence and uniqueness of solutions for the associated autonomous system.

The main theorem of this section, which extends Theorem 5.4.4 to cooperative homogeneous systems, is as follows [BMW10a].

Theorem 5.5.1. *Consider the switched system (5.1) where $f^{(i)}$, for $i = 1, \cdots, m$, is cooperative and homogeneous of degree* 0 *with respect to dilation map δ^r_λ and let each constituent subsystem have a unique GAS equilibrium at the origin. If there exists a $v \gg 0$ such that $f^{(i)}(v) \ll 0$ for all $i \in \{1, 2, ..., m\}$, then* (5.1) *has a globally D-stable equilibrium at the origin under arbitrary switching.*

Proof: We prove the theorem in a number of steps. The first step is to show Lyapunov stability of the system.

(i) Proof of Stability

Let an arbitrary switching signal $\sigma : [0, \infty) \to \{1, \ldots, m\}$ be given with switching instances $0 = t_0, t_1, t_2, \cdots$. For $x_0 \in \mathbb{R}^n_+$, let $x(t, x_0, \sigma)$ denote the solution of (5.1) corresponding to the initial condition x_0 and the switching signal σ.

To begin with, from the homogeneity of the vector fields $f^{(i)}$ it follows that for any $\lambda > 0$, $f^{(i)}(\delta^r_\lambda(v)) \ll 0$ for $i = 1, \ldots, m$. Thus as each $f^{(i)}$ is cooperative, Lemma 2.4.4 implies that the trajectory of (5.1) starting from the initial condition $x_0 = \delta^r_\lambda(v)$ is non-increasing for $0 \le t < t_1$. In particular,

$$x(t, \delta^r_\lambda(v), \sigma) < x(0, \delta^r_\lambda(v), \sigma) = \delta^r_\lambda(v), \text{ for all } 0 \le t \le t_1$$

At $t = t_1$, we switch to a new system whose initial condition is equal to $x(t_1, \delta^r_\lambda(v), \sigma)$. We know that this new system is cooperative and we also know that $f^{(i)}(\delta^r_\lambda(v)) \ll 0$, for $i = 1, \ldots, m$. Therefore, based on Lemma 2.4.4, we can conclude the trajectory of this new system is also non-increasing and we have:

$$x(t, \delta^r_\lambda(v), \sigma) < \delta^r_\lambda(v), \text{ for all } t_1 \le t \le t_2$$

Continuing in this way we can conclude that $x(t, \delta^r_\lambda(v), \sigma) < \delta^r_\lambda(v)$ for all $t \ge 0$.

For an arbitrary initial condition x_0, we can always find a $\lambda > 1$ such that $x_0 < \delta^r_\lambda(v)$. Therefore, based on monotonicity, we have:

$$x(t, x_0, \sigma) < x(t, \delta^r_\lambda(v), \sigma) < \delta^r_\lambda(v), \text{ for all } 0 \le t \le t_1$$

Let $\epsilon > 0$ be given. Then we can choose $\lambda > 0$ so that $\|\delta^r_\lambda(v)\|_\infty < \epsilon$. Now putting

$$\delta = \min_i (\delta^r_\lambda(v))_i$$

we see that if $x_0 \geq 0$ and $\|x_0\|_\infty < \delta$, then $x_0 \leq \delta_\lambda^r(v)$ and the above argument guarantees that

$$\|x(t, x_0, \sigma)\|_\infty < \epsilon, \text{ for all } t \geq 0$$

Therefore, based on Definition 5.2.3, the origin is a stable equilibrium of the system (5.1). Note that our choice of δ does not depend on the switching signal σ.

(ii) Proof of Global Asymptotic Stability

Choose $\alpha > 0$ such that

$$f^{(i)}(v) + \alpha \text{diag } (r) v \ll 0$$

for $1 \leq i \leq m$ and for $i = 1, \ldots, m$, define $g^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g^{(i)}(y) = f^{(i)}(y) + \alpha \text{diag } (r) y$$

It is evident that with our choice of α, $g^{(i)}(v) \ll 0$. Since for $i = 1, \cdots, m$, $f^{(i)}$ is cooperative, we know that for $k \neq j$:

$$\frac{\partial g_k^{(i)}}{\partial x_j}(a) = \frac{\partial f_k^{(i)}}{\partial x_j}(a) \geq 0 \quad \forall a \in \mathbb{R}^n$$

Therefore, $g^{(i)}$ is also cooperative. On the other hand, for the vector $\delta_\lambda^r(v)$, we have:

$$\begin{aligned} g^{(i)}(\delta_\lambda^r(v)) &= \alpha \text{diag } (r) \delta_\lambda^r(v) + f^{(i)}(\delta_\lambda^r(v)) \\ &= \alpha \text{diag } (r) \delta_\lambda^r(v) + \delta_\lambda^r(f^{(i)}(v)) = \delta_\lambda^r(g^i(v)) \end{aligned}$$

for $i = 1, \cdots, m$. This shows that $g^{(i)}$, for $i = 1, \cdots, m$, is homogeneous of degree 0 with respect to dilation map δ_λ^r.

Now consider the following switched system

$$\dot{y}(t) = g^{(\sigma(t))}(y(t)) \tag{5.12}$$

We have shown that for $i = 1, \cdots, m$, $g^{(i)}$ is cooperative, homogeneous of degree 0 with respect to dilation map δ_λ^r and $g^{(i)}(v) \ll 0$ for some $v \gg 0$. Therefore, using the same method as utilised in the previous step we can prove that the origin is a stable equilibrium of the switched system (5.12) and the trajectory of the system (5.12) starting from any initial condition $x_0 \in \mathbb{R}_+^n$ is uniformly bounded.

Let $x(t, x_0, \sigma)$ be a solution of (5.1) with initial condition x_0. Then it can be shown that

$$y(t) = \delta_\lambda^r(x(t)) = (e^{r_1 \alpha t} x_1, \cdots, e^{r_n \alpha t} x_n)^T \tag{5.13}$$

is a solution of (5.12) with $y(0) = x_0$ where $\lambda = e^{\alpha t}$. Note that if we show (5.13) holds for subsystems of the switched system (5.1), then it is true for the switched system itself. To this end, for all i in $\{1, \cdots, m\}$, we have:

$$\begin{aligned}
\dot{y}(t) &= (e^{r_1 \alpha t} \dot{x_1} + r_1 \alpha e^{r_1 \alpha t} x_1, \cdots, e^{r_n \alpha t} \dot{x_n} + r_n \alpha e^{r_n \alpha t} x_n)^T \\
&= (\lambda^{r_1} f_1^{(i)}(x) + r_1 \alpha \lambda^{r_1} x_1, \cdots, \lambda^{r_n} f_n^{(i)}(x) + r_n \alpha \lambda^{r_n} x_n)^T \\
&= \delta_\lambda^r(f^{(i)}(x)) + \alpha \text{diag } (r) \delta_\lambda^r(x) \\
&= f^{(i)}(\delta_\lambda^r(x)) + \alpha \text{diag } (x) \delta_\lambda^r(x) \\
&= f^{(i)}(y) + \alpha \text{diag } (r) y = g^{(i)}(y(t))
\end{aligned}$$

Now, we define $r_p = \min_i r_i$ for $i = 1, \cdots, n$ and

$$z(t) = (e^{r_p \alpha t} x_1, \cdots, e^{r_p \alpha t} x_n) = e^{r_p \alpha t} x(t)$$

It is obvious that $z(t) < y(t)$ for all $t > 0$. We proved $y(t)$ is uniformly bounded, therefore, $z(t)$ is also uniformly bounded. On the other hand, $e^{r_p \alpha t} \to \infty$ as $t \to \infty$, hence, we can conclude that $x(t, x_0, \sigma) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}_+^n$, $\sigma \in \mathcal{S}$, meaning the system (5.1) is globally uniformly asymptotically stable.

(iii) Proof of D-stability

Let matrices $D_1, \ldots, D_m$ be given such that D_i is a diagonal matrix with positive diagonal entries, for $i \in \{1, \ldots m\}$. For all $j \in \{1, \ldots m\}$, $j \neq k$, we have:

$$\frac{\partial D_i f_k^{(i)}}{\partial x_j}(a) = D_i \frac{\partial f_k^{(i)}}{\partial x_j}(a) \geq 0 \quad \forall a \in \mathbb{R}^n$$

which means all systems $\dot{x} = D_i f^{(i)}(x)$ are cooperative for $i \in \{1, \ldots m\}$.

On the other hand, we have:

$$D_i f^{(i)}(\delta_\lambda^r(x)) = D_i \delta_\lambda^r(f^{(i)}(x)) = \delta_\lambda^r(D_i f^{(i)}(x))$$

which means $D_i f^{(i)}(v)$ is homogeneous of degree 0 with respect to the dilation map $\delta_\lambda^r(x)$. Note that $D_i \delta_\lambda^r(f^{(i)}(x)) = \delta_\lambda^r(D_i f^{(i)}(x))$ follows from the fact that

D_i is diagonal with constant diagonal entries. Further, $D_i f^{(i)}(v) \ll 0$ for $1 \leq i \leq m$. It now follows from the previous arguments, that the origin is a GUAS equilibrium of the system (5.2) and hence is a globally D-stable equilibrium of the system (5.1). □

5.5.2 Commuting Positive Nonlinear Switched Systems

In this section, we consider the case where the vector fields of the subsystems of a nonlinear switched system commute. Before presenting the main results of this section, we need to define commuting vector fields.

Definition 5.5.1. Let $\mathcal{D} \subset \mathbb{R}^n$. The Lie bracket, or commutator, of two C^1 vector fields $f^{(1)}, f^{(2)} : \mathcal{D} \mapsto \mathbb{R}^n$, is the vector field defined as follows

$$[f^{(1)}, f^{(2)}](\cdot) := \frac{\partial f^{(2)}(\cdot)}{\partial x} f^{(1)}(\cdot) - \frac{\partial f^{(1)}(\cdot)}{\partial x} f^{(2)}(\cdot)$$

We say two vector fields $f^{(1)}$ and $f^{(2)}$ commute, if and only if

$$[f^{(1)}, f^{(2)}](x) \equiv 0 \quad \forall x \in \mathcal{D}$$

Note that for linear systems, we have $f^{(1)}(x) = A_1 x$ and $f^{(2)}(x) = A_2 x$ which means the two linear vector fields commute if $A_1 A_2 = A_2 A_1$. Therefore, as mentioned in Section 5.4, Definition 5.4.1 is a special case of the general definition of commuting vector fields presented here.

The following results, which is due to [MA00], states a stability condition for switched systems whose subsystems have commuting vector fields.

Theorem 5.5.2. *Let* $\{f^{(i)} : i = 1, \cdots, m\}$ *be a finite set of* C^1 *commuting vector fields and let the origin be a GAS equilibrium of all the subsystems of the switched system* (5.1)*, then the origin is a GUAS equilibrium of the switched system* (5.1)*.*

Our next result, provides a condition for D-stability for a class of commuting vector fields. To prove the next result, we make use of Theorem 2.6.5.

Theorem 5.5.3. *Consider the switched system* (5.1) *with* $m = 2$*. Assume that* $f^{(1)}$ *and* $f^{(2)}$ *commute and are irreducible, cooperative and homogeneous* *of degree* 0 *with respect to dilation map* $\delta_\lambda^r(x)$*. If the switched system* (5.1) *has a GUAS equilibrium at the origin, then the origin is a globally D-stable equilibrium of the switched system* (5.1)*.*

Proof: Since the subsystem $\dot{x} = f^{(2)}(x)$ is homogeneous, cooperative and irreducible and has a GAS equilibrium at the origin, then based on Theorem 2.6.5, we know that there exists a $v \gg 0$ such that

$$f^{(2)}(v) = \gamma \text{diag }(r)v \tag{5.14}$$

in which $\gamma < 0$ is a scalar. Now, applying Euler's formula to $f^{(2)}$ and evaluating it at v, we have:

$$\frac{\partial f^{(2)}}{\partial x}(v)\text{diag }(r)v = \text{diag }(r)f^{(2)}(v) \tag{5.15}$$

Substituting $f^{(2)}(v)$ in the right-hand side of (5.15) from (5.14), we have:

$$\frac{\partial f^{(2)}}{\partial x}(v)\text{diag }(r)v = \text{diag }(r)\gamma\text{diag }(r)v$$

$$\Rightarrow \text{diag }(r)^{-1}\frac{\partial f^{(2)}}{\partial x}(v)\text{diag }(r)v = \gamma\text{diag }(r)v$$

therefore diag $(r)v$ is an eigenvector of $(\text{diag }(r))^{-1}\frac{\partial f^{(2)}}{\partial x}(v)$. Since $\frac{\partial f^{(2)}}{\partial x}(v)$ and therefore diag $(r)^{-1}\frac{\partial f^{(2)}}{\partial x}(v)$ is irreducible and Metzler, and considering the fact that diag $(r)v \gg 0$, the Perron-Frobenius Theorem for irreducible matrices (Theorem 2.5.7) implies that γ is the right-most eigenvalue of the matrix diag $(r)^{-1}\frac{\partial f^{(2)}}{\partial x}(v)$ and diag $(r)v$ is its unique eigenvector (up to scalar multiple).

On the other hand, by evaluating the commutativity equality at $x = v$, we have:

$$\frac{\partial f^{(1)}}{\partial x}(v)f^{(2)}(v) = \frac{\partial f^{(2)}}{\partial x}(v)f^{(1)}(v)$$

By applying (5.14) and Euler's formula to the left-hand side of the above equation, we have:

$$\gamma\text{diag }(r)f^{(1)}(v) = \frac{\partial f^{(2)}}{\partial x}(v)f^{(1)}(v)$$

$$\Rightarrow \gamma f^{(1)}(v) = \text{diag }(r)^{-1}\frac{\partial f^{(2)}}{\partial x}(v)f^{(1)}(v)$$

Therefore, $f^{(1)}(v)$ is also an eigenvector corresponding to γ. Since the eigenvector corresponding to this eigenvalue is unique up to scalar multiple, then we should have:

$$f^{(1)}(v) = \kappa \text{diag }(r)v$$

where κ is a scalar. Since $\dot{x} = f^{(1)}(x)$ is homogeneous, cooperative and irreducible and has a GAS equilibrium at the origin, then based on Theorem (2.6.5), $\kappa < 0$. Thus $f^{(1)}(v) \ll 0$ and from (5.14) we know $f^{(2)}(v) \ll 0$. It now follows from Theorem 5.5.1 that the switched system (5.1) has a globally D-stable equilibrium at the origin under arbitrary switching. □

5.6 Concluding Remarks

In this chapter, we extended the notion of D-stability to positive switched systems. We presented necessary and sufficient condition for D-stability of general positive linear switched systems and unified the necessary and sufficient conditions with adding the extra assumption of irreducibility. We also extended a well-known result on stability of commuting linear switched systems and presented a condition for D-stability of commuting positive linear switched systems. We then extended the D-stability conditions to cooperative homogeneous systems of degree 0 and also nonlinear positive switched systems with commuting vector fields.

There are some possibilities to extend the presented results in different directions. Regarding the results on positive linear switched system, probably the most immediate possibility is to extend the necessary condition for D-stability of general positive linear switched systems (Theorem 5.4.7) to switched systems with more than two subsystems. Doing that, Theorem 5.4.8 would be easily extended to positive irreducible switched systems with more than two subsystems.

Another possible extension can be stating Theorem 5.5.1 for a switched system with subsystems which are homogeneous of degree $\alpha > 0$. That extension would let us easily extend Theorem 5.5.3 to commuting switched systems with subsystems which are cooperative and homogeneous of degree $\alpha > 0$.

Also, in parallel to the results presented in Chapters 3 and 4, it may be possible to state D-stability conditions for positive nonlinear switched systems with subsystems which are cooperative and subhomogeneous. In an even more general case, it might be possible to state a (probably local) D-stability results when subsystems are cooperative without imposing any other assumptions.

Finally, the definition of D-stability for switched systems (Definition 5.2.4) may be extended to include more general classes of systems. One possibility, is to extend the definition of D-stability for nonlinear systems (Definition 3.2.1) to positive nonlinear switched systems.

CHAPTER 6

Spread of Epidemics in Time-Dependent Networks

In this chapter, we study the properties of a class of epidemiological models, namely, the SIS model. Most of the work done on SIS models has been focused on models with time-invariant parameters. In this chapter, we consider a switched SIS model and present different stability conditions for such system.

6.1 Introduction

Mathematical epidemiology is the study of disease propagation in a network of individuals. Unlike most other systems, it is usually impossible and even unethical to do experiments in epidemiology. That is one of the reasons for importance of this area. As is mentioned in [Bai75]: "we need to develop models that will assist the decision-making process by helping to evaluate the consequences of choosing one of the alternative strategies available. Thus, mathematical models of the dynamics of a communicable disease can have a direct bearing on the choice of an immunisation programme, the optimal allocation of scarce resources, or the best combination of control or education technologies". The whole idea of modelling the spread of infectious diseases becomes much more important, when we note the fact that in spite of all the advancement in vaccination and prevention of disease transmission in the past few decades, infectious and parasitic diseases are the second leading cause of death worldwide (after cardiovascular diseases) [Wor08, Figure 4]. They are the leading cause of death in low-income countries [Wor08, Table 2] and in children aged under five years [Wor08, Figure 5].

In this chapter, we focus on a special class of models, which is called SIS model. This class is particularly useful when dealing with diseases that do not confer immunity. We are specially interested in cases where the parameters of the model are not time-independent. We describe the SIS model as a switching system and study the model from two different angles. First, we present conditions that guarantee the disease will be eradicated from the population. Then, we state a condition to stabilise the disease free state of the system. In other words, we study the case where the disease does not disappear in the population under arbitrary switching of the parameters of the SIS model, but we can reach that goal under a certain class of switching signals.

6.2 Literature Review

It is amazing to realize that the theory that microbes (germs) were the cause of many diseases was not really established until the 1870s. In 1876, German physician Robert Koch, published his results showing that while anthrax bacillus could not survive for long outside a host, anthrax built persisting *endospores* that could last a long time. An endospore is a dormant, tough, and temporarily non-reproductive structure produced by certain bacteria. Koch showed these endospores, embedded in soil, were the cause of unexplained 'spontaneous' outbreaks of anthrax [Nob].

Some attempts were made in the first half of the twentieth century to create models of disease propagation, most notably [Ham06, Ros10, KM27]. The model presented in [Ham06] may have been the first to consider the mass-action law in disease propagation (as discussed in the next section). Mathematical epidemiology seems to have grown exponentially starting in the middle of the 20th century [Het00]. The first edition of the Baily's book [Bai57] is considered as an important landmark in mathematical epidemiology. A tremendous variety of models have now been formulated, mathematically analysed, and applied to infectious diseases. [AM91] and [Het00] are two good references on applications of mathematical tools in modelling propagation of infectious disease and different models commonly used in this field today.

An interesting aspect of epidemiological models is that they can also be used in different areas of science and engineering. For example, in the study of spread of rumours and information in a network of individuals [Die67, HHL88, GGLNT04], in modelling spread of malignant software in computer networks [OOVM09, vMOK09], in wireless sensor networks [ASSC02] and in the analysis of social networks [CSW05, POM09, WPF+09].

In mathematical epidemiology, it is common to divide the population into different *compartments*. These compartments are given labels such as M, S, E, I, and R. Compartment M represents those infants who have temporary passive immunity which is acquired through antibodies transferred to them by their mothers. Compartment S represents the *susceptible* individuals, those healthy individuals who are not immune to the disease. Compartment I includes *infective* individuals, those who are already infected and can transfer the disease to a susceptible. Based on the nature of the disease, sometimes when there is an adequate contact of a susceptible with an infective and the transmission occurs, the susceptible may enter the *exposed* compartment E of those in the latent period, who are infected but not yet infectious. Compartment R involved those who have *recovered* and have immunity to the disease. Acronyms for epidemiological models are based on the flow of the transmission of the disease. For example, in SIR model, the individuals in the population are considered to be initially susceptible. When a disease is introduced to the population and there is adequate contact between a susceptible and an infective individual, the susceptible becomes infective (without entering any latent period) and when the infection period ends, the individual recovers from the disease with immunity. Another popular model is the SIS model. In the SIS model, all individual are considered to be either susceptible or infective. When susceptibles are in sufficient contact with infectives, they becomes infectives themselves. When infectives are restored to health, they re-join the susceptible compartment.

In this chapter, among the different models proposed in mathematical epidemiology, we exclusively deal with SIS models, which are arguably among the simplest epidemiological models. SIS models have been used to model diseases that do not confer immunity on the survivors. Tuberculosis and gonorrhoea are two much studied diseases which are mathematically described using SIS models [CY73, HY84, New03]. Computer viruses also fall into this category; they can be "cured" by antivirus software, but without a permanent virus-checking program, the computer is not protected against the subsequent attacks by the same virus.

A simple mathematical representation of an SIS model, with contacts obeying the mass action law, is as follows:

$$\begin{aligned}\dot{S}(t) &= -(\beta/N)S(t)I(t) + \gamma I(t) \\ \dot{I}(t) &= (\beta/N)S(t)I(t) - \gamma I(t) \\ S(0) &= s_0 > 0,\ I(0) = i_0 > 0\end{aligned} \tag{6.1}$$

where $S(t)$ and $I(t)$ represent the population of compartments S and I, respectively. $N = S(t) + I(t)$ is the total number of individuals in the population, which is considered to be constant. β is called the contact rate and is the average number of adequate contacts, i.e., contacts sufficient for the transmission of the disease. γ is called the transfer rate and is the average number of infectives who are cured at each time. It is been shown that $1/\gamma$ is the average infectious period for each individual [HSvdD81].

It is easy to check that the point corresponding to $I(t) = 0$ is an equilibrium of the system (6.1). Since this point represents the case where there are no infectives in the population, it is usually referred to as the *disease-free equilibrium*, DFE for short. Based on the values of the parameters of the system (6.1), it can have another equilibrium in $\operatorname{int}(\mathbb{R}^n_+)$. This equilibrium is called the *endemic* equilibrium. Like any other dynamical system, these equilibria can be unstable, stable or asymptotically stable. We will see shortly that for the enhanced SIS model we consider in this section, instability of the DFE corresponds to the existence and asymptotic stability of the endemic equilibrium. Also, asymptotic stability of the DFE means that it is the only equilibrium of the SIS model and therefore, is GAS.

Looking at the model (6.1), we see that it can be enhanced in a number of directions. Most notably, in (6.1) it is assumed that the population is completely homogeneous and the probability of contact between every susceptible and infective is the same. In the real world, age, general health condition, lifestyle, profession and whether or not a person is vaccinated against the disease are all important factors affecting the probability of the transmission of a disease to an individual and also how fast the individual is cured.

To consider such heterogeneities in the model, in the ideal case, the contact rate β should differ for each two individuals and since contacts between individuals change with time, it should be time-varying. The same applies to the transfer rate γ. Of course implementing such a model, particularly when the number of individuals is high, needs a lot of computational power and is too complicated to analyse. Apart from that, it is usually difficult to gather the parameters of the model. Also, it is not always easy to estimate contact rate for different individuals in real-life situations, although, some efforts have been made in this regard (look at [Yon09] and references therein). Therefore, it is more practical to further divide compartments I and S to smaller sub-compartments (Instead of dividing them to separate individuals) and then assume different contact and transfer rates depending on the sub-compartments.

Some attempts have been made in describing the SIS model for a heterogeneous population, most notably [FIST07, HL06, vdDW02]. Although heterogeneous models are studied extensively, most of the work done in this area does not take into account the dependence of parameters of the system on time.

In this chapter, we address this last issue by describing the SIS model as a switched system. This assumption is more relevant when we consider cases where a health policy is implemented that affects the society almost instantly. For example, when vaccination for a large group of individuals (like pupils or health-care workers) is proposed or even made obligatory and when the schools are shut down for a period of time. The switched SIS model we consider in this chapter, is based on the time-invariant model discussed in [FIST07]. In the next section, we recall the model used in [FIST07] and introduce our switched SIS model.

6.3 Problem Description

As already mentioned, we use an SIS model which is inspired by the model used in [FIST07]. In this model, the population of interest is first divided into two compartments S, susceptibles, and I, infectives, and each compartment is sub-divided into n groups. These groups can represent different age groups, different health condition, etc. Let $I_i(t)$ and $S_i(t)$ be the number of infectives and susceptibles at time t in group i for $i = 1, \cdots, n$, respectively. Also, let $N_i(t) = S_i(t) + I_i(t)$ be the total population of group i. The total population of each group is assumed to be constant; formally, $N_i(t) = N_i$.

β_{ij}, the contact rate between groups i and j, denotes the rate at which susceptibles in group i are infected by infectives in group j for $i, j = 1, \cdots, n$. Further, γ_i, the transfer rate, is the rate at which an infective individual in group i is cured. We also consider birth and death in the population, although to keep the total population constant, we assume for each group the birth and death rates are equal to the value μ_i. Using the mass-action law, the basic SIS model is then described as follows [FIST07]:

$$\begin{cases} \dot{S}_i(t) = \mu_i N_i - \mu_i S_i(t) - \sum_{j=1}^{n} \beta_{i,j} \frac{S_i(t).I_j(t)}{N_i} + \gamma_i I_i(t) \\ \dot{I}_i(t) = \sum_{j=1}^{n} \beta_{i,j} \frac{S_i(t).I_j(t)}{N_i} - (\gamma_i + \mu_i) I_i(t) \end{cases}$$

Since the population of each group is constant, it is sufficient to know $I_i(t)$. If we set $x_i(t) = I_i(t)/N_i$ and $\tilde{\beta}_{i,j} = \beta_{i,j} N_j / N_i$ and $\alpha_i = \gamma_i + \mu_i$, we obtain the following differential equation:

$$\dot{x}_i(t) = (1 - x_i(t)) \sum_{j=1}^{n} \tilde{\beta}_{i,j} x_j(t) - \alpha_i x_i(t), \quad \forall i = 1, \cdots, n \tag{6.2}$$

Based on the definition, $x \in B_n$ where $B_n := \{x \in \mathbb{R}_+^n : x \leq 1\}$. We can write the differential equation (6.2) in compact form as:

$$\dot{x} = [D + B - \text{diag }(x)B]x \tag{6.3}$$

where $D = -\text{diag }(\alpha_i)$ and $B = (\tilde{\beta}_{ij}) > 0$.

The following properties of (6.3) are easy to check.

(i) $f(x) = [D + B - \text{diag }(x)B]x$ with D and B defined as above is C^1 in $\mathbb{R}^n$, therefore, the solution for every initial condition in $\mathbb{R}^n$ exists and is unique for all $t \geq 0$.

(ii) Since D is diagonal and $B > 0$, it can be easily seen that system (6.3) is cooperative.

(iii) The origin is an equilibrium point of (6.3). This equilibrium is referred to as the disease-free equilibrium (DFE) of the system (6.3).

(iv) Since the system (6.3) is cooperative and has an equilibrium at the origin, then based on Lemma 2.6.2, the system (6.3) is positive.

(v) System (6.3) may have an equilibrium in $\text{int}\,(\mathbb{R}^n_+)$ (also referred to as an endemic equilibrium). Conditions for existence of endemic equilibrium for the system (6.3) will be stated shortly.

(v) Linearising the system (6.3) around the origin, we obtain the following linear system:

$$\dot{x}(t) = (D + B)x(t) \tag{6.4}$$

The system (6.2) (or equivalently system (6.3)) has two other important properties which are stated and proved below.

Lemma 6.3.1. *Let $B_n := \{x \in \mathbb{R}^n_+ : x \leq 1\}$. Then B_n is an invariant set for the system described in* (6.2).

Proof: As already mentioned above, the system (6.2) is positive, hence $x_i \geq 0$ for all $i \in \{1, \cdots, n\}$. On the other hand, if for some $i \in \{1, \cdots, n\}$ we have $x_i = 1$, then based on differential equation (6.2):

$$\dot{x}_i(t) = -\alpha_i < 0$$

Therefore, $x_i \leq 1$ for all $i \in \{1, \cdots, n\}$. This concludes the proof. □

Lemma 6.3.2. *The system* (6.3) *is subhomogeneous of degree* 1.

Proof: Let $f(x) = [D + B - \text{diag}\,(x)B]x$. Let $\lambda > 1$, then we have:

$$f(\lambda x) = \lambda[D + B - \lambda\text{diag}\,(x)B]x$$

and

$$\lambda f(x) = \lambda[D + B - \text{diag}\,(x)B]x$$

As $B > 0$, for all $x \geq 0$, we have:

$$[D + B - \lambda \text{diag } (x)B]x \leq [D + B - \text{diag } (x)B]x, \; \forall \; \lambda \geq 1$$

Therefore, we have $f(\lambda x) \leq \lambda f(x)$ for all $\lambda \geq 1$, $x \geq 0$ and this concludes the proof. □

One of the most important parameters in an epidemiological model is the *basic reproduction number*, R_0. There are different definitions for the basic reproduction number. Probably the most common definition is as follows. The basic reproduction number is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual during its entire period of infectiousness [DHM90]. There is also another interpretation of the reproduction number that can be found , for example, in [vdDW02].

For the SIS model (6.3), following the reference [FIST07], it can be proved that $R_0 = \rho(-D^{-1}B)$. The reproduction number can be used to characterise the existence and stability of the equilibria of (6.3). The following result is Theorem 2.3 in [FIST07].

Theorem 6.3.3. *Consider the system* (6.3)*. Assume that the matrix B is irreducible. The disease-free equilibrium of the system* (6.3) *is globally asymptotically stable if and only if $R_0 \leq 1$.*

The next result considers the existence and stability of endemic equilibria and is a restatement of Theorem 2.4 and the discussion in Section 2.2 of [FIST07].

Theorem 6.3.4. *Consider the system* (6.3) *and assume that B is irreducible. There exists a unique endemic equilibrium $\bar{x}$ in* $\text{int}\,(\mathbb{R}^n_+)$ *if and only if $R_0 > 1$. Moreover, in this case, $\bar{x}$ is asymptotically stable with the region of attraction $B_n \setminus \{0\}$.*

6.3.1 Switched SIS Model

Our main aim in this chapter is to extend Theorem 6.3.3 to switched SIS models.

Let $\{D_1, \cdots, D_m\}$ be a given set of diagonal matrices with negative diagonal entries and $\{B_1, \cdots, B_m\}$ be a given set of positive matrices. Then the switched SIS model we consider in this chapter can be described as follows:

$$\dot{x} = (D_{\sigma(t)} + B_{\sigma(t)} - \text{diag }(x)B_{\sigma(t)})x. \tag{6.5}$$

where $\sigma : \mathbb{R}_+ \mapsto \{1, \cdots, m\}$ is a piecewise constant switching signal and m is the number of subsystems. We always assume that $\sigma(\cdot) \in \mathcal{S}$, where $\mathcal{S}$ is the set of switching signals. $D_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix with negative diagonal entries and $B_i \in \mathbb{R}^{n\times n}$ with $B_i > 0$, for all $i = 1, \cdots, m$. Since as mentioned earlier, systems of the form (6.3) are continuous and C^1, then, as mentioned in Section 5.2, the existence and uniqueness of the solution of the switched system (6.5) for any initial condition in $\mathbb{R}^n_+$ and any switching signal σ is guaranteed.

Linearising (6.5) around the origin, we obtain the linear switched system

$$\dot{x} = (D_{\sigma(t)} + B_{\sigma(t)})x. \tag{6.6}$$

In the next section, we see that the linear switched system (6.6), can be very useful in studying stability properties of the nonlinear switched system (6.5).

6.4 Global Asymptotic Stability of the Disease Free Equilibrium

In this section, we present an extension of Theorem 6.3.3 to the switched SIS model (6.3), specifically, we present conditions for global uniform asymptotic stability of the disease free equilibrium (DFE) of the system (6.5). In this section, we assume the switching signal belongs to the set $\mathcal{S}$, where $\mathcal{S}$ includes all the admissible switching signals. In other words, the results presented in this section hold under arbitrary switching. We start with a general stability condition for the DFE of the nonlinear model (6.5) based on the stability properties of the linear switched system (6.5). Then, we present some more specific results.

6.4.1 A General Stability condition

The main result of this section can be stated as follows. Note that unlike Theorem 6.3.3, we do not impose an irreducibility assumption on the matrix B.

Theorem 6.4.1. *Let the DFE be a GUAS equilibrium of the linear switched system* (6.6) *under arbitrary switching. Then it is a GUAS equilibrium of the switched system* (6.5).

Proof: Let $\sigma \in \mathcal{S}$ and $x_0 \in \mathbb{R}^n_+$ be given. Let $\varphi(t, x_0)$ be the solution of the system (6.5) and $\tilde{\psi}(t, x_0)$ be the solution of the system (6.6) with the initial condition $x_0 \in B_n$. Since based on the assumption, the origin is a GUAS equilibrium of the system (6.6), then we know that $\tilde{\psi}(t, x_0) \to 0$ as $t \to \infty$. We will prove that $\varphi(t, x_0) \leq \tilde{\psi}(t, x_0)$ for all $t \geq 0$. From the positivity of (6.5), this will establish the result.

Note that if $\varphi(t, x_0) = 0$ for any $t > 0$, then we are done, therefore, we assume this is not the case.

Consider the system:

$$\dot{x} = (D_{\sigma(t)} + B_{\sigma(t)} + \hat{\epsilon} e e^T)x \text{ with } \hat{\epsilon} > 0 \tag{6.7}$$

in which e is the vector of size n with all elements equal to 1. Let $\hat{\psi}(t, x_0)$ be the solution of the system (6.7) for initial condition $x_0 \in B_n$. When $\hat{\epsilon} \to 0$, then $\hat{\psi}(t, x_0) \to \tilde{\psi}(t, x_0)$. This follows from the fact that both $\hat{\psi}(t, x_0)$ and $\tilde{\psi}(t, x_0)$ are solutions of differential equations with piece-wise continuous right-hand sides [Fil88, Section 7]. Therefore, it is enough to prove $\varphi(t, x_0)$ is always bounded by $\hat{\psi}(t, x_0)$ for all values of $\hat{\epsilon}$.

Let $x_0 \in B_n$ be an arbitrary initial condition. Looking at (6.5) and (6.7), it is evident that

$$\varphi(0, x_0) < \hat{\psi}(0, x_0)$$

We claim

$$\varphi(t, x_0) \leq \hat{\psi}(t, x_0) \text{ for all } t \geq 0$$

Suppose this is not true. Then there would be a t^* defined as below:

$$t^* := \inf\{t \geq 0 : \exists i, \varphi_i(t, x_0) > \hat{\psi}_i(t, x_0)\}$$

Based on the definition, we have:

$$\varphi(t^*, x_0) \leq \hat{\psi}(t^*, x_0) \tag{6.8}$$

and

$$\varphi_i(t^*, x_0) = \hat{\psi}_i(t^*, x_0) \quad \text{for some } i \tag{6.9}$$

Now we consider any i that satisfies (6.9). The system (6.5) is positive and we assumed $\varphi(t, x_0) \neq 0$ for all $t \geq 0$. Also $B > 0$, hence

$$-\text{diag }(\varphi(t, x_0))B_{\sigma(t)} \leq 0 \text{ for all } t \geq 0$$

and we have:

$$\begin{aligned} \dot{\varphi}_i(t^*, x_0) &= [(D_{\sigma(t^*)} + B_{\sigma(t^*)} - \text{diag }(\varphi(t^*, x_0))B_{\sigma(t^*)})\varphi(t^*, x_0)]_i \\ &< [(D_{\sigma(t^*)} + B_{\sigma(t^*)} + \hat{\epsilon}ee^T)\varphi(t^*, x_0)]_i \end{aligned} \tag{6.10}$$

Note that in (6.10) we consider the right-sided derivative to avoid complications that arise when t^* happens to be a switching instance. If we define $\hat{A} = D_{\sigma(t^*)} + B_{\sigma(t^*)} + \hat{\epsilon}ee^T$, we have:

$$\dot{\varphi}_i(t^*, x_0) < (\hat{A}\varphi(t^*, x_0))_i \tag{6.11}$$

And based on (6.8) and the fact that $\hat{A}$ is Metzler, we can conclude:

$$(\hat{A}\varphi(t^*, x_0))_i \leq (\hat{A}\hat{\psi}(t^*, x_0))_i = \dot{\hat{\psi}}_i(t^*, x_0) \tag{6.12}$$

Therefore, based on (6.11) and (6.12), we have:

$$\dot{\varphi}_i(t^*, x_0) < \dot{\hat{\psi}}_i(t^*, x_0) \tag{6.13}$$

and again, we consider right-sided derivatives. Therefore, there exists a small $\delta_i > 0$ such that

$$\varphi_i(t, x_0) < \hat{\psi}_i(t, x_0) \text{ for } t \in [t^*, t^* + \delta_i)$$

This means that there exists a $\delta = \min_i \delta_i > 0$ such that

$$\varphi(t, x_0) \leq \hat{\psi}(t, x_0) \text{ for } t \in [t^*, t^* + \delta)$$

which is a contradiction to the definition of t^*. Therefore, such t^* cannot exist and we have:

$$\varphi(t, x_0) \leq \hat{\psi}(t, x_0) \text{ for all } t \geq 0$$

We also know that $\hat{\psi}(t, x_0) \to 0$ as $t \to \infty$ and the system (6.5) is a positive system, therefore, we can conclude that $\varphi(t, x_0) \to 0$ as $t \to \infty$. This concludes the proof. $\square$

As already mentioned, Theorem 6.4.1 states that any condition that guarantees that DFE is a GUAS equilibrium of the linear switched system (6.6), guarantees that DFE is a GUAS equilibrium of the nonlinear switched SIS model (6.5). Therefore, following the result stated in Section 5.4.1, we can state the following theorem.

Theorem 6.4.2. *If there exists some vector $v \gg 0$ with $(D_i + B_i)v \ll 0$, for $i = 1, \cdots, m$, then the DFE is a globally uniform asymptotically equilibrium of the switched SIS model* (6.6).

Proof: Theorem 5.4.4 states that the if there exists some vector $v \gg 0$ with $(D_i + B_i)v \ll 0$, for $i = 1, \cdots, m$, then the DFE is a globally uniform asymptotically equilibrium of the linear switched system (6.6). Now it follows directly from Theorem (6.4.1) that DFE is a GUAS equilibrium of the nonlinear switched system (6.5). $\square$

6.4.2 Stability Condition via Joint Spectral Radius

In this section, we establish a stability condition for the DFE of the linear switched system (6.6) using a formulation of the joint spectral radius for continuous time systems as described in [Wir02].

Let $\mathcal{M} = \{A_1, \cdots, A_m\} \subset \mathbb{R}^{n\times n}$ be a set of matrices. Then for all $t \geq 0$, we can define a semigroup $\mathcal{S}_t$ as follows. $\mathcal{S}_t$ consists of all time-evolution operators $\phi_A(t, 0)$ corresponding to measurable selection maps $A : [0, \infty) \to \mathcal{M}$ and the associated time-varying differential equation $\dot{x}(t) = A(t)x(t)$. The joint spectral radius is then given by

$$\rho(\mathcal{M}) := \lim_{t\to\infty} \sup\{(\rho_t(\mathcal{M}))^{1/t}\} \tag{6.14}$$

where

$$\rho_t(\mathcal{M}) := \sup\{\rho(M) : M \in \mathcal{S}_t\}.$$

The joint spectral radius can be used to characterise uniform asymptotic stability of linear switched systems (see [Wir02] and references therein). For our purposes the following fact is sufficient.

Lemma 6.4.3. *Let $\mathcal{M} = \{A_1, \ldots, A_m\}$. If $\rho(\mathcal{M}) < 1$, then the origin is a uniformly asymptotically stable equilibrium of the linear switched system*

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$

under arbitrary switching.

The following lemma is a direct application of Lemma 6.4.3 to the system (6.6) [BMW10b].

Lemma 6.4.4. *Let $\mathcal{M} = \{D_1 + B_1, \ldots, D_m + B_m\}$. If $\rho(\mathcal{M}) < 1$, then the origin is a global uniform asymptotically stable equilibrium of* (6.6) *under arbitrary switching..*

The following Theorem, states a condition for global uniform asymptotic stability of the DFE for the nonlinear model (6.5) using joint spectral radius..

Theorem 6.4.5. *Let $\mathcal{M} = \{D_1 + B_1, \ldots, D_m + B_m\}$. If $\rho(\mathcal{M}) < 1$, then the DFE is a GUAS equilibrium of the switched SIS model* (6.5) *under arbitrary switching.*

Proof: Lemma 6.4.4 state that the condition $\rho(\mathcal{M}) < 1$ is a sufficient condition for global uniform stability of the DFE for the linear switched system (6.6). It now follows from Theorem 6.4.1, that DFE is GUAS equilibrium for the nonlinear switched system (6.5). □

6.4.3 Stability Conditions via Common Lyapunov functions

In Section 5.3, we mentioned that if all the subsystems of a switched system have a unique equilibrium at the origin and share a common Lyapunov function, then the switched system has a globally uniformly asymptotically stable equilibrium at the origin. Also, in Section 2.5 we showed that a quadratic Lyapunov function for a positive LTI system can be of the form $V(x) = x^T Px$ where P is a diagonal matrix with positive diagonal elements. Considering these two well-known results, we can state the following result for stability of the switched system (6.6).

Theorem 6.4.6. *DFE is a GUAS equilibrium of the system* (6.6)*, if all its subsystems have a common quadratic Lyapunov function of the form* $V(x) = x^T Px$ *where* P *is a diagonal matrix with positive diagonal entries.*

Based on Theorem 6.4.1, we can directly conclude that the existence of a common quadratic Lyapunov function as described in the above theorem, guarantees that DFE is GUAS equilibrium of the nonlinear switched model (6.5). In the following theorem, we show that the existence of a common Lyapunov function for the linear switched system (6.6), guarantees the existence of a common Lyapunov function for the nonlinear switched system (6.3) and hence global uniform asymptotic stability of the DFE of the switched SIS model (6.5).

Theorem 6.4.7. *The nonlinear SIS model* (6.5) *has a common quadratic Lyapunov function, if the system* (6.6) *has common quadratic Lyapunov function of the form* $V(x) = x^T Px$ *where* P *is a diagonal matrix with positive diagonal entries.*

Proof: We show that the common quadratic Lyapunov function of the system (6.6), $V(x) = x^T Px$, is also a common quadratic Lyapunov function for system (6.5). For every $f_i(x) = (D_i + B_i - \text{diag }(x)B_i)x$, with $i = 1, \cdots, m$, we have:

$$\begin{aligned}
\frac{\partial V}{\partial x} f_i(x) &= \dot{x}^T Px + x^T P\dot{x} \\
&= x^T(A_i^T - B_i^T \text{diag }(x))Px + x^T P(A_i x - \text{diag }(x)B_i)x \\
&= x^T A_i^T Px + x^T PA_i x - (x^T B_i^T \text{diag }(x)Px + x^T P\text{diag }(x)B_i x)
\end{aligned}$$

where $A_i = B_i + D_i$ for $i = 1, \cdots, m$.

We know that x, P and B_i are all element-wise nonnegative, therefore

$$-(x^T B_i^T \text{diag }(x)Px + x^T P\text{diag }(x)B_i x) < 0$$

On the other hand, since $V(x)$ is a common Lyapunov function of the system (6.6), we have:

$$x^T A_i^T Px + x^T PA_i x < 0$$

Therefore, we can conclude:

$$\frac{\partial V}{\partial x} f_i(x) < 0$$

In other words, $V(x)$ is a common quadratic Lyapunov function for (6.5). Hence, DFE is a GUAS equilibrium of the system (6.5) and this concludes the proof. $\square$

As mentioned in Section 5.3, a special class of Lyapunov functions which is particularly useful for linear positive switched systems is the class of linear copositive Lyapunov functions. The next states a similar result to Theorem 6.4.7, but for linear copositive Lyapunov functions.

Theorem 6.4.8. *Let the system* (6.6) *have a common linear copositive Lyapunov function of the form* $V(x) = v^T x$ *where* $v \in \mathbb{R}^n$ *with* $v \gg 0$. *Then the nonlinear SIS model* (6.5) *also has a common linear copositive Lyapunov function.*

Proof: The proof is similar to the proof of the previous theorem. We show that the common linear copositive Lyapunov function of the system (6.6), $V(x) = v^T x$, is also a common linear copositive Lyapunov function for system (6.5). For every $f_i(x) = (D_i + B_i - \text{diag }(x)B_i)x$, with $i = 1, \cdots, m$, we have:

$$\begin{aligned} \frac{\partial V}{\partial x} f_i(x) &= v^T \dot{x} \\ &= v^T (D_i + B_i - \text{diag }(x)B_i)x \\ &= v^T (D_i + B_i)x - v^T(\text{diag }(x)B_i)x \end{aligned}$$

We know that v^T, x, B_i are all element-wise nonnegative, therefore

$$-v^T(\text{diag }(x)B_i)x < 0$$

On the other hand, since $V(x)$ is a common Lyapunov function of the system (6.6), we have:

$$v^T(D_i + B_i)x < 0$$

Therefore, we can conclude:

$$\frac{\partial V}{\partial x} f_i(x) < 0$$

And this means $V(x)$ is a common Lyapunov function for (6.5), therefore, DFE is a GUAS equilibrium of the system (6.5). □

Remark 6.4.1. Note that Theorems 6.4.7 and 6.4.8 hold even for the case where the switched signal does not have a dwell time and the switching happens arbitrarily fast. Of course infinitely fast switching may not be meaningful but it provides us with a useful tool to expand the stability results to SIS models with parameters which are continuous functions of time.

6.5 Stabilisation of the Disease-Free Equilibrium

In the previous section we stated conditions for stability of a switched epidemiological system under arbitrary switching. In this section, we study the case where DFE is not a GUAS equilibrium of the switched system (6.5) under arbitrary switching and try to find a switching signal, that stabilises the system. The main result of this section is inspired by the results in [BS04] and [Lib03, Section 3.4]. In [BS04], a condition for stabilisation of an affine switched systems is presented.

We begin with recalling the definition of quadratically stable equilibrium points.

Definition 6.5.1. Let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let $V(x) = x^T Px > 0$ and $\dot{V}(x) \leq -\epsilon x^T x$ along the trajectory of the switched system (6.5) corresponding to some switching signal $\sigma \in \mathcal{S}$ and initial condition $x_0 \in B_n$. Then the switched system (6.5) is said to be *quadratically stabilizable.*

Remark 6.5.1. As already mentioned in Section 2.5, a necessary and sufficient condition for the global asymptotic stability of the origin for the LTI system $\dot{x} = Ax$ is the existence of a quadratic Lyapunov function of the form $V(x) = x^T Px$, where P is a symmetric positive definite matrix. We also mentioned that in the case where A is Metzler, the matrix P can be diagonal with positive diagonal entries.

To prove our main result in this section, we need the following theorem which is a restatement of Theorem 3.4 and the following discussions in [Lib03, p. 67].

Theorem 6.5.1. *Consider the linear switched system* (6.6) *and let* $A_i = B_i + D_i$ *for* $i = 1, \cdots, m$. *If there exist* $\alpha_i \in (0, 1)$ *such that*

$$\sum_{i=1}^{m} \alpha_i = 1$$

and

$$A_{eq} = \sum_{i=1}^{m} \alpha_i A_i$$

is Hurwitz, then switched system (6.6) *is quadratically stabilizable.*

Note that in the above theorem, A_i, for $i = 1, \cdots, m$, does not need to be Hurwitz. If, say, A_k is Hurwitz for some $k \in \{1, \cdots, m\}$, then we can select the switching signal to be $\sigma(t) \equiv k$ and the switched system will be quadratically stabilizable. Therefore, we exclude these trivial cases and assume A_i is not Hurwitz for $i = 1, \cdots, m$.

Definition 6.5.2. The system

$$\dot{x} = (A_{eq}x - \text{diag }(x)B_{eq}).x \tag{6.15}$$

is called an Average system for system (6.5) if

$$A_{eq} = D_{eq} + B_{eq}$$

where

$$\begin{aligned} B_{eq} &= \sum_{i=1}^{m} \alpha_i B_i \\ D_{eq} &= \sum_{i=1}^{m} \alpha_i D_i \\ \sum_{i=1}^{m} \alpha_i &= 1, \ \alpha_i \in (0, 1) \end{aligned}$$

Now we are ready to state the main result of this section.

Theorem 6.5.2. *If there exist* $\alpha_i \in (0, 1)$, *for* $i = 1, \cdots, m$ *such that* A_{eq} *is Hurwitz, then system* (6.5) *is quadratically stabilizable.*

Proof: Since A_{eq} is Metzler and Hurwitz, based on Theorem 2.5.8, there exists a diagonal matrix P with diagonal positive entries such that the matrix

$A_{eq}^T P + PA_{eq}$ is negative-definite. Hence, we select our quadratic Lyapunov function to be $V(x) = x^T Px$ where P is a diagonal matrix with positive diagonal entries. We evaluate $\dot{V}(x)$ along trajectories of the average system (6.15):

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T(A_{eq}^T - B_{eq}^T \text{diag }(x))Px + x^T P(A_{eq}x - \text{diag }(x)B_{eq})x \\ &= x^T A_{eq}^T Px + x^T PA_{eq}x - (x^T B_{eq}^T \text{diag }(x)Px + x^T P\text{diag }(x)B_{eq}x)\end{aligned}$$

We know that x, P and B_{eq} are all element-wise nonnegative, therefore

$$-(x^T B_{eq}^T \text{diag }(x)Px + x^T P\text{diag }(x)B_{eq}x) < 0$$

On the other hand, based on Theorem 6.5.1, we know

$$x^T A_{eq}^T Px + x^T PA_{eq}x \leq -\epsilon x^T x$$

Therefore, we can conclude

$$x^T A_{eq}^T Px + x^T PA_{eq}x - (x^T B_{eq}^T \text{diag }(x)Px + x^T P\text{diag }(x)B_{eq}x) < -\epsilon x^T x$$

Now, substituting the expressions for A_{eq} and B_{eq} in the above inequality, we have:

$$\sum_{i=1}^{m} \alpha_i(x^T(A_iP + PA_i)x - x^T(B_i^T \text{diag }(x)P + P\text{diag }(x)B_i)x) < -\epsilon x^T x$$

or equivalently

$$\sum_{i=1}^{m} \alpha_i(x^T(A_iP + PA_i)x - x^T(B_i^T \text{diag }(x)P + P\text{diag }(x)B_i)x + \epsilon x^T x) < 0$$

which means that for all $x \in \mathbb{R}_+^n$, the following inequality should be satisfied for some $i = 1, \cdots, m$:

$$x^T(A_iP + PA_i)x - x^T(B_i^T \text{diag }(x)P + P\text{diag }(x)B_I)x + \epsilon x^T x < 0$$

Now we define

$$\Omega_i := \{x : x^T(A_i^T P + PA_i)x < -\epsilon x^T x\} \tag{6.16}$$

for $i = 1, \cdots, m$. Note that each Ω_i is an open region and their union covers $\mathbb{R}_+^n \setminus \{0\}$.

We prove any switching signal with $\sigma(t) = i$ while the trajectory of the system (6.5) is in region Ω_i, for $i = 1, \cdots, m$, assures quadratic stability of the system (6.5).

Within the region Ω_i we have:

$$\dot{V}(x) = \dot{V}_i(x) = x^T(A_i^T P + PA_i)x - x^T(B_i^T \text{diag }(x)P + P\text{diag }(x)B_i)x$$

Since

$$x^T(A_i^T P + PA_i)x < -\epsilon x^T x$$

and

$$-x^T(B_i^T \text{diag }(x)P + P\text{diag }(x)B)x < 0$$

therefore, we have:

$$\dot{V}(x) < -\epsilon x^T x$$

Discontinuity of $\dot{V}(x)$ which is caused by the discontinuity of the switched system in the switching instance can be overcome by introducing Filippov Solutions [Fil88]. As a reminder, a continuous function $x(\cdot)$ is a solution of the switched system (6.5) in the sense of Filippov, if it is the solution of the switched system (6.5) for all $t > 0$ except the switching instances and is the solution of some average system (6.15) at the switching instances. Therefore, at switching instances, we have:

$$\dot{V}(x) = \sup_{\gamma_i}\{\sum_{i:x\in\Omega_i} \gamma_i \dot{V}_i(x)\} \leq \max_{i:x\in\Omega_i}\{\dot{V}_i(x)\} < -\epsilon x^T x$$

where $\gamma_i \in [0, 1]$ and $\sum_i \gamma_i = 1$. This means that there exists a diagonal P with positive diagonal entries and $\epsilon > 0$ that satisfy conditions stated in Definition 6.5.1. Therefore, the system (6.5) is quadratically stabilizable with the above mentioned choices of switching signals. □

Remark 6.5.2. While implementing the above mentioned switching policy, we should pay attention to two issues. Firstly, we would like to have a positive lower bound on the rate of decrease of V to make sure that $\dot{V}(x) \neq 0$ for all $x \neq 0$. This can be achieved by modifying the regions Ω_i for $i = 1, \cdots, m$. Secondly, we want to avoid chattering (infinitely fast switching on the boundaries of the regions). This can be achieved using hysteresis. For more details on both issues, you can look at [Lib03, p. 66].

6.6 Concluding Remarks

In this chapter, we studied a special class of epidemiological models, namely the SIS model. Based on existing results on SIS models with time-independent parameters we stated some results for SIS models with time-varying parameters described as a switching systems. We showed that some of those results hold for switched SIS models under arbitrarily fast switching. Hence, they can be used in stability analysis of SIS models with parameters whose values vary continuously in time. We also stated a condition for stabilisation of the disease-free equilibrium (DFE) of a switched SIS model.

There is a common criticism about epidemiological models of the kind we have used in this chapter. This criticism, which mainly comes from epidemiologists and immunologists who are more familiar with complexities of real infections in real populations is that our 'basic models' are too oversimplified. The answer to such criticisms, is best said in the introduction chapter of [AM91]:

"We see these models as having many uses: they provide insight into essential aspects of host-parasite interactions; they serve as a point of departure for adding realistic complications step-by-step, in an understandable way (so that we do not lose our way in a snowstorm of parameters), and, most important of all, they help to suggest what kinds of data need to be sought in order effectively to design and monitor programmes of control."

In fact the results presented in this chapter are a testimony to this justification. Our results are another step in dealing with the complexities of epidemiological models, enhancing the existing results based on simpler models.

The results presented in this chapter can themselves be extended in different directions. Firstly, we did not study the stability properties of the endemic equilibria. Note that unlike the DFE, each subsystem of the switched SIS model (6.5) may have different endemic equilibria. Therefore, the switched SIS model may not have a unique endemic equilibrium. In such cases, we may be interested in whether or not there exists a region that attracts the trajectory of the switched SIS system.

SIR models are another important and useful class of epidemiological models. Extending the results stated in chapter to switched SIR models would be also very useful.

CHAPTER 7

Conclusions

In this chapter, we review and summarise the results presented in this manuscript and give recommendations for possible extensions to those results.

7.1 Summary

The main theme of this manuscript was the stability analysis of different classes of positive systems subject to uncertainty. We considered three types of positive systems: nonlinear, time-delay and switched positive systems.

We started with a brief introduction to positive systems in Chapter 1 with some examples of their applications in different areas. We highlighted that relatively little is known about nonlinear positive systems subject to uncertainty.

In Chapter 2, we introduced the basic concepts and theorems which were needed in proving the results presented in the following chapters. We first reviewed some relevant concepts in matrix analysis and presented the basic definitions and theorems on dynamical systems with emphasis on their stability properties. We formally defined positive systems and presented some well-known properties of linear and nonlinear positive systems. Cooperative systems and monotonicity were also among the main topics that we discussed in that chapter. We also presented some results that link monotonicity and positivity of nonlinear positive systems. Properties of homogeneous and subhomogeneous systems were reviewed in that chapter too. Also, we reviewed the KKM lemma, which is a well-known result in fixed-point theory. We used the KKM lemma to prove some of the results in Chapters 3 and 4.

In Chapter 3, which is the first of four chapters in which we described our new results, we stated D-stability properties for different classes of monotone positive nonlinear systems. The well-known concept of D-stability for linear positive systems was extended to nonlinear positive systems and conditions for D-stability of different classes of positive nonlinear systems were presented. In particular, we presented D-stability conditions for homogeneous cooperative and subhomogeneous cooperative systems. We also presented a similar result for general cooperative systems without imposing the assumptions of homogeneity and subhomogeneity. We showed that the D-stability condition for general cooperative systems only works locally. In an attempt to obtain non-local results without adding the homogeneity or subhomogeneity assumptions, we presented an alternative result for a different class of cooperative systems which was proved only for planar case. We also dealt with subhomogeneous cooperative systems forced by a constant positive input. Built on the existing results on linear and homogeneous cooperative systems we stated a condition for positivity and asymptotic stability of such systems.

In Chapter 4, we considered positive time-delay systems with fixed but unknown values of delay. We presented conditions for stability of classes of positive time-delay systems for all positive values of delay. Similar to Chapter 3, we stated the results for homogeneous cooperative, subhomogeneous cooperative and general cooperative systems. Again, the results on general cooperative time-delay systems were shown to be true only locally.

In Chapter 5, we extended the concept of D-stability in another direction. We defined a notion of D-stability for switched systems and presented conditions for D-stability for different classes of positive switched systems. We first presented D-stability results for positive linear switched systems and then extended those results to positive nonlinear switched systems whose subsystems are cooperative and homogeneous of degree 0. We also stated D-stability conditions for positive linear and nonlinear switched systems with commuting vector fields.

And finally, in Chapter 6, as an example of the wide range of applications of positive systems, we studied stability properties of a class of epidemiological models with time-varying parameters. We reviewed the compartmental epidemiological models and focused on the SIS model. This model is useful in studying diseases that do not confer immunity. We highlighted that most of the results on different epidemiological models have been focused on models with time-invariant parameters. Clearly, such an assumption is simplistic and there is a need to extend those results to models with time-varying parameters. To address this issue, we considered an epidemiological system modelled as a positive switched system. We presented different conditions for global uniform asymptotic stability of the disease-free equilibrium of such systems. Some of those results were proved to hold for the cases where the switching happens infinitely fast.

7.2 Future Works

In the concluding sections of each of the previous four chapters, we discussed possible directions for extending the main results presented in each chapter. In this section, instead of dealing with the details of those possible extensions, we discuss some general ideas for the extension of the presented results.

As already mentioned in Section 3.9, the definition of D-stability for nonlinear systems may be extended to include more general classes of structured uncertainties. But apart from that, there are many other methodologies to deal with uncertain nonlinear systems which are studied under the broad topic of robust stability and robust control. There are some attempts made in applying robust control methods to positive linear systems, some of which were reported in Section 3.3. Some of the existing robust stability methods, may result in stronger stability conditions when considered in the context of positive systems.

In almost all the main results presented in this manuscript, we have used monotonicity methods in the proofs. Needless to say, there are positive systems which are not monotone. One obvious extension of our work is to investigate whether similar results can be obtained for positive systems not necessarily monotone. Although due to the fact that monotonicity is a powerful property, proving the same results without that assumption may not be possible.

In all of the results presented in Chapter 4, we dealt with time-delay systems with fixed but unknown values of delay. Another type of uncertainty in time-delay systems, which was absent from our discussions, are time-delay systems with unknown time-varying delays. Obtaining delay-independent stability results for such delayed systems can be of significant importance. It is actually one of the open questions in the study of time-delay systems in general.

Since we exclusively dealt with stability analysis in this manuscript, we did not consider systems with inputs and outputs (with the exception of the results presented in Section 3.8). There are some properties of dynamical systems that cannot be studied if we ignore inputs and outputs. Properties like controllability and reachability, and some properties specific to positive systems like excitability. There have been some efforts in studying such properties in the context of positive systems. Similar to the question of stability, there has been very little done in studying the effects of uncertainty in those properties.

There are different approaches in dealing with epidemiological models. Probably the most common, is the graph theoretical approach, in which each individual is considered as a node in a graph and the contact and transfer rates are the weights of the edges. This approach can be incorporated in the dynamical system approach we have adopted in Chapter 6. Matrices D and B as defined in that chapter contain contact and transfer rates for different groups of individuals. Ideally, the dimension of the considered model should be equal to the number of individuals in the population, but that is not practical for two reasons. The first is that for large populations, such a model would need a huge amount of computational power, which is not always available. The second reason is that estimating contact rate for each two individual is a very difficult task. The results presented in Chapter 6, are novel conditions for stability analysis of the disease-free equilibrium of the considered model but they have another significant property. They link stability properties of a nonlinear switched system to that of a linear switched system.

One of the issues that was completely absent from our discussion in the Chapter 6 is the stability analysis of the endemic equilibrium. Although there are some results on the properties of the endemic equilibrium for the nonlinear time-invariant models, extending these results to switched nonlinear models may not be as straightforward as extending the results on the disease-free equilibrium. Also, the same set of results may be valid for other epidemiological models, specifically for the SIR model, which is another important and common class of epidemiological models.

As already mentioned, the study of nonlinear, switched and delayed positive systems subject to any form of uncertainty, is an area with many open problems. The main aim of this manuscript was to address some of those problems, or better to say, to provide a first step in laying the foundations of the stability analysis of uncertain systems in the context of positivity. Undoubtedly, there is much left to be done, but hopefully, this manuscript will motivate other researchers to work on this fascinating subject. As is best said by Albert Einstein, “The process of scientific discovery is, in effect, a continual flight from wonder”.

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