

Essentially Negative News About Positive Systems

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Abstract

In this paper the discretisation of switched and non-switched linear positive systems using Padé approximations is considered. Padé approximations to the matrix exponential are sometimes used by control engineers for discretising continuous time systems and for control system design. We observe that this method of approximation is not suited for the discretisation of positive dynamic systems, for two key reasons. First, certain types of Lyapunov stability are not, in general, preserved. Secondly, and more seriously, positivity need not be preserved, even when stability is. Finally we present an alternative approximation to the matrix exponential which preserves positivity, and linear and quadratic stability.

Keywords: Switching positive systems, discretisation, Padé approximation

1. Introduction

Switched and non-switched linear positive systems have been the subject of much recent attention in the control engineering and mathematics literature [1], [2], [3], [4], [5], [6], [7], [8]. An important problem in the study of such systems concerns how to obtain discrete time approximations to a given continuous time system. This problem arises in many circumstances; for example, when one simulates a given system; when one approximates a continuous time system for the purpose of control system design [1]; in certain optimisation problems [9]; and in model order reduction problems [10]. While a complete understanding of this problem exists for LTI systems [11], and while some results exist for switched linear systems [12], [13], the analogous problems for positive systems are more challenging since discretisation methods must preserve not only the stability properties of the original continuous time system, but also physical properties, such as state positivity. To the best of our knowledge, this is a relatively new problem in the literature, with only a few recent works on this topic [14].

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Our objective in this paper is twofold. First, to study one specific discretisation method, namely, diagonal Padé approximations to the matrix exponential, and second to develop new approximations that are suitable for positive systems. Such a study is well motivated, as diagonal Padé approximations are a method used by control engineers. We deal with two fundamental questions. First, under what conditions are certain types of stability of the original positive switched system inherited by the discrete time approximation? Second, we also ask if and when positivity itself is inherited by the discrete-time system. We establish the following results. Under the assumption of positivity preservation first order diagonal Padé approximations preserve linear and quadratic co-positive Lyapunov functions for all choices of sampling time that preserve positivity. Thus, in a sense, linear and quadratic stability is a robust property of the first order Padé approximation. In other words, even if the approximation is bad, say, due to a poor choice of sampling time, linear co-positive stability is never lost. However in contrast, we also show that higher order Padé approximations do not, in general, preserve positivity. Counter-intuitively, this is true even for an arbitrarily small choice of sampling time h . In other words, there are examples of continuous-time positive systems for which a particular diagonal Padé approximation fails to preserve positivity for all small choices of h . We give one such example. Finally, we give sufficient conditions under which the Padé approximation is positivity preserving, and identify a new approximation method which is guaranteed to preserve both stability and positivity.

The contributions of our results are immediate. While discretisation of switched systems arises in many application domains, the question as to what properties are preserved under such discretisations appears to be a new problem and, despite its importance, has not yet received the attention it deserves. Roughly speaking, our results show that certain types of Lyapunov functions are not always preserved under Padé discretisations of LTI systems. Neither is positivity. These observations are important in many application domains; for example, the Tustin or bilinear transform is an example of a diagonal Padé approximation, while Matlab's *expm* function employs Padé approximations in computing the matrix exponential.

This paper is organised as follows: in Section 2 the notation and preliminary definitions are introduced. In Section 3, the preservation of both quadratic and linear co-positive Lyapunov functions in the discretisation process is considered. In Section 4 we consider the issue of when a diagonal Padé approximation maps a Metzler matrix to a nonnegative matrix. In Section 5 we propose an alternative approximation to the exponential matrix, which is able to avoid the situation arising in some of our examples. Section 6 concludes the paper.

2. Mathematical preliminaries

2.1. Notation

Capital letters denote matrices, small letters denote vectors. For matrices or vectors, (\prime) indicates transpose and $(*)$ the complex conjugate transpose. For matrices X or vectors

x , the notation X or $x > 0$ (≥ 0) indicates that X , or x , has all positive (nonnegative) entries and it will be called a positive (non-negative) matrix or vector. The notation $X \succ 0$ ($X \prec 0$) or $X \succeq 0$ ($X \preceq 0$) indicates that the symmetric matrix X is positive (negative) definite or positive (negative) semi-definite. The sets of real and natural numbers are denoted by \mathbb{R} and \mathbb{N} , respectively, while \mathbb{R}_+ denotes the set of nonnegative real numbers. A square matrix A_c is said to be Hurwitz stable if all its eigenvalues lie in the open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie inside the unit disc. A matrix A is said to be Metzler (or essentially nonnegative) if all its off-diagonal elements are nonnegative. A matrix B is an M-Matrix if $B = -A$, where A is both Metzler and Hurwitz; if an M-matrix is invertible, then its inverse is nonnegative [15]. The matrix I will be the identity matrix of appropriate dimensions.

2.2. Definitions

Generally speaking, we are interested in the evolution of the system

$$\dot{x}(t) = A_c(t)x(t), \quad A_c(t) \in \{A_{c1}, \dots, A_{cm}\}, \quad x(0) = x_0; \quad (1)$$

where $A_c(t) \in \mathbb{R}^{n \times n}$ is a matrix valued function, $x(t) \in \mathbb{R}^{n \times 1}$, $m \geq 1$, and where the A_{ci} are Hurwitz stable Metzler matrices. Such a system is said to be a continuous-time positive system. Positive systems [1], [16] have the special property that any nonnegative input and nonnegative initial state generate a nonnegative state trajectory and output for all times. We are interested in obtaining from this system, a discrete-time representation of the dynamics (slightly abusing notation):

$$x(k+1) = A(k)x(k), \quad A(k) \in \{A_{d1}, \dots, A_{dm}\}, \quad x(0) = x_0. \quad (2)$$

Positivity in discrete time is ensured if each A_{di} is a nonnegative matrix. One standard method to obtain A_{di} from A_{ci} is via the Padé approximation to the exponential function $e^{A_{ci}h}$, where h is the sampling time. Notice that, since (1) is a system switching according to an arbitrarily switching signal $\sigma(t) \in \{1, 2, \dots, m\}$, it is not true, even in the ideal case $A_{di} = e^{A_{ci}h}$, that $x_c(kh) = x(k)$. This property is of course recovered when $t_k = kh$, where t_k is the generic switching instant of $\sigma(t)$.

Definition 1 [17] *The $[L/M]$ order Padé approximation to the exponential function e^s is the rational function C_{LM} defined by*

$$C_{LM}(s) = Q_L(s)Q_M^{-1}(-s)$$

where

$$Q_L(s) = \sum_{k=0}^L l_k s^k, \quad Q_M(s) = \sum_{k=0}^M m_k s^k, \quad (3)$$

$$l_k = \frac{L!(L+M-k)!}{(L+M)!k!(L-k)!}, \quad m_k = \frac{M!(L+M-k)!}{(L+M)!k!(M-k)!}. \quad (4)$$

Thus, the $p - th$ diagonal Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h , is given by taking $L = M = p$; namely,

$$C_p(A_c h) = Q_p(A_c h)Q_p^{-1}(-A_c h), \quad (5)$$

where $Q_p(A_c h) = \sum_{k=0}^p c_k(A_c h)^k$ and $c_k = \frac{p!(2p-k)!}{(2p)!k!(p-k)!}$.

Much is known about the Padé maps in the context of LTI systems. In particular, it is known that diagonal Padé approximations are A-stable [18]; namely, they map the open left-half of the complex plane to the interior of the unit disc, preserving in this way the stability from the continuous-time to the discrete-time system. In this paper we shall, in part, be interested in preservation of Lyapunov functions of a certain type and we remind here the reader of two important classes of Lyapunov functions that are useful in studying positive systems.

Given a continuous time dynamic system $\Sigma_c : \dot{x} = A_c x$, $x \in \mathbb{R}^{n \times 1}$, $A_c \in \mathbb{R}^{n \times n}$, we say that the function $V(x) = x' P x$ is a quadratic Lyapunov function for Σ_c if the matrix $A_c' P + P A_c$ is negative definite for some positive definite matrix $P \in \mathbb{R}^{n \times n}$; namely if $\dot{V}(x) < 0$ for all $t \geq 0$ along all trajectories of the system.

When the system Σ_c is a positive system, then other types of Lyapunov functions are of interest. In this context, the function $V(x) = x' P x$, $P \in \mathbb{R}^{n \times n}$ positive definite, is said to be a co-positive quadratic Lyapunov function for Σ_c if $x'(A_c' P + P A_c)x < 0$ for all x in the nonnegative orthant such that $x \neq 0$; namely if $\dot{V}(x) < 0$ for all $t \geq 0$ for all trajectories starting in the non-negative orthant.

In the specific case of positive systems we may also speak of linear Lyapunov functions. Specifically, if we can find a strictly positive vector w such that $w' A_c < 0$, then the function $V(x) = w' x$ is said to be a copositive linear Lyapunov function for the system Σ_c . As before this implies that $\dot{V}(x) < 0$ for all $t \geq 0$ for all trajectories starting in the non-negative orthant.

Given a discrete time system $\Sigma_d : x(k+1) = A_d x(k)$, $x(k) \in \mathbb{R}^{n \times 1}$, $A_d \in \mathbb{R}^{n \times n}$, the definition of Lyapunov functions follow in the normal way. Namely, the function $V(x) = x' P x$, $P \in \mathbb{R}^{n \times n}$ positive definite, is said to be a co-positive quadratic Lyapunov function for Σ_d if $x'(A_d' P A_d - P)x < 0$ for all x in the nonnegative orthant such that $x \neq 0$. Similarly, if we can find a strictly positive vector w such that $w' A_d < w$, then the function $V(x) = w' x$ is said to be a copositive linear Lyapunov function for the system Σ_d .

The concept of a *common* copositive (linear/quadratic) Lyapunov function follows for finite sets of positive systems $\Sigma_{c1}, \dots, \Sigma_{cm}$ (or $\{\Sigma_{d1}, \dots, \Sigma_{dm}\}$). Such functions are useful in proving the exponential stability of various types of positive dynamic systems.

3. Lyapunov Stability

3.1. Preservation of co-positive Lyapunov functions

As noted above, an important property of Padé maps is that they map the open left half of the complex plane to the interior of the unit circle. Recently, it was shown in [13] that quadratic Lyapunov functions are preserved for sets of matrices that arise in the study of systems of the form of Equation (1). We now ask whether co-positive Lyapunov functions are preserved when discretising an LTI positive system using Padé like approximations. In the remainder of this section our attention focuses on co-positive Lyapunov functions; in particular, on linear and quadratic co-positive Lyapunov functions. Since trajectories of positive systems are constrained to lie in the positive orthant, the stability of such a system is completely captured by Lyapunov functions whose derivative is decreasing for all such positive trajectories. Such functions are referred to as co-positive Lyapunov functions, and it is known that one can always associate a linear, or a quadratic co-positive Lyapunov function, with any given stable linear time-invariant positive system [1]. With this background in mind we observe the following elementary result.

Lemma 1 *Let A_c be a Metzler and Hurwitz stable matrix and let α be a positive real number. Fix any sampling time $h > 0$, and define $A_d(h) = (\alpha(h)I + A_c) (\alpha(h)I - A_c)^{-1}$ such that $A_d(h)$ is a nonnegative matrix, where $\alpha(h) = \frac{\alpha}{h}$. Then the following statements are true.*

1. *If $v(x) = x'Px$, with $P = P' \succ 0$, is a co-positive quadratic Lyapunov function for A_c , that is*

$$x'(A_c'P + PA_c)x < 0, \quad \forall x \geq 0, \quad x \neq 0; \quad (6)$$

then $v(x)$ is a co-positive quadratic Lyapunov function for $A_d(h)$; that is

$$x'(A_d(h)'PA_d(h) - P)x < 0, \quad \forall x \geq 0, \quad x \neq 0. \quad (7)$$

2. *If $v(x) = w'x$, $w > 0$ is a linear co-positive Lyapunov function for A_c ; that is $w'A_c < 0$, then $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$; namely, $w'A_d(h) < w'$.*

Proof.

1. We begin by noting that $(\alpha(h)I - A_c)$ is an invertible M-matrix, so its inverse is nonnegative. Let $x > 0$. From (6) we find that (7) can be written as

$$\begin{aligned} x'(A_d(h)'PA_d(h) - P)x &= x'((\alpha(h)I - A_c)')^{-1}[(\alpha(h)I + A_c)'P(\alpha(h)I + A_c) \\ &\quad - (\alpha(h)I - A_c)'P(\alpha(h)I - A_c)](\alpha(h)I - A_c)^{-1}x = \\ &= x'((\alpha(h)I - A_c)')^{-1}[2\alpha(h)(A_c'P + PA_c)](\alpha(h)I - A_c)^{-1}x < 0, \end{aligned}$$

the inequality following from the fact that $(\alpha(h)I - A_c)^{-1}x > 0$.

2. Following the same rationale of the previous point we can write:

$$\begin{aligned} w' A_d(h) - w' &= [w'(\alpha(h)I + A_c) - w'(\alpha(h)I - A_c)](\alpha(h)I - A_c)^{-1} = \\ &= 2w' A_c (\alpha(h)I - A_c)^{-1} < 0. \end{aligned}$$

■

Lemma 1 is an elementary consequence of the properties of M-matrices. While we have not yet said anything about positivity preservation, these properties do have a particular meaning in the context of stability preservation. In this context Lemma 1 is a very useful result, as it says that the first order diagonal Padé approximation is a robust approximation to the original system. That is, for every $h > 0$ that preserves positivity, linear and quadratic stability is preserved. This result seems like good news since it says that the most basic Padé approximation to the matrix exponential, preserves stability, and consequently one might hope, as is the case for general matrices, that better approximations (i.e. higher orders of diagonal Padé approximations) will also preserve co-positive linear and quadratic stability. Unfortunately, rather surprisingly, this is not true, as the following example demonstrates.

Example 1 *Let*

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and let $A = -2I + N$. Now consider the vector $w' = [1 \ \frac{1}{2} \ \frac{1}{4} \ \frac{1}{8} \ \dots \ \frac{1}{2^9}]$, and note that $w'A = -2e_1$. (Here, for each $j = 1, \dots, 10$, e_j denotes the j -th standard unit basis vector in \mathbb{R}^{10} .) Now consider the second order diagonal Padé approximation (with $h = 1$) given by $C_2(A) = (I + \frac{1}{2}A + \frac{1}{12}A^2)(I - \frac{1}{2}A + \frac{1}{12}A^2)^{-1}$. It can be shown then that $C_2(A) = \sum_{k=0}^9 \alpha_k N^k$, where

$$\begin{aligned} \alpha_0 &= \frac{1}{7}, \alpha_1 = \frac{6}{7^2}, \alpha_2 = \frac{51}{2 \cdot 7^3}, \alpha_3 = \frac{213}{2^2 \cdot 7^4}, \alpha_4 = \frac{708}{2^3 \cdot 7^5}, \alpha_5 = \frac{2049}{2^4 \cdot 7^6}, \\ \alpha_6 &= \frac{5289}{2^5 \cdot 7^7}, \alpha_7 = \frac{12102}{2^6 \cdot 7^8}, \alpha_8 = \frac{23487}{2^7 \cdot 7^9}, \alpha_9 = \frac{32721}{2^8 \cdot 7^{10}}. \end{aligned}$$

In particular we find that $C_2(A)$ is a nonnegative matrix. Next, note that

$$w' C_2(A) e_{10} = \sum_{j=0}^9 \frac{\alpha_{9-j}}{2^j} = \frac{1}{2^9} \left(\frac{1}{7} + \sum_{k=1}^9 2^k \alpha_k \right).$$

Using the formulas above, it now follows that $w' C_2(A) e_{10} = \frac{1}{2^9} \frac{282497599}{282475249} > \frac{1}{2^9} = w(10)$.

Finally, observe that for all sufficiently small $\epsilon > 0$, the vector

$$w'_\epsilon = w' + \epsilon [1 \quad \dots \quad 1]$$

has the properties that: (i) $w'_\epsilon A$ is a negative vector, and (ii) $w'_\epsilon C_2(A) e_{10} > w_\epsilon(10)$. Thus we find that for each sufficiently small $\epsilon > 0$, the function $x \mapsto w'_\epsilon x$ yields an example of a linear co-positive Lyapunov function for A that fails to be a linear co-positive Lyapunov function for the (nonnegative) second order diagonal Padé approximation $C_2(A)$.

Fix an $\epsilon > 0$ so that i) and ii) above hold. Set $\rho = w'_\epsilon w_\epsilon$; we now construct a second matrix G given by $G = -\rho I + w_\epsilon w'_\epsilon$. The matrix G is marginally stable (has eigenvalues in the closed left half of the complex plane), and it is straightforward to show that the only (marginal) co-positive linear Lyapunov functions for G correspond to positive scalar multiples of w_ϵ ; namely $w'_\epsilon G \leq 0$ with $w_\epsilon > 0$.

Recall, our objective here is to show that $C_2(A)$ and $C_2(G)$ fail to have a common co-positive linear Lyapunov function. It can be shown that

$$C_2(G) = \frac{1}{1 + \frac{\rho}{2} + \frac{\rho^2}{12}} \times \left[\left(1 - \frac{\rho}{2} + \frac{\rho^2}{12}\right) I + w_\epsilon w'_\epsilon \right].$$

This in turn yields the fact that the only marginal co-positive linear Lyapunov functions for $C_2(G)$ correspond to positive scalar multiples of w_ϵ . However, as noted above, no positive scalar multiple of w_ϵ can serve as a co-positive linear Lyapunov function for $C_2(A)$. Thus, for the matrices A and G , the discrete time approximations for the exponentials corresponding to $p = 2, h = 1$ fail to have a common co-positive linear Lyapunov function.

Comment 1: Example 1 illustrates something very interesting and potentially dangerous. As one increases the order of approximation to the matrix exponential, one can in fact lose preservation of a given Lyapunov function of the original system, even when positivity is preserved.

Comment 2: Example 1 illustrates another interesting fact. We know from [13] that diagonal Padé approximations preserve quadratic Lyapunov functions. We also know that, given a stable Metzler matrix A_c , diagonal quadratic Lyapunov functions for this matrix ($D = D' \succ 0 : A'_c D + D A_c \prec 0, D$ diagonal) may be constructed from the linear co-positive Lyapunov functions associated with A_c and A'_c . The above example shows

that any such D will be preserved, but the co-positive linear Lyapunov functions may not.

Clearly, the implication of Example 1 is that some Padé approximations may result in the loss of certain co-positive Lyapunov functions. In such situations, the retort of the engineer is that the sampling rate h should be decreased to improve the approximation and to make the approximation more likely to inherit desired properties. Such an approach is reasonable since if we make the approximation order high enough, and h small enough, then we generate an improved approximation to the matrix exponential, and in some circumstances we can deduce something about preservation of Lyapunov functions. We summarise some evident results in this direction with the next lemma.

Lemma 2 *Let A_c be a Metzler and Hurwitz matrix, and suppose that $\hat{\lambda}$ is a complex number with positive real part. For each $h > 0$, let $\lambda(h) = \frac{\hat{\lambda}}{h}$, and consider the following matrices:*

$$\begin{aligned}\Theta_1 &= (\lambda(h)I + A_c)(\lambda^*(h)I + A_c); \\ \Theta_2 &= (\lambda(h)I - A_c)(\lambda^*(h)I - A_c); \\ A_d(h) &= (\lambda(h)I + A_c)(\lambda^*(h)I + A_c)(\lambda^*(h)I - A_c)^{-1}(\lambda(h)I - A_c)^{-1} = \Theta_1\Theta_2^{-1}.\end{aligned}\tag{8}$$

Suppose that there is an $h_0 > 0$ such that for all $0 < h \leq h_0$, Θ_2 is an M -matrix, and such that $A_d(h)$ is a nonnegative matrix. Then, the following statements are true.

1. *If $v(x) = x'Px$, with $P = P' \succ 0$, is a co-positive quadratic Lyapunov function for A_c , i.e.,*

$$x'(A_c'P + PA_c)x < 0, \forall x \geq 0, x \neq 0\tag{9}$$

then there is an $h_1 > 0$ such that for all $0 < h \leq h_1$, $v(x)$ is a quadratic Lyapunov function for $A_d(h)$, i.e.,

$$x'(A_d'(h)PA_d(h) - P)x < 0, \forall x \geq 0, x \neq 0.\tag{10}$$

2. *If $v(x) = w'x$, $w > 0$, is a linear co-positive Lyapunov function for A_c , that is $w'A_c < 0$ then for $0 < h \leq h_0$, $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$; namely, $w'A_d(h) < w'$.*

Proof.

1. Note that (where the dependency on h is understood, i.e. $A_d = A_d(h)$)

$$\begin{aligned}A_d &= \left(\lambda(h)\lambda^*(h)I + (\lambda(h) + \lambda^*(h))A_c + A_c^2\right)\left(\lambda(h)\lambda^*(h)I - (\lambda(h) + \lambda^*(h))A_c + A_c^2\right)^{-1}, \\ &= \Theta_1\Theta_2^{-1},\end{aligned}$$

We wish to prove that there is an $h_1 > 0$ such that for all $0 < h \leq h_1$, $x'(A'_c P + PA_c)x < 0$ implies $x'(A'_d PA_d - P)x < 0 \forall x > 0$. This follows from the fact that

$$\begin{aligned} x'(A'_d PA_d - P)x &= x' \Theta_2^{-1} [\Theta_1' P \Theta_1 - \Theta_2' P \Theta_2] \Theta_2^{-1} x \\ &= x' \Theta_2^{-1} [4|\lambda(h)|^2 \operatorname{Re}(\lambda(h))(A'_c P + PA_c) + \\ &\quad + 4\operatorname{Re}(\lambda(h))A'_c(A'_c P + PA_c)A_c] \Theta_2^{-1} x \quad (11) \\ &\leq 4\operatorname{Re}(\lambda(h)) \|\Theta_2^{-1} x\|^2 \left(\frac{|\hat{\lambda}|^2}{h^2} \gamma_1 + \gamma_2 \right) \end{aligned}$$

where

$$\gamma_1 = \sup_{y>0} \frac{y'(A'_c P + PA_c)y}{y'y}, \quad \gamma_2 = \sup_{y>0} \frac{y'A'_c(A'_c P + PA_c)A_c y}{y'y}$$

Since $\gamma_1 < 0$ by assumption, it is readily seen that $x'(A'_d PA_d - P)x < 0$ for each $x > 0$, provided that $h \in (0, h_1]$ where

$$h_1^2 = \min \left\{ \left| \frac{\hat{\lambda}^2 \gamma_1}{\gamma_2} \right|, h_0^2 \right\}.$$

(Note that here we have used the hypotheses that the real part of $\lambda(h)$ is positive, and that Θ_2 is an M-matrix.)

2. Following the same rationale of the previous point we can write:

$$\begin{aligned} w'A_d - w' &= [w'(\lambda(h)I + A_c)(\lambda^*(h)I + A_c) - w'(\lambda(h)I - A_c)(\lambda^*(h)I - A_c)] \\ &\quad \times (\lambda(h)I - A_c)^{-1} (\lambda^*(h)I - A_c)^{-1} = \\ &= 4\operatorname{Re}(\lambda(h))w'A_c(\lambda(h)I - A_c)^{-1} (\lambda^*(h)I - A_c)^{-1} < 0, \end{aligned}$$

for all $h \in (0, h_0]$ since in this interval Θ_2 is an M-matrix.

■

Comment 3: The hypotheses of Lemma 2 include the condition that Θ_2 is an M-matrix for all sufficiently small $h > 0$. It is natural to wonder when that condition holds, and we discuss that point in Theorem 3 below, as well as in the comment that follows it.

We can now state the following result, which formalises, in a certain sense, the intuition that stability, for a switched linear system, is indeed preserved provided h is chosen to be small enough (fast enough sampling), for diagonal Padé approximations. To state this result, recall the continuous-time switched linear positive system

$$\dot{x}_c(t) = A_c(t)x_c(t), \quad x_c(0) = x_0, \quad (12)$$

where $x_c(t) \in \mathbb{R}_+^n$, $x_0 \in \mathbb{R}_+^n$ is the initial condition, and $A_c(t)$ belongs to the set $\{A_{c1}, \dots, A_{cm}\}$. We then have the following result.

Theorem 1 Consider the system (12). Suppose that A_{ci} is a Metzler and Hurwitz stable matrix for each $i = 1, \dots, m$ and let $C_p(A_{ci}h)$ be the p -th order diagonal Padé approximation of $e^{A_{ci}h}$. Suppose that there is an $h_0 > 0$ such that for all $0 < h \leq h_0$, the following conditions hold:

(i) for each real pole α of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\alpha}{h}I + A_{ci})(\frac{\alpha}{h}I - A_{ci})^{-1}$ is nonnegative;

(ii) for each complex pole λ of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\lambda}{h}I - A_{ci})(\frac{\lambda^*}{h}I - A_{ci})$ is an M-matrix;

(iii) for each complex pole λ of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\lambda}{h}I + A_{ci})(\frac{\lambda^*}{h}I + A_{ci})(\frac{\lambda}{h}I - A_{ci})^{-1}(\frac{\lambda^*}{h}I - A_{ci})^{-1}$ is nonnegative.

Finally, suppose there exists a common linear co-positive Lyapunov function $x \mapsto w'x$ for system (12). Then, there is an h_1 with $0 < h_1 \leq h_0$ such that for all $0 < h \leq h_1$, the system

$$x(k+1) = A(k)x(k), \quad (13)$$

with $A(k) \in \{C_p(A_{c1}h), \dots, C_p(A_{cm}h)\}$, shares the same common linear co-positive Lyapunov function.

Proof. Consider the function $C_p(\cdot)$, denote its real poles (if any) by $\alpha_1, \dots, \alpha_l$, and denote its complex poles with positive imaginary part by $\lambda_1, \dots, \lambda_q$; we note in passing that $l + 2q = p$. Fix an index i between 1 and m .

It now follows that $C_p(A_{ci}h) = \prod_{j=1}^l (\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} \cdot \prod_{j=1}^q (\lambda_j(h) + A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1}$, where, as in Lemmas 1 and 2, $\alpha_j(h) = \frac{\alpha_j}{h}$ and $\lambda_j(h) = \frac{\lambda_j}{h}$.

From the hypothesis, each of the matrices in items (i) and (iii) is nonnegative for all $0 < h \leq h_0$. It is straightforward to show that there is an h_1 with $0 < h_1 \leq h_0$ such that for all $0 < h \leq h_1$, each of the matrices in items (i) and (iii) is nonnegative, and in addition, none of those matrices has a zero column. Suppose henceforth that $0 < h \leq h_1$.

Applying Lemma 1, we find that for each $j = 1, \dots, l$, $w'(\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} < w'$. Since $(\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1}$ is nonnegative and has no zero column for $j = 1, \dots, l$, we find that $w' \prod_{j=1}^l (\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} = w'(\alpha_1(h)I + A_{ci})(\alpha_1(h)I - A_{ci})^{-1} \cdot \prod_{j=2}^l (\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} < w' \prod_{j=2}^l (\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} < \dots < w'$.

Further, from our hypothesis, for each $0 < h \leq h_1$, and each $j = 1, \dots, q$, $(\frac{\lambda_j}{h}I - A_{ci})(\frac{\lambda_j^*}{h}I - A_{ci})$ is an M-matrix. Applying Lemma 2, we now find that $w'(\lambda_j(h) +$

$A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1} < w'$. Further, since $(\lambda_j(h) + A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1}$ is nonnegative and has no zero column for each $j = 1, \dots, q$, we find by iterating as above that for each $0 < h \leq h_1$, $w' \prod_{j=1}^q (\lambda_j(h) + A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1} < w'$.

Consequently we have $w' C_p(A_{ci}h) = w' \prod_{j=1}^l (\alpha_j(h)I + A_{ci})(\alpha_j(h)I - A_{ci})^{-1} \cdot \prod_{j=1}^q (\lambda_j(h) + A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1} < w' \prod_{j=1}^q (\lambda_j(h) + A_{ci})(\lambda_j^*(h) + A_{ci})(\lambda_j(h) - A_{ci})^{-1}(\lambda_j^*(h) - A_{ci})^{-1} < w'$.

Since the argument above applies to each index $i = 1, \dots, m$, we find that for each such i , $w' C_p(A_{ci}h) < w'$. Hence the system (14) shares a common linear co-positive Lyapunov function.

■

An immediate consequence is that if the hypotheses of Theorem 2 hold for the matrices A_{c1}, \dots, A_{cm} , then the origin of (14) is exponentially stable.

Comment 4: Using the special structure of 2×2 M-matrices, it is shown in [19] that for second order switched positive linear systems, the existence of an $h_0 > 0$ that satisfies the hypothesis of Theorem 2 is guaranteed.

An analogous Theorem may be stated for co-positive quadratic stability by imposing conditions on real and complex each factor of the Pade polynomial of the form described in Lemma 1 and Lemma 2 such that for each factor both non-negativity, and preservation of quadratic Lyapunov functions is guaranteed. Both the statement of this Theorem, and its proof closely follow Theorem 1. Consequently, we now state this result without proof.

Theorem 2 *Consider the system (12). Suppose that A_{ci} is a Metzler and Hurwitz stable matrix for each $i = 1, \dots, m$ and let $C_p(A_{ci}h)$ be the p -th order diagonal Padé approximation of $e^{A_{ci}h}$. Suppose there exists a common linear co-positive quadratic Lyapunov function $V(x) = x'Px$ for system (12). Suppose further that there is an $h_0 > 0$ such that for all $0 < h \leq h_0$, the following conditions hold:*

(i) *for each real pole α of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\alpha}{h}I + A_{ci})(\frac{\alpha}{h}I - A_{ci})^{-1}$ is nonnegative;*

(ii) *for each complex pole λ of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\lambda}{h}I - A_{ci})(\frac{\lambda^*}{h}I - A_{ci})$ is an M-matrix;*

(iii) *for each complex pole λ of $C_p(\cdot)$, and each $i = 1, \dots, m$, the matrix $(\frac{\lambda}{h}I + A_{ci})(\frac{\lambda^*}{h}I + A_{ci})(\frac{\lambda}{h}I - A_{ci})^{-1}(\frac{\lambda^*}{h}I - A_{ci})^{-1}$ is nonnegative.*

Then, for some $0 < h < h_0$, the system

$$x(k+1) = A(k)x(k), \tag{14}$$

with $A(k) \in \{C_p(A_{c1}h), \dots, C_p(A_{cm}h)\}$, shares the same common quadratic co-positive Lyapunov function as (12).

Comment on Proof: The key point in the proof follows from the fact that the existence of the interval $0 < h < h_0$ such that items (i), (ii) and (iii) hold, also implies that we find a $h < h_0$ such that each factor individually preserves the quadratic Lyapunov function $V(x)$ (see Lemma 2).

3.2. Implications of loss of Lyapunov functions

We have established in the previous section that certain types of Lyapunov functions may not be preserved when discretising switched linear systems using the diagonal Padé approximations. The implications of this observation are varied. A nice property of the Padé approximation for LTI systems is that stability in continuous time is preserved in discrete time when using this approximation. For switched linear systems, this is no longer the case, and the discrete time approximation may in fact be unstable, even if the original continuous time system is globally uniformly (with respect to switching) exponentially stable. Further details on this topic can be found in the the manuscript [13], where discretisation of general matrices is considered. Second, for positive systems, the properties of solutions are closely related to the existence of certain types of Lyapunov functions. It is shown in [19] that the existence of a common co-positive linear Lyapunov function implies a certain insensitivity to feedback delays, and also implies the stability of a certain class of switched system where the columns of the matrices switch in a certain manner [3]. If such Lyapunov functions are lost as a result of discretisation (as in Example 1), the solution of the discrete time system may not inherit these properties. Finally, we note that despite the widespread use of discrete approximations to continuous systems, and continuous approximations to discrete time systems, the questions as to what properties of the original system are inherited by the approximation, remains open. To illustrate what can happen, we give the following example of an unstable switched system, whose approximation is stable.

Example 2 Consider the Metzler and Hurwitz stable matrices:

$$A_{c1} = \begin{bmatrix} -50.5 & 49.5 \\ 49.5 & -50.5 \end{bmatrix}, A_{c2} = \begin{bmatrix} -87.08 & 8.6 \\ 129.13 & -13.9 \end{bmatrix}. \quad (15)$$

Since the matrix product $A_{c1}A_{c2}^{-1}$ has negative real eigenvalues it follows that there an unstable switching between the aforementioned matrices, and that the system $\dot{x} = A(t)x$, $A(t) \in \{A_{c1}, A_{c2}\}$ is unstable for a fast periodic switching sequence. Now consider A_{d1}, A_{d2} obtained under the 2nd order diagonal Padé approximation of $e^{A_{ci}h}$ with the discrete time step $h = 0.1$:

$$A_{d1} = \begin{bmatrix} 0.604 & 0.301 \\ 0.301 & 0.604 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.381 & 0.052 \\ 0.786 & 0.826 \end{bmatrix}.$$

It is easily verified that these matrices have a common co-positive quadratic Lyapunov function. Hence, it follows that the discretisation $x(k+1) = A_d(k)x(k)$, $A(k) \in \{A_{d1}, A_{d2}\}$ is stable, while the original continuous time system is not.

The objective of the previous example is to illustrate that the qualitative properties of the original system, and the discretisation need not match. Of course, the details of the example are somewhat trivial.

4. Nonnegativity of the diagonal Padé approximation

Our results in the previous section were concerned with the preservation of linear co-positive Lyapunov functions. That is, given a Metzler and Hurwitz matrix A_c and a vector w' that yields a linear co-positive Lyapunov function for A_c , we were concerned with whether w' also yields a linear co-positive Lyapunov function for the diagonal Padé approximation $C_p(hA)$. That makes sense only if $C_p(hA)$ is a nonnegative matrix and in this section we address the fundamental question of whether $C_p(hA)$ is nonnegative when A is Metzler and Hurwitz.

Example 3 *We begin with an alarming example which shows that in fact $C_p(hA)$ can have negative entries for all sufficiently small values of $h > 0$. To construct this example we consider a chain of first order linear systems. Such systems are ubiquitous and can be found in practically any elementary text book on control theory. They are also of some interest in the context of biological systems in the systems community [20, 4], and appear in the design of cascade filters [21].*

Consider a chain of n linear first order systems described by

$$\dot{x}_i = -\alpha_i x_i + k_i x_{i+1}, \text{ for all } i = 1, \dots, n-1, \quad (16)$$

$$\dot{x}_n = -\alpha_n x_n + k_n x_1 \quad (17)$$

More formally, the feedback system depicted can be written in state space form as $\dot{x} = Ax, x \in \mathbb{R}^{n \times n}$ where $A \in \mathbb{R}^{n \times n}$ is the matrix

$$A = \begin{bmatrix} -\alpha_1 & k_1 & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha_2 & k_2 & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_3 & k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_{n-1} \\ k_n & 0 & 0 & 0 & & -\alpha_n \end{bmatrix}. \quad (18)$$

By choosing $\alpha_i \geq 0$ and $k_i \geq 0$ one obtains that A is Metzler. (Note that with $n = 2$, Newton's second law of motion gives rise to such a system.)

Now let N be the 8×8 nonnegative (and nilpotent) matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that N is a particular example of (18), with $n = 8$, $k_1 = k_2 = \dots = k_7 = 1$, and $k_8 = \alpha_1 = \dots = \alpha_8 = 0$. In this case our system becomes a chain of homogeneous integrators connected in open loop. In spite of the fact that this is a very elementary system, we show below that preserving positivity of this elementary system is far from trivial.

Specifically, we shall consider the second order diagonal Padé approximation $C_2(hN) = (I + \frac{1}{2}hN + \frac{1}{12}h^2N^2)(I - \frac{1}{2}hN + \frac{1}{12}h^2N^2)^{-1}$. In order to do so, we begin with the function $C_2(z) = (1 + \frac{1}{2}z + \frac{1}{12}z^2)(1 - \frac{1}{2}z + \frac{1}{12}z^2)^{-1}$. Next, we write $C_2(z)$ as a power series in z as $C_2(z) = \sum_{j=0}^{\infty} \alpha_j z^j$. Computing the first few coefficients in that power series, we find that $\alpha_0 = 1, \alpha_1 = 1, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{6}, \alpha_4 = \frac{1}{24}, \alpha_5 = \frac{1}{144}, \alpha_6 = 0$, and $\alpha_7 = -\frac{1}{1728}$.

Observing that $N^8 = 0$, it now follows that the second order Padé approximation $C_2(hN)$ can be written as =

$$C_2(hN) = I + hN + \frac{1}{2}h^2N^2 + \frac{1}{6}h^3N^3 + \frac{1}{24}h^4N^4 + \frac{1}{144}h^5N^5 - \frac{1}{1728}h^7N^7.$$

Note also that for each $k, j = 1, \dots, 7$, the 1's in N^k do not overlap with the 1's in N^j whenever $k \neq j$. Consequently, $C_2(hN)$ has a negative entry in the (1, 8) position whenever $h > 0$.

Another computation shows that

$$C_2(hN)^2 = I + 2hN + 2h^2N^2 + \frac{4}{3}h^3N^3 + \frac{2}{3}h^4N^4 + \frac{19}{72}h^5N^5 + \frac{1}{12}h^6N^6 + \frac{17}{864}h^7N^7,$$

so that $C_2(hN)^2$ is a nonnegative matrix. From this, and the expression for $C_2(hN)$, we may deduce that $C_2(hN)^3$ is also a nonnegative matrix. It now follows that for any $h > 0$ and $k \geq 2$, $C_2(hN)^k$ is entrywise nonnegative. Thus we have identified an example of a nonnegative matrix N such that $C_2(hN)$ fails to be a nonnegative matrix for any $h > 0$, but all higher powers of $C_2(hN)$ are nonnegative.

Comment 5: Before proceeding, note that while the matrix N above is nonnegative, it is not Hurwitz. So, next we modify the example above to produce a related example of a matrix that is both Hurwitz and Metzler. Fix $t > 0$, and let A be the Hurwitz and

Metzler matrix $A = -tI + N$. Since $N^8 = 0$, we may write $C_2(hA) = \sum_{j=0}^7 \beta_j(h)N^j$, where each $\beta_j, j = 0, \dots, 7$, is a function of h that is continuous in a neighbourhood of 0. Further, from the fact that $C_2(hA)$ can also be written as $\sum_{j=0}^7 \alpha_j h^j (-tI + N)^j + O(h^8)$, it follows that for each $j = 0, \dots, 7, \beta_j(h) = \alpha_j h^j + O(h^{j+1})$. In particular, we find that for all sufficiently small $h > 0, \beta_7(h) < 0$. Hence, for all sufficiently small positive h , we see that $C_2(hA)$ fails to be nonnegative. An analysis similar to the above also reveals that for all sufficiently small $h > 0$, and any $k \geq 2, C_2(hA)^k$ is nonnegative.

In the context of using a Padé approximation for the purposes of simulating a continuous time positive system by a discrete time positive system, the above example has the following implication. It is possible for the first iterate of the discrete time simulation to leave the nonnegative orthant (and hence to be incompatible with the notion of a positive system), for the next iterate to return to the nonnegative orthant, and for all subsequent iterates to remain in the nonnegative orthant. While those subsequent iterates are indeed nonnegative, the fact that they are predicated on the incompatible first iterate calls their reliability into question. In particular, if one is not tracking every iterate of the discrete time simulation, but is instead sampling from the sequence of iterates over time, then there is the danger that the sampling will fail to observe the violation of nonnegativity. Thus the sampled iterates may be unreliable, but that unreliability can remain undetected.

In view of the preamble, we now turn our attention to providing sufficient conditions under which, for a given Metzler and Hurwitz matrix $A, C_p(hA)$ will be a nonnegative matrix. Our approach will be first to analyse the situation for some simple, Padé-like, rational functions, then to decompose C_p into a suitable product of such functions. We begin with the following straightforward result, which has also been noted in [22, 19] in conjunction with preservation of quadratic Lyapunov functions and is a special case of the main result in [23].

Lemma 3 *Let $A_c = \{a_{ij}\}$ be the Metzler and Hurwitz stable matrix. Fix $h > 0$ and suppose that $\alpha_0 > 0$. Set $\alpha(h) = \frac{\alpha_0}{h}$, and define A_d by*

$$A_d = (\alpha(h)I + A_c) (\alpha(h)I - A_c)^{-1}. \quad (19)$$

If

$$h \leq \min_i \frac{\alpha_0}{|a_{ii}|} \quad (20)$$

then A_d is nonnegative and Schur stable (since A_c is assumed to be Hurwitz and consequently has all diagonal entries negative).

Proof. Observing that A_d can be rewritten as a first order diagonal Padé approximation (possibly with a different value of h), the Schur stability of A_d now follows from properties of diagonal Padé approximations, see [18]. Next we note that since A_c is a Metzler and Hurwitz stable matrix then $(\alpha(h)I - A_c)^{-1} \geq 0$. If we also have $\Theta = \alpha(h)I + A_c \geq 0$

then A_d is non-negative. The i -th element on the main diagonal of Θ is $\alpha(h) - |a_{ii}|$, so if $h \leq \min_i \frac{\alpha_0}{|a_{ii}|}$, then $\alpha(h)I + A_c \geq 0$. ■

Corollary 1 *Let A_c be a Metzler and Hurwitz matrix. If $h \leq \min_i \frac{2}{|a_{ii}|}$, then $C_1(hA_c)$ is a nonnegative and Schur stable matrix.*

Proof. We have $C_1(hA) = (I + \frac{h}{2}A)(I - \frac{h}{2}A)^{-1}$, so in the notation of Lemma 3, we have $\alpha_0 = 2$. The conclusion follows immediately. ■

We now consider a Padé-like rational function where the numerator and denominator are both quadratics. Specifically, suppose that λ_0 is a complex number with $Re(\lambda_0) > 0$. Suppose that $h > 0$, set $\lambda(h) = \frac{\lambda_0}{h}$, and define A_d via

$$A_d = (\lambda(h)I + A_c) (\lambda^*(h)I + A_c) (\lambda(h)I - A_c)^{-1} (\lambda^*(h)I - A_c)^{-1}. \quad (21)$$

Set

$$\Theta_1 = (|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2), \Theta_2 = (|\lambda(h)|^2 I - 2Re(\lambda(h))A_c + A_c^2), \quad (22)$$

so that $A_d = \Theta_1 \Theta_2^{-1}$. We note in passing that taking $\lambda_0 = 3 + \sqrt{3}i$, (21) yields $C_2(hA_c)$.

Define $A_c = \{a_{ij}\}$ and $A_c^2 = \{b_{ij}\}$ then let \mathcal{P} be the set of indices i, j , $i \neq j$, such that $b_{ij} \neq 0$.

Lemma 4 *Let $A_c = \{a_{ij}\}$ be a Metzler and Hurwitz stable matrix and A_d be the matrix achieved through the transformation (21). Suppose that*

$$0 < h \leq 2Re(\lambda_0) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}. \quad (23)$$

Then Θ_1 of (22) is a nonnegative matrix, Θ_2 of (22) is an M-matrix, and A_d is nonnegative and Schur stable.

Proof. Let z be an eigenvalue of A_c , and note that $Re(z) < 0$ by hypothesis. A straightforward computation shows that

$$||\lambda(h)|^2 + 2Re(\lambda(h))z + z^2| < ||\lambda(h)|^2 - 2Re(\lambda(h))z + z^2|,$$

from which it follows that A_d is Schur stable via Lemma 2.

We claim that Θ_1 is a nonnegative matrix. To see this, first note that for each index i , the i -th diagonal entry of Θ_1 is

$$|\lambda(h)|^2 + 2Re(\lambda(h))|a_{ii}| + b_{ii}. \quad (24)$$

Since $b_{ii} = a_{ii}^2 + \sum_{j=0, j \neq i}^n a_{ij}a_{ji}$, it is easy to verify that the expression (24) is positive for all $h > 0$. Further, for any pair of distinct indices i, j , the (i, j) entry of Θ_1 is given by $2Re(\lambda(h))a_{ij} + b_{ij}$. From the hypothesis, it follows that $2Re(\lambda(h))a_{ij} - |b_{ij}| \geq 0$, which

readily yields the fact that $2\text{Re}(\lambda(h))a_{ij} + b_{ij} \geq 0$. Thus Θ_1 is nonnegative, as claimed.

Next, we claim that Θ_2 is a nonsingular M-matrix. Note that the i -th diagonal entry of Θ_2 is equal to $|\lambda(h)|^2 - 2\text{Re}(\lambda(h))|a_{ii}| + a_{ii}^2 + \sum_{j=0, j \neq i}^n a_{ij}a_{ji}$, which is readily seen to be positive. Further, for distinct indices i, j , the (i, j) entry of Θ_2 is given by $-2\text{Re}(\lambda(h))a_{ij} + b_{ij} \leq -2\text{Re}(\lambda(h))a_{ij} + |b_{ij}| \leq 0$, the last inequality following from our hypothesis. Finally, suppose that v is a nonnegative eigenvector of A_c corresponding to the eigenvalue r , say. Then $\Theta_2 v = (|\lambda(h)|^2 - 2\text{Re}(\lambda(h))r + r^2)v$; since $|\lambda(h)|^2 - 2\text{Re}(\lambda(h))r + r^2 > 0$, it now follows from Theorem 2.7 of [15] that Θ_2 is a nonsingular M-matrix, as claimed.

From the two claims above, we now find that $A_d = \Theta_1 \Theta_2^{-1}$ is a nonnegative matrix. ■

Lemmas 3 and 4 will now yield the following result regarding the nonnegativity of a p -th order diagonal Padé approximation.

Theorem 3 *Let A_c be a Metzler and Hurwitz stable matrix and $A_d(h) = C_p(A_c h)$ be the p -th order diagonal Padé approximation to $e^{A_c h}$. Let α_l , denote the m real poles of $C_p(\cdot)$, and let $\lambda_k, \lambda_k^*, k = 1, \dots, \frac{n}{2}$ denote the $\frac{n}{2}$ complex conjugate pairs of non-real poles of $C_p(\cdot)$. If $m \geq 1$, we define $\hat{\alpha} = \min_{l=1, \dots, m} \alpha_l$, and if $n \geq 2$, we define $\hat{\lambda} = \min_{k=1, \dots, \frac{n}{2}} \text{Re}(\lambda_k)$.*

Then $A_d(h)$ is nonnegative and Schur stable for every $h \leq h^$, where*

$$h^* = \min_i \frac{\hat{\alpha}}{|a_{ii}|}, \text{ if } n = 0, m \geq 1 \quad (25)$$

$$h^* = 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}, \text{ if } m = 0, n \geq 2 \quad (26)$$

$$h^* = \min \left(\min_i \frac{\hat{\alpha}}{|a_{ii}|}, 2\hat{\lambda} \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|} \right), \text{ if } m \geq 1, n \geq 2 \quad (27)$$

where a_{ij} and b_{ij} denote the (i, j) element of A_c and A_c^2 respectively.

Proof. We begin by noting that $\alpha_l > 0, l = 1, \dots, m$ and $\text{Re}(\lambda_k) > 0, k = 1, \dots, \frac{n}{2}$, and that $m + n = p$. Decomposing the p -th order diagonal Padé approximation into real and complex conjugate pairs of poles [13], we have:

$$\begin{aligned} A_d(h) &= \prod_{l=1}^m (\alpha_l(h)I + A_c) \times \prod_{k=1}^{n/2} (|\lambda_k(h)|^2 I + 2\text{Re}(\lambda_k(h))A_c + A_c^2) \\ &\quad \times \prod_{l=1}^m (\alpha_l(h)I - A_c)^{-1} \prod_{k=1}^{n/2} (|\lambda_k(h)|^2 I - 2\text{Re}(\lambda_k(h))A_c + A_c^2)^{-1}, \end{aligned} \quad (28)$$

where $\alpha_l(h) = \frac{\alpha_l}{h}, l = 1, \dots, m$ and $\lambda_k(h) = \frac{\lambda_k}{h}, \lambda_k^*(h) = \frac{\lambda_k^*}{h}, k = 1, \dots, \frac{n}{2}$. For each l , we may apply Lemma 3 to the factor $(\alpha_l(h)I + A_c)(\alpha_l(h)I - A_c)^{-1}$ to deduce that it is nonnegative. Similarly, for each k we apply Lemma 4 to the factor $(|\lambda_k(h)|^2 I +$

$2\operatorname{Re}(\lambda_k(h))A_c + A_c^2(|\lambda_k(h)|^2I - 2\operatorname{Re}(\lambda_k(h))A_c + A_c^2)^{-1}$ to find that it is also nonnegative. We find immediately that $A_d(h)$ is nonnegative. Finally, since $A_d(h)$ is a diagonal Padé approximation, it is necessarily Schur stable [18]. ■

Comment 6: Recall that for a $q \times q$ matrix M , the *directed graph of M* , $D(M)$, is the directed graph on vertices labeled $1, \dots, q$ such that for any pair of vertices i, j , $D(M)$ contains the arc $i \rightarrow j$ if and only if $m_{ij} \neq 0$. (See [15] for further background.) In this remark, we provide an interpretation of the condition that $h^* > 0$ in Theorem 3 in terms of $D(A_c)$. Note that $h^* > 0$ if and only if for each nonzero offdiagonal entry in A_c^2 , the corresponding entry in A_c is also nonzero. We claim that the condition that $h^* > 0$, is equivalent to the condition that for any pair of distinct indices i, j , either $i \rightarrow j$ in $D(A_c)$, or there is no path from i to j in $D(A_c)$. To see the claim, first suppose that $h^* > 0$, and suppose that we have a pair of distinct indices i and j such that there is a path from i to j in $D(A_c)$. Select a shortest such path, say $i \equiv i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_l \equiv j$, and note that necessarily the vertices i_0, \dots, i_l must be distinct. Suppose that $l \geq 2$, and note that in that case, the (i_0, i_2) entry of A_c^2 is positive, since that entry is equal to $\sum_k a_{i_0k}a_{ki_2} \geq a_{i_0i_1}a_{i_1i_2} > 0$. Since $h^* > 0$, it must be the case that $a_{i_0i_2} > 0$, but then $i \equiv i_0 \rightarrow i_2 \rightarrow \dots \rightarrow i_l \equiv j$, is a path from i to j of length less than l , a contradiction. We conclude that $l = 1$, so that $i \rightarrow j$ in $D(A_c)$. Conversely, suppose that for any pair of distinct indices i, j , either $i \rightarrow j$ in $D(A_c)$, or there is no path from i to j in $D(A_c)$. Select a pair of distinct indices i, j such that the (i, j) of A_c^2 is nonzero - i.e. $\sum_k a_{ik}a_{kj} \neq 0$. Then for some index k we have a_{ik} and a_{kj} both nonzero. If k is either i or j , then certainly $a_{ij} > 0$; if k is neither i nor j , then $D(A_c)$ contains the path $i \rightarrow k \rightarrow j$, and so by hypothesis $D(A_c)$ must contain the arc $i \rightarrow j$, and again we see that $a_{ij} > 0$. In either case we find that the (i, j) entry of A_c is positive, and it follows readily that $h^* > 0$. This completes the proof of the claim. Observe that in the special case that A_c is irreducible, so that for any pair of distinct vertices i, j , $D(A_c)$ contains a path from i to j , we find that if $h^* > 0$, then necessarily every off-diagonal entry of A_c must be positive.

Comment 7: We note that in the case that $n \geq 2$, the quantity h^* in Theorem 3 is positive if and only if, for each nonzero offdiagonal entry in A_c^2 , the corresponding entry in A_c is also nonzero. It is important to stress here that this condition is always met for 2×2 matrices, see [19]. This follows as a consequence of the following lemma [19].

Lemma 5 *Let $A_c \in \mathbb{R}^{2 \times 2}$ be a Metzler and Hurwitz matrix. Denote the entries of this matrix by a_{ij} and the entries of A_c^2 by b_{ij} , $i, j \in \{1, 2\}$. Then $b_{ij} = 0$ if and only if $a_{ij} = 0$.*

The proof of Theorem 3 shows that for each complex pole λ of C_p , the matrix Θ_2 of (22) (and Lemma (2)) is an M-matrix whenever $h \leq h^*$. Next, we discuss a method for finding the quantity h_0 in Theorem 2. Suppose that we are given Metzler and Hurwitz matrices A_{c1}, \dots, A_{cm} and suppose that (12) has a common linear co-positive Lyapunov function. For each $i = 1, \dots, m$, compute the value h^* for the matrix A_{ci} according to (25), (26), (27), and denote the corresponding value by h_i^* . Now set $h_0 = \min\{h_1^*, \dots, h_m^*\}$; observe

that $h_0 > 0$ if and only if for each $i = 1, \dots, m$, A_{ci}^2 has nonzero off-diagonal entries only in positions where A_{ci} is positive. An inspection of the proof of Theorem 2 now shows that for all $0 < h \leq h_0$, the discretised system (14) shares the same common linear co-positive Lyapunov function as the continuous-time system (12).

The expression for h^* given in Theorem 3 depends in part on the functions $\hat{\alpha}$ and $\hat{\lambda}$ that are computed from the poles of the Padé approximation C_p . In order to give the reader some sense of the magnitude of those quantities, the following table provides computed values for $\hat{\alpha}$ and $\hat{\lambda}$ for a few small values of p . Note that a * entry in the $\hat{\alpha}$ or $\hat{\lambda}$ column denotes the fact that C_p has no real roots, or no complex roots, respectively.

p	$\hat{\alpha}$	$\hat{\lambda}$
1	0.5	*
2	*	0.25
3	0.2153	0.1423
4	*	0.0916
5	0.1371	0.0640
6	*	0.0474
7	0.1006	0.0367
8	*	0.0293
9	0.0794	0.0240
10	*	0.0201
11	0.0656	0.0171
12	*	0.0147
13	0.0559	0.0128
14	*	0.0113
15	0.0487	0.0100

5. An alternative approximation to the exponential matrix

The previous sections illustrate some disadvantages of the Padé approximation and motivates the search for other approximations to the matrix exponential. In this section we present a Padé-like approximation that has the following properties: one can always find a sampling time such that positivity is preserved, and in addition, for any h , both linear and quadratic co-positive Lyapunov functions are preserved.

The basic idea is to divide the time window of length h into $\frac{h}{p}$ intervals and, in every time interval to use an approximation that has some desired properties. As our interval approximation we use the generalised first order diagonal Padé, since from the discussion in Section 3, for such approximations one can always find an upper bound of the sampling time h , below which positivity is guaranteed. Hence we introduce the following approximation to the exponential matrix:

$$A_{ad} = \left[\left(I + \frac{A_c h}{2p} \right) \left(I - \frac{A_c h}{2p} \right)^{-1} \right]^p, \quad p \in \mathbb{N}. \quad (29)$$

We claim that as $p \rightarrow \infty$, $A_{ad} \rightarrow e^{A_c h}$. To see the claim, we first note that A_{ad} can be rewritten as $\left(I + \frac{A_c h}{2p}\right)^p \left(I - \frac{A_c h}{2p}\right)^{-p}$. Let N_k be the $k \times k$ matrix with ones on the superdiagonal and zeros elsewhere, and let z be a complex number. A straightforward exercise shows that as $p \rightarrow \infty$, $\left(I + \frac{(zI + N_k)h}{2p}\right)^p \rightarrow e^{(zI + N_k)h/2}$ while $\left(I - \frac{(zI + N_k)h}{2p}\right)^{-p} \rightarrow e^{-(zI + N_k)h/2}$. Appealing to the Jordan canonical form for A_c , it now follows that as $p \rightarrow \infty$, $\left(I + \frac{A_c h}{2p}\right)^p \left(I - \frac{A_c h}{2p}\right)^{-p} \rightarrow e^{A_c h/2} \cdot (e^{-A_c h/2})^{-1} = e^{A_c h}$, as claimed. Consequently, A_{ad} converges to $e^{A_c h}$ as $p \rightarrow \infty$. We then have the following result.

Theorem 4 *Let $\{A_{c,1}, \dots, A_{c,m}\}$ be a set of Metzler and Hurwitz stable matrices. For each $i = 1, \dots, m$, let $A_{ad,i}(h) = C_{ap}(A_{c,i}h)$ be the p -th order of the approximation to exponential matrix $e^{A_{c,i}h}$ defined in Equation (29). Then the following properties hold*

1. *Fix an i between 1 and m , and suppose that*

$$0 < h \leq h_i = \min_j \frac{2}{|a_{jj,i}|}, \quad (30)$$

where $a_{jj,i}$ are the elements on the main diagonal of the matrix $A_{c,i}$. Then $A_{ad,i}$ is both nonnegative and stable.

2. *Consider the following continuous-time switching positive system*

$$\dot{x}(t) = A_c(t)x(t), \quad x(0) = x_0, \quad (31)$$

where $x(t) \in \mathbb{R}_+^n$, $x_0 \in \mathbb{R}_+^n$ is the initial condition and $A_c(t)$ belongs to $\{A_{c,1}, \dots, A_{c,m}\}$. Suppose that (30) holds. Then the discretised system

$$x(k+1) = A(k)x(k) \quad (32)$$

is positive, where $A(k) \in \{C_{ap}(A_{c,1}h), \dots, C_{ap}(A_{c,m}h)\}$. Moreover, if there exists a common quadratic or linear co-positive Lyapunov function for system (31), then the origin $x = 0$ is globally uniformly exponentially stable for system (32).

Proof. 1. Write the matrix $A_{ad,i}$ as:

$$A_{ad,i} = \left[\left(I + A_c \frac{\hat{h}}{2} \right) \left(I - A_c \frac{\hat{h}}{2} \right)^{-1} \right]^p = A_{d1,i}^p, \quad (33)$$

where $\hat{h} = \frac{h}{p} \leq 2 \min_j \frac{1}{|a_{jj,i}|}$. Therefore, from Corollary 1, $A_{ad,i}$ is a nonnegative matrix. Schur stability follows from Lemma 3 since $A_{c,i}$ is Metzler and Hurwitz.

2. Nonnegativity of the $A_{ad,i}$ is evident from the definition of h . Now, assume that there exists a $w > 0$ is such that $w' A_{c,i} < 0$, for all $i \in \{1, 2, \dots, m\}$. We now show that this implies that $w' A_{ad,i} < w'$ for all $i \in \{1, 2, \dots, m\}$. As in the previous part of the proof we can write $A_{ad,i} = A_{d1,i}^p$ for all $i \in \{1, 2, \dots, m\}$. From Lemma 1, we know that

$$w' A_{d1,i} < w', \quad (34)$$

from which it follows that

$$w' A_{ad,i} = w' A_{d1,i}^p = w' A_{d1,i} A_{d1,i}^{p-1} < \dots < w' A_{d1,i} < w', \quad \forall i \in \{1, 2, \dots, m\}. \quad (35)$$

Hence linear common co-positive Lyapunov functions are preserved by this approximation to the exponential matrix.

Now, assume that there exists a positive definite $P = P' \succ 0$ such that for all non-negative x , $x'Px > 0$, and $x'(A'_{c,i}P + PA_{c,i})x < 0$ for all $i \in \{1, 2, \dots, m\}$. Following the same rationale used above we will prove that the same P satisfies also

$$x'(A'_{ad,i}PA_{ad,i} - P)x < 0, \quad \forall x > 0, \quad i \in \{1, 2, \dots, m\}. \quad (36)$$

Recalling that $A_{ad,i} = A_{d1,i}^p$, with $A_{d1,i}$ nonnegative and Schur for each $i \in \{1, 2, \dots, m\}$, and that $x'A'_{d1,i}PA_{d1,i} < x'Px$, $\forall x > 0$, $i \in \{1, 2, \dots, m\}$ according to Lemma 1, we can write

$$x'A'_{ad,i}PA_{ad,i} = x'A'_{d1,i} \left[\left(A_{d1,i}^{p-1} \right)' P \left(A_{d1,i}^{p-1} \right) \right] A_{d1,i} x < \dots < x'Px, \quad \forall i \in \{1, 2, \dots, m\}. \quad (37)$$

Hence both positivity linear/quadratic common co-positive Lyapunov function are preserved through this approximation.

The conclusion now follows readily. ■

Comment 9: The approximation to the exponential matrix given by (29) is in fact a minor variation on the well-known scaling and squaring method for computing the matrix exponential (see [24]). The scaling and squaring method exploits the fact that for a square matrix M and $j \in \mathbb{N}$, $e^M = (e^{M/2^j})^{2^j}$. Accordingly, the scaling and squaring method proceeds by scaling the original matrix by a power of two, computing a Padé approximant of the resulting matrix, and then successively squaring that approximant to produce an approximation to the exponential of the original matrix.

Thus, if p is chosen as a power of 2, then (29) coincides exactly with the scaling and squaring method, where the Padé approximant computed is the first order diagonal Padé approximant. Following the analysis given in section 11.3.1 of [24], we find that if $p = 2^j$ is chosen so that $\|hA_c\|_\infty \leq 2^{j-1}$, then the matrix A_{ad} of (29) has the property that

$$\frac{\|e^{A_c} - A_{ad}\|_\infty}{\|e^{A_c}\|_\infty} \leq \frac{h}{6} \|A_c\|_\infty e^{\frac{h}{6} \|A_c\|_\infty}.$$

In particular, for small values of h , A_{ad} approximates e^{hA_c} with high relative accuracy, in addition to the above mentioned features that A_{ad} preserves both positivity and linear/quadratic co-positive Lyapunov functions.

With $p = 2^j$ chosen as above, the algorithm presented in [24] for implementing the scaling and squaring method using the first order diagonal Padé approximant requires about $2(j + \frac{4}{3})n^3$ flops, where n is the order of the matrix in question. Thus we find that if A_c is $n \times n$, then the matrix A_{ad} of (29) can be computed in $2(j + \frac{4}{3})n^3$ flops.

6. Conclusions

In this paper we examine the suitability of diagonal Padé transformations for discretising positive systems. Unfortunately, the results of this investigation are uniformly bad. In particular, a number of problems with this transformation are noted, and an alternative method is presented that avoids these pitfalls.

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