# Reversibility Questions in Groups arising in Analysis

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*Dedicated to Paul Gauthier*

Abstract. An element $g$ of a group is called *reversible* if it is conjugate in the group to its inverse. In this paper we review some results about the structure of groups involving the reversible elements and we pose some questions about groups associated to a Banach algebra.

## 1. Introduction

Definition 1.1. An element $g$ of a group is called *reversible* if it is conjugate in the group to its inverse, i.e. there exists some map $h$, belonging to the group, such that the conjugate $g^h = h^{-1}gh$ equals $g^{-1}$. We say that $h$ *reverses* $g$, in this case.

This concept had its origins in the study of dynamical systems [**Bir15**]. Classical conservative systems such as the harmonic oscillator and a system of $n$ bodies moving under their mutual (Newtonian) gravitational attraction, and billiards (on a table without pockets or corners!) have what is called a time-reversal symmetry: a bijection of the phase space which conjugates the dynamical system to its inverse.

I became interested in reversibility because I encountered reversible dynamical systems when studying a number of different problems related to approximation:

- Approximation by functions of the form $F(x,y) = f(x)+g(y)$ on compact sets in $\mathbb{R}^2$ [**MO79, MO83**].
- Approximation by polynomials of the form $p(z^2, \bar{z}^2+\bar{z}^3)$ on a disk in the complex plane [**OSG02**].

I was not the first to find that an apparently undynamical problem had an essential connection to some discrete dynamical system. In fact, this phenomenon has been named *hidden dynamics* by V.I. Arnold [**Vor81**]. Other problems of this type are:

- Biholomorphic classification of a pair of tangent real-analytic arcs in the plane [**AG05, Kas15, Nak98, Vor81, Vor82**].
- The polynomial hull of a disk having an isolated complex tangent [**MW83, SG00**].

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Each of these problems involves a pair of non-commuting involutions, and so relates to a reversible element in some group of maps. Moreover, it turned out that this connection to dynamics provided the key to resolving the problem.

With this in mind, I decided to come at the phenomenon from the other end, and to try to understand the phenomenon of reversibility in general. The natural context is the theory of groups, and I have been calculating examples, assembling what is known from the literature, and encouraging people to examine reversibility in their favourite groups. In fact, it turns out that there is a gigantic literature on reversibility in various different groups. Much of this falls into islands (or continents) of work, where the workers are unaware of the connection between this aspect of their field and of other fields. The work cuts right across the various fields of mainstream mathematics. See [**AR95b, BR97, BR01, BR03, BR06, Bri96, Bul88, Dev76, Djo67, GM03, GM04, Gon96, Goo99, Gow75, HK58, Kan01, Lam92, LRC93, LR98, QC89, Rad81, Sar07, Sev86, Web96, Web98, Won66**], and further references therein and below.

In this short paper, I review some results about reversibility and particularly factorisation into reversible factors, and pose some questions about groups related to Banach algebras, specifically.

I have drawn upon material in the draft of a book on aspects of reversibility that is in preparation with Ian Short, and wish to acknowledge his help with this.

## 2. Notation

Let $G$ be a group. We use the following notation:

$I = I(G) := \{f \in G : f^2 = \mathrm{id}\}$ — the set of *involutions* of $G$ (including the identity, id).

$R_f = R_f(G) := \{h \in G : f^h = f^{-1}\}$ (where $f^h = h^{-1}fh$) —the set of *reversers of* $f$.

$R = R(G) := \{f \in G : R_f \neq \emptyset\}$ — the set of *reversible elements*.

For $A \subset G$, $A^n := \{f_1 \cdots f_n : f_j \in A\}$, and $A^\infty := \bigcup_{n=1}^\infty A_n$.

Elements of $I^2$ are called *strongly-reversible.* They are reversed by an involution.

Membership in $I^n$ or $R^n$ is a conjugacy invariant, and $I^2 \subset R$.

$I^\infty$ and $R^\infty$ are normal subgroups of $G$.

## 3. Example: $\mathsf{GL}(n, \mathbb{C})$

Classification of linear reversible maps on $\mathbb{C}^n$ is simple. Suppose $F \in \mathsf{GL}(n, \mathbb{C})$ (the general linear group over the field $\mathbb{C}$ of complex numbers) is reversible. Since the Jordan normal form of $F^{-1}$ consists of blocks of the same size as $F$ with reciprocal eigenvalues, the eigenvalues of $F$ that are not $\pm 1$ must split into groups of pairs $\lambda, 1/\lambda$. Furthermore, we must have the same number of Jordan blocks of each size for $\lambda$ as for $1/\lambda$. Vice versa, if the eigenvalues of $F$ are either $\pm 1$ or split into groups of pairs $\lambda, 1/\lambda$ with the same number of Jordan blocks of each size, then both $F$ and $F^{-1}$ have the same Jordan normal form and are therefore conjugate to each other.

Incidentally, each matrix $A \in \mathsf{GL}(n, \mathbb{C})$ is mapped to a conjugate of its inverse by some outer automorphism of the group. In fact $A \mapsto (A^t)^{-1}$ is an automorphism, and $A^t$ is conjugate to $A$. There is a wider context concerning "outer reversibility" in a group.

## 4. The Basic Questions

In each group, $G$, we ask:

- Which $f$ are reversible in $G$?
- Which $h$ reverse a given $f$?
- Describe $I^\infty$.
- Describe $R^\infty$.
- Is $I^n = I^\infty$ for some $n$?
- Is $R^n = R^\infty$ for some $n$?
- Does every nonempty $R_g$ have an element of finite order? If so, what orders occur? Is $\min\{\mathrm{o}(h) : h \in R_g\}$ bounded, for $g \in R$?

If $g$ is reversible by some element of finite order, then it is the product of two elements of that (even) order. Thus results about $R^n$, combined with results about the order of reverses, also give information about factorizing elements of $G$ as a product of elements of at most a given order. Some find this interesting [**HOR01**].

We wish to encourage people to investigate the questions above in their favourite groups. Many groups have been analysed, and in the next section we survey some of these results. But many groups remain to be investigated. In the final section, we shall draw particular attention to groups associated to Banach algebras.

## 5. Survey of Known Results

We now give a summary of some answers, in examples of various categories of groups, with a sampling of relevant sources (by no means complete). Here, $G$ always denotes the group under consideration. We give no detail about the derivation of these results, just references to the related literature. Some of the proofs are quite deep, and they draw on diverse branches of mathematics.

**5.1.** If $G$ is abelian, then obviously $R = I = I^\infty$.
If $G$ is free, then it is not hard to see that $R = I = \{1\}$.

**5.2. Finite Groups.** [**Cox47, KN05, ST08, Ste98, TZ05**]

**Dihedral $D_n$:** $I^2 = G = R$.
**Symmetric $S_n$:** $I^2 = G = R$.
**Finite Coxeter:** $I^2 = G = R$.
**Alternating $A_n$:** $I^2 = R$.
$R \neq G$, except when $n \in \{1, 2, 5, 6, 10, 14\}$.
**Quaternion 8-group:** $I^2 \neq R = G$.
**Finite, simple $G$:**
$R = \{1\}$ if $|G|$ is odd.
$G = R^2$, if $|G|$ is even, except for $\mathrm{PSU}(3, 3^2)$.
In general, $G \neq I^2$; when it happens is known.

**5.3. Classical Groups.** [**Bak02, Bal78, BKN97, BR97, Djo86, DM82, Ell77, EN82, Ell93, EM90, EV04, Goo97, HP71, Liu88, KN87, Knü88, LOS07, Nie87**]

**General Linear $\mathsf{GL}(n, F)$ ($n > 1$):** $I^2 = R$.
$I^4 = I^\infty = R^2$.
**Special Linear $\mathsf{SL}(n, \mathbb{C})$:** $I^2 = R$ unless $n = 2$ (mod4).
$R^2 = G$.

**Orthogonal** $\mathsf{O}(n,\mathbb{R})$**(≈ spherical isometries):** $I^2 = G$.
**Special Orthogonal** $\mathsf{SO}(n,\mathbb{R})$**:** $I^2 = R$.
$I^3 = G$ if $n \geq 3$.
$I^2 = G$ if $n \neq 2 \pmod 4$.
**Unitary** $sfU(n,\mathbb{C})$**:** $I^2 = R$
$I^4 = I^\infty$.
**Special Unitary** $\mathsf{SU}(n,\mathbb{C})$**:** $I^2 \neq R$.
$I^3 \neq I^6 = G = R^2$.
**Unitary Quaternionic** $\mathsf{Sp}(n,\mathbb{C}) = \mathsf{Symp}(2n,\mathbb{C}) \cap sfU(2n,\mathbb{C})$**:**
$I^2 \neq R = G = I^6$.
**Spinor** $\mathsf{Spin}(n,\mathbb{C})$**:** $I^2 = G$ if $n = 0, 1, 7 \mod 8$.
$R = G$ unless $n = 2 \mod 4$.
$I^4 = G$ if $n \geq 5$.

**5.4. Discrete Matrix Groups.** [**BR97, BR06, Ish95, Laf97**]
$\mathsf{GL}(n,\mathbb{Z})$**:** $I^{3n+9} = G$. $I^{41} = G$ for $n \geq 84$.
$I^2 \neq R \subset I^4$ when $n = 2$.
**Modular** $\mathsf{PSL}(2,\mathbb{Z})$**:** $I^2 = R$.

**5.5. Finite-dimensional Isometry Groups.** [**Sho08**]
**Euclidean** $\mathsf{Isom}(\mathbb{R}^n)$**:** $I^2 = G$.
**Orientation-preserving** $\mathsf{Isom}^+(\mathbb{R}^n)$**:** $I^3 = G$ if $n \geq 3$.
$I^2 = G$ if $n = 0$ or $3 \pmod 4$.
**Hyperbolic** $\mathsf{Isom}(\mathsf{H}^n)$**:** $I^3 = G$ if $n \geq 2$.

**5.6. Homeomorphism Groups.** [**And62, Cal71, FS55, FZ82, GOS09, Jar02a, Jar02b, O'F04, You94**]
$\mathsf{Homeo}(\mathbb{R})$**:** $I^2 \neq R$. $I^3 \neq I^4 = G = R^2$.
$\mathsf{Homeo}^+(\mathbb{R})$**:** $R^4 = G$.
$\mathsf{Homeo}(\mathbb{S}^1)$**:** $I^2 \neq R$. $I^3 = R^2 = G$.
$\mathsf{Homeo}^+(\mathbb{S}^1)$**:** $I^2 \neq R$. $I^3 = R^2 = G$.
$\mathsf{Homeo}(\mathbb{S}^n)$**:** $G = I^6$ when $n = 2$ or $3$. (Open for $n > 3$).
**Compact surface MCG:** $I^n \neq G = I^\infty$, $\forall n \in \mathbb{N}$, if genus $> 2$.

**5.7. Maps with extra Structure.** [**AO09, AR95a, Bir39, BY09, GS10, OS09**]
**Diffeomorphism** $\mathsf{Diffeo}(\mathbb{R})$**:** $I^2 \neq R$.
$I^3 \neq I^4 = G = R^2$.
$\mathsf{Diffeo}^+(\mathbb{R})$**:** $R^4 = G$.
**Formal germs on** $(\mathbb{C}^n, 0)$**:** [**O'F08, AO09**] $I^4 = I^\infty$ when $n = 1$.
$R^2 = R^\infty$ when $n = 1$.
$R^k = R^\infty$ with $k = 3 + 2 \cdot \text{ceiling}(\log_2 n)$
when $n \geq 2$.
$I^{15} = R^\infty$ when $n = 2$.
**Piecewise-linear** $\mathsf{PL}(\mathbb{R})$**:** [**BS01, BS85**] $I^2 \neq R$.
$I^3 \neq I^4 = G = R^2$.
$\mathsf{PL}^+(\mathbb{R})$**:** $I = \{1\}$. $R^4 = G$.
**PL with finitely many nodes** $\mathsf{PLF}(\mathbb{R})$**:** $I^2 = R$.
$\mathsf{PLF}^+(\mathbb{R})$**:** $R^4 = G$.

## 6. Banach Algebras

Let $A$ be a Banach algebra. We may associate two collections of groups to $A$. In each case, we ask the usual questions.

**6.1. $A^{-1}$ and its subgroups.** Suppose $A$ has identity (or adjoin one, if not) and $\|1\| = 1$.

Reversibility in $A^{-1}$ is not interesting unless $A$ is noncommutative. Also central reversibles are just central involutions, so the real problems are about the quotient

$$\frac{A^{-1}}{Z(A^{-1})} \equiv \mathsf{Inn}(A).$$

One interesting subgroup is

$$\mathsf{Iso}(A) = \{x \in A : \|x\| = \|x^{-1}\| = 1\}.$$

This coincides with the subgroup (often denoted $sfU(A)$ [**All11**]) of unitary elements, in case $A$ is a $C^*$ algebra.

One may also focus on the connected component of 1 in either group, $G$, and on the intersection $G \cap E^c$ with the commutator of any subset $E \subset A$.

One also has the normal subgroup
$\{x \in A : \|a - 1\| < 1\}^\infty$, which lies in the group $(\exp A)^\infty$.

For instance, Gustafson, Halmos, and Radjavi [**GHR76**] showed that for finite-dimensional (real or complex) Hilbert spaces $H$, the group $G = \mathsf{GL}(H)$ has $I^4 = I^\infty$, and they noted that for infinite-dimensional $H$, we have $I^4 \neq I^7 = \mathsf{GL}(H)$. I don't know whether 7 is the best possible value in that statement. What happens with other $C^*$ algebras?

It is known that for finite-dimensional $\mathsf{GL}(H)$, we have $I^2 = R$ in $G$, and also in the unitary subgroup. What about other $C^*$ algebras?

**6.2. Aut$(A)$ and its subgroups.** The main interesting subgroup (apart from the inner automorphism group, already mentioned) is the group of isometric isomorphisms. This is often the same as $\mathrm{Aut}(A)$.

As an example, when $X$ is a locally-compact Hausdorff space and $A = C_0(X, \mathbb{C})$, then $\mathrm{Aut}(A)$ is isomorphic to $\mathsf{Homeo}(X)$, so we have seen answers in case $X = \mathbb{R}^1$ and $X = \mathbb{S}^1$. For the disk algebra, the automorphism group is isomorphic to $\mathsf{PSL}(2, \mathbb{R})$, so all elements are reversible. Similarly, for the polydisk algebra, the automorphism group is isomorphic to the group of conformal automorphisms of the polydisk, which is isomorphic to the wreath product $S_n \wr \mathsf{PSL}(2, \mathbb{R})$, and again all elements are reversible.

We remark that the algebra of all formal power series in $n$ indeterminates, with complex coefficients, has a natural Frechet algebra structure, and embeds in some Banach algebras [**All11**]. Its automorphism group is isomorphic to the group of formally-invertible formal germs mentioned already.

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