Preservation of Common Quadratic Lyapunov Functions and Padé Approximations

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***Abstract*—It is well known that the bilinear transform, or first order diagonal Padé approximation to the matrix exponential, preserves quadratic Lyapunov functions between continuous-time and corresponding discrete-time linear time invariant (LTI) systems, regardless of the sampling time. It is also well known that this mapping preserves common quadratic Lyapunov functions between continuous-time and discrete-time switched systems. In this note we show that while diagonal Padé approximations do not in general preserve other types of Lyapunov functions (or even stability), it is true that diagonal Padé approximations of the matrix exponential, of any order and sampling time, preserve quadratic stability. A consequence of this result is that the quadratic stability of switched systems is robust with respect to certain discretization methods.**

I. Introduction

The diagonal Padé approximations to the exponential function are known to map the open left half of the complex plane to the open interior of the unit disk [1]. Considering the diagonal Padé approximations to the matrix exponential function, this gives rise to a correspondence between continuous-time stable LTI (linear time invariant) systems and their discrete-time stable counterparts, a fact that is often exploited in the systems and control community [2]. Perhaps the best known map of this kind is the first order diagonal Padé approximant (also known as the bilinear or Tustin map [1]). The bilinear map is known not only to preserve stability, but also preserve quadratic Lyapunov functions. That is, a positive definite matrix P satisfying $A_c^*P+PA_c<0$ will also satisfy $A_d^*PA_d-P<0$ where A_d is the mapping of A_c under the bilinear transform [2] with some sampling time h [3]. This makes it extremely useful when transforming a continuous-time switching system:

$$\dot{x}=A_c(t)x, \qquad A_c(t)\in\mathscr{A}_c \tag{1}$$

into an approximate discrete-time counterpart,

$$x(k+1)=A_d(k)x(k), \quad A_d(k)\in\mathscr{A}_d \tag{2}$$

because, the existence of a common positive definite matrix P satisfying $A_c^*P+PA_c<0$ for all $A_c\in\mathscr{A}_c$ implies that the same P satisfies $A_d^*PA_d-P<0$ for all $A_d\in\mathscr{A}_d$. Thus quadratic stability of the continuous-time switching system implies quadratic stability of the discrete-time counterpart. This property is useful in obtaining results in discrete-time from their continuous-time counterparts [2], and in providing a robust method to obtain a stable discrete-time switching system from a continuous-time one. With this latter application in mind, and motivated by the work initiated in [2], our objective in this note is to determine whether this property is preserved by higher order Padé approximants. As we shall see, for any order of approximation, and for any sampling time h, such approximations preserve quadratic stability.

II. Mathematical Preliminaries

The following definitions and results are useful in developing the main theorem which is given in Section III.

Notation : A square matrix A_c is said to be Hurwitz stable if all of its eigenvalues lie in open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie in the open interior of the unit disc. The notation M^* is used to denote the complex conjugate transpose of a general matrix M; M is hermitian if $M^*=M$. A hermitian matrix P is said to be positive (negative) definite if $x^*Px>0$ ($x^*Px<0$) for all non-zero x and we denote this by $P>0$ ($P<0$). In all of the following definitions, $P=P^*>0$.

A matrix P is a Lyapunov matrix for a Hurwitz stable matrix A_c if $A_c^*P+PA_c<0$. In this case, $V(x)=x^*Px$ is a quadratic Lyapunov function (QLF) for the continuous-time LTI system $\dot{x}(t)=A_cx(t)$

A matrix P is a Stein matrix for a Schur stable matrix A_d if $A_d^*PA_d-P<0$. In this case, $V(x)=x^*Px$ is a quadratic Lyapunov function for the discrete-time LTI system $x(k+1)=A_dx(k)$.

Given a finite set of Hurwitz stable matrices $\mathscr{A}_c$ a matrix P is a common Lyapunov matrix (CLM) for

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$\mathscr{A}_c$ if $A_c^*P + PA_c < 0$ for all A_c in $\mathscr{A}_c$. In this case, we say that the continuous-time switching system (1) is quadratically stable (QS) with Lyapunov function $V(x) = x^*Px$ and V is a common quadratic Lyapunov function (CQLF) for $\mathscr{A}_c$.

Given a finite set of Schur stable matrices $\mathscr{A}_d$ a matrix P is a common Stein matrix (CSM) for $\mathscr{A}_d$ if $A_d^*PA_d - P < 0$ for all A_d in $\mathscr{A}_d$. In this case, we say that the discrete-time switching system (2) is quadratically stable (QS) with Lyapunov function $V(x) = x^*Px$ and V is a common quadratic Lyapunov function (CQLF) for $\mathscr{A}_d$.

Whenever a continuous-time or discrete-time system is quadratically stable then, it is globally exponentially stable about the origin. This fact is sometimes very useful in demonstrating stability of switching systems. The converse result is not true, that is, global exponential stability does not imply quadratic stability for a switching system.

Our primary interest in this note is to determine the invariance of quadratic Lyapunov functions under diagonal Padé approximations to the matrix exponential.

Definitions and basic results : The following definitions and Lemmas will be useful to derive the main results.

Definition 1: (Diagonal Padé Approximations) [1] [4]: The p^{th} order diagonal Padé approximation to the exponential function e^s is the rational function C_p defined by

$$C_p(s) = \frac{Q_p(s)}{Q_p(-s)} \tag{3}$$

where

$$Q_p(s) = \sum_{k=0}^{p} c_k s^k \quad \text{and} \quad c_k = \frac{(2p-k)!p!}{(2p)!k!(p-k)!} \tag{4}$$

Thus the p^{th} order diagonal Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h, is given by

$$C_p(A_c h) = Q_p(A_c h) Q_p^{-1}(-A_c h) \tag{5}$$

where $Q_p(A_c h) = \sum_{k=0}^{p} c_k (A_c h)^k$.

Much is known about diagonal Padé maps in the context of LTI systems. In particular, the fact that such approximations map the open left half of the complex plane to the interior of the unit disc is widely exploited in systems and control. This implies the well known fact that these maps preserve stability as stated formally in the following lemma.

Lemma 1: [1] (Preservation of stability) Let A_c be a Hurwitz stable matrix and $A_d = C_p(A_c h)$ be the p^{th} order diagonal Padé approximation of $e^{A_c h}$. Then A_d is Schur stable.

A special diagonal Padé approximation is the first order approximation. This is also sometimes referred to as the bilinear or Tustin transform.

Definition 2: (Bilinear transform) [1] [4]: The first order diagonal Padé approximation to the matrix exponential with sampling time h is defined by:

$$C_1(A_c h) = \Big(I + A_c \frac{h}{2}\Big)\Big(I - A_c \frac{h}{2}\Big)^{-1}. \tag{6}$$

This approximation is known not only to preserve stability, but also to preserve quadratic Lyapunov functions [2], [3], [5]; namely if P is a Lyapunov matrix for A_c, then it is also a Stein matrix for $A_d = C_1(A_c, h)$. The converse is also true. Actually, we have the following known result which is a special case of Lemma 3 below

Lemma 2: [5] (Preservation of Lyapunov functions) Let A_c be a Hurwitz stable matrix and $A_d = C_1(A_c h)$ be the first order diagonal Padé approximation (bilinear transform) of $e^{A_c h}$ for some $h > 0$. Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for A_d.

As we shall see the bilinear transform plays a key role in studying general diagonal Padé approximations. In particular, a *complex* version of this map that inherits some of the above properties will be very useful in what follows.

Lemma 3: [12] (The complex bilinear transform) Let A_c be a Hurwitz stable matrix and for any complex number λ with $Re(\lambda) > 0$, define the matrix

$$A_d = (\lambda I + A_c)(\lambda^* I - A_c)^{-1} \, . \tag{7}$$

Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for A_d.

Proof : Consider any matrix $P = P^* > 0$. When A_d is given by (7), the Stein inequality $A_d^*PA_d - P < 0$ can be expressed as

$$(\lambda^* I - A_c)^{-*}(\lambda I + A_c)^* P (\lambda I + A_c)(\lambda^* I - A_c)^{-1} - P < 0 \, .$$

Post-multiplication by $\lambda^* I - A_c$ and pre-multiplication by $(\lambda^* I - A_c)^*$ results in the following equivalent inequality

$$(\lambda I + A_c)^* P(\lambda I + A_c) - (\lambda^* I - A_c)^* P(\lambda^* I - A_c) < 0,$$

which simplifies to

$$(\lambda + \lambda^*)(PA_c + A_c^* P) < 0.$$

Since $\lambda + \lambda^* > 0$ this last inequality is equivalent to the Lyapunov inequality $PA_c + A_c^* P < 0$. Thus P is a Lyapunov matrix for A_c if and only if it is a Stein matrix for A_d. ■

The final basic result that we shall need concerns common quadratic Lyapunov functions for discrete-time systems. A proof of this (well known) lemma is given in the Appendix.

Lemma 4: [12] If P is a CSM for $A_1, \cdots, A_m$ then P is a Stein matrix for the matrix product $\prod_{i=1}^m A_i$.

III. Main Results

We now present the main result of the paper. A main consequence of this result is that common quadratic Lyapunov functions are preserved by all diagonal Padé discretizations. Thus, quadratic stability is preserved under all diagonal Padé discretizations of a quadratically stable continuous-time switched system. This fact is stated formally in Corollary 1.

Theorem 1: Suppose that A_c is a Hurwitz stable matrix and A_d is any p^{th} order diagonal Padé approximation to $e^{A_c h}$ for any $h > 0$. If P is a Lyapunov matrix for A_c then, P is a Stein matrix for A_d.

Proof: Consider any matrix P which is a Lyapunov matrix for A_c. Recall that $A_d = Q_p(A_c h) Q_p^{-1}(-A_c h)$. Since the coefficients of the polynomial Q_p are real,

$$Q_p(sh) = kh^p \prod_{j=1}^{m} (\alpha_j + s) \prod_{i=1}^{n} (\lambda_i + s)(\lambda_i^* + s)$$

for some $k \neq 0$, where $m + 2n = p$, the real numbers $-h\alpha_1, \cdots, -h\alpha_m$ are the real roots of Q_p and the complex numbers $-h\lambda_i, -h\lambda_i^*, i = 1, \cdots, n$ are the non-real roots of Q_p. Since all the roots of Q_p have negative real parts ([1] [4]) we must have $\alpha_j > 0$ for all j and $Re(\lambda_i) > 0$ for all i. It now follows that A_d can be expressed as

$$A_d = \left(\prod_{j=1}^{m} (\alpha_j I + A_c)\right)\left(\prod_{i=1}^{n} (\lambda_i I + A_c)(\lambda_i^* I + A_c)\right)$$
$$\left(\prod_{i=1}^{n} (\lambda_i I - A_c)(\lambda_i^* I - A_c)\right)^{-1}\left(\prod_{j=1}^{m} (\alpha_j I - A_c)\right)^{-1}$$

which, due to commutativity of the factors, can be expressed as

$$A_d = \left(\prod_{j=1}^{m} (\alpha_j I + A_c)(\alpha_j^* I - A_c)^{-1}\right)\left(\prod_{i=1}^{n} (\lambda_i I + A_c)(\lambda_i^* I - A_c)^{-1}\right)$$
$$\left(\prod_{i=1}^{n} (\lambda_i^* I + A_c)(\lambda_i I - A_c)^{-1}\right).$$

Hence A_d is a product of bilinear terms of the form $(\lambda I + A_c)(\lambda^* I - A_c)^{-1}$ where $Re(\lambda) > 0$. Since P is a Lyapunov matrix for A_c, it follows from Lemma 3 that P is a Stein matrix for each of the bilinear terms. Thus A_d is a product of a bunch of matrices each of which have P as a Stein matrix. It now follows from Lemma 4 that P is a Stein matrix for A_d. ■

The following corollary is easily deduced from the main theorem. This is probably the most useful result in the paper.

Corollary 1: Suppose that $P = P^* > 0$ is a CLM for a finite set of matrices $\mathscr{A}_c$. Then P is CSM for any finite set of matrices $\mathscr{A}_d$, where each A_d in $\mathscr{A}_d$ is a diagonal Padé approximation of $e^{A_c h}$ of any order for some A_c in $\mathscr{A}_c$ and $h > 0$.

Proof : If P is a CLM for $\mathscr{A}_c$ then, P is an Lyapunov matrix for every A_c in $\mathscr{A}_c$. It now follows from Theorem 1, that P is a Stein matrix for every A_d in $\mathscr{A}_d$. Hence P is a CSM for $\mathscr{A}_d$. ■

The last corollary shows that the Padé approximation of e^{Ah} preserves quadratic stability. Thus, the discrete time system is quadratically stable if the continuous one is. However, the theorem does not imply the converse. In fact the converse is not true as the following example illustrates.

Example 1: Consider the Hurwitz matrices:

$$A_{c1} = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Since the matrix product $A_{c1}A_{c2}$ has negative real eigenvalues it follows that there is no CLM [8]. Now consider the matrices A_{d1}, A_{d2} obtained under the 2^{nd} order diagonal Padé approximation of $e^{2A_{ci}}$ with the discrete time step $h = 2$:

$$A_{d1} = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0.8187 \end{bmatrix}.$$

These matrices have a CSM which is

$$P_d = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}.$$

Comment: Example 1, together with Corollary 1, illustrate the following facts. Let $\mathscr{A}_c$ be a finite set of Hurwitz matrices and $\mathscr{A}_d$ a corresponding finite set of Schur stable matrices obtained under a diagonal Padé approximation. If P is a CLM for $\mathscr{A}_c$ then P is a CSM for $\mathscr{A}_d$. However, as the above example demonstrates, the existence of a CSM for $\mathscr{A}_d$ does not imply the existence of a CLM for $\mathscr{A}_c$. In order to achieve a converse result, additional conditions have to be imposed and this issue has been discussed in [12].

IV. Implications of Main Result

The starting point for our work was the recently published paper [2]. One of the main results of that paper was the fact that the bilinear transform preserves quadratic stability when applied to continuous-time switched systems. We have shown that this property also holds for general diagonal Padé approximations (although the converse statement is not true). This is an important observation due to the fact that while the bilinear transform is stability preserving, it is not always a good approximation to the matrix exponential. Our result says that "more accurate" approximations are also stability preserving.

Two potential applications of this result are immediate. First, stable discrete-time systems may be obtained from their continuous-time counterparts in a manner akin to that described in [2]. Secondly, our results may provide a method to discretize quadratically stable linear switched systems; see [7] for a recent paper on this topic. That is, given a quadratically stable switched linear system, a discrete-time counterpart obtained using diagonal Padé approximations to the matrix exponential, will also be quadratically stable.

In the context of the previous comment, it is important to realise that the stability preserving property of Padé approximations is very important. It was recently shown that non-quadratic Lyapunov functions may not be preserved under the bilinear transform. This fact was first demonstrated in [2], where it was proven that unlike quadratic Lyapunov functions (QLFs), ∞-norm and 1-norm type Lyapunov functions are not necessarily preserved under the bilinear mapping. In fact the situation may be worse as the following example illustrates.

Example 2: Consider a continuous-time switching system described by (1) with $\mathscr{A}_c = \{A_{c1}, A_{c2}, A_{c3}\}$ where

$$
\begin{aligned}
A_{c1} &= \begin{bmatrix} -19.00 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -0.10 \end{bmatrix}, \\
A_{c2} &= \begin{bmatrix} -19 & 0 & 0 \\ -10 & -9 & 0 \\ -18.75 & 0 & -0.10 \end{bmatrix}, \\
A_{c3} &= \begin{bmatrix} -19.00 & 0 & 18.75 \\ 0 & -9 & 8.75 \\ 0 & 0 & -0.10 \end{bmatrix}.
\end{aligned}
$$

Using the ideas in [10] it can be shown that the continuous-time switching system is globally exponentially stable. It follows from the results of Dayawansa and Martin [11] that this switching system has a Lyapunov function (though this is not necessarily quadratic). Now consider a discrete-time approximation to the above system. We assume that switching is restricted to only occur at multiples of the sampling time $h = 0.25$. Using the first order Pad'e approximation, we obtain a discrete-time switching system described by (2) with $\mathscr{A}_d = \{A_{d1}, A_{d2}, A_{d3}\}$ where

$$A_{di} = (I - \frac{1}{8}A_{ci})^{-1}(I + \frac{1}{8}A_{ci}), \qquad i = 1, 2, 3,$$

that is,

$$
\begin{aligned}
A_{d1} &\approx \begin{bmatrix} -0.40 & 0 & 0 \\ 0 & -0.06 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}, \\
A_{d2} &\approx \begin{bmatrix} -0.40 & 0 & 0 \\ -0.35 & -0.06 & 0 \\ -1.37 & 0 & 0.98 \end{bmatrix}, \\
A_{d3} &\approx \begin{bmatrix} -0.40 & 0 & 1.37 \\ 0 & -0.06 & 1.01 \\ 0 & 0 & 0.98 \end{bmatrix}.
\end{aligned}
$$

We now claim that the discrete-time switching system is unstable. To see this we simply consider the incremental switching sequence $A_{d3} \to A_{d2} \to A_{d1}$; then the dynamics of the system evolve according to the product

$$A_d = A_{d1} A_{d2} A_{d3}.$$

Since the eigenvalues of A_d are approximately $\{-0.002, -0.060, -1.035\}$, then with one eigenvalue outside the unit disc, this switching sequence, repeated periodically results in an unstable system.

Comment : It is important to put the above example in context. The bilinear transform of A_{ci} used is a first order Padé approximation of $e^{A_i ht}$, where $h = 0.25$ is the sampling time. The point of our example is to illustrate that, when a QLF does not exist for a switched linear continuous time system, it is possible that stability may be lost via the Padé approximation. Our sampling time h is chosen to illustrate this point. Clearly, by selecting a faster sampling time one obtains a better approximation to the continuous-time system and stability may be preserved.

Our example is consistent with the results reported in a recent paper [2], where it is noted that while quadratic Lyapunov functions are preserved under the bilinear transform, other non-quadratic Lyapunov functions are not. Unfortunately, the example demonstrates that matters are much worse than reported in this paper; namely, that not only are non-quadratic functions not preserved under this mapping, but also stability need not be.

V. Conclusions

In this paper we have shown that diagonal Padé approximations to the matrix exponential preserves quadratic Lyapunov functions between continuous-time and discrete-time switched systems. We have also shown that the converse is not true. Namely, it does not follow that the original continuous-time system is quadratically stable even if the discrete-time system has a quadratic Lyapunov function. Furthermore, it is easily seen that such approximations do not (in general) preserve stability when used to discretize switched systems that are stable (but not quadratically stable).

Our results suggest a number of interesting research directions. An immediate question concerns discretization methods that preserve other types of stability. Since general Padé approximations can be thought of as products of complex bilinear transforms, an immediate question in this direction concerns the equivalent map for other types of Lyapunov functions. Namely, given a continuous-time system with some Lyapunov functions, what are the mappings from continuous-time to discrete-time that preserve the Lyapunov functions. A natural extension of this question concerns whether discretization methods can be developed for exponentially stable switched and nonlinear systems but which do not have a quadratic Lyapunov function. An important question also concerns systems in which stability criteria and replaced with optimality criteria. In other words, how does one discretise a system and preserve certain types of optimality criteria. Finally, a important question concerns the analysis of the stability of feedback systems with both discrete-time and continuous-time subsystems. Such systems arise when a discrete-time control is used to regulate a switched system. These and other topics will be the subject of future publications.

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