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Research Article

The Spherical Boundary and Volume Growth

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We consider the *spherical boundary*, a conformal boundary using a special class of conformal distortions. We prove that certain bounds on volume growth of suitable metric measure spaces imply that the spherical boundary is “small” (in cardinality or dimension) and give examples to show that the reverse implications fail. We also show that the spherical boundary of an annular convex proper length space consists of a single point. This result applies to l^2-products of length spaces, since we prove that a natural metric, generalizing such “norm-like” product metrics on a (possibly infinite) product of unbounded length spaces, is annular convex.

1. Introduction

There are various notions of “boundaries at infinity” of metric spaces in the literature. One of these is the spherical boundary $\partial_S X$ of certain unbounded metric spaces, as introduced in [1]. This is defined in detail in Section 2, but let us mention here that it is a byproduct of the concept of sphericalization, which replaces an unbounded metric l by a conformally distorted bounded metric σ. This allows one to interpret results in [2] concerning the quasihyperbolizations of bounded length spaces in the context of certain unbounded spaces. To this end, the relationship between l and σ, together with the relationship between their associated quasihyperbolizations, was studied in [1].

The spherical boundary $\partial_S X$ is a key ingredient in studying the invertibility of the sphericalization process as is clear from the results in [1], but no detailed study of the links between features of X and features of $\partial_S X$ was carried out there. The current paper aims to throw more light on one such link by proving results of the following type: if the spherical boundary of a suitable metric measure space is sufficiently large, then X has rapid volume growth. For instance, contrast Euclidean space $\mathbb{R}^n$ whose spherical boundary is a singleton set (if $n > 1$) with the hyperbolic plane whose spherical boundary has infinite Minkowski dimension.

After some preliminaries in Section 2, Section 3 examines annular convexity and conditions under which the spherical boundary is a singleton set and annular convexity, and Section 4 contains the main results.

2. Preliminaries

We denote by $a \wedge b$ and $a \vee b$ the minimum and maximum, respectively, of numbers a, b.

2.1. Metric Spaces and Paths

Below, (X, d) is always a metric space which may have additional properties as specified. We denote by $\overline{X}_d$ the metric closure of (X, d) and, viewing X as a subset of $\overline{X}_d$, we write $\partial X_d = \overline{X}_d \setminus X$. Given $x, y \in \overline{X}_d$, $\Gamma(x, y)$ denotes the class of rectifiable paths $\lambda : [0, T] \to \overline{X}_d$ for which $\lambda|_{(0,T)}$ is a rectifiable path in X, $\lambda(0) = x$, and $\lambda(T) = y$. We also define $\Gamma_d(x, y)$ to be the subset of $\Gamma(x, y)$ consisting of paths that are parametrized by d-arclength. We write $\Gamma(x, y; X)$ or $\Gamma_d(x, y; X)$ if the space needs to be specified.

Suppose (X, d) is rectifiably connected and, only for this paragraph, let us write $d'(x, y) = \inf_{\gamma \in \Gamma_d(x,y)} \mathrm{len}_d(\gamma)$, $x, y \in \overline{X}_d$. When restricted to $X \times X$, d' defines the *inner metric* associated with d. We say that d is a *length metric*, and that (X, d) is a *length space*, if $d(x, y) = d'(x, y)$, for all $x, y \in X$; this equality clearly extends to points $x, y \in \overline{X}_d$. More generally, we say that a rectifiably connected metric space (X, d) is a *local length space* if $d(x, y) = d'(x, y)$ whenever $x \in X$, $y \in \overline{X}_d$, and $d(x, y) \le d(x, \partial X_d)$. (X, d) is a *geodesic space* if, for all $x, y \in X$, there exists a path $\gamma \in \Gamma_d(x, y)$ of length $d(x, y)$.

Every domain $\Omega \subset \mathbb{R}^n$ is a local length space when equipped with the Euclidean metric, and a slit disk in the Euclidean plane is a simple example of a local length space that is not a length space.

Given a local length space (X, d), we define the *length boundary of* (X, d), $\partial_0 X_d$, to be the set of all points $y \in \partial X_d$ for which $\Gamma_d(x, y)$ is nonempty for some (and hence all) $x \in X$. Equivalently $\partial_0 X_d$ is the set of all $y \in \overline{X}_d$ whose inner distance from some (and hence all) $x \in X$ is finite. If d is a length metric, then $\partial_0 X_d = \partial X_d$, but equality may fail if d is merely a local length space. For instance, if d is the Euclidean metric on a domain $\Omega \subset \mathbb{R}^n$ which spirals sufficiently tightly near some point $y \in \partial \Omega_d$, then $y \notin \partial_0 \Omega_d$.

The rest of our notation is quite standard. We denote by $B_d(x, r)$, $\overline{B}_d(x, r)$, and $S_d(x, r)$, the *open ball*, *closed ball*, and *sphere of radius r about* $x \in X$; we omit the d-subscript if the metric is understood. If $r \le 0$, $B_d(x, r)$ is the empty set. A metric space is *proper* if all its closed balls are compact.

An *arc* in X is an injective path $\gamma : I \to X$. We do not distinguish notationally between paths and their images. If γ is an arc in X, and $u, v \in \gamma$, $\gamma[u, v]$ is the subarc of γ with endpoints u, v. Given two d-rectifiable arcs γ, γ' in metric spaces (X, d) and (X', d') with $\mathrm{len}_d(\gamma) \le \mathrm{len}_{d'}(\gamma')$, we define the *initial length map* $f : \gamma \to \gamma'$ by the requirement that f maps each initial segment $\gamma[u, v]$ of γ to the initial segment $\gamma'[u', v']$ of γ' that satisfies $\mathrm{len}_{d'}(\gamma'[u', v']) = \mathrm{len}_d(\gamma[u, v])$.

2.2. Metric Measure Spaces

A *metric measure space* (X, d, μ) is a metric space with an attached positive Borel measure μ which gives positive finite measure to all balls; if (X, d) is a (local) length space we call (X, d, μ) a *(local) length measure space*.

Suppose (X,d,μ) is a metric measure space and $C \geq 1$. We say that (X,d,μ) is *C-doubling* if $\mu(2B) \leq C\mu(B)$ whenever $tB = B(x,tr)$ for fixed but arbitrary $x \in X$, $r > 0$. We say that (X,d,μ) is *C-translate doubling* if instead $\mu(B') \leq C\mu(B)$ whenever B, B' are overlapping balls of the same radius or *weak C-translate doubling* if we merely have $\mu(B') \leq C(1+r)\mu(B)$ whenever B, B' are overlapping balls of radius r.

A measure is doubling if and only if it is translate doubling and the underlying space has finite Assouad dimension (equivalently, all balls can be covered by a bounded number of balls of half the radius). Thus, doubling and translate doubling are (quantitatively) equivalent in Euclidean space and examples of length spaces with translate doubling measures that fail to be doubling including Hausdorff n-measure on hyperbolic n-space and arclength measure on the Cayley graph of an n-generator free-group.

The *lower Minkowski dimension* $\underline{\dim}_M E$ of a subset E of a metric space (X,d) is defined by

$$\underline{\dim}_M(E) = \liminf_{\epsilon \to 0^+} \frac{\log N(E,\epsilon)}{\log(1/\epsilon)}, \tag{2.1}$$

where $N(E,\epsilon)$ is the maximum cardinality of a collection of disjoint open balls of radius ϵ and centers in E.

2.3. The Spherical Boundary

A Borel function $g : [0,\infty) \to (0,\infty)$ is said to be a *C-sphericalizing function*, $C > 2$, if it has the following properties:

(S1) $g(r) \leq Cg(s)$ whenever $r, \ s \geq 0, \ r \leq 2s+1$, and $s \leq 2r+1$;

(S2) $\int_r^\infty g(t)dt \leq Crg(r), \ r \geq 1$.

We recall the following property of a sphericalizing function, taken from [1].

Lemma 2.1. *If $g : [0,\infty) \to (0,\infty)$ is a C-sphericalizing function then*

(S3) $g(s)/g(r) \leq C^2(r/s)^{1+1/C}$, *for all* $1 \leq r \leq s$.

In particular, $tg(t) \to 0$ *as* $t \to \infty$.

Suppose (X,l,o) is an unbounded pointed local length space, and let us write $|x| = l(x,o)$, $x \in \overline{X}_l$. Given a sphericalizing function $g : [0,\infty) \to (0,\infty)$, we define a new metric $\mathfrak{S}(l,o,g)$ on X by the equation:

$$\mathfrak{S}(l,o,g)(x,y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma g(|z|)dl(z), \quad x,y \in X. \tag{2.2}$$

We usually write σ in place of $\mathfrak{S}(l,o,g)$.

If $\gamma \in \Gamma_l(x,y)$, $x \in X$, $y \in \partial X_l$, then it is clear that γ is also of finite σ-length, so the length boundary $\partial_0 X_l$ can be viewed as a subset of ∂X_σ. We define the *spherical boundary of* X, $\partial_S X$ to be $\partial X_\sigma \setminus \partial_0 X_l$, and the *spherical closure of* X to be $\overline{X}_\sigma$. Since any point in $x \in X \cup \partial_0 X_l$ is at a finite l-distance from o, and since g is bounded away from zero on bounded intervals, it follows that a sequence in X is σ-convergent to a point in $X \cup \partial_0 X_l$ if and only if it is l-convergent. It also follows that if $x \in X$, $y \in \partial_S X$, and $\gamma \in \Gamma_\sigma(x,y)$, then γ cannot be contained in any ball $B_l(o,r)$.

We record some useful elementary estimates involving the two metrics l and $\sigma = \mathfrak{S}(l,o,g)$. Below, $G(t) \equiv (1+t)g(t)$, $|x| \equiv l(x,o)$, $\delta_\infty(x) = \sigma(x,\partial_S(X))$, and C_g is the sphericalization constant of g. It follows by standard analysis (as in [1, Proposition 2.14]) that $\delta_\infty(x) \ge \int_{|x|}^\infty g(t)dt$, for all $x \in X$. Using (S2) and (S3), we readily deduce that

$$\delta_\infty(x) \ge C_g^{-2} G(|x|), \quad x \in X. \tag{2.3}$$

Suppose $\gamma \in \Gamma_l(o,x)$ for some $x \in X$, with $\mathrm{len}_l(\gamma) = L < |x| + 1$, and let $\nu = \gamma|_{[s,L]}$, where $\gamma(s) = v$, $|v| \ge 1$. Then $|\gamma(t)| \in (t-1,t)$, $0 \le t \le L$, and so using (S1) and (S2), we deduce that

$$\mathrm{len}_\sigma(\nu) = \int_s^L g\left(\left|\gamma(t)\right|\right)dt \le C_g^2 \int_{|v|}^L g(t)dt \le C_g^3 G(|v|). \tag{2.4}$$

3. Spherical Boundaries and Annular Convexity

A metric space (X,d) is said to be C-*annular convex* for some $C \ge 2$ if for every $o \in X$, $r > 0$, and every pair of points $x_1, x_2 \in B(o,r) \setminus B(o,r/2)$ there exists a path γ from x_1 to x_2 in the annulus $B(o,Cr) \setminus B(o,r/C)$ of length at most $Cd(x_1,x_2)$.

Annular convex spaces, introduced by Herron et al. in [3], form a large class of spaces that include all Banach spaces (with the exception of one-dimensional real Banach spaces) and most spaces equipped with a doubling measure that supports a Poincaré inequality (as follows from the results in [4]).

In this section, we show that finite and countable l^2 products of metric spaces are annular convex. This is of interest to us because of the following simple result.

Proposition 3.1. *The spherical boundary of an unbounded proper annular convex pointed length space* (X,l,o) *is a singleton set.*

Proof. Since X is proper, it follows from [5, Theorem 2.4] that $\partial_S X$ is nonempty. We write $|a| = l(a,o)$, $a \in X$, and let $\sigma = \mathfrak{S}(l,o,g)$, where g is a given C_g-sphericalizing function.

Fixing a fix a pair of points $z, w \in \partial_S X$, we pick sequences (z_n) and (w_n) in X converging to z and w, respectively. Since necessarily $|z_n| \to \infty$ and $|w_n| \to \infty$, we may assume that $|z_n| > n$ and $|w_n| > n$, $n \in \mathbb{N}$.

Joining z_n to o by a path of length at most $|z_n| + 1$, and picking a point z_n' on this path with the property that $|z_n'| = n$, it follows from (2.4) that $\sigma(z_n, z_n') \le C_g^3 n g(n)$. Using Lemma 2.1, we deduce that $\sigma(z_n, z_n') \to 0$ as $n \to \infty$. We similarly find points w_n' such that $|w_n'| = n$ and $\sigma(w_n, w_n') \to 0$ as $n \to \infty$.

But by annular convexity and the properties of sphericalizing functions, it is easy to see that $\sigma(z_n', w_n') \le C' n g(n)$, where $C' = C'(C, C_g)$ and so $\sigma(z_n', w_n') \to 0$. Thus $\sigma(z_n, w_n) \to 0$ and so $z = w$, as required. □

Note that the assumption that X is proper in Proposition 3.1 was needed only to show that the spherical boundary is nonempty. Some such condition is needed to deduce this fact: for instance if X is a bouquet of line segments of length n for each $n \in \mathbb{N}$, joined together by identifying with each other the left endpoints of all such intervals, then it is easy to show that $\partial_S X$ is empty.

It is clear from the proof of Proposition 3.1 that C-annular convexity can be replaced by the following formally weaker condition: a metric space (X, d) is *weakly C-annular convex*, where $C \geq 1$, if for every $o \in X$, $r > 0$, and every pair of points $x_1, x_2 \in X$ such that $d(o, x_i) = r$, $i = 1, 2$, there exists a path γ from x_1 to x_2 in $X \setminus B(o, r/C)$ of length at most $2Cr$. However, replacing annular convexity by weak annular convexity is of no real benefit in Proposition 3.1, since for length spaces the two conditions are quantitatively equivalent. We record the simple argument for completeness.

Proposition 3.2. *If a length space is weak C-annular convex, then it is $(3C)$-annular convex.*

Proof. Consider distinct points $x_1, x_2 \in B(o, r) \setminus B(o, r/2)$. Let $d_i := d(o, x_i)$, $i = 1, 2$. Join x_1, x_2 by a path λ of length less than $d(x_1, x_2) + \epsilon$, where $\epsilon > 0$ is so small that $d(x_1, x_2) + \epsilon < 2d(x_1, x_2)$. Clearly λ remains inside $B(o, 3Cr)$, so it certainly verifies the $(3C)$-annular convexity condition if it remains outside $B(o, r/3)$. Assume therefore that λ ventures inside this ball. Let z_1, z_2 be the first and last points z on λ such that $d(z, o) = r/3$, let γ_1 be the initial segment of λ from x_1 to z_1, let γ_2 be the final segment of λ from z_2 to x_2, and let γ_3 be a path given by weak annular convexity for the pair z_1, z_2 and center point o. Let γ be the concatenation of γ_1, γ_3, and γ_2.

Then $\mathrm{len}_d(\gamma_1) + \mathrm{len}_d(\gamma_2) \leq \epsilon + d(x_1, x_2)$ and weak annular convexity gives $\mathrm{len}_d(\gamma_3) \leq 2Cr/3$, so $\mathrm{len}_d(\gamma) \leq \epsilon + d(x_1, x_2) + 2Cr/3$. The path λ intersects $B(o, r/3)$ and its endpoints lie outside $B(o, r/2)$, so $r/3 = 2(r/2 - r/3) < d(x_1, x_2) + \epsilon$. Thus

$$\mathrm{len}_d(\gamma) < (2C + 1)(d(x_1, x_2) + \epsilon) \leq 3Cd(x_1, x_2), \tag{3.1}$$

when $\epsilon > 0$ is sufficiently small. By construction, γ remains outside $B(o, r/3C)$ and, since $\lambda \subset B(o, 3Cr)$, it suffices to verify that γ_3 also lies in this ball. But $\mathrm{len}(\gamma_3) \leq 2Cr/3$, and the endpoints of γ_3 are a distance $r/3$ from o, so

$$\gamma_3 \subset \overline{B}\left(o, \frac{(C + 1)r}{3}\right) \subset B(o, 3Cr), \tag{3.2}$$

as required. □

It turns out that product spaces are annular convex.

Proposition 3.3. *Let (X, d) be the Cartesian product of unbounded length spaces (X_1, d_1) and (X_2, d_2), with d being the l^2 product of d_1 and d_2. Then (X, d) is a 4-annular convex length space.*

We get the following immediate corollary of Propositions 3.1 and 3.3.

Corollary 3.4. *Let (X, d) be the Cartesian product of unbounded length spaces (X_1, d_1) and (X_2, d_2), with d being the l^2 product of d_1 and d_2. Then the spherical boundary of (X, d) is a singleton set.*

Rather than proving Proposition 3.3, we prove a much more general result that is modeled on the previously mentioned fact that Banach spaces of dimension at least 2 are annular convex. We will generalize this to a large class of what could roughly be termed normed spaces with values in unbounded length spaces (which can vary from point to point). More precisely, we look at metrics constructed in the following manner.

We begin with a real normed vector space $(V, \|\cdot\|)$ which we assume to consist of functions defined on an index set I. We say that the norm $\|\cdot\|$ on V is *monotonic on* I if

(a) V is a space of real-valued functions on I, that is, $V \subseteq \mathbb{R}^I$.

(b) If $f \in V$, $g : I \to \mathbb{R}$, and $|g(i)| \le |f(i)|$ for $i \in I$, then $g \in V$ and $\|g\| \le \|f\|$.

Assume now that (X_i, d_i), $i \in I$ are metric spaces and let $P := \prod_{i\in I} X_i$. We define as follows the *metric subproduct* (X, d) of (X_i, d_i), $i \in I$, relative to some fixed $a \in P$ and the norm $\|\cdot\|$, which is monotonic on I: if $x = (x_i)_{i\in I} \in P$, then $x \in X$ whenever $f(i) := d_i(a_i, x_i)$ defines a function $f \in V$, and we define $d(x, y) = \|i \mapsto d_i(x_i, y_i)\|$, which makes sense by monotonicity of the norm. We write $(X, d) = \prod_{i\in I}^{V;a}(X_i, d_i)$.

Theorem 3.5. *Suppose V is a normed vector space of dimension at least 2 which is monotonic on I, and suppose $(X, d) = \prod_{i\in I}^{V;a}(X_i, d_i)$, where each (X_i, d_i) is an unbounded length space. Then (X, d) is 4-annular convex.*

Before proving Theorem 3.5, we discuss metric subproducts X and give some examples. If $x, x' \in X$, it is clear that $d(x, x') \le d(x, a) + d(a, x')$, so certainly $d(x, x') < \infty$. It is routine to verify that d is a metric on X. Although X is defined with respect to some $a \in P$, it is clear that we get the same metric space (X, d) if a is replaced by any $b \in X$. However, we get a metric subproduct disjoint from our original X if we replace a by any $b \in P \setminus X$.

Lemma 3.6. *Suppose V is a normed vector space of dimension at least 2 which is monotonic on I, and suppose (X_i, d_i) is a length space for all $i \in I$. Then $(X, d) = \prod_{i\in I}^{V;a}(X_i, d_i)$ is also a length space.*

Proof. Suppose $x, x' \in X$ and let $\epsilon > 0$ be fixed but arbitrary. We define a path $\lambda : [0,1] \to X$ such that $\lambda(t) = (\lambda_i(t))_{i\in I}$, $0 \le t \le 1$, with the following important properties which we record for later reference:

$$\lambda_i : [0,1] \longrightarrow X_i \text{ is a constant speed path from } x_i \text{ to } x_i', \quad i \in I,$$
$$\operatorname{len}_d(\lambda_i) \le (1+\epsilon) d_i(x_i, x_i'), \quad i \in I, \tag{3.3}$$

$$d(\lambda(s), \lambda(t)) \le (1+\epsilon)(t-s) d(x, x'), \quad 0 \le s \le t \le 1. \tag{3.4}$$

In fact this is quite easy to do: since X_i is a length space, we can certainly pick λ_i satisfying (3.3). Then

$$d_i(\lambda_i(s), \lambda_i(t)) \le \operatorname{len}_{d_i}\left(\lambda_i|_{[s,t]}\right) = (t-s)\operatorname{len}_{d_i}(\lambda_i) \le (1+\epsilon)(t-s) d_i(x_i', y_i'), \tag{3.5}$$

for all $i \in I$ and $0 \le s \le t \le 1$. Assembling together these paths λ_i to get a path $\lambda : [0,1] \to X$, (3.4) follows from the last estimate and the fact that d is defined via a monotone norm. In particular, (3.4) implies that $\operatorname{len}_d(\lambda) \le (1+\epsilon) d(x, x')$. Since $x, x' \in X$ and $\epsilon > 0$ are all arbitrary, the result follows. □

Suppose we fix $o \in X$, where $o = (o_i)_{i\in I}$. For every $i \in I$ and $R > 0$ we can find a point $z_i(R) \in X_i$ such that $d_i(o_i, z_i(R)) = R$: in fact the unboundedness of X_i ensures that there exists $z \in X_i$ such that $d_i(o_i, z) \geq R$, and then we use continuity to pick the required $z_i(R)$ on a path from z to o_i. It follows that if $v \in V$, we can find a point $x^v = (x_i^v)_{i \in I} \in X$ such that $d_i(x_i^v, o_i) = |v_i|$ and so $d(x^v, o) = \|v\|$.

We use function notation $f(i)$ for V versus subscript notation x_i for X to emphasize the difference between the normed space $(V, \|\cdot\|)$ and the subproduct (X, d).

The simplest examples of monotonic normed spaces are l^p spaces associated with finite or countably infinite I, for $1 \leq p \leq \infty$. In the case of finite I, the subproduct X coincides as a set with the full Cartesian product $\prod_{i \in I} X_i$. In the special case where I has cardinality 2 and $p = 2$, we deduce Proposition 3.3.

Beyond the above l^p spaces, other examples of monotonic-normed spaces include l^p sums over uncountable index sets, but more interesting examples are normed sequence spaces of Orlicz or variable exponent $l^{p(\cdot)}$ type.

Note that if $I = \mathbb{N}$ and each X_i is the real line, then the subproduct is merely the normed space V translated by a sequence $a \in \mathbb{R}^{\mathbb{N}}$: thus these subproducts are all cosets of V, and any two such subproducts for different choices of a can be put into a natural 1-1 correspondence.

However, there is not always such a natural 1-1 correspondence. Consider for instance the case where X_i is the metric subspace of the real line given by $X_i = \{0\} \cup (\bigcup_{n \in \mathbb{N}} \{n, 1/n\})$ for each $i \in \mathbb{N}$ and $V = l^2$. If $a = (0)_{i \in \mathbb{N}}$, then it is readily verified that the metric subproduct $X := \prod_{n \in \mathbb{N}}^{V;a} X_i$ has the cardinality of the continuum, whereas if $a = (c)_{i \in \mathbb{N}}$ for any fixed $c \in X_i \setminus \{0\}$, then X is a countable space.

It can be shown that this dependence of the cardinality of X on our choice of a does not occur when the spaces (X_i, d_i), $i \in \mathbb{N}$ are length spaces (essentially because its cardinality is at least that of the continuum if V is nontrivial). However, the above example suggests that there is in general no natural map from one metric subproduct to another.

Example 3.7. The constant 4 cannot be improved in Theorem 3.5. For instance if X is the closed first quadrant of the l^1-plane; this choice of X corresponds to taking V to be the l^1 plane, with $X_1 = X_2$ being the Euclidean half line $[0, \infty)$. Let $r = 4$, $x = (0, 0)$, $y = (2, 0)$, and $o = (1 + \delta, 1 + \delta)$ where $0 < \delta < 1$. Then every path from x to y intersects $B(o, cr)$ for any $c > 1/4$, as long as $\delta < 4c - 1$.

We now move on to the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $x, y \in B(o, r) \setminus B(o, r/2)$ be the pair of points for which we want to verify the 4-annular convexity condition (with other data r, o as usual), and let $S := d(x, o) + d(y, o)$, so that $S > r$. As in the proof of Proposition 3.2, a path γ connecting x and y of length at most $(1 + \epsilon) d(x, y)$, where $\epsilon > 0$, verifies the 4-annular convexity condition for data (x, y, o, r) unless γ intersects $B(o, r/4)$. We may therefore assume that this intersection occurs and so

$$(1 + \epsilon) d(x, y) \geq \operatorname{len}_d(\gamma) > \left(d(x, o) - \frac{r}{4}\right) + \left(d(y, o) - \frac{r}{4}\right). \tag{3.6}$$

Taking a limit as $\epsilon \to 0$ we get $d(x, y) \geq S - r/2$. In particular, $d(x, y) > S/2$.

Let us write $e_A \in \{0, 1\}^I$ for the characteristic function of any $A \subset I$: thus $e_A(i) = 1$ if and only if $i \in A$. For $x \in X$ and $A \subset I$, let $x^A \in X$ be defined by $x_i^A = x_i$ if $i \in A$, and $x_i^A = o_i$ if $i \notin A$. For convenience, we write $e_i := e_{\{i\}}$ and $x^i := x^{\{i\}}$ for any $i \in I$.

Since V has dimension at least 2, monotonicity readily implies that there are distinct indices $j, k \in I$ such that the basic functions e_j, e_k lie in V.

We now define “scalar multiplication” on X, restricted to scalar values $0 \le \alpha \le 1$. Choosing $\lambda : [0,1] \to X$ to be as in the proof of Lemma 3.6, with $(x, x', \epsilon) = (x, o, \epsilon)$ and $0 < \epsilon \le 1/3$, we let $\alpha \cdot x := \lambda(s)$, where s is the minimal $s' \in [0,1]$ such that $d(\lambda(s'), o) = \alpha d(x, o)$. This definition is typically not unique since λ is not unique, but note that if $x_i = o_i$ then $(\alpha \cdot x)_i = o_i$ for all $0 \le \alpha \le 1$.

Suppose first that we can find some such as A such that $d(x^A, o) \ge r/4$ and $d(y^B, o) \ge r/4$, where $B := I \setminus A$. Let $z^x := \alpha \cdot x^A$ and $z^y := \beta \cdot y^B$ for $\alpha = r/4d(x^A, o)$ and $\beta = r/4d(y^B, o)$, so that $d(z^x, o) = d(z^y, o) = r/4$. Let λ_i, $i \in A$, and s be the associated coordinate paths and argument for $\alpha \cdot x^A$, as in the last paragraph.

Now join x and y by a path γ defined in the following piecewise manner by concatenating, in the natural order, paths $\gamma^m : [0,1] \to X$, $1 \le m \le 4$. First γ^1 is a path from x to z^x which has component paths $\gamma_i^1 : [0,1] \to X_i$, where γ_i^1 is a rescaled copy of $\lambda_i|_{[0,s]}$ if $i \in A$ (and $\lambda = (\lambda_i)$, s are as above), while γ_i^1 is a constant speed path from x_i to o_i of length at most $(1+\epsilon)d_i(x_i, o_i)$ if $i \in B$. Thus, each γ_i is a constant speed path of length at most $(1+\epsilon)d_i(x_i, o_i)$, and so $\text{len}_d(\gamma^1) \le (1+\epsilon)d(x, o)$.

Next let $z^{xy} = (z_i^{xy})_{i \in I}$, $z_i^{xy} = z_i^x$ for $i \in A$, and $z_i = z_i^y$ for $i \in B$. Then $d(z^x, z^{xy}) = d(z^y, z^{xy}) = r/4$. Let $\gamma^2 : [0,1] \to X$ be any path z^x to z such that the coordinate paths γ_i^2 are stationary paths for $i \in A$ and are of d_i-length at most $(1+\epsilon)d_i(z_i^x, z_i^{xy})$ for $i \in B$. Since $\|\cdot\|$ is a monotonic norm, we deduce that $\text{len}_d(\gamma_2) \le (1+\epsilon)r/4$.

Finally γ^3 and γ^4 are analogues of γ_2 and γ^1, respectively, but with (x, A, B) replaced by (y, B, A), and with the directions of the paths reversed. It follows that the d-length of our concatenated path γ is at most $(1+\epsilon)(S + r/2) \le 3(1+\epsilon)S/2$. Since $d(x, y) \ge S/2$ and $\epsilon \le 1/3$, we deduce that $\text{len}(\gamma) \le 4d(x, y)$, as required.

We next need to show that γ^m, $1 \le m \le 4$, stays outside $B(o, r/4)$. It follows from the definition of $\alpha \cdot x$ that $d(\lambda_1(t), o) \ge r/4$ for all $0 \le t \le 1$, the same estimate follows for γ^2 by monotonicity of the norm and symmetry with γ^2 and γ^1 then gives the same estimates for γ^3 and γ^4, respectively.

Finally we need to show that each γ^m is contained in $B(o, 4r)$. The triangle inequality ensures that $d(\gamma^1(t), o) < r + (\epsilon/2)r < 4r$ and $d(\gamma^2(t), o) < (r/4) + (1+\epsilon)(r/4) < 4r$ for all $0 \le t \le 1$. The same estimates for γ^3 and γ^4 follow by symmetry.

We may therefore make the added assumption that there is no way to split I into complementary subsets A and B such that $d(x^A, o) \ge r/4$ and $d(y^B, o) \ge r/4$. Note though that for any $w \in X$ and $A \subset I$, we either have $d(w^A, o) \ge d(w, o) - r/4 > r/4$ or $d(w^{I \setminus A}, o) \ge r/4$. By our added assumption, it follows that for every $A \subset I$ and $B := I \setminus A$, either $d(w^A, o) \ge d(w, o) - r/4$ and $d(w^B, o) < r/4$ both hold for $w \in \{x, y\}$ or $d(w^B, o) \ge d(w, o) - r/4$ and $d(w^A, o) < r/4$ both hold for $w \in \{x, y\}$.

In particular, one of these last pairs of conditions holds for a set A that contains j but not k. By switching the definitions of (A, j) and (B, k) if necessary, we assume that $d(w^A, o) \ge d(w, o) - r/4$ and $d(w^B, o) < r/4$ both hold for $w \in \{x, y\}$, and that $j \in A$. We choose $w \in X$ such that $w_i = o_i$ for $i \ne k$ and $d(w, o) = r/4$.

As in the previous case, we let $z^x := \alpha \cdot x^A$ for $\alpha = r/4d(x^A, o)$, but now we choose $z^y := \beta \cdot y^A$ for $\beta = r/4d(y^A, o)$. As before $d(z^x, o) = d(z^y, o) = r/4$. Also let w^x, w^y be the points satisfying $w_i^x = z_i^x$ and $w_i^y = z_i^y$ if $i \ne k$, and $w_k^x = w_k^y = w_k$.

We now join x and y by a path γ defined in the following piecewise manner by concatenating, in the natural order, paths $\gamma^m : [0,1] \to X$, $1 \le m \le 5$, where γ^1 is a path from x to z^x defined as in the previous case, γ^2 is a path of length at most $(1+\epsilon)r/4$ from z^x to w^x which is stationary except in coordinate k, γ^3 is a path of length at most $(1+\epsilon)r/2$ from w^x to w^y which is stationary in coordinate k, γ_4 is a path of length at most

$(1+\epsilon)r/4$ from w^y to z^y which is stationary except in coordinate k, and γ^5 is analogous to γ^1 in reverse, but from z^y to y. As in the previous case, we see that $\text{len}(\gamma) \le (1+\epsilon)S + r$. Since $d(x,y) > S/2 > r/2$, by taking ϵ to be sufficiently small we get $\text{len}(\gamma) \le 4d(x,y)$, as required. The fact that $\gamma \subset B(o,4r) \setminus B(o,r/4)$ can be verified as before, so we leave it to the reader. □

The examples that we have so far include the cases where V is an l^p or related space, but we cannot handle general L^p spaces because the requirement that the norm is monotonic restricts us to spaces V where nonnegative functions that are pointwise less than a given function in V must also lie in V. This is incompatible with spaces of measurable functions (unless the sigma algebra is the power set), let alone spaces of continuous or smooth functions. To get similar results for such spaces, the basic problem is getting fine control over the relationship between $d_i(\lambda_i(t), o_i)$ for different values of $i \in I$ and fixed $0 < t < 1$, where $\lambda = (\lambda_i)$ is as in Proposition 3.2. One way to get such control is to assume that each (X_i, d_i) is a geodesic space, so that we can assume that $\lambda_i : [0,1] \to X_i$ is a constant speed geodesic. Then $\lambda = (\lambda_i)$ is also a constant speed geodesic and $d(\lambda(t), o) = (1-t)d(x,o)$, allowing us to get analogues of Theorem 3.5 for more general spaces. The assumptions of monotonicity and dimension at least 2 would need to be replaced by assumptions appropriate to the context.

4. Large Spherical Boundary and Fast Volume Growth

A metric measure space, even a proper one, can have very fast volume growth and small spherical boundary, in the sense that its spherical boundary is a singleton set. For instance the product Riemannian manifold $X = H^2 \times H^2$ has exponential volume growth and constant negative Ricci curvature, but Corollary 3.4 implies that $\partial_S X$ is a one-point space.

However, implications in the reverse direction are possible. Our *Guiding Principle* is that for *reasonably general classes* of pointed length measure spaces (X, l, o, μ), a *large* spherical boundary forces (X, l) to have *rapid volume growth*. By making appropriate choices for the vague italicized phrases in our Guiding Principle, we get some theorems. We state and prove three such results in this section and discuss some relevant examples. In all instances, the *reasonably large class of spaces* consists of spaces satisfying a doubling condition or some weak variant thereof.

Throughout this section, (X, l, o) is a pointed length space, g is a C_g-sphericalizing function, with associated spherical metric $\sigma = \mathfrak{S}(l, o, g)$ and spherical boundary $\partial_S X$. Also $G(t) = (1+t)g(t)$, $|x| = l(x,o)$, and $\delta_\infty(x) = \text{dist}_\sigma(x, \partial_S X)$.

In our first result, we assume that our metric measure space is doubling. This is a rather strong condition and it implies slow (meaning polynomial rate) volume growth so, without any explicit mention of volume growth, we deduce that the spherical boundary is quite small in the sense of having finite cardinality.

Theorem 4.1. *Suppose (X, l, μ) is C_μ-doubling. Then $\partial_S X$ is a finite set whose cardinality is bounded by a number dependent only on C_g and C_μ.*

In our other results, we replace doubling by translate doubling or weak translate doubling. Unlike doubling, (weak) translate doubling puts no real constraint on volume growth, so volume growth enters the statements of our results explicitly.

Theorem 4.2. *Suppose (X, l, μ) is C_μ-translate doubling. If $\underline{\dim}_M \partial_S X > 0$, then $f(r) \equiv \mu(B_l(o, r))$ grows faster than any polynomial. In fact,*

$$\liminf_{r\to\infty} \frac{\log f(r)}{\log^2 r} > 0. \tag{4.1}$$

Theorem 4.3. *Suppose (X, l, μ) is weak C_μ-translate doubling. If $\underline{\dim}_M \partial_S X > 0$, then $f(r) := \mu(B_l(o, r))$ grows at a polynomial rate or faster, that is,*

$$\liminf_{r\to\infty} \frac{\log f(r)}{\log r} > 0. \tag{4.2}$$

Proof of Theorem 4.1. Suppose $\partial_S X$ has at least N points $z_1, \ldots, z_N$. Choose $0 < \epsilon < (\delta_\infty(o)/5) \wedge (G(1)/9C_g^3)$ so small that the balls $4B_i \equiv B_\sigma(z_i, 4\epsilon)$ are all disjoint and choose points $u_i \in B_i$.

Suppose $\delta_\infty(z) \le 3\epsilon$. Using (2.3), we get

$$\frac{G(|z|)}{C_g^2} \le \frac{G(1)}{3C_g^3}. \tag{4.3}$$

It now follows from (S1) and the definition of G that $|z| \ge 2$.

We carry out the following construction for each index i, $1 \le i \le N$. Choose $\gamma_i \in \Gamma_l(o, u_i)$ with $\mathrm{len}_l(\gamma) = L < |u_i| + 1$, and let $B_i' := B_\sigma(v_i, \epsilon)$, where v_i is the first point at which γ_i meets $\overline{3B_i}$. Now $d_\infty(v_i) \le 3\epsilon$, and so $|v_i| \ge 2$. Using (2.3) and (2.4), we see that $G(|v_i|) \approx \epsilon$. In view of (S3), we see that the distances $|v_i|$ are mutually comparable, so let us choose a pair of mutually comparable radii r, R such that $2 \le r \le |v_i| \le R$. By (S1), $B_l(v_i, t) \subset B_\sigma(v_i, C_g t g(t))$ for every $0 < t \le r$. We can therefore fix $t \le r$, $t \approx r$, so that $B_i'' := B_l(v_i, t) \subset B_i'$.

Every B_i'' is contained in the single ball $B_0 = B_l(o, R + t)$ and in turn B_0 is contained in each of the balls sB_i'', $s = (2R + t)/t$. Since t and R are comparable, doubling ensures that $\mu(B_0) \le C_1 \mu(B_i'')$, where C_1 depends only on C_μ and $(2R + t)/t \lesssim 1$. Since B_0 contains N disjoint balls of measure at least $\mu(B_0)/C_1$, it follows that $N \le C_1$, as required. □

Proof of Theorem 4.2. Part of the proof is similar to that of Theorem 4.1, so we will be sketchy. Since $\underline{\dim}_M \partial_S X > 0$, there are constants $c, Q > 0$ such that $\partial_S X$ contains $c\epsilon^{-Q}$ disjoint σ-balls $B_{\epsilon,i}$ of radius ϵ for all $0 < \epsilon \le \mathrm{dia}_\sigma(X)$. Taking $\epsilon_j = A^{-j}\mathrm{dia}_\sigma(X)$ for a fixed number $A > 1$, we can associate radii t_j such that each of the $c\epsilon_j^{-Q}$ balls $B_{\epsilon_j,i}$ contains an l-ball $B_{j,i}''$ whose radius is t_j and whose distance from the origin is contained in the interval $[r_j, R_j]$ for some numbers r_j, R_j are comparable with t_j; the constants of comparability can be taken to depend only C_g. We assume, as we may, that A is chosen so large that $c\epsilon_j^{-Q} \ge 2^j$, $r_1 \ge 1$, and $R_j + 2t_j < r_{j+1}$, $j \in \mathbb{N}$. Note also that the ratios r_{j+1}/r_j are uniformly bounded by a constant dependent only on A and C_g, so that $\log r_j \approx j$.

Translate doubling ensures that the balls $B_{j,i}''$ are of comparable measure with $B_j \equiv B_l(o, r_j)$, so there exists a constant $C > 0$ such that $f(r_{j+1}) \ge 2^j f(r_j)/C$ for each $j \in \mathbb{N}$. Iterating this, we get that $f(r_j) \ge 2^{j(j-1)/2} f(r_1)/C^j$. Since $\log r_j \approx j$, the result follows. □

We omit the proof of Theorem 4.3 as it is so similar to that of Theorem 4.2. In fact it differs from it only in the last paragraph above, and the required modifications are straightforward.

We now consider some examples. All of our examples are either n-dimensional Riemannian manifolds or one-point joins of a finite number of n-dimensional Riemannian manifolds (meaning that the distinguished points o in these manifolds are all identified with each other). In all these cases, the associate measure is the usual measure on a Riemannian manifold (or equivalently Hausdorff n-measure).

It is easy to give examples relevant to Theorem 4.1. Euclidean space $\mathbb{R}^n$ has spherical boundary of cardinality 2 for $n = 1$, and 1 for $n > 1$: the $n = 1$ case follows easily from the definition, while the $n > 1$ case follows for instance from Corollary 3.4. The one-point join at 0 of $k \in \mathbb{N}$ copies of the half-line $[0, \infty)$ is a doubling space whose spherical boundary has cardinality k.

We do not know whether or not there exists a space (X, l, μ) that satisfies the assumptions of Theorem 4.2 and has sharp volume growth rate

$$\liminf_{r \to \infty} \frac{\log \mu(B_l(o, r))}{\log^2 r} < \infty. \tag{4.4}$$

However, hyperbolic space (H^n, l, μ) is an example of a translate doubling space with much faster volume growth whose spherical boundary has infinite Minkowski dimension, as follows from the following more precise result.

Proposition 4.4. *Let $\sigma = \mathfrak{S}(l, o, g)$ be the sphericalized metric on H^n for the standard sphericalizing function $g(t) = (1 + t)^{-2}$. The minimum number N_r of σ-balls of radius $r > 0$ required to cover $\partial_S H^n$ satisfies*

$$C_n^{-1} r^{n-1} \exp\left(\frac{(n-1)}{r}\right) \le N_r \le C_n r^{n-1} \exp\left(\frac{2(n-1)}{r}\right), \tag{4.5}$$

where C_n depends only on n. For a general sphericalizing function g, N_r grows faster than $c_g \cdot \exp((n-1)/r^{c_g})/C_n$, where $c_g, C_n > 0$ depend only on their subscripted parameters.

Proof. We assume that $n = 2$: this does not change anything essential in the proof but it simplifies the notation. It suits us to think of H^2 as the warped product $B \times_f F$, where $B = [0, \infty)$, $F = S^1$, and the warping function is $f(t) = \sinh t$. We identify $\partial_S H^2$ with F as a set and view F as the set of points in the complex plane of the form $\exp(i\theta)$, $\theta \in \mathbb{R}$. For the moment, assume that $g(t) = (1 + t)^{-2}$. Due to the symmetry of H^2, to get the lower bound on N_r, it suffices to show that $\sigma(\exp(i\theta), 1) \ge 2r$, whenever $\theta \approx \exp(-1/r)/r$, is sufficiently small.

Suppose we join $\exp(i\theta)$ with 1 via a path γ whose B-coordinate achieves a minimum value $s \ge 0$. Considering only the horizontal component of arclength, we deduce from that $\mathrm{len}_\sigma(\gamma) \ge 2\int_s^\infty g(t)dt = 2/(1 + s)$. Considering only the vertical component of arclength, we have $\mathrm{len}_\sigma(\gamma) \ge H(s) := \theta g(s) \sinh s$. Thus $\sigma(\exp(i\theta), 0) \ge m$, where m is the minimum over all $s \ge 0$ of $M(s) := 2/(1+s) \vee H(s)$. Since we may take θ to be less than θ_0 for any $\theta_0 > 0$ of our choice, we may assume that the minimum of $M(s)$ occurs when $s > 1$. But then m equals the minimum over all $s \ge 1$ of $2(1+s)^{-1} \vee (\theta(1+s)^{-2} \sinh s)$, which occurs when $1+s = \theta \sinh(s)/2$. Taking $r = 1/(1 + s)$ in this last equation gives $\theta = 2\sinh(1 - 1/r)/r$, as required.

To obtain an upper bound on N_r, it suffices to consider the path γ consisting of a horizontal segment from $\exp(i\theta)$ to the point with first coordinate s where $s = \theta \sinh(s)$, then the shorter vertical segment to the point with second coordinate 1, and finally a horizontal segment to $1 \in F = \partial_S H^2$. Then $\sigma(\exp(i\theta), 1) \leq \text{len}_\sigma(\gamma) = 2/(1+s) + H(s)$, where $H(s)$ is as above. Taking s as above gives the required upper bound for N_r.

For a general sphericalizing function g, we obtain as above that

$$\text{len}_\sigma(\gamma) \gtrsim G(s) \vee (\theta g(s) \sinh s), \tag{4.6}$$

where $G(s) = (1+s)g(s)$. This lower bound is minimal when $1 + s = \theta \sinh s$. Using the fact that $G(s)$ decays at a polynomial rate as $s \to \infty$, it is a routine matter to obtain the desired conclusion. □

Finally we show that Theorem 4.3 is sharp by considering the warped product (X, l), where $X = B \times_f F$, $B = [0, \infty)$, $F = S^1$, the warping function is $f(t) = t^2$, and the sphericalizing function is $g(t) = (1+t)^{-2}$. We take $o \in X$ to be the (unique) point with first coordinate 0.

Proposition 4.5. *If (X, l, o) is as above then $(\partial_S X, \sigma)$ is bilipschitz equivalent to the arclength metric on S^1. Moreover, (X, l, μ) is weak translate doubling, where μ is Hausdorff 2-measure.*

Proof. As in the proof of Proposition 4.4, a lower bound on $\sigma(\exp(i\theta), 0)$ is given by the minimum m over all $s \geq 0$ of $2/(1+s) \vee H(s)$, where $H(s) = \theta f(s) g(s)$. Taking $\theta > 0$ to be small, we may assume that the minimum occurs when $s > 1$. But then m is comparable with the minimum over all $s \geq 1$ of $s^{-1} \vee \theta$, which occurs when $s = 1/\theta$ and equals θ. Thus $\sigma(\exp(i\theta), 0) \gtrsim \theta$.

On the other hand, as in Proposition 4.4, we see that

$$\sigma(\exp(i\theta), 1) \leq \frac{2}{1 + 1/\theta} + \theta g\left(\frac{1}{\theta}\right) f\left(\frac{1}{\theta}\right) \approx \theta, \quad 0 < \theta \leq \pi. \tag{4.7}$$

The fact that X is weak translate doubling follows from the fact that there exists constants c, $C > 0$ such that $c(r^2 \wedge r^3) \leq \mu(B_r) \leq C(r^2 \vee r^3)$ whenever B_r is a ball of radius $r > 0$. We leave this as an exercise to the reader. □

Note that the space X in Proposition 4.5 and Euclidean 3-space have the same volume growth rate, but $\partial_S X$ is topologically S^1 (at least for g decaying no faster than the standard sphericalizing function) whereas $\partial_S \mathbb{R}^3$ is a one-point space. This again emphasizes that although the size of the spherical boundary constrains volume growth (for a large class of spaces), volume growth does not determine the size of the spherical boundary. We have also seen that restrictions such as negative Ricci curvature in the case of Riemannian manifolds is also not sufficient to ensure a nontrivial boundary. We would need more detailed curvature conditions, like an upper bound on the decay rate of Alexandrov curvature, in order to obtain results in that direction.

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