A HOMOMORPHIC INTERPRETATION OF THE COMPLEX FM EXPANSION

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A Complex Frequency Modulation (FM) signal is one whose instantaneous phase is time-varying according to a complicated dynamic function. This paper commences with the standard expansion for the spectrum of a Complex FM signal. It then explains how this can be interpreted in terms of a series of convolutions. The Homomorphic processing framework, in essence, provides a means by which a convolution operation can be related to a product operation which can then be transformed into an addition. This is very useful as it offers an approach for the fast computation of the theoretical spectra of complex FM signals, and further then leads to a cepstrum-like representation that will only display the modulation indices of the FM components. 'Liftering' of this representation can be carried out to alter the proportion of modulation components in the FM signal. Examples of the various stages of this processing will be given to illustrate its usefulness in the analysis and synthesis Complex FM signals.

INTRODUCTION

Complex Frequency modulation (Complex FM) describes an advanced form of the standard FM equation where the modulation signal is non-sinusoidal ([1], [2]). Many textbooks in the fields of communications studies and music synthesis present a derivation of the frequency spectrum for sinusoidal-modulator FM signals [3], [4]. Standard texts, such as [5], and research papers that have considered the non-sinusoidal case are generally older. Within these works expressions have been presented for the expression of a Complex FM wave where the modulator signal had a simple geometric description, which facilitated the mathematical derivation ([6], [7]). In computer music, Complex FM was introduced as a sound synthesis tool using a modulator composed of a fixed number of sinusoids [8]. The well-known DX7 synthesizer from Yamaha had 32 oscillator configurations known as algorithms that essentially were combinations of Complex FM oscillators [9]. From an academic viewpoint, the usefulness of Complex FM oscillators was also investigated for FM analysis/synthesis applications. One example is a 'double-modulator/single carrier arrangement' that was proposed as an effective approach for synthesizing the sounds of real acoustic instruments ([10], [11]). More recently, the power and flexibility of Complex FM oscillators has been exploited for new sound synthesis applications. First of all, they have been used to generate low-aliasing digital versions of the classical waveforms of analog subtractive synthesis ([12], [13]). Secondly, a Complex FM oscillator description was necessary for analysing the bandwidth of Exponential FM signals [14] in their digital form so as to prevent aliasing distortion [15]. Additionally, they have been applied in the evaluation of the output spectrum of audio-rate modulated PLTV filters, an example being the time-varying allpass described in [16]. Lastly, they appeared again to describe the spectra of Vector PhaseShaping (VPS) oscillators [18]. In all cases, the theoretical spectrum derived from a Complex FM oscillator description has been used to inform about the potential for aliasing distortion appearing in the oscillator output. This theoretical evaluation is required as a straightforward FFT-based analysis will not explicitly identify those frequency components in the oscillator signal that will fold back into the signal's spectrum below whichever Nyquist rate that was chosen and be of significant magnitude such that they appear as aliasing distortion to the listener. The theoretical spectrum thus allows an evaluation of such components prior to the selection of the sampling frequency, and thus it can be suitably set to minimize any aliasing.

A key issue though is that the expression for the expansion of an arbitrary Complex FM equation is costly to implement, however, particularly as the number of modulators, and their modulation indices, increases, even if care is taken to optimise the implementation [19]. This was found to be a serious restriction to its applicability. In an effort to overcome this, this paper will present an alternative approach to implementing the expression for the spectrum of a Complex FM signal. It is based on the observation that FM can be represented as a spectral convolution [20]. This can immediately be exploited to create a much faster and more flexible implementation of the Complex

FM expansion that, unexpectedly, can exploit the FFT algorithm. Secondly, by adopting this approach, it opens up other possibilities for the interpretation of FM signals. Specifically, homomorphic processing is well-known by the speech community for transforming from a convolution, to a multiplication, to an addition [21]. It produces a representation known as the cepstrum [22]. Additionally, signal properties such as pitch and spectral envelope are more easily extracted using simple operations the cepstral domain. Thus, a similar form of processing will be developed here to see whether it will reveal particular features of the Complex FM signal.

Section 1 of the paper will introduce the relevant expressions and illustrate the computational gains that can be achieved, while section 2 will examine the outputs of a homomorphic-inspired transformation of the complex FM signal. Section 3 will then conclude the paper.

1 THE COMPLEX FM EXPANSION

The Fourier series for an equation of an FM signal with a sinusoidal modulator can be written in terms of Bessel functions of the first kind. This is available in many computer music textbooks, ([4] for example). It is

$$\cos(\omega_c t + I\sin(\omega_m t)) = \sum_{k=-\infty}^{\infty} J_k(I)\cos(\omega_c t + k\omega_m t) \qquad (1)$$

where ω_c is the carrier frequency, ω_m is the modualtion frequency, I is the modulation index and $J_k(.)$ is a Bessel function of order k.

It is possible to expand eq. (1) to demonstrate its connectivity with spectral convolution. Noting that for a Bessel function

$$J_{-k}(I) = (-1)^k J_k(I) \qquad (2)$$

then, eq. (1) it can be written

$$\begin{aligned}\cos(\omega_c t + I\sin(\omega_m t)) = {} & J_0(I)\sin(\omega_c t) + \\ & -2J_1(I)\sin(\omega_m t)\sin(\omega_c t) \\ & +2J_2(I)\cos(2\omega_m t)\cos(\omega_c t) \\ & -2J_3(I)\sin(3\omega_m t)\sin(\omega_c t) \\ & +2J_4(I)\cos(4\omega_m t)\cos(\omega_c t) \\ & -\ldots\end{aligned} \qquad (3)$$

This expression for an FM signal shows the sidebands are the result of a convolution between the carrier and the sign-adjusted, phase-shifted, bessel-function weighted, low-to-high order harmonics. These are related in frequency to the original frequency of the modulation signal. Thus, even the simple case produces a complicated spectral description.

When the modulator is not a simple sinusoidal signal as in eq. (1) the signal spectrum grows in complexity in relation to the number of partial components that can describe that modulation. For example, if a purely sinusoidal expansion, with a maximum of K harmonic components, is used to describe the modulation then we can write the time-domain complex FM signal as

$$d(t) = \cos\left(\omega_c t + \sum_{i=1}^{K} I_i \sin(i\omega_m t + \varphi_i) + \theta\right) \qquad (4)$$

where θ is an arbitrary phase shift of the carrier, and the modulation is described by a number of harmonically related components of fundamental frequency ω_m, each one with a phase shift of φ_i and magnitude I_i. From [1], this signal will have a frequency spectrum given by

$$\begin{aligned}&\cos\left(\omega_c t + \sum_{i=1}^{K} I_i \sin(i\omega_m t + \varphi_i) + \theta\right) \\ &= \sum_{k_k}\ldots\sum_{k_1}\left(\prod_{i=1}^{K} J_{k_i}(I_i)\right)\cos\left(\omega_c t + \left(\sum_{i=1}^{K} k_i(i\omega_m t + \varphi_i)\right) + \theta\right)\end{aligned} \qquad (5)$$

According to [23], the sidebands produced by the modulation of the carrier by the first modulating oscillator are modulated again as a carrier by the next modulation term. This process repeats for each successive modulation term. This was further elaborated in [20] where it was explained that the spectrum of $d(t)$ can also be written as a series of spectral convolutions. This can be written using complex notation as

$$d(t) = \Re\left\{\Im^{-1}\left\{\begin{array}{l}\Im\left\{e^{j(\omega_c t)}\right\} * \\ \Im\left\{\sum_{k_1} J_{k_1}(I_1) e^{j(k_1(\omega_m t + \varphi_1))}\right\} * \\ \ldots * \Im\left\{\sum_{k_K} J_{k_K}(I_K) e^{j(k_K(N\omega_m t + \varphi_K))}\right\}\end{array}\right\}\right\}, \qquad (6)$$

where $\Im()$ denotes the Fourier transform, $\Im^{-1}()$ denotes the inverse Fourier transform, $\Re()$ is the real part, and the convolution operator is denoted by an asterisk (*).

Considering the properties of the Fourier transform, the spectrum term within the brackets on the right hand side of eq. (6) (i.e. prior to the inverse Fourier transform and real part operations) can be rewritten as single inverse Fourier transform of the product of the Fourier transform of the individual spectral, that is,

$$D(\omega)=\Im^{-1}\left\{\begin{array}{l}\Im\left\{\Im\left\{e^{j(\omega_c t)}\right\}\right\} \\ \Im\left\{\Im\left\{\sum_{k_1} J_{k_1}(I_1)e^{j(k_1(\omega_m t+\varphi_1))}\right\}\right\}. \\ \ldots\Im\left\{\Im\left\{\sum_{k_K} J_{k_K}(I_K)e^{j(k_K(N\omega_m t+\varphi_K))}\right\}\right\}\end{array}\right\}, \tag{7}$$

where $D(\omega)$ is the spectrum of $d(t)$.

Eq. (7) illustrates how the FFT can be utilised to convert the set of spectral convolutions into a set of products. This then suggested that an alternative implementation of the theoretical spectrum of a Complex FM signal should be possible over one based on eq. (5).

1.1 Computational Advantages

Adopting this interpretation of eq. (5) facilitates the development of a fast algorithm for the computation of the theoretical spectrum of a Complex FM signal. This can be demonstrated by a simple experiment that compares an FFT-based implementation based on eq. (7) with the one proposed in [19]. The method in [19] was an effort to improve on a direct implementation of eq. (7) that requires multiple nested loops. In the experiment both algorithms were used to generate the spectra of 100 different Complex FM signals. A fixed carrier frequency of 200Hz and modulation frequency of 100Hz were set. The modulation was chosen to be formed from two harmonics, one of frequency f_m and the other at $2f_m$. The modulation indices (i.e. the amplitudes) of these modulators were varied over the range 1 to 10 for both modulators, resulting in 100 test signals. The implementation of eq. (7) was achieved by simply creating the line spectra for the carrier and modulators, taking their FFTs, multiplying them together and then finding the IFFT. The number of frequency components associated with the spectrum of each modulator was limited according to the criterion provided in [19]. This ensures that the size of the highest component in each spectrum is less than -80dB. A matlab [24] m-file listing of the algorithm is given in Table 1.

The main program FMspec takes as input the carrier frequency in hertz, `fc`, the modulation frequency `fm`, a vector of modulation indices `Imod`, a vector of modulation phases, `theta`, a carrier phase shift, `pshift`, the `CarrierType`, which can be cos or sin, the `ModulatorType`, which again can be cos or sin. A value for sampling frequency in Hertz, `Fs`, should be picked to ensure that the effective bandwidth of the Complex FM signal is below the Nyquist rate.

```
[FullSpec,faxis]=FMspec(fc,fm,Imod,theta,pshift
,CarrierType,ModType,Fs);
if CarrierType=='sin'
   shift=-pi./2;
else
   shift=0;
end

faxis=0:Fs-1;
SpecCarrier=zeros(1,Fs);
I=find(faxis==fc);
SpecCarrier(I)=1.*exp(j.*shift).*exp(j.*pshift)
;
FT=fft(SpecCarrier);
len=length(Imod);
for index=1:len
   [Spec,faxis]= BSp
(index.*fm,Imod(index),theta(index),pshift,Fs,M
odtype);
   SpecT(index,:)=Spec;
   FT=FT.*fft(Spec);
   end
FullSpec=(ifft(FT));
faxis=-0.5.*Fs:0.5.*Fs-1;

[Sp,fax]=BSp(fm,Imod,theta,pshift,Fs,ModType);
p=[1.1023 10.3016];
Kmax=Imod.*p(1)+p(2);
K=2.*round(Kmax)-1;
Kvals=-floor(K./2):floor(K./2);
Amp=besselj(Kvals,Imod);
if ModType=='cos'
   Phase=-(Kvals.*(-pi./2-theta));
else
   Phase=Kvals.*theta;
end
Hfreq=Kvals.*fm;
faxis=-(Fs./2):(Fs./2)-1;
Spec=zeros(1,Fs);
for i=1:length(Hfreq)
   I=find(faxis==Hfreq(i));
   Spec(I)=Amp(i).*exp(j.*Phase(i));
end
```

Table 1 Matlab listing of the algorithm for computation of the theoretical Complex FM spectrum

The spectrum of the carrier is given by the variable `SpecCarrier`, and it is phase shifted appropriately by an amount `shift` depending on whether the carrier is sinusoidal or cosinusoidal. In this program it is also assumed that the carrier (and sideband components) will be exactly located at its frequency value(s) in the spectrum. The grid of indices along the frequency axis, i.e. the array size, for `SpecCarrier` must allow for this. In the program in Table 1 it is assumed that the spectral frequencies are integer values and are spaced at 1 Hz intervals.

The FFT of the spectrum of the carrier is given by `FT`. The individual spectra of the modulators are found using the function `BSp`. The orders at which the Bessel function for each modulation index will be computed is given by `K`. The maximum value, `Kmax`, is found using a line fit expression given in [19] whose coefficients are given in array `p`. This leads to the amplitudes of the sidebands, `Amp`. The phases of the sidebands are given by `Phase`, and they must be shifted correctly depending on whether the modulator is sinusoidal or cosinusoidal. The resulting spectrum, `Spec`, is returned to the main program, and its FFT is multiplied with that of the carrier. Successive modulator spectra are combined with the product in this manner. The final spectrum of the complex FM signal, `FullSpec`, is given by the IFFT of the variable `FT`, and it is centred around 0Hz.

The experiments to test the computational efficiency of this program were were carried out using the Matlab software environment on an Intel core 2 1.66Ghz processor. Figure 1 shows the computation times.

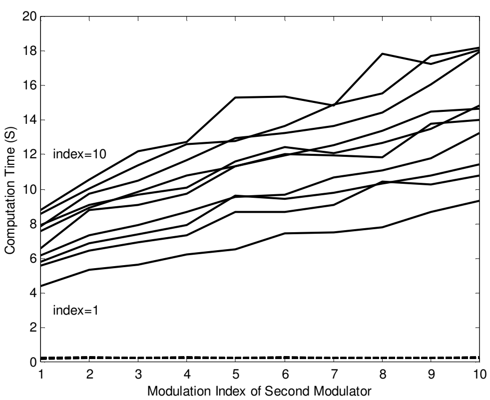

Figure 1 Comparison of computation time of the method in [11] (solid line) with the FFT based method (dashed line)

The solid lines give the computation times for the algorithm of [19]. These increase with respect to both modulation indices in an approximately linear fashion. For example, for a modulation index of 1 for the first modulator and a modulation index of 10 for the second modulator the computation time was just over 9.2 seconds, while for a modulation index of 10 for the first modulator and a modulation index of 10 for the second modulator the computation time was about 18 seconds. In contrast using the FFT-based method the computation time is much lower in all cases, as shown by the dashed line, and does not change with respect to the modulation indices. It gives an average computation time of 0.25 seconds. Thus, it clearly appears from this experiment to be far more advantageous to use this implementation approach.

2 HOMOMORPHIC INTERPRETATION

Homomorphic processing was introduced as a nonlinear processing technique. using a combination of the logarithmic operation and Fourier transforms, that could convert signals that had been convolved together into an addition [22]. This would provide a means of deconvolution. It has been successfully applied in many fields, most notably in speech processing for both formant analysis via spectral envelope extraction, and pitch detection [21]. Here, it is proposed to use a combination of logarithmic processing and IFFT processing on the Complex FM signal. It is a natural link as in the previous section the Fourier transform was exploited to convert the spectral convolution into a product. Returning to eq. (6) it can be seen that a quadrature version of the FM signal of eq. (4), as indicated by the hat, can be given by

$$\hat{d}(t) = e^{j(\omega_c t)} \cdot \sum_{k_1} J_{k_1}(I_1) e^{j(k_1 \omega_m t + \varphi_1)} \ldots \sum_{k_K} J_{k_K}(I_K) e^{j(k_K (K\omega_m t + \varphi_K))} \tag{8}$$

Note that the phase shift θ of eq. (4) has been omitted in eq. (8) for convenience. The quadrature signal can be obtained directly by finding the real and imaginary parts of eq. (6) or the Hilbert transform of the FM signal can be used instead. However, it should be noted that the quadrature version of the signal and its Hilbert transform are not exactly equivalent [25] but except for some effects at the signal boundaries they will be the same..

The logarithm can be found of eq. (8), however, it must be a complex logarithm. This is given by

$$\log\left(\hat{d}(t)\right) = \log\left|\hat{d}(t)\right| + j\angle\hat{d}(t) \tag{9}$$

where phase unwrapping must be carried out on the imaginary part of eq. (9) to ensure that there are no phase discontinuities at π or $-\pi$ radians. The phase unwrapping can be performed by a standard unwrapping followed by removal of the linear phase term [26]. This linear phase term is directly related to the carrier phase. Then, a homomorphic representation, $c(t)$, can be obtained if the IFFT of the output is performed

$$c(t) = \Im^{-1}\left\{\log\left(\hat{d}(t)\right)\right\} \tag{10}$$

If the magnitude of $c(t)$ is found, scaled by a factor of 2, and then plotted, it will display the modulation indices of each modulator as a series of peaks. An example is given in Figure 2 for a Complex FM signal whose carrier frequency is 1000Hz, and has a complex modulator with three harmonic components with a modulator fundamental frequency of 20Hz. The modulation indices are 4, 5.5 and 2.3. The initial phases of the modulators are π/3, 7π/4 and 6π/5. Figure 2 shows a plot of the representation

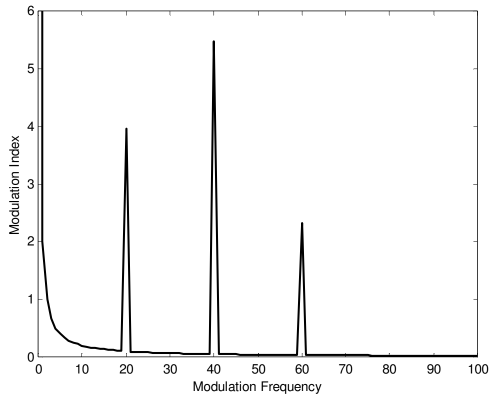

Figure 2: The magnitude of $c(n)$ whose peaks correspond to the modulation indices.

The peaks in Figure 2 lie at the frequencies of the three modulators and the magnitude of the peaks correspond exactly to the values of the modulation indices.

To find the initial phases of the modulators the angle of $c(t)$ can be found at the bins corresponding to the modulation frequencies. However, depending on whether the carrier and modulators are sinusoidal or cosinusoidal different rules apply to get the correct phase value. It is also assumed that the modulator phase is defined from 0 to 2π. The rules are given in Table 2, where the mod operation is denoted by '%'

Carrier	**Modulator**	**Rule**
cos	cos	$\pi/2 - \angle c(t)$
cos	sin	$(\pi - \angle c(t))\%2\pi$
sin	cos	$\pi/2 - \angle c(t)$
Sin	sin	$(\pi - \angle c(t))\%2\pi$

Table 2 Rules for phase extraction from $c(t)$

Figure 3 plots the phase angle of $c(n)$. Peaks can be seen that correspond to the initial phases of the modulation components.

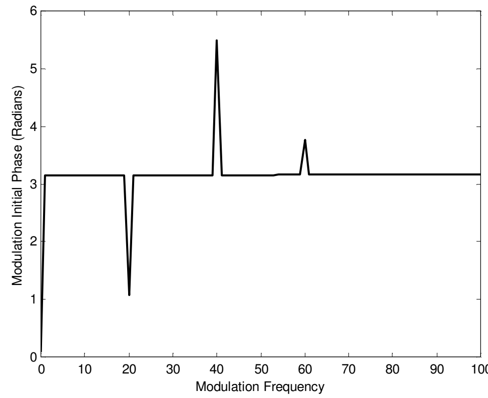

Figure 3: The phase of $c(n)$ whose peaks that correspond to the modulations' initial phase.

The average difference between the peak values at the modulation frequencies in Figure 3 and the original phases is 2×10^{-4}.

It should be noted that the information found using this homomorphic representation could also be found by taking the Fourier transform of the instantaneous frequency of the Complex FM signal. It is presented here as an alternative, however.

2.1 Filtering of *c*(*t*)

Another very interesting feature of this representation is that it allows the filtering of components in the modulation signal. By liftering [22] or adjusting the magnitude of any of the peaks in the homomorphic representation, it is possible to change the proportion of that component (i.e. its modulation index) in the frequency modulation itself. This liftering operating is similar in principle to that used normally in homomorphic processing. Once this liftering is done, the FFT is carried out, which is then followed by the rearranging of the real and imaginary parts as a complex exponential to retrieve the filtered spectrum. This last action is to undo the homomorphic operation. Figure 4 shows an example where the modulation index of the second modulator at 40Hz has been reduced to zero. The upper panel in the figure shows the spectrum of the original signal, while in the lower panel is the spectrum of the filtered version.

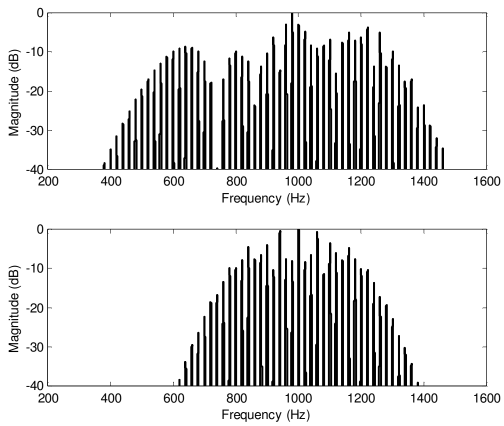

Figure 4 Comparison of Spectra of original Complex FM signal (upper panel) and signal with the second modulator removed (lower panel)

It is clear from the plot that the filter has reduced the bandwidth of the Complex FM signal significantly. This is to be expected as the modulation index of the filtered modulator was 5.5 which is relatively large in comparison to the others.

3 CONCLUSION

This paper has considered complex FM signals and their expansion as a series of spectral convolutions. A fast FFT-based algorithm for the computation of the theoretical complex FM spectrum was introduced. This led to the interpretation of the complex FM expression using the ideas of Homomorphic processing. This offered an alternative way to create a visualisation of the modulation indices and initial phases of the modulation components of the complex FM signal. Filtering of the modulation was shown to be possible using a liftering-like procedure. A copy of the matlab source code and an example is available at [22].

Future work will expand the analysis to included modulator feedback and nested modulators. It will also consider the application of the representation to the optimisation of modulation parameters for the modelling of acoustic instrument tones in a manner similar to [27]. Lastly, it will seek to enhance its application further in the design of PLTV filter structures [17] for sound synthesis.

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