Instantons in 4-dimensional gauged O(5) Skyrme models

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Abstract

We consider a family of four-dimensional non-linear sigma models based on an O(5) symmetric group, whose fields take their values on the 4-sphere S^4. An SO(4)-subgroup of the model is gauged. The solutions of the model are characterised by two distinct topological charges, the Chern-Pontryagin charge of the gauge field and the degree of the map, i.e. the winding number, of the S^4 field. The one dimensional equations arising from the variation of the action density subjected to spherical symmetry are integrated numerically. Several properties of the solutions thus constructed are pointed out. The only solution with *unit* Chern-Pontryagin charge are the usual BPST instantons with zero S^4 winding number, while solutions with *unit* S^4 winding number have zero Chern-Pontryagin charge.

1 Introduction

The model considered in this work is described by the Lagrangian on $\mathbb{R}^4$

$$\mathcal{L} = \frac{\lambda_0}{24}|F_{\mu\nu}^{[\alpha\beta]}|^2 \quad + \quad \frac{\lambda_1}{2}|D_\mu\phi^a|^2 + \frac{\lambda_2}{24}|D_\mu\phi^a \times D_\nu\phi^b|^2 \\ \quad + \quad \frac{\lambda_3}{72}|D_\mu\phi^a \times D_\nu\phi^b \times D_\rho\phi^c|^2 + V(\phi^5, \cos\omega) \ , \qquad (1)$$

in terms of the S^4 valued fields $\phi^a = (\phi^\alpha, \phi^5)$, ($\alpha$=1,2,3,4) satisfying the constraint

$$\phi^a\phi^a \ = \ 1 \ ,$$

and the $SO(4)$ gauge connection $A_\mu^{[\alpha\beta]}$ with curvature $F_{\mu\nu}^{[\alpha\beta]}$. The covariant derivatives in (1) are defined by

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha + A_\mu^{[\alpha\beta]}\phi^\beta \ , \quad D_\mu\phi^5 = \partial_\mu\phi^5 \ . \qquad (2)$$

(The brackets [..] imply antisymmetrisation of indices throughout.)

The Lagrangian (1) differs from that of the various models considered in [1], in the physically important[1] respect that the kinetic term *quadratic* in the S^4 field ϕ^a, which was absent there [1], is present here. As a result of the Derrick scaling requirement, the system (1) features also a kinetic term *sextic* in the S^4 valued field. (1) differs from that in [1] also in the presence of the generic potential $V(\phi^5, \cos\omega)$

$$V(\phi^5, \omega) = \lambda(\cos\omega - \phi^5)^n \ , \quad 0 \leq \omega \leq \pi \ , \quad n = \text{integer} \ , \qquad (3)$$

whose role it is to fix the asymptotic value of the field ϕ^a, rather like the pion–mass potential in the usual 3 dimensional $O(4)$ Skyrme [2] model. Like in that case [2], this term serves only the purpose of fixing the asymptotics, and will be considered only in this limited context. Moreover, anticipating our conclusions in Subsection 2.1, namely that only for $\omega = 0$ is it possible to construct finite action solutions, the relevant $V(\phi^5, \omega = 0)$ is

$$V(\phi^5, \omega = 0) = \lambda(1 - \phi^5) \ , \qquad (4)$$

in analogy with the pion–mass [2] potential. Notwithstanding, in Subsection 2.2 we have described a model differing from (1) and characterised by nonzero ω (in particular $\omega = \frac{\pi}{2}$), which can support topologically stable finite action solutions. As will be seen there, the Lagrangian of such a model does not feature the usual YM term and is therefore not studied further here.

Our topologically stable finite action solutions are interpreted as instantons, although the latter are not always characterised by the second Chern–Pontryagin density, but also by the $S^4 \to S^4$ degree of the S^4 valued field (which may or may not be integer).

[1]Without a *quadratic* kinetic term, it is not possible to infer from *finite action conditions*, that that the matter field (in this case the S^4 valued field) becomes asymptotically a constant and hence consistent with *vacuum* field. It would then be impossible to interpret the resulting topologically stable finite action solution as an instanton.

By adapting the methods formulated in [3] used in establishing the topological lower bounds given by the degree of the map $S^4 \to S^4$, a suitable such lower bound can be established for the action (1), for all positive values of $\lambda_1, \lambda_2, \lambda_3$, and λ_0. This will be established in Section 2. In Section 3, the system (1) will be subjected to spherical symmetry, and thenceforth we will restrict to the study of the ensuing one dimensional equations numerically. The results of our numerical work is presented in Section 4, and Section 5 is devoted to the summary and discussion of our results.

In the work of [1], which employs the above model (1) with $\lambda_1 = \lambda_3 = 0$, it was found that the action of the main $SO(4)$ gauged $O(5)$ model decreased with the Skyrme coupling λ_2 and exhibited a bifurcation at a value very close to and above the action of the (pure) Yang–Mills (YM) instanton [4]. Thus beyond a critical value of the coupling λ_2, the system did not support a finite action solution, and more importantly the action could be made smaller by decreasing λ_2. Our main aim in the present work is the verification of these two properties of the solutions when $\lambda_1 > 0$ and $\lambda_3 > 0$. We have found that these properties of the solution persist, namely that instantons can be constructed for values of λ_2 (holding λ_3 constant) up to some value λ_2^{cr}, and for values of λ_3 (holding λ_2 constant) up to some value λ_3^{cr}. Also, the actions of these instantons decrease with λ_2 and λ_3 respectively consistently with the Derrick scaling requirement. We find surprisingly that the action at $\lambda_2 = 0$ is nonzero, inspite of the vanishing of the topological lower bound at that point. We defer discussion of the possible physical significance of these properties to Section 5.

Another objective here is to probe the nature of the topological lower bounds. In [1], we were exclusively concerned with lower bounds stated in terms of the degree of the map of the S^4 valued field, such that the corresponding solutions supported *vanishing* Chern–Pontryagin charge. Here we attempt to construct numerically, instantons with *nonzero* Pontryagin charge when the S^4 field is nontrivial. We do not find such solutions and offer an analytic argument to support their nonexistence.

The boundary conditions employed for the S^4 valued field are

$$\lim_{r\to 0} \phi^5 = -1 \ , \qquad \lim_{r\to\infty} \phi^5 = \cos\omega \ , \tag{5}$$

but we will be restricting our numerical investigations to the $\omega = 0$ case. For the gauge field we will adopt the vacuum behaviours

$$\lim_{r\to 0} A_\mu = 0 \ , \qquad \lim_{r\to\infty} A_\mu = qg\ \partial_\mu g^{-1} \ . \tag{6}$$

$q = 0$ leads to zero Pontryagin charge. Pure-gauge $q = 1$, and half–pure-gauge $q = \frac{1}{2}$, both lead to nonzero Pontryagin charges. The half–pure-gauge case $q = \frac{1}{2}$ pertains to $\omega \neq 0$, which we do not study numerically. The pure-gauge cases $q = 0$ and $q = 1$ both pertain to $\omega = 0$, and are studied numerically. It turns out that instantons with nontrivial (S^4 valued) matter field can only be constructed for the $q = 0$ case.

2 Lower bounds

The work in this Section follows very closely that in [3] and that in Section 2 of [1]. The analysis in [3, 1] was adapted to the definition of topological charges presenting lower bounds on the actions, of systems supporting asymptotics with $\omega = 0$ in (5). Here, we extend this analysis to include values of $0 \leq \omega \leq \pi$. In the generic case therefore, we would expect solutions with a *fractional* analogue of the Baryon number in 3 dimensions [5], while in the limiting case [3, 1] $\omega = 0$ this will be the degree of the map of the S^5 valued field taking an *integer* value.

We start with the definition of the winding number density

$$\varrho_0 = \frac{1}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{abcde}\partial_\mu\phi^a\partial_\nu\phi^b\partial_\rho\phi^c\partial_\sigma\phi^d\phi^e \ , \tag{7}$$

which is inadequate for our purposes here since it is *gauge variant*, and its *gauge invariant* version

$$\varrho_G = \frac{1}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{abcde}D_\mu\phi^a D_\nu\phi^b D_\rho\phi^c D_\sigma\phi^d\phi^e \ , \tag{8}$$

whose volume integral cannot be evaluated by stating the asymptotic conditions, *i.e.* it is not useful as a topological charge density.

The volume integral of (7) can indeed be evaluated by stating the asymptotic conditions (5), such that for $\omega = 0$ the value of this integral is integer, and is fractional for $\omega \neq 0$. The normalisation ensures that in the spherically symmetric case this is the unit charge, or winding number. The task is to define a suitable density which is **(a)** gauge invariant, and **(b)** its volume integral equals the volume integral of the density ϱ_0, (7).

To this end we find the relation between the two densities (7) and (8), in suitable form, such that *all gauge–variant terms appear as total divergences*. Thus,

$$\begin{aligned} \varrho_G &= \varrho_0 + \partial_\mu\left(\phi^5\partial_\nu\Omega_{\nu\mu} + \tilde{\Omega}_\mu\right) \\ &\quad + \frac{3}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left[\phi^\alpha D_\nu\phi^\beta D_\mu\phi^5 F_{\rho\sigma}^{\gamma\delta} - \frac{1}{8}\phi^5\left(1-\frac{1}{3}(\phi)^2\right)F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\right] \end{aligned} \tag{9}$$

where

$$\Omega_{\nu\mu} = \frac{3}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}A_\sigma\phi^\alpha\left(2\partial_\rho\phi^\beta + (A_\rho\phi)^\beta\right) \tag{10}$$

$$\tilde{\Omega}_\mu = \frac{3}{128\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left\{\phi^5\left(1-\frac{1}{3}(\phi)^2\right)A_\nu^{\gamma\delta}\left[\partial_\rho A_\sigma^{\alpha\beta} - \frac{2}{3}(A_\rho A_\sigma)^{\alpha\beta}\right]\right\} \ . \tag{11}$$

The volume integral of the (gauge–variant) total divergence terms in (9) can, after conversion to two surface integrals, be evaluated using the asymptotic values (5) and (6). If these surface integrals vanished, then the volume integral of the remaining gauge invariant terms could be expressed in terms of the volume integral of ϱ_0, namely the (integral or fractional) winding number, leading to the definition of a gauge-invariant topological charge density.

This can be checked by substituting the spherically symmetric Ansatz (31) and (32) in (10) and (11). One then sees immediately that the density (10) vanishes asymptotically by symmetry, while the density (11) yields a non-vanishing contribution to the corresponding surface integral, which depends on the asymptotic parameter ω. Thus the definition of the charge density to be given below is ω-dependent.

Making use of the relation

$$\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}=\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\partial_\mu\left\{A_\nu^{\gamma\delta}\left[\partial_\rho A_\sigma^{\alpha\beta}-\frac{2}{3}(A_\rho A_\sigma)^{\alpha\beta}\right]\right\} \tag{12}$$

one can add and subtract the gauge–invariant density

$$\cos\omega\left(\cos\omega-\frac{1}{3}\cos^2\omega\right)\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}$$

to (9), such that the non-vanishing surface contribution of $\tilde{\Omega}_\mu$ is cancelled. After some rearrangement, the natural definition for the charge density ϱ pertaining to a system characterised by the asymptotic parameter ω is

$$\begin{aligned}\varrho=\varrho_G \quad &- \quad \frac{3}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\Big\{\phi^\alpha D_\nu\phi^\beta D_\mu\phi^5 F_{\rho\sigma}^{\gamma\delta}\\ &- \quad \frac{1}{8}\left[\phi^5\left(1-\frac{1}{3}(\phi)^2\right)-\cos\omega\left(\cos\omega-\frac{1}{3}\cos^2\omega\right)\right]F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\Big\}\ ,\end{aligned} \tag{13}$$

which is the manifestly gauge–invariant definition that is employed in establishing (Bogomol'nyi like) lower bounds, and which is equivalent to the definition

$$\varrho=\varrho_0+\partial_\mu\left(\phi^5\partial_\nu\Omega_{\nu\mu}+\hat{\Omega}_\mu\right)\ , \tag{14}$$

in which $\Omega_{\nu\mu}$ is defined by (10) while $\hat{\Omega}_\mu$ is

$$\begin{aligned}\hat{\Omega}_\mu=\frac{3}{128\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\Big\{ \quad &\left[\phi^5\left(1-\frac{1}{3}(\phi)^2\right)-\cos\omega\left(\cos\omega-\frac{1}{3}\cos^2\omega\right)\right]\times\\ &A_\nu^{\gamma\delta}\left[\partial_\rho A_\sigma^{\alpha\beta}-\frac{2}{3}(A_\rho A_\sigma)^{\alpha\beta}\right]\quad\Big\}\ .\end{aligned} \tag{15}$$

For the spherically symmetric fields (31)-(32), $x_\mu\hat{\Omega}_\mu$ vanishes asymptotically, and since we already know that $\Omega_{\nu\mu}$ vanishes asymptotically, we see that the volume integral of (14) equals the (generic fractional) winding number. The (topological charge) density is gauge–invariant, and its volume integral is just the winding number of the S^4 valued field.

Not surprisingly the definition (13) for the two most prominent cases $\varrho(\omega=0)$ and $\varrho(\omega=\frac{\pi}{2})$, simplifies somewhat. After some manipulations one has

$$\varrho(0) \;=\; \varrho_G+\frac{3}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left[\phi^5 D_\mu\phi^\alpha D_\nu\phi^\beta F_{\rho\sigma}^{\gamma\delta}+\frac{1}{12}\left((\phi^5)^3-1\right)F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\right] \tag{16}$$

$$\varrho(\frac{\pi}{2}) \;=\; \varrho_G+\frac{3}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi^5\left[D_\mu\phi^\alpha D_\nu\phi^\beta F_{\rho\sigma}^{\gamma\delta}+\frac{1}{12}(\phi^5)^2F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\right] \tag{17}$$

In the next two Subsections 2.1 and 2.2 we shall analyse models whose actions are bounded from below by the topological charges corresponding to (16), pertaining to $\omega=0$, and (17), pertaining to $\omega=\frac{\pi}{2}$.

2.1 Model with $\omega = 0$

The relevant gauge–invariant charge density in this case is (16). The first term, ϱ_G can be reproduced as a consequence of the inequality

$$\left|\kappa_1 D_\mu\phi^a-\frac{\kappa_3^3}{3!}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{abcde}D_{[\nu}\phi^b D_{[\rho}\phi^c D_{\sigma]]}\phi^d\phi^e\right|^2\ge 0\ ,$$

leading to

$$\kappa_1^2|D_\mu\phi^a|^2+\kappa_3^6|D_{[\nu}\phi^b D_{[\rho}\phi^c D_{\sigma]]}\phi^d|^2\ge\kappa_1\kappa_3^3\varrho_G\ , \tag{18}$$

where κ_1 and κ_3 are constants with the dimension of length. The next term in (16),

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi^5 D_\mu\phi^\alpha D_\nu^\beta F_{\rho\sigma}^{\gamma\delta}\ ,$$

is reproduced by the inequality

$$\left|\kappa_0^2 F_{\mu\nu}^{\alpha\beta}-\frac{\kappa_2^2}{2!^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi^5 D_{[\rho}\phi^\gamma D_{\sigma]}\phi^\delta\right|^2\ge 0\ ,$$

expanding which and adding the appropriate positive terms to the left hand side yields

$$\kappa_0^4|F_{\mu\nu}^{\alpha\beta}|^2+\kappa_2^4|D_{[\mu}\phi^a D_{\nu]}\phi^b|^2\ge\kappa_0^2\kappa_2^2\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi^5 D_\rho\phi^\gamma D_\sigma\phi^\delta F_{\mu\nu}^{\alpha\beta}\ . \tag{19}$$

Finally the last term in (16) can be reproduced by adding the two inequalities

$$\begin{aligned}(\phi^5)^2\left|\phi^5 F_{\mu\nu}^{\alpha\beta}-\frac{1}{2!^2}F_{\rho\sigma}^{\gamma\delta}\right|^2 &\ge 0\ ,\\ \left|F_{\mu\nu}^{\alpha\beta}+\frac{1}{2!^2}F_{\rho\sigma}^{\gamma\delta}\right|^2 &\ge 0\ ,\end{aligned}$$

and then adding suitable positive quantities to the right hand side to yield

$$\bar\kappa_4^4|F_{\mu\nu}^{\alpha\beta}|^2\ge\frac{1}{3!}\bar\kappa_4^4\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}[(\phi^5)^2-1]F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\ . \tag{20}$$

The constants κ_0, in κ_1, κ_3, and $\bar\kappa_0$ in (18), (19) and (20), all have the dimension of length.

Adding (18), (19) and (20), we end up with an inequality whose left hand side, up to some redefinitions, is precisely the system (1), without the potential term V. This Lagrangian is bounded from below by the right hand side, which will be a topological bound if the latter coincides with the topological charge density (16) (up to a constant multiple). This turns out to be the case, provided that the constants κ_0, in κ_1, κ_3, and $\bar{\kappa}_0$ satisfy the following constraints

$$\kappa_0^2\kappa_2^2 = 3\kappa_1\kappa_3^3 \ , \qquad 2\bar{\kappa}_0^4 = 3\kappa_1\kappa_3^3 \ , \tag{21}$$

with the constant multiplying ϱ (16), equal to $\kappa_1\kappa_3^3$. Thus the action (before redefining the constants) is bounded from below as

$$S \ \geq \ \kappa_1\kappa_3^3 \ N \ , \tag{22}$$

where N is the winding number. The action S is the four-volume integral of the Lagrange density

$$\begin{aligned}\hat{\mathcal{L}} \ &= \ (\kappa_0^4+\bar{\kappa}_0^4)|F_{\mu\nu}^{[\alpha\beta]}|^2+\kappa_1^2|D_\mu\phi^a|^2 \\ &+\kappa_2^4|D_\mu\phi^a\times D_\nu\phi^b|^2+\kappa_3^6|D_\mu\phi^a\times D_\nu\phi^b\times D_\rho\phi^c|^2 \ ,\end{aligned} \tag{23}$$

subject to the restrictions (21), which is up to some redefinitions coincides with (1) without the potential term V.

It is seen from (22) that the condition that this lower bound be nontrivial is that neither one of κ_1 and κ_3 should vanish. It could be thought that this means κ_2 can be set equal to zero without violating this bound, but from the first member of the constraint equations (21) we see that this is impossible. We conclude therefore that **the lower bound remains valid only as long as all the constants** $(\kappa_0,\kappa_1,\kappa_2,\kappa_3,)$**, and hence also the couplings** $(\lambda_0,\lambda_1,\lambda_2,\lambda_3,)$**, remain positive and nonzero**.

As will be shown in Section 3, the constants λ_0 and λ_1 can be scaled away leaving only two independent coupling constants λ_2 and λ_3, both of which have to be positive and greater than zero if the lower bound (22) is to be preserved.

Before proceeding to the next Subsection, we note that the Lagrangian (1) is not unique in being bounded from below by the topological charge density (16). Rather, it is the simplest system motivated by the requirements that it features the YM term and the *quadratic* kinetic term of the scalar field. An inspection of the spherically symmetric restriction of the YM term, eqn (33) below, implies the finite action condition on the gauge field function $a(r)$

$$\lim_{r\to\infty} a(r) = \pm 1 \ ,$$

which in the language of the asymptotic conditions (6) means that $q=0$ and $q=1$, respectively. In the first case, the Pontryagin charge vanishes, while in the second case it is equal to 1.

2.2 Model with $\omega = \frac{\pi}{2}$

The relevant gauge–invariant charge density in this case is (17). Unlike in the previous Subsection however, here we do not proceed straightforwardly to construct the simplest density which is bounded from below by (17).

The reason is that when $\omega \neq 0$ (as in the case case at hand with $\omega = \frac{\pi}{2}$) the gauge group $SO(4)$ breaks down to $SO(3)$ at infinity. This can easily be seen by rotating the asymptotic field ϕ^α $(\alpha = 1,2,3,4)$ of length $|\phi^\alpha(\infty)| = \sin\omega$, to the constant vector field along the x_4 axis, by means of an appropriate $SO(4)$ gauge transformation. The effect of this transformation on the $so(4)$ gauge connection is, that it develops a line singularity along the x_4 axis, and its non-vanishing components then take their values in the residual $so(3)$. We do not give the details of the passage to this 'Dirac gauge' here, because this has been given in detail previously in Refs. [6, 7], in the context of the $SO(4) \times U(1)$ gauged Higgs model[2]

The relevant information that follows from the preceding discussion is, that the asymptotic connection field $\tilde{A}_\mu^{ab} = (\tilde{A}_i^{ab}, \tilde{A}_4^{ab})$, $(a = \alpha, 5)$ in the Dirac gauge decays exactly as $\frac{1}{r}$, and its only non-vanishing component is

$$\tilde{A}_i^{[\alpha\beta]} = \frac{1}{r(1-\hat{x}_4)}(\delta_i^\alpha \hat{x}^\beta - \delta_j^\beta \hat{x}^\alpha) \ , \tag{24}$$

which takes its values in $so(3)$.

It follows that the corresponding asymptotic curvature field has the Coulomb decay $\frac{1}{r^2}$, and as a consequence the integral of the YM action density will diverge logarithmically in four dimensions. This simple fact can also be seen by inspecting the spherically symmetric YM action density in (33). Thus, in constructing the density bounded from below by the topological charge density (17), it is not legitimate to employ the usual YM action density.

The remedy is to use instead the YM density constructed from the antysymmetrised product of two curvature two-forms, namely

$$|F_{\mu\nu\rho\sigma}^{abcd}|^2 = |(F_{\mu[\nu}^{\alpha[\beta} F_{\rho\sigma]}^{\gamma\delta]} + F_{\mu[\nu}^{[\delta\gamma} F_{\rho\sigma]}^{\beta]\alpha})|^2 \ .$$

This term arises naturally in reproducing the last term in the charge density (17). To reproduce the second term in the charge density, it is not legitimate to make use of inequality (19) since the latter features the usual YM density. Given that for the instanton (vacuum) interpretation of the solution we need to have the quadratic kinetic term of the scalar field, this necessitates the appearance of the term

$$|F_{[\mu\nu}^{\alpha\beta} D_{\lambda]}\phi^\gamma|^2 \ .$$

Finally, to reproduce the first term, ϱ_G, in (17), the most economical option is to adopt the inequality (18). (This avoids the introduction of the additional and unnecessary term $|D_{[\mu}\phi^a D_{\nu]}\phi^b|^2$ in the Lagrangian.)

Following the above arguments, we write down the three topological inequalities corresponding to (18), (19), (20) for the present case with $\omega = \frac{\pi}{2}$. With (18)

[2]The analysis here is identical to that in [6, 7], with the 4 component field ϕ^α here replacing the Higgs field $\Phi = \gamma_5\gamma_\mu\hat{x}_\mu$ of [6, 7]. Indeed this is the case for all d-dimensional $(d \geq 3)SO(d)$ Higgs models with d-vector Higgs fields[8], of which the most familiar is the Wu-Yang monopole in $d = 3$.

unchanged, we just give the second two

$$\bar{\kappa}_1^2|D_{[\mu}\phi^a|^2+\bar{\kappa}_3^6|F_{[\mu\nu}^{\alpha\beta}D_{\lambda]}\phi^\gamma|^2 \ \ \geq \ \ 3\bar{\kappa}_1\bar{\kappa}_3^3\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\phi^5 D_\rho\phi^\gamma D_\sigma\phi^\delta F_{\mu\nu}^{\alpha\beta} \tag{25}$$

$$\bar{\kappa}_4^8|F_{\mu\nu\rho\sigma}^{abcd}|^2+\tau^2(\phi^5)^6 \ \ \geq \ \ \tau\bar{\kappa}_4^4\frac{3}{2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}(\phi^5)^3 F_{\mu\nu}^{\alpha\beta}F_{\rho\sigma}^{\gamma\delta}\ , \tag{26}$$

where the constants $\bar{\kappa}_1$, $\bar{\kappa}_3$ and $\bar{\kappa}_4$ all have the dimension of length, while τ is dimensionless.

Adding (18), (25) and (26) results in an inequality whose right hand side can be identified (up to a numerical factor) with the topological charge density (17), provided that

$$\bar{\kappa}_1\bar{\kappa}_3^3=3\kappa_1\kappa_3^3\ , \qquad 6\tau\bar{\kappa}_4^4=\kappa_1\kappa_3^3\ . \tag{27}$$

The resulting topological inequality bounding the action from below, analogous to (22), is

$$\tilde{S}\geq\kappa_1\kappa_3^3\ N\ , \tag{28}$$

in which the action $\tilde{S}$ is the four-volume integral of

$$\begin{aligned}\tilde{\mathcal{L}}=\bar{\kappa}_4^8|F_{\mu\nu\rho\sigma}^{abcd}|^2 \ &+\ \bar{\kappa}_3^6|F_{[\mu\nu}^{ab}D_{\lambda]}\phi^c|^2+\kappa_3^6|D_\mu\phi^a\times D_\nu\phi^b\times D_\rho\phi^c|^2\\ &+\ (\kappa_1^2+\bar{\kappa}_1^2)|D_{[\mu}\phi^a|^2+\tau^2(\phi^5)^6\ ,\end{aligned} \tag{29}$$

subject to the constraints (27). Note that the potential (3) with $\omega=0$ and $n=6$ appears quite naturally in (29), and in this case its presence is mandatory if the lower bound on the action is to be preserved.

Because (29) does not feature the usual YM term besides the F^4 term, it is not likely to be of any physical interest. Hence we do not analyse it numerically. We note that in the case of the $SO(4)\times U(1)$ Higgs model [6, 7], which also features the F^4 YM term to the exclusion of the usual F^2, it could be argued that at high energies that system reduced to a conventional $SO(4)\times U(1)$ Higgs system

$$\mathrm{Tr}\left(\lambda_2 F_{\mu\nu}^2+\lambda_1 D_\mu\Phi^2+\lambda_0(\Phi^2+\eta^2)^2\right)\ , \tag{30}$$

where the constant η is the VEV of the Higgs field, and all fields are antihermitian. In other words, the $SO(4)\times U(1)$ Higgs model was interpreted as the low energy effective action of (30). Unfortunately, we do not have such an interpretation for the system (29).

3 Spherical symmetry

The spherically symmetric Ansatz employed is

$$A_\mu^{[\alpha\beta]}=\frac{a(r)-1}{r}(\delta_\mu^\alpha\hat{x}^\beta-\delta_\mu^\beta\hat{x}^\alpha) \tag{31}$$

$$\phi^\alpha=\sin f(r)\hat{x}^\alpha,\quad \phi^5=\cos f(r) \tag{32}$$

As explained in Section 2.2, we will restrict our numerical analysis to the case of $\omega = 0$, and hence give the spherically symmetric reduction only of the terms in the system (1), or (23), pertaining to $\omega = 0$.

Substituting (31)-(32) into the component terms of (1) we have, for each term

$$
\begin{aligned}
|F_{\mu\nu}^{\alpha\beta}|^2 &= 12\left[\left(\frac{a'}{r}\right)^2+\left(\frac{a^2-1}{r^2}\right)^2\right] &(33)\\
|D_\mu\phi^a|^2 &= f'^2+3\left(\frac{a^2\sin^2 f}{r^2}\right) &(34)\\
|D_\mu\phi^a\times D_\nu\phi^b|^2 &= 12\left(\frac{a^2\sin^2 f}{r^2}\right)[f'^2+\left(\frac{a^2\sin^2 f}{r^2}\right)] &(35)\\
|D_\mu\phi^a\times D_\nu\phi^b\times D_\rho\phi^c|^2 &= 36\left(\frac{a^2\sin^2 f}{r^2}\right)^2\left[3f'^2+\left(\frac{a^2\sin^2 f}{r^2}\right)\right] &(36)
\end{aligned}
$$

In the following we will study the classical solutions of the model (1) and characterize them by the classical action S defined by means of

$$S=\frac{1}{8\pi^2}\int d^4x\mathcal{L} \tag{37}$$

The reduced 1-dimensional Lagrangian is r^3 times the sum, with the appropriate numerical coefficients in (1), of all the above 4 terms. We do not display this one-dimensional Lagrangian, nor the ordinary differential equations that follow.

The asymptotic values of the function $f(r)$ corresponding to (5) translate to

$$\lim_{r\to 0} f(r)=\pi\ , \qquad \lim_{r\to\infty} f(r)=\omega\ , \tag{38}$$

with $\omega = 0$, while the asymptotic values of the function $a(r)$ for the cases $q = 0$ and $q = 1$ translate respectively, to

$$
\begin{aligned}
\lim_{r\to 0} a(r) &= 1\ , \qquad \lim_{r\to\infty} a(r) = 1 &(39)\\
\lim_{r\to 0} a(r) &= 1\ , \qquad \lim_{r\to\infty} a(r) = -1\ . &(40)
\end{aligned}
$$

Let us first point out that the embedded charge-one-BPST-instanton solutions [4]

$$a_{BPST}(r)=\frac{k^2-r^2}{k^2+r^2}\quad ,\quad f(r)=n\pi\ \ \text{(everywhere)} \tag{41}$$

(k is a real constant, n is an integer) exist irrespectively of the values of $\lambda_{1,2,3}$ and leads to $S_{BPST}=\frac{4}{3}$, corresponding to the action of the charge-one-instanton solution of the Yang–Mills theory [4]. Here we are interested in classical solutions with non constant $f(r)$.

The number of four coupling constants can be reduced to two by using the following scaling argument. Transforming $r \to \sigma r$, we have

$$S(\lambda_1,\lambda_2,\lambda_3,\lambda_0)=S(\lambda_1\sigma^2,\lambda_2,\lambda_3\sigma^{-2},\lambda_0)=\lambda_1\sigma^2 S\left(1,\frac{\lambda_2}{\lambda_1\sigma^2},\frac{\lambda_3}{\lambda_1\sigma^4},\frac{\lambda_0}{\lambda_1\sigma^2}\right). \tag{42}$$

Choosing $\sigma^2 = \frac{\lambda_0}{\lambda_1}$ this gives

$$S(\lambda_1, \lambda_2, \lambda_3, \lambda_0) = \lambda_0 S\left(1, \frac{\lambda_2}{\lambda_0}, \frac{\lambda_3\lambda_1}{\lambda_0^2}, 1\right). \tag{43}$$

In the following we will make use of the above scaling property and set $\lambda_1 = \lambda_0 = 1$.

4 Numerical results

We have studied numerically the solutions of the classical equations associated with (1) for the $\omega = 0$ model, restricting to the one dimensional spherically symmetric fields given by (33)-(36). Most of the work is carried out with the potential (4) decoupled i.e. with $\lambda = 0$.

In [1] the above equations have been studied in detail in the case $\lambda_1 = \lambda_3 = 0$. Here we want to study the classical solutions for non-vanishing $\lambda_1, \lambda_2, \lambda_3$. Using the standard Derrick scaling argument, it is easily seen that regular classical solutions will exist only if the coupling constants λ_1, λ_3 are **both** nonzero. On the other hand, the topological lower bound (22) derived in the previous Section states that in addition to λ_1 and λ_0 (which we have already set to $\lambda_0 = \lambda_1 = 1$ by scaling), both λ_2 and λ_3 must be positive and nonzero. On the basis of the last statement, there is no guarantee that the solution persists when λ_2 vanishes, even though this is consistent with the Derrick scaling requirement.

As a result of our numerical studies, we have learnt that with the asymptotics (39), the solution persists when λ_2 vanishes. In this case there remains only one coupling constant to vary, λ_3, which is a simpler case to study and this is presented in Subsection 4.1. In Subsection 4.2, again with the asymptotics (39), we study the cases where λ_2 is varied for fixed nonzero value of λ_3, and also where λ_3 is varied for fixed nonzero value of λ_2. These families of solutions all have Pontryagin charge equal to zero. In Subsection (4.3), we present the results of our numerical search for solutions with *unit* Pontryagin charge and nontrivial scalar field, with asymptotics (40). The result is negative, and we have supplied an analytic construction in support of the nonexistence of such instantons.

4.1 Solutions with $a(0) = a(\infty) = 1$ and $\lambda_2 = 0$

With these boundary conditions, the Chern–Pontryagin charge would be zero and the topological lower bound would be stated in terms of the degree of the map only.

Integrating the equations for small values of λ_3 we were able to construct solutions with

$$a(0) = 1 \quad , \quad a(\infty) = 1 \quad , \quad f(0) = \pi \quad , \quad f(\infty) = 0 \tag{44}$$

The profiles of the functions a, f of this solution are presented in Fig.1 for $\lambda_3 = 0.425$ by the solid lines. In the limit $\lambda_3 = 0$ the classical action tends to zero and the function $a(r)$ tends to a constant : $a(r) = 1$. When λ_3 increases, the function $a(r)$ develops a local minimum (at $r = r_m$) which becomes progressively deeper as indicated in Fig. 2. The general dependence of r_m on λ_3 is also reported on Fig. 2. At the same time the classical action of the solution increases with λ_3, and this is illustrated in Fig. 3.

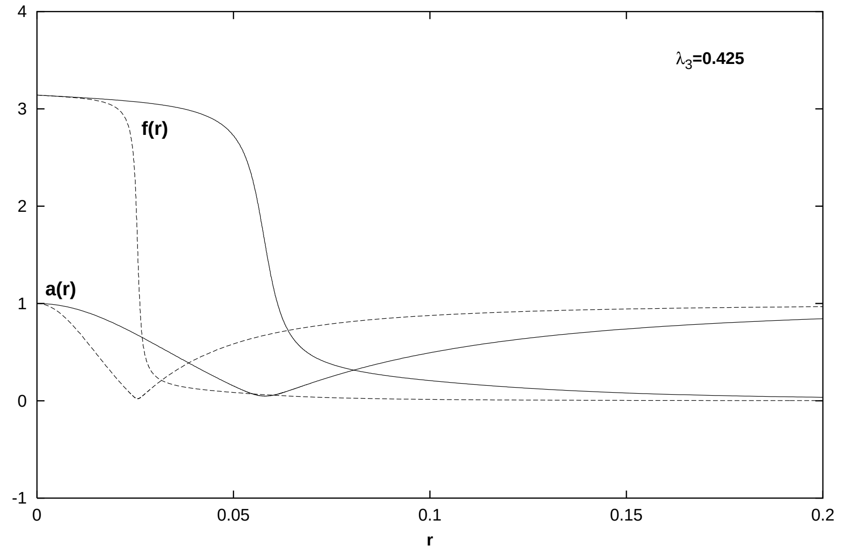

Figure 1: The profiles of two solutions $a(r)$ and $f(r)$ as functions of r for $\lambda_2 = 0$ and $\lambda_3 = 0.425$ for the first branch (solid line) and the second branch (dotted line).

This situation persists up to a critical value of λ_3, say λ_3^c, and numerically we found

$$\lambda_3^c \approx 0.42661 \ . \tag{45}$$

Corresponding to this critical value, we find $r_m \approx 0.041$, $a(r_m) \approx 0.034$ and $S \approx 1.3345$. In particular the value of the action is slightly higher than the value 4/3 corresponding to the action of the instanton solution of Yang–Mills theory [4].

In fact, a large part of this action comes from the Yang-Mills part of the Lagrangian and the contribution due to the S^4 valued matter field is rather tiny (less than one percent) because, as indicated by Fig. 1, the function $f(r)$ becomes very steep in the region where the function $a(r)$ has its minimum.

We found no solutions for $\lambda_3 > \lambda_3^c$; however, a careful analysis of the equations strongly suggests that a second branch of solution exists, as illustrated on Fig. 2. As far as the classical action of the two branches is concerned they terminate into a cusp at $\lambda_3 = \lambda_3^c$, in a way very similar to Fig. 9 of [1].

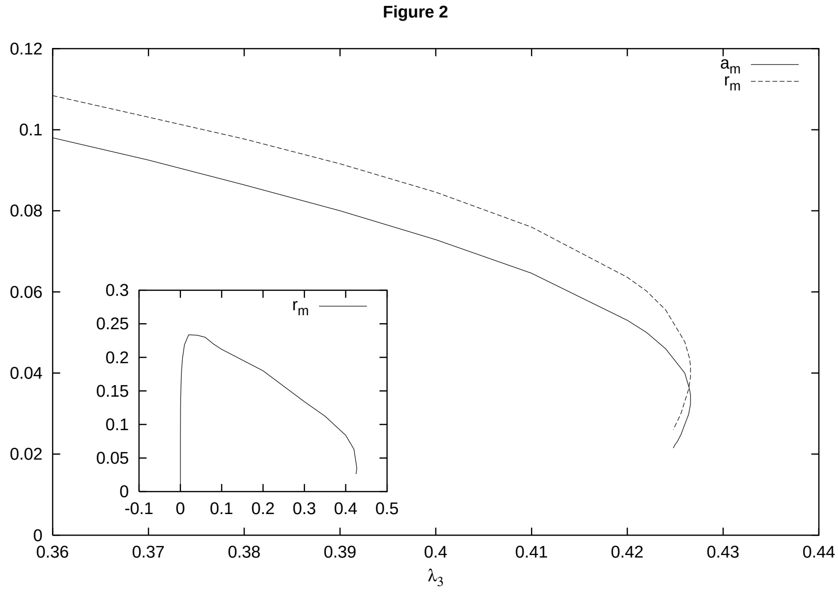

Figure 2: λ_3 dependence of r_m and $a(r_m)$ for the two branches near the critical value of λ_3 when $\lambda_2 = 0.1$. The global dependence on r_m is displayed in the window.

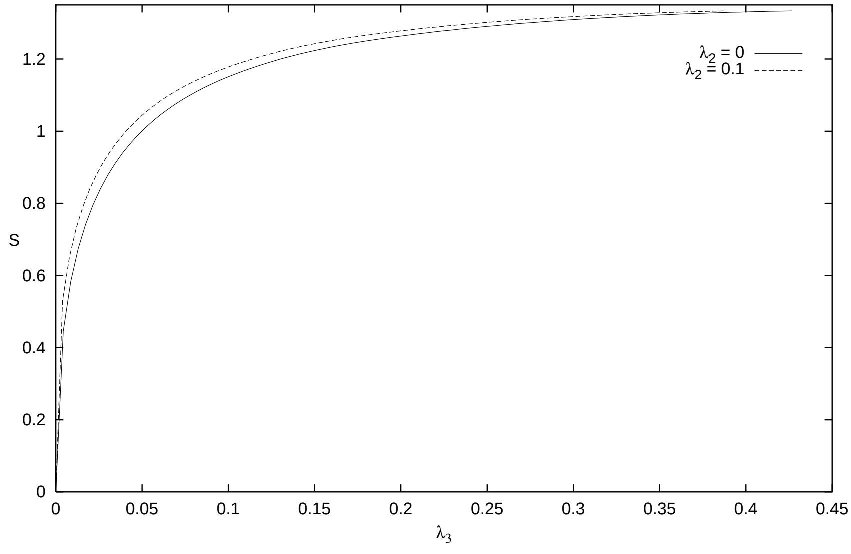

Figure 3: λ_3 dependence of the action for $\lambda_2 = 0$ (solid line) and $\lambda_2 = 0.1$ (dotted line).

As suggested by Fig. 2, it is very likely that when λ_3 decreases to below λ_3^c on the second branch, the minimum $a(r_m)$ has a tendency to approach zero while the derivative $f'(r_m)$ of the function $f(r)$ becomes infinite. For that reasons, the construction of the classical solution along this branch is numerically difficult and we we had to stop it at $\lambda_3 \approx 0.4248$.

Nevertheless, it seems that the profile $a(r)$ of the solution on the second branch is such that

$$\lim_{\lambda_3 \to \lambda_3^*} a(r) = \mid a_{BPST}(r) \mid \tag{46}$$

a_{BPST} being the profile of the charge-1 instanton (41) for an appropriate value of the scaling constant k. The numerical difficulties prevented the evaluation of λ_3^* but, according to Fig. 2, one can expect $\lambda_3^* \approx 0.42$.

The solutions were constructed with the subroutine COLSYS [9] (see Appendix of [10] for a short description) and with a high degree of accuracy: typically with an error less than 10^{-8}.

To finish this Subsection we mention that the pattern of solutions presented above for $\lambda_2 = 0$ seems to persist for $\lambda_2 > 0$. For instance, for $\lambda_2 = 1$ we find $\lambda_3^c \approx 0.14$, i.e. a much lower value than in the case $\lambda_2 = 0$. More details are given in the next Subsection.

4.2 Solutions with $a(1) = a(\infty) = 1$ with $\lambda_2 > 0, \lambda_3 > 0$

In this Subsection we present numerical results for solutions with the same asymptotics as in Subsection 4.1 above, but **(a)** holding λ_2 fixed at $\lambda_2 = 0.1$ and varying λ_3, and **(b)** holding λ_3 fixed at $\lambda_3 = 0.1$ and varying λ_2.

As in Subsection 4.1 above, the solutions do not appear to persist for arbitrarily large λ_3 (when $\lambda_2 = 0.1$), and arbitrarily large λ_2 (when $\lambda_3 = 0.1$). Unlike in Subsection 4.1 however, we have not endeavoured to find accurate critical values for the λ_2 and λ_3, respectively. The general features of the solutions remain unchanged whether or not λ_2 vanishes.

The action versus λ_3 (with $\lambda_2 = 0.1$) is plotted on Fig. 3. The action versus λ_2 (with $\lambda_3 = 0.1$) is plotted on Fig. 4. In both cases we note the remarkable feature that the action rises to just above the BPST instanton action $\frac{4}{3}$. This feature is shared with the restricted model with $\lambda_1 = \lambda_3 = 0$ model studied in Ref. [1].

Fig. 5 is the analogue of Fig. 2, where the progress of r_{min} and a_{min}, the position of the minimum and the value of the minimum of the function $a(r)$, are plotted against increasing λ_2 (with $\lambda_3 = 0.1$ fixed). Figs. 2 (resp. Fig. 5) describes the manner in which the solutions disappear as the value of the λ_3 (resp. λ_2) approaches a critical value. (The analysis confirming the existence of two distinct branches is given only in Fig. 2.)

In addition to the above results, we have made a study of the generic system (1) with the potential (4) included. It turns out that decoupling the potential (4) results

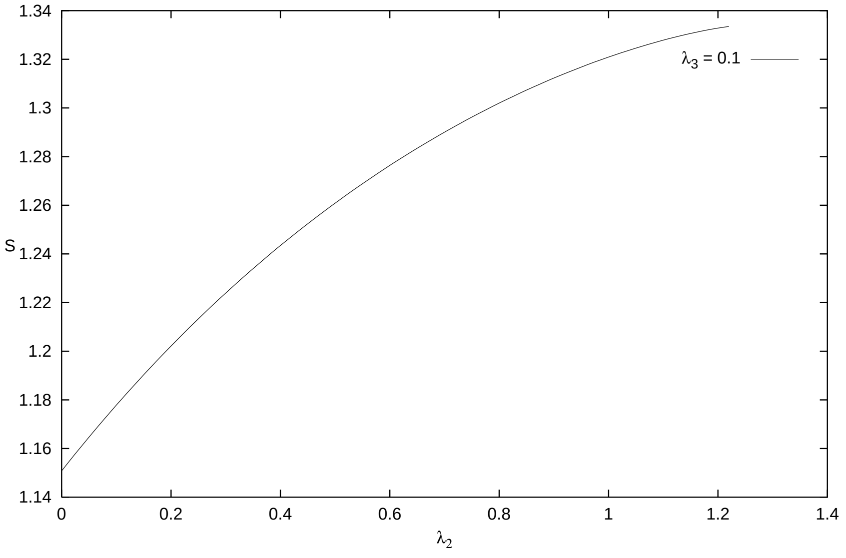

Figure 4: λ_2 dependence of the action for $\lambda_2 = 0.1$.

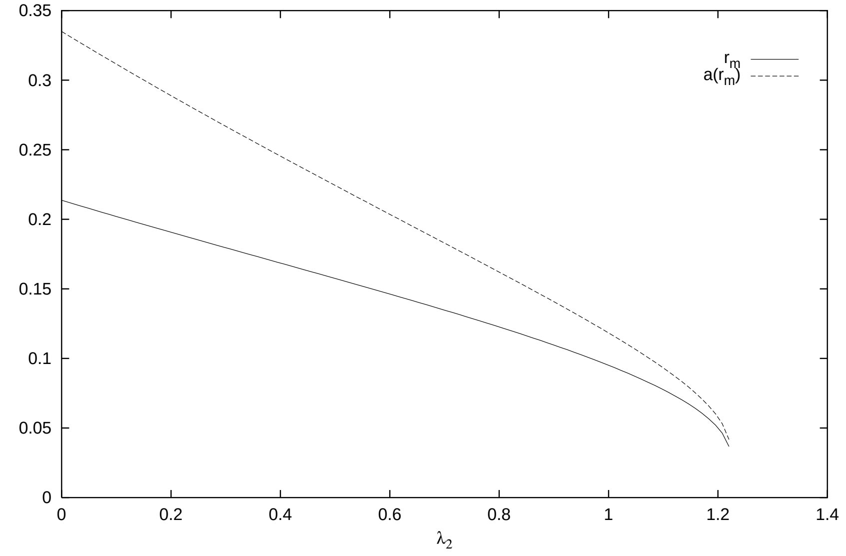

Figure 5: λ_2 dependence of r_m and a_m for $\lambda_3 = 0.1$.

in no appreciable qualitative changes in the values of the action, for all values of the coupling constants λ_2 and λ_3. This property of the present model is shared by the usual (ungauged) Skyrme model [2].

4.3 Solutions with $a(0) = -a(\infty) = 1$

Should solutions of this type exist, their Chern–Pontryagin charge would be nonzero. As we shall see below, the only such solutions are those with trivial S^4 valued field, i.e. only the pure YM [4] instantons.

The set of boundary conditions

$$a(0) = 1 \quad , \quad a(\infty) = -1 \quad , \quad f(0) = \pi \quad , \quad f(\infty) = 0 \ , \tag{47}$$

pertain to *unit* Chern–Pontryagin charge for the spherically symmetric configuration at hand, and provide a natural alternative to the solution discussed above.

Although we could numerically exhibit regular configurations obeying these conditions and with an action slightly higher than 4/3, we have not succeeded in constructing a solution of the equations with these conditions. We used both COLSYS and a relaxation method to find solutions satisfying the boundary conditions (47) and all the numerical results accumulated leads us to the formulation of the following statement: the solutions of the equations of motion obeying (47) are constituted by the functions

$$a(r) = \frac{k^2 - r^2}{k^2 + r^2} \quad , \quad k \in \mathbb{R} \quad , \quad f(0) = 0 \quad , \quad f(r) = \pi \quad , r > 0 \tag{48}$$

considered in the limit $k \to 0$. The limiting configuration, which is determined in terms of step functions, has action $S = 4/3$.

We now give an analytic construction to show that no such smooth solutions exist. To do this we will consider a suitably chosen one parameter family of field configurations and show that in the limit where the parameter vanishes, the field f becomes a step function shrunk near the origin and that the action reduces to the action of the instanton.

First of all let us write the expression for the action after performing the scaling $r = \sigma x$

$$\begin{aligned} S &= \frac{1}{2}\int \left\{ \lambda_0 \left(\left(\frac{a_x}{x}\right)^2 + \left(\frac{a^2-1}{x^2}\right)^2 \right) + \sigma^2 \lambda_1 \left(f_x^2 + 3\left(\frac{a^2 \sin^2 f}{x^2}\right)\right)\right. \\ &+ \left. \lambda_2 \frac{a^2 \sin^2 f}{x^2}\left(f_x^2 + \frac{a^2 \sin^2 f}{x^2}\right) + \frac{\lambda_3}{\sigma^2}\left(\frac{a^2 \sin^2 f}{x^2}\right)^2 \left(3 f_x^2 + \frac{a^2 \sin^2 f}{x^2}\right)\right\} x^3 dx \end{aligned} \tag{49}$$

We then consider the following configuration: for a we take the usual instanton solution and we call x_0 the point where $a(x_0) = 0$. For f we take

$$f \ = \ \pi \quad \text{if} \qquad x \leq x_0 - \frac{\epsilon}{2} \tag{50}$$

$$f = (x_0+\frac{\epsilon}{2}-x)\frac{\pi}{\epsilon} \quad \text{if} \qquad x_0-\frac{\epsilon}{2}\leq x\leq x_0+\frac{\epsilon}{2} \tag{51}$$

$$f = 0 \quad \text{if} \qquad x\geq x_0+\frac{\epsilon}{2}. \tag{52}$$

We then notice that the support of the action density for the first 3 terms is the interval $[x_0-\frac{\epsilon}{2}, x_0+\frac{\epsilon}{2}]$. Moreover in that interval we can write

$$a = K(x-x_0)+O((x-x_0)^2) \tag{53}$$

where K is a constant. Defining

$$S_a = \int \frac{\lambda_0}{2}\left((\frac{a_x}{x})^2+(\frac{a^2-1}{x^2})^2\right)r^3dr \tag{54}$$

we can write

$$\begin{aligned} S &= S_a + \frac{1}{2}\int_{x_0-\epsilon/2}^{x_0+\epsilon/2}\left\{\sigma^2\lambda_1 x^3\left(\frac{\pi^2}{\epsilon^2}+\frac{3}{x^2}K^2(x-x_0)^2\sin^2 f\right)\right. \\ &+ \lambda_2 x^3\left(\frac{K^2}{x^2}(x-x_0)^2\sin^2 f(\frac{\pi^2}{\epsilon^2}+\frac{K^2}{x^2}(x-x_0)^2\sin^2 f)\right) \\ &+ \left.\frac{\lambda_3}{\sigma^2}x^3\left(\frac{K}{x}(x-x_0)\right)^2\left(\frac{3}{\epsilon^2}\pi^2+\frac{K^2}{x^2}(x-x_0)^2\sin^2 f\right)\right\}dx \end{aligned} \tag{55}$$

We can the replace $\sin f$ by 1 and perform each of the integrals to the lowest order in ϵ which gives

$$\begin{aligned} S &\leq S_a+\sigma^2\frac{\lambda_1}{2}(\frac{1}{\epsilon}x_0\pi^2+K^2x_0\epsilon^3)+\lambda_1\sigma^2 O(\epsilon)+\frac{\lambda_2}{2}K^2\left(\frac{x_0}{3}\pi^2\epsilon+\frac{K^2\epsilon^5}{5(x_0+\epsilon/2)}\right) \\ +\lambda_2 O(\epsilon^3) &+ \frac{\lambda_3}{2\sigma^2}K^4\left(\frac{3}{(x_0+\epsilon/2)^5}\epsilon^3+\frac{K^2}{7(x_0+\epsilon/2)^3}\epsilon^7\right)+\frac{\lambda_3}{\sigma^2}O(\epsilon^5) \end{aligned} \tag{56}$$

choosing $\sigma=\epsilon$ we have

$$\lim_{\epsilon\to 0} S = S_a \tag{57}$$

and in that limit the field f becomes singular, showing that there are no regular solutions with this boundary conditions. This is indeed what we observed when we tried to compute such solution numerically.

5 Summary and discussion

The coupling of non-linear sigma models to gauge fields often leads to sets of classical equations whose solutions obey various types of critical phenomena like bifurcations and/or pairs of solutions terminating into a cusp. The classical equations associated with the Lagrangian (1), in the spherically symmetric Ansatz, are of this type. These solutions seem to follow the same pattern, irrespectively of the different Skyrme terms added, i.e. these patterns seem to be independent of the dynamical details.

In this paper, we have studied the classical solutions ensuing from the Lagrangian (1) for three different sets of the two independent coupling constants (λ_2, λ_3). Inspite of the fact that our analysis in Section 2 (specifically in Subsection 2.1) leads to the establishment of a topological lower bound on the action provided that **both λ_2 and λ_3 be positive**, we have found that in fact solutions persist at $\lambda_2 = 0$. A similar situation arises in the three dimensional Skyrme model augmented by a sextic Skyrme term. In that case too, when the usual (quartic) Skyrme term is decoupled, thus invalidating the topological lower bound, the solution persists [11] notwithstanding. This most probably means that our (Bogomol'nyi type) analysis in Section 2 is not far reaching enough for the model at hand. For example, in the case of Hopf solitons, there exists no Bogomol'nyi type lower bound on the energy, but instead one finds that a bound nevertheless can be established in terms of Sobolev type inequalities [12]. We have not carried out the appropriate analysis here, but expect that this can be done. Accordingly we have treated the simplified model (1) with $\lambda_2 = 0$ as legitimate and have carried out the detailed analysis of exhibiting the cusp structure alluded to in the previous paragraph, for that model in Subsection 4.1, which we summarise in the next paragraphs.

For the $\lambda_2 = 0$ model, clearly the two branches of solutions (the ones with non trivial S^4 valued matter field) terminate into a cusp at $\lambda_3 = \lambda_3^c$. This is a typical situation met in catastrophy theory.

The spherically symmetric (BPST) instanton of the pure Yang–Mills theory [4] plays a major role and behaves as an attractor (at least when one of the coupling constants approaches a certain value) of the solution which excites both the matter and the gauge fields. It is very likely that the second branch of solutions bifurcates from the BPST-branch at the critical value $\lambda_3 = \lambda_3^* < \lambda_3^c$. However, due to the absolute value in the limit (46), the bifurcating solution does not occur in a standard, i.e. continuous way.

The qualitative features of the instanton of the $\lambda_2 = 0$ model just described were confirmed also in the generic model with non-vanishing λ_2 and λ_3, in Subsections 4.2, where the cusp resulting from the existence of two distinct branches was not explicitly constructed.

In Subsection 4.3, we verified that there existed no instantons with *unit* Pontryagin charge in this model, irrespective of the value of $S^4 \to S^4$ winding number. This is important since it tells us that the zero Pontryagin charge instantons of this model are not the analogues of the triangle anomaly, and hence that the nonperturbative quasiclassical effects they describe must be given a new physical interpretation. (We return to this in the last paragraph.)

Before alluding to the possible physical relevance of the model, we note that non-vanishing Pontryagin charge instantons can readily be constructed by changing the model to feature a symmetry breaking potential (3) as opposed to (4). We have presented the simplest such model in Subsection 2.2, but did not carry out a numerical study in that case because the model involved was rather remote from known physically relevant models.

In short, we have seen that the system consisting of the interacting YM and $O(5)$ sigma models supports instantons with vanishing Pontryagin charge, which do not describe quasiclassical effects analogous to the triangle anomaly, but which have the $S^4 \to S^4$ winding number as the topological charge. Besides, the gauge group for this model is not that of the Standard Model. On the other hand, it is quite straightforward to construct an $O(5)$ model interacting with the $SO(3) \times SO(2)$ YM system that supports such instantons, by adapting the analysis of Subsection 2 to that case. (That remains a future project.) Moreover, the number of independent S^4 valued fields is equal to four, just as the number of the Higgs doublet in the Standard Model, thus an $SO(3) \times SO(2)$ gauged $O(5)$ model added to the Standard Model, could be regarded as a complicated low energy version of the latter, whose (axially symmetric) instantons can describe new nonperturbative effects. In this sense, the $SO(4)$ gauged $O(5)$ model studied here can be regarded as a prototype of a physically more relevant model.

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