

Rotating regular solutions in Einstein-Yang-Mills-Higgs theory

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Abstract

We construct new axially symmetric rotating solutions of Einstein-Yang-Mills-Higgs theory. These globally regular configurations possess a nonvanishing electric charge which equals the total angular momentum, and zero topological charge, representing a monopole-antimonopole system rotating around the symmetry axis through their common center of mass.

Introduction.— Rotation is an universal phenomenon, which seems to be shared by all objects, at all possible scales. For a gravitating Maxwell field, the Kerr-Newman black hole solutions represent the only asymptotically flat configurations with nonzero angular momentum. However, no regular rotating solution is found in the limit of zero event horizon radius.

The inclusion of a larger (non Abelian) gauge group in the theory leads to the possibility of regularising these configurations, as evidenced by the Bartnick-McKinnon (BM) solution of the Einstein-Yang-Mills (EYM) equations [1]. However, to date no explicit example of an asymptotically flat regular rotating solution with non Abelian matter fields is known [2]. Although predicted perturbatively [3], no rotating generalisations of the BM solution seem to exist [4, 5]¹.

The situation is more complicated in a spontaneously broken gauge theory. As discussed in [6, 4, 2] for a Higgs field in the adjoint representation (the case considered in this letter), the Julia-Zee dyons do not present generalisations with a nonvanishing angular momentum. In fact the general result presented in [9] proves that the angular momentum of any regular solution with a nonvanishing magnetic charge is zero². This however leaves open the possibility of the existence of rotating Einstein-Yang-Mills-Higgs (EYMH) solutions in the topologically trivial sector of the theory. We have in mind solutions described by an equal number of monopoles and antimonopoles situated on the z -axis with zero net magnetic charge like those in [11] and [12], gravitating and in flat space respectively; but unlike the latter [11, 12], with nonzero electric charge. Although the density of the magnetic field is locally nonzero, the magnetic charge of these configurations measured at infinity would vanish. This, in the presence of an electric charge, results in nonzero angular momentum.

Despite the presence of some comments in the literature on the possible existence of such solutions, no explicit construction has been attempted. Here we construct numerically the simplest example of a regular rotating solution in a spontaneously broken gauge theory. It represents an asymptotically flat, electrically charged monopole-antimonopole (MA) system rotating around their common center of mass. For a vanishing electric field, the solution reduces to the static axially symmetric MA configurations discussed in [11].

Axially symmetric ansatz and general relations.— Our study of the SU(2)-EYMH system is based upon the action

$$S = \int \left(\frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu\Phi D^\mu\Phi) - \frac{1}{4} \lambda \text{Tr}(\Phi^2 - \eta^2)^2 \right) \sqrt{-g} d^4x, \quad (1)$$

with Newton's constant G , the Yang-Mills coupling constant e and Higgs self-coupling constant λ .

We consider the usual Lewis-Papapetrou ansatz [13] for a stationary, axially symmetric spacetime with two Killing vector fields $\partial/\partial\varphi$ and $\partial/\partial t$. In terms of the spherical coordinates r , θ and φ , the isotropic metric reads

$$ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2\theta (d\varphi - \frac{\omega}{r} dt)^2, \quad (2)$$

¹Regular EYM configurations with a nonzero angular momentum have been found only in the presence of a negative cosmological constant [7]. However, in this case the properties of the spherically symmetric solutions differ substantially from those of the BM counterparts [8].

²Rotating black holes with a global magnetic charge may exist and the first set of such solutions have been recently presented in [10].

where f, m, l and ω are only functions of r and θ .

For the matter fields, we use a suitable parametrization of the axially symmetric ansatz derived by Rebbi and Rossi [14], with a SU(2) gauge connection

$$A_\mu dx^\mu = \vec{A} \cdot d\vec{r} + A_t dt = \frac{1}{2er} [\tau_\phi (H_1 dr + (1 - H_2) r d\theta) - (\tau_r H_3 + \tau_\theta (1 - H_4)) r \sin \theta d\phi + (\tau_r H_5 + \tau_\theta H_6) dt], \quad (3)$$

and a Higgs field of the form

$$\Phi = (\Phi_1 \tau_r + \Phi_2 \tau_\theta). \quad (4)$$

The SU(2) matrices $(\tau_r, \tau_\theta, \tau_\phi)$ are defined in terms of the Pauli matrices $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ by $\tau_r = \vec{\tau} \cdot (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, $\tau_\theta = \vec{\tau} \cdot (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$, $\tau_\phi = \vec{\tau} \cdot (-\sin \phi, \cos \phi, 0)$.

The six gauge field functions H_i and the two Higgs field function Φ_i depend only on the coordinates r and θ ³. We fix the residual gauge degree of freedom by choosing the usual gauge condition $r \partial_r H_1 - \partial_\theta H_2 = 0$ [5, 10, 11].

Asymptotically flat, regular MA solutions are found by imposing the boundary conditions

$$f = m = l = 1, \quad \omega = 0, \quad H_1 = H_3 = 0, \quad H_2 = H_4 = -1, \quad (5)$$

$$H_5 = \gamma \cos \theta, \quad H_6 = \gamma \sin \theta, \quad \Phi_1 = \eta \cos \theta, \quad \Phi_2 = \eta \sin \theta,$$

at infinity and

$$\partial_r f = \partial_r m = \partial_r l = \omega = 0, \quad H_1 = H_3 = 0, \quad H_2 = H_4 = 1, \quad (6)$$

$$\cos \theta \partial_r H_5 - \sin \theta \partial_r H_6 = 0, \quad \sin \theta H_5 + \cos \theta H_6 = 0,$$

$$\cos \theta \partial_r \Phi_1 - \sin \theta \partial_r \Phi_2 = 0, \quad \sin \theta \Phi_1 + \cos \theta \Phi_2 = 0,$$

at the origin. The functions H_1, H_3 and the derivatives $\partial_\theta f, \partial_\theta l, \partial_\theta m, \partial_\theta \omega, \partial_\theta H_2$ and $\partial_\theta H_4$ have to vanish for both $\theta = 0$ and $\theta = \pi/2$. The other matter functions satisfy the boundary conditions $\partial_\theta H_5 = H_6 = \partial_\theta \Phi_1 = \Phi_2 = 0$ on the z -axis ($\theta = 0$) and $H_5 = \partial_\theta H_6 = \Phi_1 = \partial_\theta \Phi_2 = 0$ on the ρ -axis ($\theta = \pi/2$).

The constants γ, η in (5) correspond to the asymptotic magnitude of the electric potential and Higgs field, respectively. The field equations imply the following expansion as $r \rightarrow \infty$

$$f \sim 1 - \frac{2M}{r}, \quad \omega \sim \frac{2J}{r^2}, \quad H_5 \sim \gamma \cos \theta \left(1 - \frac{Q_e}{r}\right), \quad H_6 \sim \gamma \sin \theta \left(1 - \frac{Q_e}{r}\right). \quad (7)$$

The expression for the electric and magnetic charges derived by using the 't Hooft field strength tensor is

$$\mathbf{Q}_e = \frac{1}{4\pi} \oint_\infty dS_\mu Tr \{ \hat{\Phi} F_{\mu t} \}, \quad \mathbf{Q}_m = \frac{1}{4\pi} \oint_\infty dS_\mu \frac{1}{2} \epsilon_{\mu\nu\alpha} Tr \{ \hat{\Phi} F_{\nu\alpha} \}, \quad (8)$$

where $\hat{\Phi} = \Phi/|\Phi|$. As implied by the asymptotic behavior (5), (7) these configurations carry zero net magnetic charge, $\mathbf{Q}_m = 0$ (although locally the magnetic charge density is nonzero) and a nonvanishing electric charge $\mathbf{Q}_e = \gamma Q_e$. Therefore they will present a magnetic dipole moment C_m which can be obtained from the asymptotic form of the non-abelian gauge field, after choosing a gauge where the Higgs field is asymptotically constant $\Phi \rightarrow \tau_3$ [12]

$$\vec{A} \cdot d\vec{r} = C_m \frac{\sin^2 \theta}{r} \frac{\tau_3}{2} d\varphi. \quad (9)$$

The mass M of these regular solutions is obtained from the asymptotic expansion (7) or equivalently from $M = - \int (2T_t^t - T_\mu^\mu) \sqrt{-g} dr d\theta d\varphi$ [13]. The constant J appearing in (7) corresponds to the total angular

³The static MA solutions discussed in [11, 12] have been obtained for a different parametrization of the same ansatz, imposed by a different choice of the SU(2) matrices $(\tau_r, \tau_\theta, \tau_\phi)$. Note that $H_5 = H_6 = 0$ for static solutions.

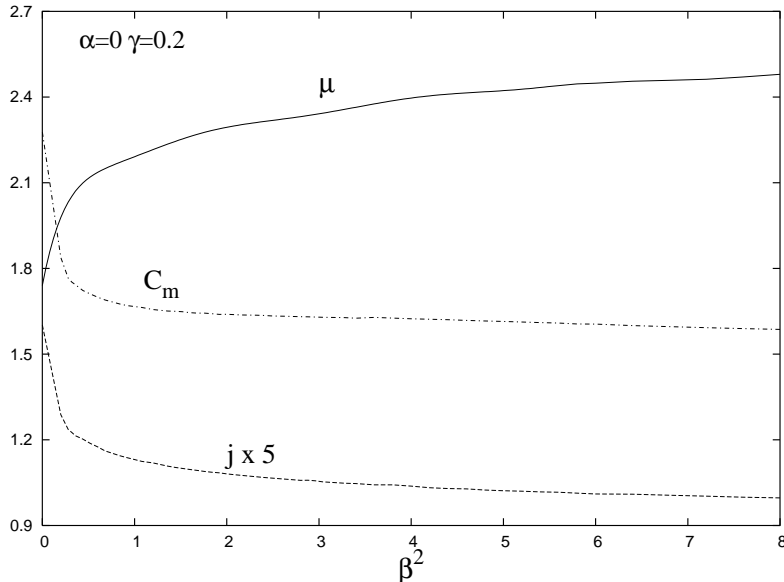


Figure 1. The mass, angular momentum and magnetic dipole moment are plotted as a function of Higgs self-coupling constant β^2 for flat space rotating MA solutions.

momentum of a solution which can also be written as a volume integral $J = \int T_\varphi^t \sqrt{-g} dr d\theta d\varphi$. As proven in [4], another form of this expression, in terms of asymptotics of the gauge potentials, is

$$J = \oint_\infty dS_\mu 2Tr\{WF^{\mu t}\}, \quad (10)$$

(with $W = A_\varphi - \tau_z/2$), which, from the asymptotic expression (7) is just the electric charge, $J = \mathbf{Q}_e$. Introducing the dimensionless coordinate $x = r\eta e$ and the Higgs field $\phi = \Phi/\eta$, the equations depend on the coupling constants $\alpha = \sqrt{4\pi G}\eta$ and $\beta^2 = \lambda/e^2$, yielding the dimensionless mass and angular momentum $\mu = \frac{e}{4\pi\eta}M$, $j = \frac{e\eta^2}{4\pi}J$.

Numerical results.– We solve numerically the set of twelve coupled non-linear elliptic partial differential equations, subject to the above boundary conditions, employing a compactified radial coordinate $\bar{x} = x/(1+x)$. As initial guess we use the static MA regular solutions, corresponding to $\gamma = 0$. For any MA configuration, increasing γ leads to rotating regular solutions with nontrivial functions H_5 , H_6 and ω .

For $\alpha = 0$, we find rotating MA solutions in a flat spacetime background. As remarked in [15], for vanishing Higgs potential these solutions can be generated, from the pure magnetic MA configuration (\vec{A}, Φ_0) by using the transformation $\vec{A} \rightarrow \vec{A}$, $\Phi \rightarrow \Phi_0 \cosh \xi$, $A_t \rightarrow \Phi_0 \sinh \xi$ (no similar relation exists for gravitating solutions although for small enough values of α the time component of the gauge field and the Higgs field are still almost proportional). Their properties can also be deduced from the $\lambda = 0$ MA configuration [15]. To demonstrate the dependence of the flat space MA rotating solutions on the Higgs self-interaction, we plot in Figure 1 the mass/energy, angular momentum and magnetic dipole moment as a function of β . A similar qualitative picture is found for gravitating solutions. However, all $\alpha \neq 0$ solutions presented here have no Higgs potential, $\beta^2 = 0$.

When α is increased from zero, while keeping γ fixed, a branch of rotating solutions emerges from the flat spacetime configurations. This branch ends at a critical value α_{cr} which depends on the value of γ , the numerical errors increasing dramatically for $\alpha > \alpha_{cr}$ for the solutions to be reliable. As $\alpha \rightarrow \alpha_{cr}$, the geometry remains regular with no event horizon appearing, and, the mass and angular momentum approach finite values. Along this branch, the MA pair move closer to the origin and the mass, angular momentum and magnetic dipole moment of the solutions decrease to some limiting values (see Figure 2).

As discussed in [11], apart from the fundamental branch, the static MA solutions admit also an infinite sequence of excited configurations, emerging in the $\alpha \rightarrow 0$ limit (after a rescaling) from the spherically

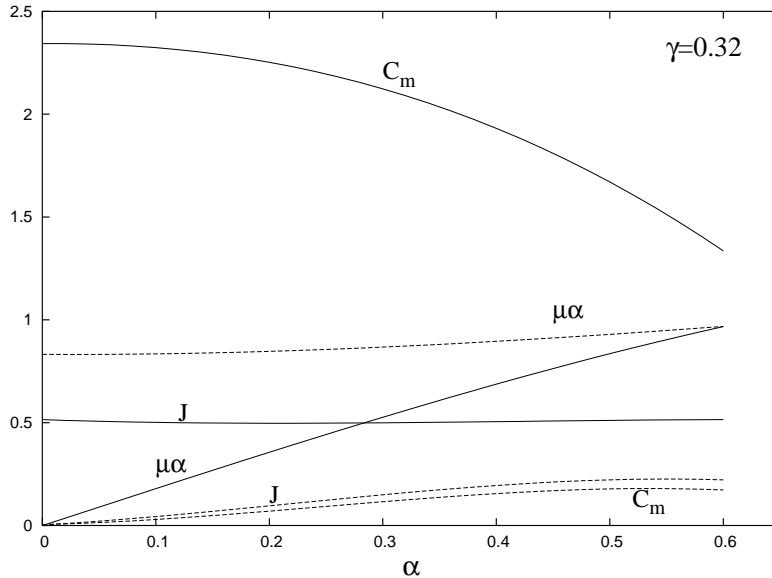


Figure 2. The scaled mass $\alpha\mu$, the angular momentum j and the magnetic dipole moment C_m are shown as a function on α for a fixed value of the electric potential magnitude at infinity $\gamma = 0.32$. The solid and the dotted lines correspond to the fundamental and the second branch of solutions, respectively.

symmetric BM solutions⁴. The lowest excited branch, originating from the one-node BM solution, evolves smoothly forward from $\alpha = 0$ to α_{cr} where it bifurcates with the fundamental branch.

These static excited MA solutions present also rotating generalizations. For the considered range of γ , we find that a second branch of rotating MA solutions emerges at $\alpha_{cr}(\gamma)$, extending backwards to $\alpha = 0$. As seen in Figure 2, although for a given γ the mass of the α_{cr} solutions is the same - within the limit of the numerical accuracy, other quantities such as angular momentum and magnetic dipole moment present a discontinuity. It is likely however that the complete picture is more complicated, as in contrast to the static MA case, when $J \neq 0$ there is a black hole solution with degenerate horizon for the regular solutions to merge into. Thus, we expect more solutions to exist, representing a branch section possibly bending backwards in α , and merging for some α_m into the extremal Kerr-Newman solution with $M^2 = 2J^2$. The numerical construction of such configurations presents a considerable numerical challenge beyond the scope of the present work.

The excited solutions become infinitely heavy as $\alpha \rightarrow 0$ while the locations of the monopole and anti-monopole approach the origin. The angular momentum/electric charge and the magnetic dipole moment of the solutions vanish in the same limit. The rescaling $x \rightarrow x\alpha$, $\Phi \rightarrow \Phi/\alpha$ reveals that, similar to the static MA solutions, the limiting solution on the upper branch is the first spherically symmetric BM solution. In this case, the limit $\alpha \rightarrow 0$ corresponds to $\eta \rightarrow 0$, for a nonzero value of G . The limiting value of the scaled mass $\hat{\mu} = \alpha\mu$ corresponds also to the mass of the one-node BM solution, with $H_1 = H_3 = 0$, $H_2 = H_4 = w(r)$, $H_5 = H_6 = 0$. Thus, no rotating limiting EYM solution is found in this way. Without a Higgs field, the regularity conditions force the electric potentials to vanish identically⁵.

Indeed, for any value of α , we could not find solutions with an asymptotic magnitude of electric potential greater than that of the Higgs field (i.e. $\gamma > 1$ after rescaling). In this situation, similar to the case of dyon configurations [15], the asymptotic behavior of some gauge field functions became oscillatory, failing to satisfy the required boundary conditions.

In Figure 3 we show the mass, angular momentum and magnetic dipole moment as a function of γ for a fixed value of α . Both fundamental and second branch solutions are displayed. These quantities increase always with γ and stay finite as $\gamma \rightarrow 1$.

⁴For the fundamental branch solutions, $\alpha \rightarrow 0$ corresponds to $G \rightarrow 0$.

⁵Spherically symmetric EYM solutions with $A_t \neq 0$ cannot exist [16], while the rotating black hole solutions have $\gamma = 0$ [5], the electric field being supported by the rotating event horizon contribution.

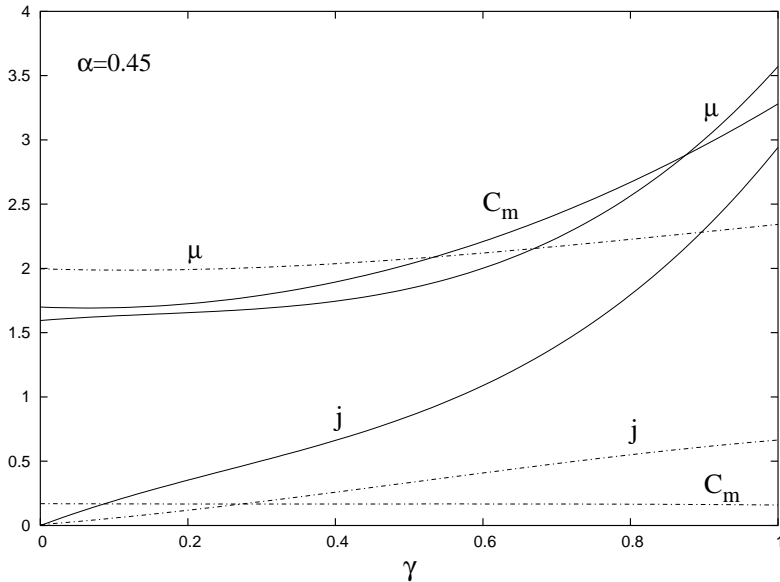


Figure 3. The mass μ , the angular momentum j and the magnetic dipole moment C_m are shown as a function on the magnitude of the electric potential at infinity η for a fixed value of $\alpha = 0.45$. The solid and the dotted lines correspond to the fundamental and the second branch of solutions, respectively.

The rotating solutions share a number of common properties with the purely magnetic MA configurations. The modulus of the Higgs field possesses always two zeros at $\pm d/2$ on the z -symmetry axis, corresponding to the location of the monopole and antimonopole, respectively. Both \vec{A} and Φ_a present a shape similar to the static case. The energy density $\epsilon = -T_t^t$ possesses maxima at $z = \pm d/2$ and a saddle point at the origin, and presents the typical form exhibited in the literature on MA solutions. A different picture is found for the angular momentum density. As seen in Figure 4, the MA system rotates as a single object and the T_φ^t -component of the energy momentum tensor associated with rotation presents a maximum in the $z = 0$ plane and no local extrema at the locations of the monopole and the antimonopole.

Although we have restricted the analysis here to the simplest sets of solutions, rotating MA configurations have been found also starting with excited MA branches with $A_t = 0$. These solutions do not possess counterparts in flat spacetime and their $\alpha \rightarrow 0$ limit corresponds always to (higher node-) BM solutions. **Further remarks.**— We have presented here the first set of globally regular solutions of EYM theory possessing a nonvanishing angular momentum. These asymptotically flat configurations carry mass, angular momentum, electric charge and no net magnetic charge. The electric charge is induced by rotation and equals the total angular momentum.

The excited rotating solutions do not possess a counterpart in flat space, and their angular momentum vanishes in the no-Higgs field limit, which corresponds to the BM configurations. The nonexistence of a rotating generalisation of the BM solution can be viewed as a consequence of the impossibility to obtain regular, electrically charged nonabelian solutions without a Higgs field.

The situation here differs from the electrically neutral gravitating case [11], where there is no available black hole solution (e.g. Reissner-Nordström) for the fundamental branch to end in, due to the absence of a global (magnetic) charge. Here by contrast we have an electric charge, so one might expect the fundamental branch to end in the corresponding rotating black hole, namely the Kerr-Newman solution. Our numerical results indicate tentatively that this might well be the case, as the metric functions seem to be decreasing towards zero with α increasing beyond the bifurcation point α_{cr} . Unfortunately we cannot make this claim reliably here, because the numerical accuracy of these results is not sufficiently good. The complexity of this numerical task is beyond the scope of the present work, and remains an outstanding matter to be disposed of in future.

Concerning the stability of these solutions, in the absence of a topological charge, we expect them to be unstable, similar to the electrically uncharged MA configurations.

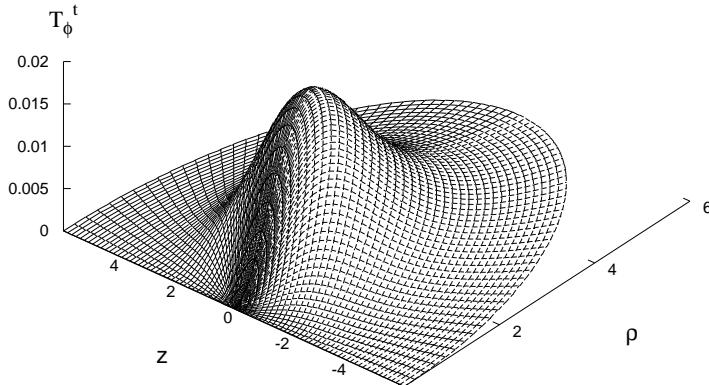


Figure 4. The angular momentum density T_ϕ^t is shown for a typical fundamental branch MA rotating solution, with $\alpha = 0.42$, $\gamma = 0.26$.

By including an integer n (the winding number) in the ansatz (3), rotating MA chains and rotating vortex rings can be found for $n > 1$. We expect also that EYMH theory possesses a whole sequence of rotating solutions, obtained within the ansatz (3), for an asymptotic behaviour of the Higgs field with $\Phi_1 = \eta \cos(2k+1)\theta$, $\Phi_2 = \eta(2k+1)\sin\theta$, where k is a positive integer (the static limit of these solutions is discussed recently in [17]). These solutions would possess again an angular momentum equal to the electric charge but no net magnetic charge. The $\alpha \rightarrow 0$ limit of the excited solutions will correspond to the recently discovered sequence of EYM static axially symmetric configurations [18].

Rotating MA black hole solutions, generalizing the static, axially symmetric black holes with magnetic dipole hair [19], should exist as well. However, these solutions will not satisfy the intriguing relation $\mathbf{Q}_e/J = 1$ which is a unique property of the regular configurations with zero topological charge.

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