Some Metric Theorems on Polynomials.

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Abstract

The theory of metric Diophantine approximation can be studied from many different perspectives. The problems studied in this thesis all concern questions on integer polynomials. Simultaneous rational approximation to integer polynomials is studied in the p-adic metric. Next, the nature of the closest root to an argument of a leading polynomial is studied in the Euclidian and p-adic metrics. Finally the nature of regular systems for third degree polynomials is investigated.

Contents

Notation

Notation that is used extensively throughout this document is listed below to assist the reader.

$P(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{Z}\ i = 1..n$	an integer polynomial of degree n
$H(P) = \max_{0\leq j\leq n} \lvert a_j \rvert$	the height of an integer polynomial
$R(P,Q) = a_n^m b_m^n \prod_{1\leq i\leq n} \prod_{1\leq j\leq m} (\alpha_i - \beta_j)$	the resultant of two polynomials
$D(P) = a_n^{2n-2} \prod_{1\leq i<j\leq n} (\alpha_i - \alpha_j)^2$	the discriminant of a polynomial
$a \ll b$	$a < Kb\ K > 0$ constant
$a \gg b$	$a > Kb\ K > 0$ constant
$a \asymp b$	$a \ll b$ and $a \gg b$
$S_P(\alpha)$	the set of all numbers closest to a root α of P
$P_n(H)$	$P \in \mathbb{Z}[z], \deg P = n, H(P) = H$

Chapter 1

Introduction and notation

1.1 Introduction

One of the main goals of Diophantine approximation is to investigate the quantity $|x - \frac{p}{q}|$, where x is a real number and $p, q \in \mathbb{Z}, q \neq 0$. This was initially investigated in the 19th century by Dirichlet and Liouville who proved results on rational approximation.

Theorem 1.1 (Dirichlet)**.** *Let x and Q be real numbers with $Q \geq 1$. There exists a rational number $\frac{p}{q}$ with $1 \leq q \leq Q$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{qQ}.$$

If x is irrational, then there exist infinitely many rational numbers $\frac{p}{q}$ such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Classical results in Diophantine approximation have been adapted and extended to cover a wide range of different perspectives including approximation by algebraic numbers, approximation on manifolds and approximation under different metrics. Some open questions in these topics are investigated in this thesis.

In this Chapter is introduced the necessary background information required in the thesis. The first section will give some Lemmas due to Sprindžuk [85] on the topic of metric Diophantine approximation. The topic is introduced and some of the historical development of the subject is explained. Also the framework is provided to develop the ideas presented in later chapters. The second section will define the p-adic numbers, and some essential concepts of p-adic number theory including the p-adic field and the completeness of the p-adic field. Again, some Lemmas that are useful are stated. Of particular interest is Hensel's Lemma. The third section will provide the definitions of different types of measure and dimension which are used extensively throughout, including Hausdorff Dimension and some related Lemmas necessary for Chapter 3.

Chapter 2 provides a brief history of the subject and in Chapter 3 the p-adic version of a Theorem on simultaneous Diophantine approximation on polynomials, proved in [32], is studied.

In Chapter 4 different versions of a problem of Nesterenko are considered. This problem was first introduced by Y.V. Nesterenko and presented at the International Conference of Number Theory in Shaulyai (Lithuania, 2008). The question is to determine for an integer polynomial P, whether the root α_1 of P belongs to the p-adic field or is in the extension.

For some important problems in transcendental number theory it is necessary to know whether the root of a polynomial α_1 is a real or complex number. Knowledge of the nature of α_1 admits the use of regular systems in tackling the following problems: the Hausdorff dimension of the set of $x \in \mathbb{R}$, for which, for $w > n$, the inequality $|P(x)| < H^{-w}$ has infinitely many solutions in polynomials P (see [3], [20]); generalising the divergence case of Khintchine's Theorem to polynomials (see [6],[24]); solving the inequality $|x - \alpha_1| < \varepsilon_0$ for almost all x and integer algebraic numbers α_1 (see [37]). For the latter problem if $\alpha_1 \in \mathbb{R}$ the solutions lie in an interval of length $2\varepsilon_0$. On the other hand, if $\alpha_1 \in \mathbb{C} \setminus \mathbb{R}$, we know nothing about the set of solutions as it could be a disc in the complex plane with centre α_1 and radius ε_0, which need not intersect the real axis at all.

In the second section of Chapter 4 the results of Nesterenko in the p-adic domain are improved. The approach used here uses the discriminant of the polynomial.

A small result that generalizes Nesterenko's problem to $\mathbb{R} \times \mathbb{Q}_p^*$ is also proved. Specifically, if an integer polynomial P simultaneously satisfies

$$|P(x)| < H^{-w_1}, \quad |P(w)|_p < H^{-w_2} \tag{1.1}$$

what can we say about the roots of P?.

Finally, a problem of Bugeaud [35] is studied in Chapter 5. The question posed concerns the length of intervals for which a regular system of real algebraic numbers of degree 3 can be constructed.

1.2 Definitions and notation

In all cases unless otherwise stated, $P \in \mathbb{Z}[x]$ is the polynomial

$$P(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{Z} \;\; i = 1 \ldots n, a_n \neq 0. \tag{1.2}$$

The *height* $H = H(P)$ of a polynomial of degree n is defined as

$$H = H(P) = \max_{0 \le j \le n} |a_j|. \tag{1.3}$$

Throughout this document, it will further be assumed that when used, $x \in \mathbb{R}$; if α is a root of a polynomial P then $\alpha \in \mathbb{C}$ and if w is a root of P, then $w \in \mathbb{Q}_p^*$.

Also, $\mathrm{hcf}(a, b)$ will be used to denote the highest common factor of the non-zero integers a and b.

The *resultant* of two non-constant integer polynomials, $P(x) = \sum_{l=0}^{n} a_l x^l$, and $Q(x) = \sum_{k=0}^{m} b_k x^k$, is defined as

$$R(P, Q) = a_n^m b_m^n \prod_{1 \le i \le n} \prod_{1 \le j \le m} (\alpha_i - \beta_j) \tag{1.4}$$

where $P(\alpha_i) = 0$ and $Q(\beta_j) = 0$. It should be clear that $R(P, Q) = 0$ if and only if P and Q have a common root. A special case of the resultant $R(P, P')$ where P' is the derivative of P is called the discriminant, and is defined below. The *discriminant* of the polynomial P will be written as $D(P)$, and defined as

$$D(P) = a_n^{2n-2} \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2. \tag{1.5}$$

The discriminant $D(P) = 0$ if and only if P' and P have a common root, that is, if P has a root of multiplicity larger than 1.

In their recent article, Johnson and Kollár [64] noted that the discriminant as a tool had moved to the periphery of the study of polynomials of a single variable from its central position in the mid-nineteenth century. They state "for example, resultants were removed from the fourth edition of van der Waerden's classic *Algebra* in 1959, and have not appeared in subsequent editions." Recently both have again been used in proofs in the theory of metric Diophantine approximation.

Given positive real numbers, a and b, the Vinogradov notation $a \ll b$ ($a \gg b$) is used when there exists a positive constant K such that $a < Kb$, (respectively $a > Kb$). If $a \ll b$ and $a \gg b$ then a and b are said to be comparable, which is denoted $a \asymp b$.

A polynomial P will be called *leading* if it satisfies

$$|a_n| \gg H(P). \tag{1.6}$$

For each $t \in \mathbb{R}$ let

$$||t|| = \min\{|t - r| : r \in \mathbb{Z}\} = \text{dist}(t, \mathbb{Z})$$

and for each $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, let

$$||\mathbf{x}|| = \max\{||x_1||, \ldots, ||x_k||\}.$$

The supremum norm will be denoted by $| \, . \, |$, that is, for a vector $\mathbf{x} \in \mathbb{Z}^n$,

$$|\mathbf{x}| = \max\{|x_1|, \ldots, |x_n|\}.$$

1.3 Lemmas on polynomials

Consider integer polynomials as defined in (1.2). Fix $\varepsilon_0 = \varepsilon_0(n, H(P)) > 0$. If $|P(x)| < \varepsilon_0$, then it is possible to obtain an upper bound for $|x - \alpha_1|$, see [85] where α_1 is the closest root of P to x. For each zero α_k of P we associate the set $S_P(\alpha_k)$ as

$$S_P(\alpha_k) = \{x \in \mathbb{C} : |x - \alpha_k| = \min_{1 \le i \le n} |x - \alpha_i| (k = 1, 2, \ldots, n)\}, \tag{1.7}$$

i.e. the set of all numbers whose distance to α_k is closer than to any other root of P. The sets S_P partition $\mathbb{C}$ for each polynomial except at the boundary.

In this thesis the following Lemmas are used. When they are listed without proof, their proofs can be found in the cited texts. The first Lemma is often referred to as Gelfond's Lemma.

Lemma 1.1 ([35], Lemma A.3). *Let $P_1, P_2, \ldots, P_k$ be polynomials of degree $n_1, \ldots, n_k$ respectively, and let $P = P_1P_2 \ldots P_k$. Let $n = n_1 + n_2 + \ldots + n_k$. Then*

$$2^{-n}H(P_1)H(P_2)\ldots H(P_k) \leq H(P) \leq 2^n H(P_1)H(P_2)\ldots H(P_k).$$

Lemma 1.2 ([85], Lemma 2). *Let P be an integer polynomial of degree n and let x be a number (real or complex) such that $x \in S_P(\alpha_1)$. Then*

$$|x - \alpha_1| < 2^n |P(x)||P'(\alpha_1)|^{-1}. \tag{1.8}$$

Proof. The proof is short so it will be included. As $x \in S_P(\alpha_1)$, it follows that

$$|\alpha_1 - \alpha_i| \leq |\alpha_1 - x| + |\alpha_i - x| \leq 2|x - \alpha_i|, \text{ for } i = 2, \ldots, n. \tag{1.9}$$

Hence

$$|P'(\alpha_1)| = a_n \prod_{2 \leq i \leq n} (\alpha_1 - \alpha_i)| < 2^n a_n \prod_{2 \leq i \leq n} |x - \alpha_i| = \frac{2^n |P(x)|}{|x - \alpha_1|}$$

and the result follows. □

Lemma 1.3. *Let $x \in S_P(\alpha_1)$ where α_1 is a root of order s of the polynomial P, an integer polynomial of degree n. Then*

$$|x - \alpha_1| < 2^{\frac{n-s}{s}} (|P(x)||a_n|^{-1} \prod_{j \geq s+1} |\alpha_1 - \alpha_j|^{-1})^{1/s}. \tag{1.10}$$

Proof. Again, as the proof is short it is included for convenience. Since $x \in S_P(\alpha_1)$, using (1.9)

$$|\alpha_1 - \alpha_j| \leq 2|x - \alpha_j|, \ \ 2 \leq j \leq n.$$

Then from the decomposition $P(x) = a_n(x-\alpha_1)^s(x-\alpha_{s+1})\dots(x-\alpha_n)$ we obtain

$$|x-\alpha_1|^s = |P(x)|(|a_n| \prod_{j \geq s+1} |x-\alpha_j|)^{-1} \leq 2^{n-s}|P(x)||a_n|^{-1} \prod_{j \geq s+1} |\alpha_1 - \alpha_j|^{-1}$$

and the result follows directly. □.

Lemma 1.4. *Suppose $P \in \mathbb{Q}[x]$ is an irreducible polynomial over $\mathbb{Q}$. Then P does not have repeated roots in $\mathbb{C}$.*

Proof. Suppose $P \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$, but has a repeated root $\beta \in \mathbb{C}$. Consider the derivative P' of P. Clearly $P' \in \mathbb{Q}[x]$. Since P is irreducible and P' has degree less than P, it must be that P and P' are coprime. Thus, by Euclid's algorithm there exist polynomials, S and T in $\mathbb{Q}[x]$ such that

$$S(x)P(x) + T(x)P'(x) = 1. \tag{1.11}$$

However, $\beta \in \mathbb{C}$ is a repeated root of P. Thus

$$P(x) = (x-\beta)^2 G(x),$$

where $G(x)$ is a polynomial with coefficients in $\mathbb{C}$. Differentiating we see that

$$P'(x) = (x-\beta)^2 G'(x) + 2(x-\beta)G(x).$$

It is clear that $P(\beta) = P'(\beta) = 0$. Substituting β for x in (1.11) gives a contradiction. □

1.4 Definition of p-adic numbers and introductory concepts

The p-adic numbers were first described by Kurt Hensel in 1897 [59]. A comprehensive introduction to the p-adic numbers can be found in many texts; see [55] for example.

Definition 1.1. *Fix a prime number p. A p-adic number is defined as*

$$w = \sum_{r=-\infty}^{+\infty} c^r p^r \tag{1.12}$$

where $c \in \{0, \ldots, p-1\}$.

Every non-zero rational can be expressed uniquely in the form $p^m a^{'}$ where $m \in \mathbb{Z}$ and $a^{'}$ is a rational number whose numerator and denominator are coprime to p.

Using this, the p-adic metric is defined below. The notion of 'distance' in this metric measures how many times p divides either the numerator or denominator of $a \in \mathbb{Q}$. Before defining the metric, the concepts of valuations and absolute values, in a p-adic sense, must be introduced.

Definition 1.2. *A valuation $v_p : k \to \mathbb{R} \cup \{\infty\}$ is a function from a field k to the extended real line such that*

$$\begin{aligned} &i) && v_p(ab) && = v_p(a) + v_p(b); \\ &ii) && v_p(a+b) && \geq \min(v_p(a), v_p(b)); \\ &iii) && v_p(0) && = \infty. \end{aligned}$$

An immediate consequence of i) above is that $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$.

Definition 1.3. *Fix a prime number $p \in \mathbb{Z}$. The p-adic valuation on $\mathbb{Z}$ is the function*

$$v_p : \mathbb{Z} \setminus \{0\} \to \mathbb{R}$$

where $v_p(n)$ *is the unique integer satisfying*

$$n = p^{v_p(n)} n' \text{ with } p \nmid n'.$$

This can be extended to the rationals; if $a = p^n \frac{u}{v} \in \mathbb{Q}$ and p does not divide uv then $v_p(a) = n$.

Definition 1.4. *An absolute value is a function* $|\,\boldsymbol{.}\,|_p : k \to \mathbb{R}_+$ *such that*

$$\begin{aligned} i) \quad & |x|_p = 0 \text{ if and only if } x = 0; \\ ii) \quad & |xy|_p = |x|_p |y|_p \end{aligned}$$

and either

$$\begin{aligned} iii_a) \quad & |x + y|_p \leq |x|_p + |y|_p \textbf{ or} \\ iii_b) \quad & |x + y|_p \leq \max(|x|_p, |y|_p) \text{ hold.} \end{aligned}$$

Definition 1.5. *For* $x \in \mathbb{Q}$*, the p-adic absolute value of* x *is*

$$|x|_p = p^{-v_p(x)}$$

for $x \neq 0$*, and we use the convention* $|0|_p = 0$*.*

If two absolute values define the same topology they are said to be *equivalent.*

Definition 1.6. *The p-adic field is the completion of* $\mathbb{Q}$ *with respect to the p-adic metric.*

Theorem 1.2 (Ostrowski)**.** *Every non-trivial absolute value on* $\mathbb{Q}$ *is equivalent to one of the absolute values* $|\ |_p$*, where either* p *is a prime number or* $p = \infty$*. The case* $p = \infty$ *corresponds to* $\mathbb{C}$*.*

1.4.1 Hensel's Lemma

The Theorem known as "Hensel's Lemma" describes one of the most important algebraic properties of the p-adic numbers. Basically, it says that in many circumstances one can decide quite easily whether a polynomial has roots in the set of p-adic numbers, $\mathbb{Q}_p$. The test involves finding an "approximate" root of the polynomial, and then verifying a condition on the derivative.

Theorem 1.3 (Hensel's Lemma, [27], Page 134, Lemma 6.17)**.** *Let P be a polynomial with coefficients in $\mathbb{Z}_p$, let $x_0 \in \mathbb{Z}_p$ and $|P(x_0)|_p < |P'(x_0)|_p^2$. Then as $n \to \infty$ the sequence*

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$$

tends to some root $w \in \mathbb{Q}_p$ of the polynomial P and

$$|w - x_0|_p \leq \frac{|P(x_0)|_p}{|P'(x_0)|_p^2} < 1.$$

1.5 Measure and dimension

1.5.1 Hausdorff dimension and measure

A more refined measure than the Lebesgue measure is frequently required in the investigation of number theoretic problems. For example, the Liouville numbers are of Lebesgue measure zero, as is the set of very well-approximable numbers (which are defined in 1.8). It is known that the set of Liouville numbers is a strict sub-set of the set of very well approximable numbers but Lebesgue measure cannot distinguish between the size of these sets.

The Hausdorff dimension which was introduced by F. Hausdorff in 1912, is sufficiently refined to distinguish between the two sets used in the example above.

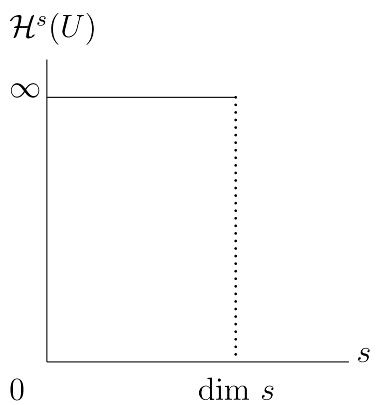
Definition 1.7 (Hausdorff Dimension)**.** *Let E be a set in $\mathbb{R}^n$ and s a non–negative real number. Given $\delta > 0$, a δ–cover of E is a countable collection of sets C_i, each with diameter less than δ, such that $E \subseteq \cup_{i=1}^{\infty} C_i$. Define*

$$\mathcal{H}^s_\delta(E) = \inf \sum_{C_i \in \mathbb{C}} (\operatorname{diam} C_i)^s$$

where the infimum is taken over all δ–covers of E. The Hausdorff outer s–measure $\mathcal{H}^s(E)$ is $\lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$ and the Hausdorff dimension $\dim E$ *is defined as*

$$\dim E = \inf\{s : \mathcal{H}^s(E) = 0\}.$$

Further details can be found in [27, 53]. Diagrammatically, the graph of measure against dimension can be represented as follows:



So at the Hausdorff dimension s, the measure changes from ∞ to 0.

1.5.2 Borel-Cantelli Lemma

The Borel-Cantelli Lemma is an important tool in proving many Theorems in metrical Diophantine approximation. For convenience the convergence half is stated and proved here.

Lemma 1.5 (Borel-Cantelli)**.** *Let* (Ω, μ) *be a measure space with* $\mu(\Omega)$ *finite and let* $A_j, j \in \mathbb{N}$ *be a family of measurable sets. Let*

$$A_\infty = \{\omega \in \Omega : \omega \in A_j \textit{ for infinitely many } j \ \in \mathbb{N}\},$$

and suppose the sum

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty. \tag{1.13}$$

Then $\mu(A_\infty) = 0$.

Proof.

It is readily verified that A_∞ can be written as

$$A_\infty = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j$$

or

$$A_\infty = \limsup A_j.$$

Clearly A_∞ is measurable since it is a countable intersection of measurable sets. We have

$$\mu(A_\infty) \leq \mu(\bigcap_{j=n}^{\infty} A_j)$$

for every $n \geq 1$. Hence

$$\mu(A_\infty) \leq \sum_{j=n}^{\infty} \mu(A_j) \tag{1.14}$$

for every $n \geq 1$. As the sum converges, the right hand side can be made arbitrarily small by taking n sufficiently large. Thus

$$\mu(A_\infty) = 0 \tag{1.15}$$

as required. □

1.5.3 Well-approximable numbers and regular systems

Diophantine approximation began as a study of how closely real numbers could be approximated by rational numbers. Classical results arising from this study have been generalised to approximation by algebraic numbers and Diophantine approximation on manifolds.

Definition 1.8. *A number x is said to be very well-approximable if there exists a positive real number $\varepsilon > 0$ such that,*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

for infinitely many rational numbers $\frac{p}{q}$.

A more general error function ψ defines the set of ψ-approximable points as follows:

Definition 1.9. *The set $W(m, n, \psi)$ of ψ-approximable points $\boldsymbol{x} \in \mathbb{R}^{mn}$ is defined for a positive valued function ψ as,*

$$W(m, n, \psi) = \{X \in \mathbb{R}^{mn} : |\boldsymbol{q}X - \boldsymbol{p}| < \psi(|\boldsymbol{q}|), \text{ for i.m. } \boldsymbol{q} \in \mathbb{Z}^m, \boldsymbol{p} \in \mathbb{Z}^n\}.$$

Here X is an $m \times n$ matrix, $\mathbf{q}$ is a row vector, $\mathbf{p}$ is a column vector, and ψ is the approximating function. If ψ is of the form

$$\psi(r) = r^{-\tau}, \tau > 0$$

then the set will be denoted by $W(m,n,\tau)$, and referred to as the set of τ-approximable points.

In [3] Baker and Schmidt introduced the idea of regular systems and proved the regularity of real algebraic numbers of given degree. This allowed them to obtain the lower bound for the Hausdorff dimension of the set of real numbers which are approximated by algebraic numbers with a given order of approximation. A regular system will now be defined and will then be referred to in Chapter 5.

Definition 1.10. *Let Γ be a countable set of real numbers and $N : \Gamma \to \mathbb{R}$ be a positive function. The pair (Γ, N) is called a regular system if there exist constants $c_1 = c_1(\Gamma, N) > 0$ such that for any interval $I \subset \mathbb{R}$ there exists a sufficiently large number $T_0 = T_0(\Gamma, N, I) > 0$ such that for any integer $T > T_0$ there exist $\gamma_1, \gamma_2, \ldots, \gamma_t$ in $\Gamma \cap I$ such that*

$$\begin{aligned} N(\gamma_i) \leq T; \;\; 1 \leq i \leq t; \\ |\gamma_i - \gamma_j| > T^{-1}; \;\; 1 \leq i < j \leq t; \\ t > c_1 |I| T. \end{aligned} \tag{1.16}$$

Given a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, monotonic decreasing, with $\psi(r) \leq \frac{1}{2r}$ for large r, and a set

$$\Lambda(\Gamma, N, \psi), = \xi \in \mathbb{R} : |\xi - \frac{p}{q} < \psi(q^2) \text{ for i.m. } \frac{p}{q} \in \Gamma,$$

modifying Lemma 1 from [3], Rynne [80] showed that a lower bound for the Hausdorff Dimension of $\Lambda(\Gamma, N, \psi)$ can be established.

Theorem 1.4 ([80]). *Suppose that the system (Γ, N) is regular. Let ψ : $\mathbb{R}^+ \to \mathbb{R}^+$ monotonic decreasing, with $\psi(x) \leq \frac{1}{2x}$ for large x. Let $s_0 =$ $\sup s : \lim_{s\to\infty} x\psi(x)^s = \infty$, then*

$$\dim \Lambda(\Gamma, N, \psi) \geq s_0.$$

In [3] Baker and Schmidt showed that the set of rational numbers $\frac{p}{q}$, $\gcd(p,q) = 1$, together with the function $N(\frac{p}{q}) = q^2$ is a regular system. Further results will be discussed in Chapter 2. Regular systems will also be used in Chapter 5 and the relationship between the interval I and T_0 will be investigated.

Chapter 2

Historical overview

2.1 Introduction

The history of Diophantine approximation is well recorded in general number theory texts such as Hardy and Wright, [57] which cover the classical results of Gauss, Dirichlet, Liouville and Kronecker. Dedicated books on the topic have also been written, such as Cassels' tract [40] with an emphasis on rational approximation to a single real number and simultaneous rational approximation, and other specialist books such as [58] in which is discussed the general metric theory of Diophantine approximation, and Bernik and Dodson's book, [27] where the metrical theory of approximation on manifolds is considered. Bugeaud [35] has also written on the topic, but focuses on approximation to algebraic numbers. Waldschmidt [88] recently published a comprehensive overview of the recent developments in metric Diophantine approximation. Related topics in number theory such as continued fractions, the geometry of numbers and p-adic number theory are important tools in the investigation of problems in Diophantine approximation, and have numerous texts available, for example [40], [55], [57], [58], [60].

The relationship between real numbers x and rationals $\frac{p}{q}$ is well understood. When x is algebraic, the τ-approximable theory is essentially complete, through many classical results which stem from Dirichlet's result (1.1), culminating with the results of K.F. Roth [79].

Theorem 2.1 (Roth)**.** *Let α be a real algebraic number and let $\varepsilon > 0$. Then there are only finitely many rational numbers $\frac{p}{q}, q \geq 1$ such that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\varepsilon}}.$$

Another approach is to investigate relationships which hold for almost all numbers, which started with Khintchine [65] and evolved into the theory of metric Diophantine approximation. In metric Diophantine approximation the solution sets of Diophantine inequalities is considered in terms of the measure on that set. If a set X has measure 0, it's complement X^c is said to have full measure, and almost all points of the solution set lie in X^C.

Khintchine's famous result states that given a decreasing function $\psi(q)$ then for almost all x the inequality

$$\left|x - \frac{p}{q}\right| < \frac{\psi(q)}{q}$$

has at most finitely or infinitely many solutions $\frac{p}{q}$ depending on whether the sum $\sum_{q=1}^{\infty} \psi(q)$ converges or diverges.

Theorem 2.2 (Khintchine)**.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that ψ is decreasing. Then the Lebesgue measure $|W(\psi)|$ of $W(\psi)$ satisfies*

$$\left|(W(\psi))\right| = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r) < \infty, \\ full & \text{if } \sum_{r=1}^{\infty} \psi(r) = \infty. \end{cases}$$

In 1931 Jarnik published a refinement of Khintchine's Theorem [61] where the Lebesgue measure was replaced with the Hausdorff f-measure, $\mathcal{H}^f$.

Theorem 2.3 (Jarnik). *Let f be a dimension function such that $r^{-1}f(r) \to \infty$ as $r \to 0$ and $r^{-1}f(r)$ is decreasing. Let ψ be a real positive decreasing function. Then*

$$\mathcal{H}^f(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} rf(\psi(r)) < \infty, \\ \infty & \text{if } \sum_{r=1}^{\infty} rf(\psi(r)) = \infty. \end{cases}$$

where $m(w(\psi))$ denotes the Lebesgue measure of the ψ-approximable numbers.

After rational approximation to a single number, one may investigate the algebraic approximation properties of real or complex numbers. In this context, the problems may focus on either the distance $|x-\alpha|$ between a given real or complex number x, and algebraic numbers α, or on the size of $|P(x)|$ where P is a non-zero integer polynomial. The two different perspectives give rise to two different classifications of the real numbers.

For a real number x, a given positive integer n, and a real number $H \geq 1$, Mahler [72] defined the quantity,

$$w_n(x, H) := \min\{|P(x)| : P \in \mathbb{Z}[x], H(P) \leq H, \deg(P) \leq n, P(x) \neq 0\}. \tag{2.1}$$

Let

$$w_n(x) = \limsup_{H\to+\infty} \frac{-\log w_n(x, H)}{\log H}. \tag{2.2}$$

and

$$w(x) = \limsup_{n\to+\infty} \frac{w_n(x)}{n}. \tag{2.3}$$

Thus $w_n(x)$ is the largest real number w for which there exist infinitely many integer polynomials P of degree at most n satisfying

$$0 < |P(x)| \leq H(P)^{-w}. \tag{2.4}$$

With this notation, Mahler set up the following classification of the real numbers:

Definition 2.1 (Mahler)**.** *Let $x \in \mathbb{R}$. Define x to be an*

$$\begin{aligned}
&A-\textit{number}, && \textit{if} \quad w(x) = 0;\\
&S-\textit{number}, && \textit{if} \quad 0 < w(x) < +\infty;\\
&T-\textit{number}, && \textit{if} \quad w(x) = +\infty \textit{ and } w_n(x) < +\infty \textit{ for any } n \geq 1;\\
&U-\textit{number}, && \textit{if} \quad w(x) = +\infty \textit{ and } w_n(x) = +\infty \textit{ for any } n \geq n_0.
\end{aligned}$$

In 1962, Sprindžuk [85] extended this classification to the complex numbers. Let $\zeta \in \mathbb{C}$ and define

$$\tilde{w_n}(\zeta, H) := \min\{|P(\zeta)| : P \in \mathbb{Z}[X], H(P) \leq H, \deg(P) \leq n, P(\zeta) \neq 0\};$$

$$\tilde{w}(\zeta, H) = \limsup_{n \to +\infty} \frac{\log\log(\frac{1}{w_n(\zeta,H)})}{\log n}$$

and

$$\tilde{w}(\zeta) = \sup_{H \geq 1} \tilde{w}(\zeta, H).$$

If $\tilde{w}(\zeta) = +\infty$, then let H_0 denote the smallest *integer* such that $\tilde{w}(\zeta, H_0) = +\infty$. If no such H_0 exists, define H_0 to be $+\infty$. Next, let

$$\tilde{\mu}(\zeta, H) = \limsup_{n \to +\infty} \frac{-\log w_n(\zeta, H)}{n^{\tilde{w}(\zeta)}},$$

and

$$\tilde{\mu}(\zeta) = \limsup_{H \to +\infty} \frac{\tilde{\mu}(\zeta, H)}{\log H}.$$

Using these, he gave the following definition

Definition 2.2 (Sprindžuk). *A complex number ζ is defined as an*

$$\begin{array}{ll} \tilde{A}-number & \text{if } \tilde{w}(\zeta)<1 \text{ or } \tilde{w}(\zeta)=1 \text{ and } \tilde{\mu}(\zeta)=0; \\ \tilde{S}-number & \text{if } 1<\tilde{w}(\zeta)<+\infty \text{ or if } \tilde{w}(\zeta)=1 \text{ and } \tilde{\mu}(\zeta)>0; \\ \tilde{T}-number & \text{if } \tilde{w}(\zeta)=+\infty \text{ and } H_0(\zeta)=+\infty; \\ \tilde{U}-number & \text{if } \tilde{w}(\zeta)=+\infty \text{ and } H_0(\zeta)<+\infty. \end{array}$$

Both classifications given by 2.1 and 2.2 are based on two parameters, the degree and height of the polynomial, and both parameters approach infinity. The differences between the classifications relate to the implementation where Mahler first let the height H approach infinity and then the order n of the polynomial. Sprindžuk's approach was to let the order of the polynomial approach infinity before the height of the polynomial. Sprindžuk also established [83] that the $\tilde{A}-numbers$ are precisely the algebraic numbers.

In 1932, Koksma proposed an alternative classification to Mahler's that preceded Sprindžuk's. Koksma's classification of numbers is based on the idea of approximation of a real number ξ by algebraic numbers. For a given positive integer n, and a real number $H \geq 1$, the value w_n^* is defined as the distance of the closest algebraic number to ξ of degree less than n and height less than H, i.e.

$$w_n^*(\xi, H) = \min\{|\xi - \alpha| : \alpha \text{ real algebraic, } \deg(\alpha) \leq n, H(\alpha) \leq H, \alpha \neq \xi\}.$$

Here, $H(\alpha)$ denotes the height of the algebraic number, which is the maximum coefficient of the minimal polynomial of α.

If

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(w_n^*(\xi, H))}{\log H}$$

and

$$w^*(\xi) = \limsup_{H\to\infty} \frac{w_n^*(\xi)}{n},$$

then $w_n^*(\xi)$ is the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w-1}. \tag{2.5}$$

The relationship between Mahler's and Koksma's classifications is discussed in Section 3.4 of [35], and Sprindžuk's classification in Section 8.1 of the same book.

Algebraic and polynomial approximations are closely related. It is well understood that the value of an irreducible polynomial close to an algebraic number ζ has a small value, and also that a polynomial taking a small value at ζ, is likely to have a root close to ζ. This relationship is not fully understood yet however, and specific problems of this nature are investigated later in the thesis.

2.2 Polynomial and simultaneous approximation to a single number

A simple application of Dirichlet's box principle yields the existence of polynomials with small values at a given real point. For example:

Lemma 2.1 ([35] Lemma 8.1)**.** *Let ξ be a complex number, n be an integer with $n \geq 2$ and H be a real number. Then there exists a positive constant c, depending only on ξ, such that for any sufficiently large H there is a non-zero integer polynomial P with* $\deg P \leq n$*, $H(P) \leq H$, satisfying $|P(\xi)| \leq H^{-cn}$.*

There are variants and special cases of this Lemma. The fact that the exponent n above cannot be improved was shown by Sprindžuk [85] who proved that for each $\varepsilon > 0$ there are only finitely many non-zero integer polynomials of degree at most n satisfying

$$|P(\xi)| \leq H(P)^{-n-\varepsilon} \tag{2.6}$$

except on a set of Lebesgue measure zero.

For each real number ξ two exponents $\omega_n(\xi)$ and $\hat{\omega}_n(\xi)$ are defined: $\omega_n(\xi)$ is defined in (2.3) above and $\hat{\omega}_n(\xi)$ is a generalisation of $\omega(\xi)$ where $H^{-\omega}$ is replaced with $N^{-\omega}$ for $N \geq 1$.

From Lemma 2.1 it follows that for any $n \geq 1$ and for any $\xi \in \mathbb{R}$, which is not algebraic of degree $\leq n$,

$$n \leq \omega_n(\xi) \leq \hat{\omega}_n(\xi) \tag{2.7}$$

In fact in [85] Sprindžuk proved that

Theorem 2.4. *For all real numbers ξ,*

$$n \leq \omega_n(\xi) \leq \hat{\omega}_n(\xi) \tag{2.8}$$

and for almost all real numbers ξ

$$n = \omega_n(\xi) = \hat{\omega}_n(\xi) \tag{2.9}$$

for all $n \geq 1$.

If ξ is an algebraic irrational, Schmidt [82] proved

Theorem 2.5. *Let $n \geq 1$ be an integer and let ξ an algebraic number of degree $d > n$. Then*

$$\omega_n(\xi) = \hat{\omega}_n(\xi) = n.$$

Finally, Davenport and Schmidt [41] proved

Theorem 2.6. *For any real number ξ which is transcendental or algebraic of degree at least $n+1$,*

$$\hat{\omega}_n(\xi) \leq 2n - 1.$$

2.3 Metrical results on polynomial curves.

In 1932 Mahler [72], following his fundamental study of the theory of transcendental numbers, formulated the conjecture that for any $\varepsilon > 0$ the inequality

$$|P(x)| < H(P)^{-n-\varepsilon} \tag{2.10}$$

has at most a finite number of solutions in integer polynomials P of degree n for almost all $x \in \mathbb{R}$, where $H(P)$ is the height of P. This famous conjecture motivated a lot of research which developed both the theory of transcendental numbers and metric Diophantine approximation on manifolds. Some important results are described below.

2.3.1 Results connected to Mahler's conjecture.

In 1980, Bernik [19] proved certain conjectures posed by Sprindžuk ([85, pp 159–160]), while proving Mahler's conjecture. Sprindžuk asked three questions which are described below. Let $v(\Omega_1, \Omega_2, \ldots, \Omega_n)$ denote the supremum of the set of numbers v such that

$$|a_1\Omega_1 + \ldots + a_n\Omega_n| < H^{-v}, \text{ where } H = \max(|a_1|, \ldots, |a_n|).$$

Problem A. Let $m_1, m_2, \ldots, m_n$ be distinct natural numbers. Let ω be a transcendental number and let $v_n(\omega)$ be the function $v(\Omega_1, \Omega_2, \ldots, \Omega_n)$

defined above, with the parameters

$$\Omega_1 = \omega^{m_1}, \Omega_2 = \omega^{m_2}, \dots, \Omega_n = \omega^{m_n}.$$

Does the equation $v_n(\omega) = n$, $(n = 1, 2 \dots)$ hold for almost all real ω regardless of the choice of the numbers $m_1, m_2, \dots, m_n$?

Problem B.1. Let $n_1, n_2, \dots, n_k$ be arbitrary natural numbers; ω_i be transcendental numbers for $i = 1, \dots, k$ and let $v_n(\omega_1, \omega_2, \dots, \omega_k)$ be the function $v(\Omega_1, \Omega_2, \dots, \Omega_k)$ defined above, with $\Omega_j = \omega_1^{i_j}$ for $j = 1, 2, \dots, k$ where $i_1, i_2, \dots, i_k$ satisfy the conditions

$$0 \le i_1 \le n_1,\ 0 \le i_2 \le n_2, \dots,\ 0 \le i_k \le n_k$$

with $i_1 + i_2 + \dots + i_k \neq 0$. Let $n = (n_1 + 1)(n_2 + 1) \dots (n_k + 1) - 1$. Does the equation

$$v(\omega_1, \dots, \omega_k) = n$$

hold for almost all $\bar{\omega}$ regardless of the choice of numbers $n_1, n_2, \dots, n_k$?

Problem B.2. Let m be an arbitrary natural number and let $v_n(\omega_1, \omega_2, \dots, \omega_k)$ be the function $v(\Omega_1, \Omega_2, \dots, \Omega_k)$ defined above, with

$$\Omega_j = \omega_1^{i_1} \omega_2^{i_2} \dots \omega_k^{i_k}\ (j = 1, 2, \dots, k)$$

where $i_1, \dots, i_k$ satisfy the conditions

$$0 \neq i_1 + i_2 + \dots + i_k \le m, \text{ with } i_j \ge 0, j = 1, \dots, k.$$

Let $n = \binom{m+k}{k} - 1$. Does the equation

$$v(\omega_1, \dots, \omega_k) = n$$

hold for almost all $\bar{\omega} = (\omega_1, \omega_s, \dots, \omega_k)$?

Further work to [19] gave rise to generalisations and applications [49],[78]. In 1989 V. Bernik [22], in considering these problems, established a generalisation of Mahler's question.

Theorem 2.7 (Bernik 89)**.** *Given a monotonic function* $\Psi : \mathbb{N} \to \mathbb{R}^+$ *such that*

$$\sum_{h=1}^{\infty} \Psi(h) \tag{2.11}$$

converges, then for almost all $\psi \in \mathbb{R}$*,*

$$|P(\psi)| < H(P)^{-n+1}\Psi(H(P)) \tag{2.12}$$

has only finitely many solutions in $P \in \mathbb{Z}[x]$ *with* $\deg P \leq n$*.*

For $n = 1$, Theorem 2.7 is equivalent to Khintchine's Theorem and the divergence case holds as well as the convergence case. The divergence case for any n was established by Beresnevich [8] who showed that if

$$\sum_{h=1}^{\infty} \Psi(h) \tag{2.13}$$

diverges, then for almost all real ξ, (2.12) has infinitely many solutions $P \in \mathbb{Z}[x]$ with $\deg P = n$.

2.4 Metrical results on manifolds

Many of the metrical results concerning polynomials have been generalised to manifolds. In their book in 1999, Bernik and Dodson [27] influenced research on Diophantine approximation on manifolds. This text was in turn, an extension of Sprindžuk's book [86] which gave the first systematic account of the then emerging theory of metric Diophantine approximation on manifolds.

Many results in Diophantine approximation such as Khintchine's Theorem are of a metrical nature, that is they hold on a set of full or zero measure. As embedded manifolds are of measure zero in the ambient space $\mathbb{R}^n$, it is appropriate to work with the relative measure induced by the manifold. For any $S \subset M$, the induced Lebesgue measure of S relative to M will be denoted by $|S|$.

The set $S_\tau(M)$ of simultaneously τ–approximable points lying on an m–dimensional manifold M embedded in $\mathbb{R}^n$ is defined by

$$S_\tau(M) = \{\mathbf{x} \in M : ||q\mathbf{x}|| < |q|^{-\tau} \text{ for infinitely many } q \in \mathbb{Z}\},$$

i.e. $S_\tau(M) = M \bigcap W(1, n, \tau)$. There is a natural dual to this set, namely $L_\tau(M)$ where

$$L_\tau(M) = \{\mathbf{x} \in M : ||\mathbf{q}.\mathbf{x}|| < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}.$$

Obviously any element of $\mathbb{Q}^n$ lying on M is in $S_\tau(M)$ for all τ. Correspondingly, the intersection of M with a rational hyperplane given by the equation $\mathbf{q}.\mathbf{x} = p$ (for $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$) is contained in $L_\tau(M)$ for all τ. Any other points in either $S_\tau(M)$ or $L_\tau(M)$ lie "close" to infinitely many of these points or planes.

A manifold M embedded in $\mathbb{R}^n$ is said to be *extremal* if $|S_\tau(M)| = 0$ for $\tau > 1/n$ or equivalently (using Khintchine's Transference principle [27]) if $|L_\tau(M)| = 0$ for $\tau > n$. Manifolds satisfying various geometric, analytic and number theoretic properties have been shown to be extremal.

Using trigonometric sums, Sprindžuk proved the following Theorem on extremal manifolds [85].

Theorem 2.8. *Given integers m, n, $1 \le n \le m$, let Ω be a domain in $\mathbb{R}^m$,* *and let* $f_j = f_j(t_1, \ldots, t_m) : \Omega \to \mathbb{R}$ $(1 \le j \le n)$ *be real functions defined on* Ω *that satisfy the following conditions:*

a) The partial derivatives $\frac{\partial^2 f_j}{\partial t_i \partial t_k}$ *are continuous in* Ω *(*$1 \le j \le n$, $1 \le i, k \le m$*);*

b)The Jacobian

$$\det\left(\frac{\partial^2 f_j}{\partial t_i \partial t_k}\right)_{j,k=1,2,\ldots,m} \neq 0$$

almost everywhere in Ω*;*

c) Every linear combination

$$\phi(t_k) = c_1 \frac{\partial^2 f_1}{\partial t_1 \partial t_k} + \ldots + c_n \frac{\partial^2 f_n}{\partial t_n \partial t_k}$$

with $c_i \in \mathbb{Z}$*, is locally monotonic. If the conditions a), b) and c) hold then the manifold* Γ *containing the set of points* $(t_1, \ldots, t_m, f_1(t), \ldots, f_n(t))$ *is extremal.*

A more general result is due to Kleinbock and Margulis [67] who proved that a non–degenerate manifold is extremal. This has been extended by Kleinbock to a larger class of manifolds in [66]. Non–degeneracy is a generalisation of the idea of non-zero curvature and means that for almost all points on the manifold there exists $l \in \mathbb{N}$ such that the partial derivatives of an appropriate parametrisation up to order l span $\mathbb{R}^n$. If the error function $q^{-\tau}$ is replaced with a general non–increasing function ψ then the dual set is denoted $L_\psi(M)$. It has been shown (see [10, 14, 28]) that for any non–degenerate manifold M the set $L_\psi(M)$ satisfies a 'zero–one' law. That is, depending on the divergence or convergence of a certain sum, the set has full or zero Lebesgue measure respectively. (This proves the Baker–Sprindžuk conjecture.)

One would expect that as τ increases the size of the sets L_τ and S_τ should decrease and this leads naturally to questions concerning Hausdorff dimension in the case of zero Lebesgue measure. It was proved by R. C. Baker in [5] that for any planar curve C with non–zero curvature everywhere except on a set of Hausdorff dimension zero, the Hausdorff dimension, $\dim L_\tau(C)$ of $L_\tau(C)$ for $\tau \geq 2$ is

$$\dim L_\tau(C) = \frac{3}{\tau + 1}.$$

(When $\tau \leq 2$, $L_\tau(C) = C$ by Dirichlet's Theorem.) In higher dimensions, Bernik [20] obtained the Hausdorff dimension $\frac{n+1}{\tau+1}$ for $L_\tau(C)$ when C is the Veronese curve, $V_n = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$. Also, the Hausdorff dimension of $L_\tau(M)$ was shown to be $m - 1 + \frac{n+1}{\tau+1}$ for m–dimensional C^3 manifolds M with $m \geq 2$ on which there are two non–vanishing principal curvatures except on a set of Hausdorff dimension $m - 1$ [46]. This dimension is a lower bound when M is extremal and C^1 [44]. The upper bound is still an open question. On the other hand, very little is known about the set $S_\tau(M)$ although there does exist a Khintchine type Theorem for 2-convex C^3 manifolds [47] and an asymptotic formula holds under fairly restrictive curvature conditions [48], for further details see [27].

In 2006 [15] Beresnevich, Dickinson and Velani refined the requirements on Khintchines's Theorem [65], allowing the removal of one condition on the error function, (that $x \to \Psi(x)$ is decreasing). In the same paper they also established Khintchine-type results for the Hausdorff measure of the lim-sup sets

$$\mathcal{K}_{\mathcal{S}}(\Psi) = \limsup_{j \to +\infty} \{\xi \in E : |\xi - \alpha_j| < \Psi(j)\} \tag{2.14}$$

where $\mathcal{S} = (\alpha_j)$ is an optimal regular system on E, a bounded open real interval.

In [17] Beresnevich and Velani established a general inhomogeneous mass transference principle allowing results on the Hausdorff measure of Diophantine approximation problems to be inferred from similar results regarding the Lebesgue measure.

Subsequent to this, in 2007, Beresnevich, Dickinson and Velani proved the following Theorem in [16]

Theorem 2.9. *Let* $\psi : \mathbb{N} \to (0, \infty)$ *be monotonic. Let* Γ *be a* C^3 *planar curve with non-vanishing curvature, defined on a bounded domain, i.e. has finite length* $\mathcal{L}$*. If*

$$\mathcal{A}_2(\psi, \Gamma) = \{(x, y) \in \Gamma : \max\{\|qx\|, \|qy\|\} \leq \psi(q) \text{ holds for i. m. } q \in \mathbb{N}\}, \tag{2.15}$$

then the length $|\mathcal{A}_2(\psi, \Gamma)|$ *of* $\mathcal{A}_2(\psi, \Gamma)$ *satisfies*

$$\begin{cases} 0 & \text{if } \Sigma_{h=1}^{\infty} h\psi(h) < \infty \\ \mathcal{L} & \text{if } \Sigma_{h=1}^{\infty} h\psi(h) = \infty. \end{cases}$$

For sufficiently large τ there also exist results for the unit circle centred at the origin [42], the parabola [9] and quadric surfaces [50]. Unlike R. C. Baker's result [5] which holds for all polynomial curves in $\mathbb{R}^2$ there is no single formula for all $\tau > \frac{1}{n}$ for the Hausdorff dimension of $S_\tau(M)$.

2.5 Results in the p-adic metric

In 1932 Mahler [72], following his fundamental study of the theory of transcendental numbers described above, also proposed a classification of p-adic numbers that coincides with the classification (2.1) above but where x is in $\mathbb{Q}_p$ rather than $\mathbb{R}$. This is discussed extensively in [35, Section 9.3].

In 1945 Jarnik [63] proved a p-adic generalisation of Khintchine's Theorem [65], and in 1955 Lutz [71] extended Jarink's result to systems of Linear forms. In 1965 Sprindžuk [84] also considered the p–adic analogue of Mahler's conjecture and proved the following Theorem:

Theorem 2.10 (Mahler-Sprindžuk)**.** *The inequality*

$$|P(w)|_p < H(P)^{-1-n-\varepsilon}$$

has only a finite number of solutions in rational integer polynomials P of degree n for almost all $w \in \mathbb{Q}_p$.

For general n, Bernik, Dickinson and Yuan [26] proved the p-adic inhomogeneous analogue of Theorem 2.10. They showed that:

Theorem 2.11. *For any $d \in \mathbb{R}$,*

$$|P(w) + d|_p < H(P)^{-n-1-\varepsilon} \tag{2.16}$$

has only a finite number of solutions in rational integer polynomials P of degree n for almost all $w \in \mathbb{Q}_p$.

In 2006, using the ubiquity frameworks constructed in [15] Beresnevich, Dickinson and Velani were able to establish the p-adic equivalent of the earlier results established by Dickinson and Velani [45] discussed above.

2.6 Summary.

This brief overview of the development of the theory of metric Diophantine approximation contains essential results that are used in the following chapters. The development of these results however also provide many of the tools and techniques that are used in obtaining the results that follow.

Chapter 3

On simultaneous rational approximation to a p-adic number and its integral powers

3.1 Introduction

This work on the Hausdorff dimension of p-adic approximable numbers on polynomials was undertaken during a visit to the Institut de Recherche Mathématique Avancée, Université de Strasbourg, in January 2010. Support for the visit was provided by a Ulysess grant, and the material presented here forms part of a paper published in the Edinburgh Math. Journal, [33].

For a positive integer n and a p–adic number w, let $\lambda_n(w)$ denote the supremum of the real numbers λ such that there are arbitrarily large positive integers q such that $||qw||_p, ||qw^2||_p, \ldots, ||qw^n||_p$ are all less than $q^{-\lambda}$. Here, $||x||_p$ denotes the infimum of $|x-n|_p$ as n runs through the integers. The set of values taken by the function λ_n was studied in [33].

Some Lemmas that are important in the development of Theorem 3.2 below are stated first. These were first published in [43], and as the approach used is informative for the proof of Theorem 3.2, they are also proved here for convenience. Some definitions are needed:

Definition 3.1. *Let* $\Gamma = \{(x, y) \in [0, 1] \times I : y = P(x)\}$ *where* P *is an* n*th degree polynomial and* $I \subset \mathbb{R}$ *is some suitable interval.*

Let

$$\omega(\alpha) = \sup(\tau : \alpha \in W(1, 1, \tau)).$$

Hence, if $\tau > \omega(\alpha)$, then $\alpha \notin W(1, 1, \tau)$.

Now define $\Gamma(\alpha)$, and $S_\tau(\Gamma(\alpha))$ as

$$\Gamma(\alpha) = \{(x, y) \in [0, 1]^2 : y = x^2 + \alpha\},$$

and

$$S_\tau(\Gamma(\alpha)) = \{(x, y) \in \Gamma(\alpha) : \left|x - \frac{p}{q}\right| < q^{-\tau}, \left|y - \frac{r}{q}\right| < q^{-\tau} \text{for i.m. } p, q, r \in \mathbb{Z}\}$$

Lemma 3.1 ([43]). *Assume* $\tau > 1$.

$$S_\tau(\Gamma(\alpha)) = \emptyset \text{ for } \tau > 2\omega(\alpha) + 1.$$

Proof. Let $(x, y) \in S_\tau(\Gamma(\alpha))$, so that there exists $\rho, \varepsilon > 0$ where

$$x = \frac{p}{q} + \varepsilon$$

$$y = \frac{r}{q} + \rho$$

where $\varepsilon = \varepsilon(\frac{p}{q})$ and $\rho = \rho(\frac{r}{q})$, and $\varepsilon, \rho = O(q^{-\tau-1})$, for infinitely many $p, q, r \in \mathbb{Z}$. Then

$$\frac{r}{q} + \rho = \frac{p^2}{q^2} + 2\varepsilon\frac{p}{q} + \varepsilon^2 + \alpha.$$

Hence

$$q^2\alpha - rq + p^2 = O((q)^{\frac{(1-\tau)}{2}})$$

which is impossible for infinitely many $p, q, r \in \mathbb{Z}$ if $\tau > 2\omega(\alpha) + 1$, and the result follows directly. $\square$

The ideas from Lemma 3.1 were subsequently generalised, and then used in [32] to prove a more general result, to calculate the Hausdorff Dimension of the set of simultaneously τ-approximable points on polynomial curves in $\mathbb{R}^n$.

Let

$$\Gamma = \{(x, P_1(x), \ldots, P_{n-1}(x)) \in \mathbb{R}^n : P_j \in \mathbb{Z}[x]\}$$

be a polynomial curve in $\mathbb{R}^n$. Let $d_j = \deg P_j$ and let $d = \max_{j=1,\ldots,n-1} d_j$ then for polynomial curves and more general polynomial surfaces the following Lemma, originally proved in [32] applies:

Lemma 3.2. *Let Γ represent any polynomial curve or surface of the form*

$$\Gamma = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : y_1 = P_1(\mathbf{x}), \ldots, y_{n-m} = P_{n-m}(\mathbf{x})\}$$

where $P_i \in \mathbb{Z}[\mathbf{x}]$. Let $d_i = \deg P_i$ and assume without loss of generality that $d_1 \leq d_2 \leq \cdots \leq d_{n-m} = d$. Define

$$S_\tau(\Gamma) = \{(\mathbf{x}) \in \Gamma : ||q\mathbf{x}|| < |q|^{-\tau} \textit{for infinitely many } q \in \mathbb{Z}\}.$$

Let $(\mathbf{x}, \mathbf{y}) \in S_\tau(\Gamma)$. If

$$|Dx_i - t_i| < D^{-\tau} \quad \textit{and} \quad |Dy_j - r_j| < D^{-\tau}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n-m$, $\tau > d-1$, D a sufficiently large integer and $t_i, r_j \in \mathbb{Z}$, then the point $\left(\frac{t_1}{D}, \ldots, \frac{t_m}{D}, \frac{r_1}{D}, \ldots, \frac{r_{n-m}}{D}\right)$ lies on Γ.

Proof. Let $(\mathbf{x}, \mathbf{y}) \in \Gamma$ so that $y_j = P_j(\mathbf{x})$, $j = 1, \ldots, n-m$. If $(\mathbf{x}, \mathbf{y}) \in S_\tau(\Gamma)$ then

$$|x_i - t_i/D| < D^{-\tau-1} \quad \text{for} \quad i = 1, \ldots, m$$
$$|y_j - r_j/D| < D^{-\tau-1} \quad \text{for} \quad j = 1, \ldots, n-m.$$

Hence, $x_i - t_i/D = \varepsilon_i$ and $y_j - r_j/D = \eta_j$ for some ε_i and η_j with $|\varepsilon_i|, |\eta_j| < D^{-\tau-1}$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$. As

$$y_j = P_j(\mathbf{x}) = P_j(\mathbf{t}/D + \boldsymbol{\varepsilon}) \text{ for } j = 1, \ldots, n-m$$

it follows that

$$\frac{r_j}{D} + \eta_j = P_j\left(\frac{\mathbf{t}}{D}\right) + R_j(\boldsymbol{\varepsilon})$$

where $R_j(\boldsymbol{\varepsilon}) = O(|\boldsymbol{\varepsilon}|)$ is the sum of the remaining terms. Multiplying through by D^{d_j}, where d_j is the degree of P_j, gives

$$r_j D^{d_j-1} + D^{d_j}\eta_j = D^{d_j} P_j(\mathbf{t}/D) + D^{d_j} O(|\boldsymbol{\varepsilon}|)$$

so that

$$|r_j D^{d_j-1} - D^{d_j} P_j(\mathbf{t}/D)| = |D^{d_j} O(|\boldsymbol{\varepsilon}|) - D^{d_j}\eta_j| = O(D^{d_j-\tau-1})$$

and the LHS is an integer. For sufficiently large D the RHS is less than 1 which implies that the LHS must equal zero. Therefore

$$\frac{r_j}{D} = P_j\left(\frac{\mathbf{t}}{D}\right)$$

and the point $(\mathbf{t}/D, \mathbf{r}/D)$ lies on Γ for $\tau > d-1$. □

With this Lemma, the authors were then able to prove the following Theorem.

Theorem 3.1. *For* $\tau > \max(d-1, 1)$ *the Hausdorff dimension of* $S_\tau(\Gamma)$ *is*

$$\dim S_\tau(\Gamma) = \frac{2}{d(\tau+1)}.$$

Theorem 3.1 was proved by obtaining the upper and lower bounds separately. The upper bound were found using covering and counting arguments and the lower bound by adapting the classical set of well approximable numbers. The latter is not best possible for $\tau < d-1$, but holds for all $\tau > 2/d - 1$.

Define the curve $\Gamma \subset \mathbb{Z}_p^n$ as $\Gamma = \{(w, w^2, \ldots, w^n) \in \mathbb{Z}_p^n\}$. The set of points $(w, \eta_2, \ldots, \eta_n) \in \Gamma$ which satisfy the inequalities $|qw - r|_p \leq |q, r, \mathbf{t}|^{-\tau}$ and $|q\eta_i - t_i|_p \leq |q, r, \mathbf{t}|^{-\tau}$ for infinitely many $q, r \in \mathbb{Z}$ and $\mathbf{t} \in \mathbb{Z}^{n-1}$ will be denoted by $W_\tau(\Gamma)$.

Theorem 3.2. *Let $n \geq 2$ be an integer. Then,*

$$\dim W_\tau(\Gamma) = \frac{2}{n\tau}.$$

Theorem 3.2 is a p-adic analogue of Theorem 3.1, and quite similar to that of Theorem 3.1. The proof of Theorem 3.2 is restricted to the Veronese curve as opposed to more general integer poloynomial curves. It is expected that the proof also holds for general integer polynomial curves.

3.2 Proof of Theorem 3.2

We will use the notation $|a, b, c|$ to denote the maximum of $|a|$, $|b|$ and $|c|$. If a is a vector then $|a|$ is the maximum of the vector entries. To prove the Theorem, it is first necessary to determine the Hausdorff dimension and measure of $W_\tau(\Gamma)$. The proof relies on the following Lemma which shows that if $(w, \boldsymbol{\eta}) \in W_\tau(\Gamma)$ then the rational approximants $(r/q, \mathbf{t}/q)$ also lie on Γ for τ sufficiently large. This Lemma is a p-adic version of Lemma 3.2.

Lemma 3.3. *Let $(w, \boldsymbol{\eta}) \in W_\tau(\Gamma)$ so that there exist infinitely many $D, r \in \mathbb{Z}$, $\mathbf{t} \in \mathbb{Z}^{n-1}$ such that $|Dw - r|_p < |D, \mathbf{t}, r|^{-\tau}$ and $|D\eta_i - t_i|_p < |D, r, \mathbf{t}|^{-\tau}$. Then $(r/D, \mathbf{t}/D) \in \Gamma$.*

Proof. Let $(w, \boldsymbol{\eta}) \in W_\tau(\Gamma)$. Hence $\eta_i = w^i$ and there exist integers t_i, r and D such that $|Dw - r|_p < |D, \mathbf{t}, r|^{-\tau}$ and $|D\eta_i - t_i|_p < |D, r, \mathbf{t}|^{-\tau}$. Therefore, $|w - r/D|_p < |D, \mathbf{t}, r|^{-\tau}|D|_p^{-1}$ and $|\eta_i - \mathbf{t}/D|_p < |D, \mathbf{t}, r|^{-\tau}|D|_p^{-1}$ and there exist $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Q}_p$, such that $w - r/D = \varepsilon_1$ and $\eta_i - t_i/D = \varepsilon_i$ for $i = 2, \ldots, n$ with $|\varepsilon_i|_p < |D, \mathbf{t}, r|^{-\tau}|D|_p^{-1}$. Then,

$$\eta_i = t_i/D + \varepsilon_i = w^i = (r/D + \varepsilon_1)^i = (r/D)^i + R(\varepsilon_1)$$

where R is a polynomial with each term containing ε_1. Hence, $t_i/D - (r/D)^i = R(\varepsilon_1) - \varepsilon_i$ so that

$$D^{i-1}t_i - D^i(r/D)^i = D^i(R(\varepsilon_1) - \varepsilon_i).$$

Clearly $|R(\varepsilon_1)|_p \leq |\varepsilon_1|_p < |D, \mathbf{t}, r|^{-\tau}|D|_p^{-1}$. Thus,

$$|D^{i-1}t_i - D^i(r/D)^i|_p \leq |D|_p^{i-1}|D, \mathbf{t}, r|^{-\tau}.$$

The LHS is a rational integer and therefore has a finite p–adic expansion. Thus, if τ is sufficiently large then the LHS will be zero. Let α be the largest power of p occurring in the p-adic expansions of $r, \mathbf{t}$ and D. Then the maximum power of p in the p-adic expansion of $D^{i-1}t_i$ is $i\alpha$. Similarly, the maximum power of p in $D^i(r/D)^i$ is $i\alpha$. Note that $|D, \mathbf{t}, r| \asymp p^\alpha$ so that if $\tau > n$, we have $|D, \mathbf{t}, r|^{-\tau}|D|_p < p^{-n\alpha}$ which is enough to prove the Lemma. □

The proof of the Theorem also uses the following Theorem from [15]. This Theorem, which is the p-adic analogue of the main result in [45], is the p-adic equivalent of a generalised Jarnik Theorem 2.3.

Theorem 3.3 (Theorem 16, [15]). *Let f be a dimension function such that $r^{-mn}f(r) \to \infty$ as $r \to 0$ and $r^{-mn}f(r)$ is decreasing. Furthermore, suppose* *that* $r^{-(m-1)n}f(r)$ *is increasing. Let* ψ *be a real, positive, decreasing function. Then*

$$\mathcal{H}^f(W_p(m,n,\ \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} f(\psi(r))\psi(r)^{-(m-1)n}r^{m+n-1} < \infty \\ \infty & \text{if } \sum_{r=1}^{\infty} f(\psi(r))\psi(r)^{-(m-1)n}r^{m+n-1} = \infty. \end{cases} \tag{3.1}$$

Now the structure of W_τ is considered. Define the point P_{rq} as

$$P_{rq} = \left(\frac{r}{q},\ldots,\frac{r^n}{q^n}\right) = \left(\frac{rq^{n-1}}{q^n},\ldots,\frac{r^n}{q^n}\right).$$

If the highest common factor of r and q is 1 then the common denominator of P_{rq} is q^n. Let $h=(r,q)$ with $r=r_1h$ and $q=q_1h$. Then

$$P_{rq} = \left(\frac{r_1q_1^{n-1}}{q_1^n},\ldots,\frac{r_1^n}{q_1^n}\right) = P_{r_1q_1}.$$

We may therefore assume without loss of generality that $(r,q) = 1$. If $\Xi = (w,\eta_2,\ldots,\eta_n) \in W_\tau(\Gamma)$ and $\tau > n$, then Ξ must be approximated by infinitely many points P_{rq} with $(r,q)=1$ and must satisfy the inequalities

$$|q^n\xi - rq^{n-1}|_p < |q^n,r^n|^{-\tau},$$

$$|q^n\eta_2 - r^2q^{n-2}|_p < |q^n,r^n|^{-\tau},\ldots,$$

$$|q^n\eta_n - r^n|_p < |q^n,r^n|^{-\tau}.$$

The proof of the Theorem now follows that in [32]. First, we move from the set $W_\tau(\Gamma)$ to the set

$$V_\tau(\Gamma) = \{\xi \in I : (\xi,\boldsymbol{\eta}) \in W_\tau(\Gamma)\}.$$

For all $\xi \in \mathbb{Z}_p$,

$$|\xi_1 - \xi_2|_p = \max_i |\xi_1^i - \xi_2^i|_p$$

for all $i = 1, 2, \ldots, n$. Thus, there is a bi–Lipschitz transformation between any ball $B(\xi, r) \subset \mathbb{Z}_p$ and the image of that ball on Γ. To determine the Hausdorff dimension and measure of $W_\tau(\Gamma)$ it is therefore enough to find the Hausdorff dimension and measure of $V_\tau(\Gamma)$. It can be readily verified that the inclusions

$$\begin{array}{l} \bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B(r/q, |r^n, q^n|^{-\tau}) \subset V_\tau(\Gamma) \quad \subset \\ \bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B(r/q, |r^n, q^n|^{-\tau} |q^n|_p^{-1}) \end{array} \tag{3.2}$$

hold. (Note that $|D|_p^{-1} \geq 1$.)

The fact that $\dim W_\tau(\Gamma) \geq \dim V_\tau(\Gamma) \geq \frac{2}{n\tau}$ and the fact that the Hausdorff $2/n\tau$ measure is infinite follow directly from Theorem 3.3 by putting $\psi(r) = r^{-n\tau}$ and $f(r) = r^s$. It is therefore only necessary to prove the upper bound for the Hausdorff dimension.

Lemma 3.4.

$$\dim V_\tau(\Gamma) \leq \frac{2}{n\tau}.$$

Proof. Using the RHS of (3.2) gives a cover of $V_\tau(\Gamma)$ for each n so that

$$\begin{aligned} \mathcal{H}^s(V_\tau(\Gamma)) &\ll \sum_{k>N} \sum_{r,q:\max(r,q)=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} \\ &\ll \sum_{k>N} \left(\sum_{r,q:\max(r,q)=q=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} + \sum_{r,q:\max(r,q)=r=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} \right) \\ &\ll \sum_{k>N} \left(k k^{-n\tau s} |k|_p^{-ns} + k^{-\tau ns} \sum_{q=1}^{k} |q|_p^{-ns} \right). \end{aligned}$$

Consider the second sum $\sum_{k>N} k^{-\tau ns} \sum_{q=1}^{k} |q|_p^{-ns}$ first and choose α be such that $p^\alpha \leq k < p^{\alpha+1}$. Then, as $|k|_p = 1$ if p does not divide k, we have

$$\begin{aligned}
\sum_{q=1}^{k} |q|_p^{-ns} &= \sum_{q\leq k, p\not| q} 1 + \sum_{q\leq k: p|q \text{ and } p^2\not| q} p^{ns} + \cdots + \sum_{q\leq k: p^\alpha | q} p^{\alpha ns} \\
&\ll k + \frac{k}{p}p^{ns} + \frac{k}{p^2}p^{2ns} + \cdots + \frac{k}{p^\alpha}p^{\alpha ns} \\
&\ll k\sum_{i=0}^{\alpha} p^{i(ns-1)} \ll k
\end{aligned}$$

for $s > \frac{2}{n\tau}$ and $\tau > n \geq 2$. Now, using the same arguments consider the first sum $\sum_{k>N} kk^{-n\tau s}|k|_p^{-ns}$ to obtain

$$\begin{aligned}
\sum_{k>N} kk^{-n\tau s}|k|_p^{-ns} &\ll \sum_{k>N: p\not| k} k^{1-n\tau s} + \sum_{r>N: p\not| r} (pr)^{1-n\tau s}p^{ns} + \sum_{r>N: p\not| r} (p^2 r)^{1-n\tau s}p^{2ns} + \ldots \\
&\ll \sum_{k>N} k^{1-n\tau s} \sum_{i=0}^{\infty} p^{i(1+ns-n\tau s)}.
\end{aligned}$$

The last geometric series converges if $s > \frac{1}{n\tau - n}$. For $\tau > n \geq 2$ it is easy to show that $\frac{2}{n\tau} > \frac{1}{n\tau-n}$. Thus if $s > \frac{2}{n\tau}$ then both the sums converge which is enough to prove $\dim W_\tau(\Gamma) = \dim V_\tau(\Gamma) \leq \frac{2}{n\tau}$ for $\tau > n$. This implies that $\dim E_\lambda \geq \frac{2}{n\lambda}$ which completes the proof of Theorem 3.2. □

It is now possible to obtain the dimension of E_λ, where

$$E_\lambda = \lim_{k\to\infty} W_\lambda(\Gamma) \setminus W_{\lambda+1/k}(\Gamma).$$

Clearly, $E_\lambda \subset W_\lambda(\Gamma)$ so that $\dim E_\lambda \leq \frac{2}{n\lambda}$. Also, $\mathcal{H}^{2/n\lambda}(W_\lambda(\Gamma)) = \infty$, and $\mathcal{H}^{2/n\lambda}(W_{\lambda+1/k}(\Gamma)) = 0$ for all $n \geq 1$. Thus, $\mathcal{H}^{2/n\lambda}(W_\lambda(\Gamma) \setminus W_{\lambda+1/k}(\Gamma)) = \infty$.
□

Chapter 4

On a problem of Nesterenko: examining the closest root to an argument of a polynomial

4.1 Introduction

In this chapter, a result originally considered by Y.V. Nesterenko is examined. The material presented in the first section forms the main part of a paper published in the International Journal of Number Theory, [34]. The problem is to determine, for an integer polynomial P, which roots α of P belong to the real numbers $\mathbb{R}$. In the second section, the same problem is examined for the p-adic field. The results slightly improve Nesterenko's earlier work. The final section studies the same problem for simultaneous approximation in the real and p-adic fields.

4.2 Main results and remarks

Recall the sets

$$S_P(\alpha_j) \quad = \quad \{x \in \mathbb{R} : |x - \alpha_j| = \min_{1 \le m \le n} |x - \alpha_m|\},$$

where $1 \le j \le n$. From now on it will be assumed without loss of generality that $x \in S_P(\alpha_1)$.

The following two Theorems concern results which hold for $x \in \mathbb{R}$ and $x \in S_P(\alpha_1)$.

Theorem 4.1. *Let $P \in \mathbb{Z}[x]$ be a leading polynomial of degree n, $n \ge 2$, with discriminant $D(P) \ne 0$. If*

$$|P(x)| < H^{-w} \tag{4.1}$$

for $w > 2n - 3$ and sufficiently large H and $x \in S_P(\alpha_1)$, then $\alpha_1 \in \mathbb{R}$ and

$$|x - \alpha_1| \ll H^{-w+n-2}. \tag{4.2}$$

Corollary 4.1. *If $P(x) = \prod_{i=1}^{k} T_i(x)^{s_i}$, where the T_i are irreducible polynomials, and degree $T_i \le n_i$, and $D(P) = 0$ then Theorem 4.1 holds with (4.2) replaced by*

$$|x - \alpha_1| \ll H(T_i)^{-w+n_i-2} \ \ \textit{for some} \ \ i, \ 1 \le i \le k, \tag{4.3}$$

where $w > 2n_i - 3$.

Note that the condition $w > 2n - 3$ in Theorem 4.1 cannot be arbitrarily improved. To illustrate this, consider the following example.

Example 1. Let P_n be the leading polynomial

$$P_n(x) = x^{n-2}((b^2 + 1)x^2 + 2bx + 1) = x^{n-2}R_2(x), \ b \in \mathbb{Z}, \ b > 1.$$

The height of P_n is $H(P_n) = b^2 + 1$. The polynomial P_n has complex roots $\frac{-b \pm i}{b^2+1}$ and a real root 0 of order $n-2$. Let $x_0 = -\frac{b}{b^2+1}$. Then

$$R_2(x_0) = (b^2+1)x_0^2 + 2bx_0 + 1 = \frac{1}{b^2+1} = H^{-1},$$

where $|x_0| = \frac{b}{b^2+1} \leq \frac{\sqrt{b^2+1}}{b^2+1} = H^{-1/2}$. Hence,

$$|P_n(x_0)| = |x_0|^{n-2}H^{-1} \leq H^{-n/2}.$$

Since the distance from x_0 to 0 is $\frac{b}{b^2+1}$, the roots $\frac{-b \pm i}{b^2+1}$ are the closest roots to x_0. The upshot is that in Theorem 1 we cannot take $w \leq n/2$. It would be interesting to know if either of the bounds $\frac{n}{2}$ and $2n-3$ is sharp; this is the subject ongoing research.

When $D(P) = 0$, if $H(T_i)$ has a very small value then (4.3) is a poor upper bound. We are able to prove a more general result than Corollary 1 but the downside is that the resulting upper bound is not as strong as (4.2).

Theorem 4.2. *Let P be a leading integer polynomial of degree n, $n \geq 2$, and $D(P) = 0$. Let $|P(x)| < H^{-w}$, then for $w > 2n-3$ and sufficiently large H, the closest root α_1 to x belongs to $\mathbb{R}$ and*

$$|x - \alpha_1| \ll H^{-(w+1)/n}. \tag{4.4}$$

Preliminaries

As P is a leading polynomial, by definition

$$|a_n| \gg H. \tag{4.5}$$

From this and the well known property $|\alpha_i| \ll \frac{H}{|a_n|}$ it further follows that

$$|\alpha_j| \ll 1, \quad j = 1, \dots, n; \tag{4.6}$$

i.e. the roots of P are bounded.

Reorder the other roots of P so that

$$|\alpha_1 - \alpha_2| \;\; \leq \;\; |\alpha_1 - \alpha_3| \;\; \leq \ldots \leq \;\; |\alpha_1 - \alpha_n|.$$

Proof of Theorem 4.1

Consider a polynomial P satisfying $D(P) \neq 0$ and (4.1). Since $D(P)$ is always an integer,

$$|D(P)|^{1/2} \geq 1.$$

By definition,

$$D(P) = (a_n)^{2n-2} \prod_{1 \leq j \leq n} (\alpha_1 - \alpha_j)^2 \prod_{2 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$$

and using

$$P'(\alpha_1) = a_n(\alpha_1 - \alpha_2) \ldots (\alpha_1 - \alpha_n)$$

it follows that

$$\begin{aligned} D(P) \;\; &= (a_n)^{2n-4}((a_n)^2 \textstyle\prod_{1 \leq j \leq n}(\alpha_1 - \alpha_j)^2)(\prod_{2 \leq i < j \leq n}(\alpha_i - \alpha_j)^2) \\ &= (a_n)^{2n-4}|P'(\alpha_1)|^2 \textstyle\prod_{2 \leq i < j \leq n}(\alpha_i - \alpha_j)^2 \end{aligned}$$

which implies that

$$D(P)^{\frac{1}{2}} \;\; = |a_n|^{n-2}|P'(\alpha_1)| \textstyle\prod_{2 \leq i < j \leq n} |\alpha_i - \alpha_j| \ll |a_n|^{n-2}|P'(\alpha_1)|,$$

so that

$$|P'(\alpha_1)| \gg H^{-n+2} \tag{4.7}$$

and, using (1.8),

$$|x - \alpha_1| \ll H^{-w+n-2}. \tag{4.8}$$

Assume that α_1 is a complex root of P. Then its conjugate is also a root of P. For simplicity, let $\alpha_2 = \bar{\alpha_1}$. Using (4.8) and the fact that $|x-\alpha_1| = |x-\alpha_2|$, it follows that

$$|\alpha_1 - \alpha_2| \leq |x-\alpha_1| + |x-\alpha_2| \ll H^{-w+n-2}. \tag{4.9}$$

Using (4.9) and (4.6), it further follows that

$$\begin{aligned} 1 \leq |D(P)|^{1/2} &= |a_n|^{n-1}|\alpha_1-\alpha_2| \textstyle\prod_{j=3}^{n} |\alpha_1-\alpha_j| \prod_{2\leq i<j\leq n} |\alpha_i-\alpha_j| \\ &\ll |a_n|^{n-1}H^{-w+n-2} \ll H^{-w+2n-3}. \end{aligned} \tag{4.10}$$

If $w > 2n-3$ clearly 4.10 is false. Thus α_1 is a real root of P and satisfies (4.8). □

Proof of Corollary 4.1

Consider the polynomial P satisfying $D(P) = 0$ and (4.1). If $D(P) = 0$, then P has repeated roots, and Lemma 1.4 implies that P is reducible. Write P as a product of irreducible polynomials $T_i(x) \in \mathbb{Z}[x]$:

$$P(x) = \prod_{i=1}^{k} T_i^{s_i}(x).$$

Since $D(P) = 0$ and T_i is an irreducible polynomial there exists an index l, $1 \leq l \leq k$ such that $s_l \geq 2$.

The next objective is to show that for some index j, $1 \leq j \leq k$, the inequality

$$|T_j(x)| < 2^{nw/2}H^{-w}(T_j) \tag{4.11}$$

holds. Assume the contrary, so

$$|T_j(x)| \geq 2^{nw/2}H^{-w}(T_j) \text{ for all } j,\ 1 \leq j \leq k.$$

By Lemma 1.1,

$$|P(x)| \geq \prod_{j=1}^{k} (2^{nw/2} H^{-w}(T_j))^{s_j} \geq 2^{nw(\sum_{j=1}^{k} s_j/2-1)} H(P)^{-w} \geq H(P)^{-w}$$

which contradicts (4.1). Thus (4.11) holds.

Hence as $D(T_j) \neq 0$, by the same argument as in the proof of Theorem 4.1, there exists a real root α_1 of T_j when $w \geq 2n_j - 3$ satisfying (4.3). Clearly $P(\alpha_1) = 0$. □

Proof of Theorem 4.2

Let $P \in \mathbb{Z}[x]$ satisfying $D(P) = 0$, and (4.1) and write

$$P(x) = \prod_{i=1}^{k} T_i^{s_i}(x), \quad s_i \geq 1, \tag{4.12}$$

where T_i is an irreducible polynomial, of degree $n_i, i = 1, \ldots, k$.

Case 1. If $k = 1$ then $P(x) = T_1^{s_1}$ and $s_1 \geq 2$ since $D(P) = 0$ and T is irreducible. In this case deg $T_1 = n/s_1 \in \mathbb{N}$, and $H(T_1) \asymp H(P)^{1/s_1}$. From (4.1), we get $|T_1(x)| \ll H(T_1)^{-w}$.

For $n_1 = 1$, (i.e. T_1 is a linear polynomial,) the estimate for $|x - \alpha_1|$ can be calculated directly as follows. Let $T_1(x) = d_1 x + d_0$ so that $a_n = d_1^n$. By (4.5) it is clear that $|d_1| \gg H(P)^{1/n}$ and

$$|x + d_0/d_1| < H(T_1)^{-w} |d_1|^{-1} \ll H(P)^{-(w+1)/n}. \tag{4.13}$$

Now, assume that $n_1 \geq 2$, and let α_1 be the closest root of T_1 to x. Since $D(T_1) \neq 0$ and $2 \leq s_1 \leq n/2$ the same method as in the proof of Theorem 1 can be used for T_1 to show that for $w > 2n/s_1 - 3$ the root α_1 belongs to $\mathbb{R}$ and the estimate

$$|x - \alpha_1| \ll H(T_1)^{-w-2+n/s_1} \ll H(P)^{-(w+2)/s_1 + n/s_1^2} := H(P)^{f(s_1,w)}$$

holds. Maximising $f(s_1, w)$ over the domain $w > 2n-3$ and $2 \le s_1 \le n/2$, gives

$$|x - \alpha_1| \ll H(P)^{-2w/n}. \tag{4.14}$$

Case 2. Assume that $k \ge 2$ and suppose that $s_1 = 1$. Rewrite (4.12) in the form

$$P(x) = T_1(x) \prod_{i=2}^{k} T_i^{s_i}(x) = T_1(x) P_1(x) \tag{4.15}$$

where each T_i is irreducible. Again the approach used is to assume α_1 is non-real and establish a contradiction.

Let $\alpha_1 \in \mathbb{C} \setminus \mathbb{R}$ be a root of the polynomial T_1. As α_1 is complex, its conjugate is also a root. For simplicity, let $\alpha_2 = \bar{\alpha_1}$. Thus deg $T_1 \ge 2$. Clearly deg$P_1 = n - n_1$. And, since $D(P) = 0$ and T_1 does not have repeated roots, P_1 has at least two common roots, so $n \ge 4$ and $n_1 \le n-2$. Let $H(T_1) = H(P)^{\lambda}$, $0 \le \lambda \le 1$, so that by Lemma 1.1, $H(P_1) \ll H(P)^{1-\lambda}$.

By definition the polynomials $T_1(x) = t_{n_1} x^{n_1} + \ldots + t_1 x + t_0$ and $P_1(x) = p_{n-n_1} x^{n-n_1} + \ldots + p_1 x + p_0$ do not have common roots. Denote (from 1.4) by $R(T_1, P_1)$ the resultant of T_1 and P_1. Then $R(T_1, P_1) \ne 0$ and $R(T_1, P_1) \in \mathbb{Z}$. In this case,

$$R(T_1, P_1) = t_{n_1}^{n-n_1} p_{n-n_1}^{n_1} \prod_{1 \le i \le n_1,\, n_1+1 \le j \le n} (\alpha_i - \alpha_j).$$

Hence,

$$1 \le |R(T_1, P_1)| \ll H^{\lambda(n-n_1)+(1-\lambda)n_1} \prod_{1 \le i \le n_1,\, n_1+1 \le j \le n} |\alpha_i - \alpha_j|. \tag{4.16}$$

Since $|\alpha_j| \ll 1$, $1 \le j \le n$,

$$\prod_{3 \le i \le n_1,\, n_1+1 \le j \le n} |\alpha_i - \alpha_j| \ll 1. \tag{4.17}$$

From (4.16) and (4.17) it follows that

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j||\alpha_2-\alpha_j| \gg H^{-\lambda(n-n_1)-(1-\lambda)n_1} \tag{4.18}$$

and for the same reason

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j| \gg H^{-\lambda(n-n_1)-(1-\lambda)n_1}. \tag{4.19}$$

The following facts will also be needed:

$$\text{if } \alpha_j\in\mathbb{R} \text{ then } |\alpha_1-\alpha_j|=|\bar\alpha_1-\alpha_j|;$$
$$\text{if } \alpha_j\in\mathbb{C}\setminus\mathbb{R} \text{ then } |\bar\alpha_1-\alpha_j|=|\alpha_1-\bar\alpha_j|.$$

Using these, (4.18) gives

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j|^2 \gg H^{-\lambda(n-n_1)-(1-\lambda)n_1}. \tag{4.20}$$

Now Lemma 1.3, for $s_1=1$, implies that

$$|x-\alpha_1| \ll |P(x)||a_n|^{-1} \underbrace{\prod_{2\le k\le n_1} |\alpha_1-\alpha_k|^{-1}}_{(4.21')} \underbrace{\prod_{n_1+1\le k\le n} |\alpha_1-\alpha_k|^{-1}}_{(4.21'')}. \tag{4.21}$$

An upper bound for (4.21″) follows from (4.20). The discriminant $D(T_1)$ of T_1 is used together with (4.6) to estimate (4.21′) so that

$$1\le |D(T_1)|^{1/2} = t_{n_1}^{n_1-1} \prod_{1\le i<j\le n_1} |\alpha_i-\alpha_j| \ll H^{\lambda(n_1-1)} \prod_{2\le k\le n_1} |\alpha_1-\alpha_k|. \tag{4.22}$$

Thus,

$$\prod_{2\le k\le n_1} |\alpha_1-\alpha_k| \gg H^{-\lambda(n_1-1)} \tag{4.23}$$

and using (4.6) again, $\prod_{3\le k\le n_1} |\alpha_1-\alpha_k| \ll 1$ so that

$$|\alpha_1-\alpha_2| \gg H^{-\lambda(n_1-1)}. \tag{4.24}$$

In several places the following approach is used to establish upper bounds on a function of the height of P. A domain from the minimum to maximum allowable values of both λ and n is defined. The maximum value of the function being considered is evaluated on this domain which gives the required maximum value of the bounds.

On combining (4.1), (4.20)–(4.24) and the fact that $|\alpha_1 - \alpha_2| \leq 2|x - \alpha_1|$ we have

$$1 \leq 2|\alpha_1 - \alpha_2|^{-1}|x - \alpha_1| \ll H^{f_1(\lambda, n_1)} := H^{-w-1+\lambda(n_1-2)+(\lambda n+n_1)/2}.$$

Define the domain

$$D_1 = \{(\lambda, n_1) : 0 \leq \lambda \leq 1 \text{ and } 2 \leq n_1 \leq n-2\}$$

for $n \geq 4$. Then, the maximum of f_1 on D_1 is $-w + 2n - 6$; i.e.

$$1 \leq 2|\alpha_1 - \alpha_2|^{-1}|x - \alpha_1| \leq H^{-w+2n-6}$$

which is a contradiction for $w > 2n - 6$ and sufficiently large H.

Now as $\alpha_1 \in \mathbb{R}$, no conjugate root exists in (4.15). Hence the index n_1 satisfies $1 \leq n_1 \leq n - 2$.

In a similar approach to above, define the function

$$f_2(\lambda, n_1) = -w - 1 - \lambda(n_1 - 1) + \lambda(n - n_1) + (1 - \lambda)n_1$$

on the domain

$$D_2 = \{(\lambda, n_1) : 0 \leq \lambda \leq 1 \text{ and } 1 \leq n_1 \leq n-2\}.$$

Using (4.1), (4.18), (4.23), and (4.21), we obtain

$$|x - \alpha_1| \ll H^{f_2(\lambda, n_1)} \leq H^{-w+n-2}, \tag{4.25}$$

since the maximum value of $f_2(\lambda, n_1)$ on D_2 is $-w+n-2$.

Case 3. Assume $k \geq 2$ and $s_1 \geq 2$. Rewrite the polynomial P as

$$P(x) = T_1^{s_1}(x)P_2(x), \text{ where } P_2(x) = \prod_{i=2}^{k} T_i^{s_i}(x).$$

Let α_1 be a root of T_1 and $\deg T_1 = n_1$ so that $\deg P_2 = n - n_1 s_1$. Again suppose that $\alpha_1 \in \mathbb{C} \setminus \mathbb{R}$. For simplicity, let $\alpha_2 = \bar{\alpha_1}$, the conjugate of α_1. Then $n \geq 2s_1 + 1 \geq 5$ and $4 \leq 2s_1 \leq n_1 \leq n-1$. Let $H(T_1) = H(P)^{\lambda/s_1}$, $0 \leq \lambda \leq 1$, so that, by Lemma 1.1, $H(P_2) \ll H(P)^{1-\lambda}$.

Since T_1 is irreducible over $\mathbb{Q}$, then T_1 has no multiple roots over $\mathbb{C}$. Thus,

$$\prod_{j=2}^{n} |\alpha_1 - \alpha_j| = \prod_{2 \leq j \leq n_1/s_1} |\alpha_1 - \alpha_j|^{s_1} \prod_{n_1+1 \leq j \leq n} |\alpha_1 - \alpha_j|.$$

This, with Lemma 1.3, for $s = s_1$ gives

$$|x - \alpha_1| \ll H(P)^{\frac{-w-1}{s_1}} \prod_{2 \leq j \leq n_1/s_1} |\alpha_1 - \alpha_j|^{-1} \prod_{n_1+1 \leq j \leq n} |\alpha_1 - \alpha_j|^{-1/s_1}. \quad (4.26)$$

By definition, the discriminant of T_1 satisfies

$$1 \leq |D(T_1)|^{1/2} = H(P)^{\lambda(n_1/s_1 - 1)/s_1} \prod_{1 \leq i < j \leq n_1/s_1} |\alpha_i - \alpha_j|.$$

It follows via (4.6) that

$$\prod_{2 \leq j \leq n_1/s_1} |\alpha_1 - \alpha_j| \gg H(P)^{-\lambda(n_1/s_1 - 1)/s_1} \quad (4.27)$$

and in particular that,

$$|\alpha_1 - \alpha_2| \gg H(P)^{-\lambda(n_1/s_1 - 1)/s_1}. \quad (4.28)$$

Since the resultant of T_1 and P_2 does not equal zero, it follows that

$$1 \leq |R(T_1, P_2)| \ll H^{\lambda(n-n_1)/s_1} H^{(1-\lambda)n_1/s_1} \prod_{1 \leq i \leq n_1/s_1,\, n_1+1 \leq j \leq n} |\alpha_i - \alpha_j|. \quad (4.29)$$

Therefore, from (4.6) and (4.29)

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j||\alpha_2-\alpha_j| \gg H^{-\lambda(n-n_1)/s_1-(1-\lambda)n_1/s_1} \tag{4.30}$$

and also,

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j| \gg H^{-\lambda(n-n_1)/s_1-(1-\lambda)n_1/s_1}. \tag{4.31}$$

In a similar manner to (4.20) in Case 2, the following holds

$$\prod_{n_1+1\le j\le n} |\alpha_1-\alpha_j| \gg H^{\frac{-\lambda(n-n_1)-(1-\lambda)n_1}{2s_1}}. \tag{4.32}$$

Define a function

$$f_3(\lambda,n_1) = -\frac{w+1}{s_1} + \frac{2\lambda(n_1-s_1)}{s_1^2} + \frac{\lambda(n-n_1)+(1-\lambda)n_1}{2s_1^2}$$

on the domain

$$D_3 = \{(\lambda,n_1):\ 0\le\lambda\le 1 \text{ and } 2s_1\le n_1\le n-1\}$$

and a function

$$f_4(s_1) = -\frac{w+1}{s_1} + \frac{4n-3-4s_1}{2s_1^2}$$

on the interval $I_1 = \left[2, \frac{n-1}{2}\right]$. Note that the function $f_3(\lambda,n_1)$ has a maximum value of $f_4(s_1)$ on D_3 and, for $w > 2n-3$, the maximum value of $f_4(s_1)$ on I_1 is

$$\frac{-2w-2}{n-1} + \frac{4n-2}{(n-1)^2}.$$

Then, using the formulae (4.1), (4.26), (4.27), (4.28), (4.32) and the fact that $|\alpha_1-\alpha_2| \le 2|x-\alpha_1|$, it follows that

$$1 \le 2|\alpha_1-\alpha_2|^{-1}|x-\alpha_1| \ll H^{f_3(\lambda,n_1)} \le H^{f_4(s_1)} \le H^{\frac{-2w-2}{n-1}+\frac{4n-2}{(n-1)^2}}. \tag{4.33}$$

This is a contradiction for $w > 2n-3$, $n \ge 3$, and sufficiently large H

so again $\alpha_1 \in \mathbb{R}$. Again, therefore, to find the distance of x to α_1, as in case 2, we know that no conjugate root exists, and the index n_1 runs from $s_1 \leq n_1 \leq n-1$.

Let

$$f_5(\lambda, n_1) = -\frac{w+1}{s_1} + \frac{\lambda(n-n_1-s_1)}{s_1^2}.$$

Using the formulae (4.1), (4.26), (4.27), and (4.31), we obtain

$$|x-\alpha_1| \ll H(P)^{f_5(\lambda,n_1)} \leq H(P)^{-\frac{w+1}{s_1}+\frac{n-1}{s_1^2}} < H(P)^{\frac{-w}{n-1}}, \tag{4.34}$$

where the right hand side of (4.34) is a straightforward consequence of maximizing $f_5(\lambda, n_1)$ on

$$D_4 = \{(\lambda, n_1) : 0 \leq \lambda \leq 1 \text{ and } s_1 \leq n_1 \leq n-1\}.$$

Now, by (4.34), for $w > 2n-3$ and $2 \leq s_1 \leq n-1$ we get

$$|x-\alpha_1| \ll H^{\frac{-w}{n-1}}. \tag{4.35}$$

Combining (4.14), (4.13), (4.25) and (4.35), for $w > 2n-3$ and sufficiently large H, the estimates

$$|x-\alpha_1| \ll \max\{H^{-w+n-2}, H^{\frac{-w}{n-1}}, H^{-2w/n}, H^{-(w+1)/n}\} = H^{-(w+1)/n}$$

hold, for $x \in S_P(\alpha_1)$. This completes the proof of the Theorem.

□

4.3 Nesterenko's problem in $\mathbb{Z}_p$.

In [77] Y. Nesterenko discussed the solvability of the equation $P(x) = 0$ in the ring of p-adic integers $\mathbb{Z}_p$ and proved the following result:

Theorem 4.3 (Nesterenko). *If* $|P(x)|_p \le e^{-8n^2}H^{-4n}$, *where* $n = degP$, $H = H(P)$, *then there exists a* p*-adic number* γ *such that* $P(\gamma) = 0$, $|x-\gamma|_p < 1$.

This result can be improved for p–adic leading polynomials. Such a polynomial satisfies

$$|a_n|_p \gg 1. \tag{4.36}$$

Theorem 4.4. *Let* P *be a* p*–adic leading integer polynomial of degree* n. *Then if*

$$|P(w)|_p < H^{-w_2} \tag{4.37}$$

for $w_2 > 2n-2$, *and for sufficiently large* $H > H_0(n)$, *it follows that the closest root* γ_1 *of* P *to* $w \in \mathbb{Z}_p$ *belongs to* $\mathbb{Q}_p$ *and*

$$|w-\gamma_1|_p < 1. \tag{4.38}$$

Preliminary setup and auxilliary Lemmas

Let $P \in \mathcal{P}_n$ have roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. Then, from (4.36) it follows that

$$|\gamma_i|_p \ll 1, \quad i = 1, \ldots, n; \tag{4.39}$$

i.e. the roots are bounded. This follows from Lemma (6.6) in [27].

Define the p-adic equivalent of the previously defined sets $S_P(\alpha)$ as

$$T_p(\gamma_k) \;=\; \{w \in \mathbb{Z}_p : |w-\gamma_k|_p = \min_{1\le i\le n} |w-\gamma_i|_p\}, \;\; 1 \le k \le n.$$

Consider the set $T_p(\gamma_k)$ for a fixed k and for ease of notation assume that $k = 1$. Next, reorder the other roots so that

$$|\gamma_1-\gamma_2|_p \;\le\; |\gamma_1-\gamma_3|_p \;\le \ldots \le\; |\gamma_1-\gamma_n|_p.$$

A Lemma proved by Bernik, which is a generalisation of a Lemma by Sprindžuk [85] is also needed.

Lemma 4.1 (Bernik)**.** *[19] Let $w \in T_P(\gamma_1)$. Then*

$$|w-\gamma_1|_p < \min_{1\le j\le n}(|P(w)|_p|P'(\gamma_1)|_p^{-1}\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p)^{1/j}.$$

The Theorem can now be proved.

Proof of Theorem 4.4. Two cases must be dealt with separately: $D(P) \neq 0$ and $D(P) = 0$.

Case I. First consider a polynomial P satisfying $D(P) \neq 0$ and (4.37), and assume that $|P'(w)|_p^2 \le |P(w)|_p$. We will obtain a contradiction. Using (4.39), we get $|P'(w)|_p < H^{-w_2/2}$.

It is well known that $|D(P)| = \frac{|\Delta|}{|a_n|}$, where

$$\Delta = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & \dots & 0 & \dots & 0 \\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 \end{pmatrix}.$$

Hence the determinant,

$$\begin{aligned} |\Delta| &\le |a_n|((2n-2)!(nH)^{2n-2} + n(2n-2)!(nH)^{2n-2}) \\ &= |a_n|(2n-2)!(n+1)(nH)^{2n-2} \le 2n^{2n-1}(2n-2)!H^{2n-2}|a_n|, \end{aligned}$$

using the fact that $|a_i| \le H$, $i = 0, 1, \dots, n$. Thus, $|D(P)| \le c_1(n)H^{2n-2}$, where $c_1(n) = 2n^{2n-1}(2n-2)!$. This implies that

$$|D(P)|_p \ge c_1^{-1}(n)H^{-2n+2}. \tag{4.40}$$

At this point, for convenience, define the number s_j as

$$\prod_{k=j+1}^{n}|\gamma_1-\gamma_k|_p^{-1} = H^{s_j}. \tag{4.41}$$

Using Lemma 4.1, $|a_n|_p \gg 1$ and (4.37),

$$\begin{array}{rcl} |w-\gamma_1|_p & \leq & \min_{1\leq j\leq n}(|P(w)|_p |P'(\gamma_1)|_p^{-1} \prod_{k=2}^{j} |\gamma_1-\gamma_k|_p)^{1/j} \\ & < & \min_{1\leq j\leq n}(H^{-w_2}|a_n|_p^{-1} \prod_{k=j+1}^{n} |\gamma_1-\gamma_k|_p^{-1})^{1/j} \\ & \leq & \min_{1\leq j\leq n}(H^{-w_2}|a_n|_p^{-1} H^{s_j})^{1/j} \\ & \ll & \min_{1\leq j\leq n} H^{\frac{-w_2+s_j}{j}}. \end{array}$$

Define $\sigma(P)$ as the cylinder of points w satisfying

$$|w-\gamma_1|_p \ll \min_{1\leq j\leq n} H^{\frac{-w_2+s_j}{j}}.$$

Let $\theta_j = \frac{w_2-s_j}{j}$ and denote by θ_0 the maximum value of θ_j, $j=1,\ldots,n$.

Now the polynomial P' is expanded as a Taylor series and each term is estimated on $\sigma(P)$. Thus

$$\begin{array}{rcl} P'(w) & = & P'(\gamma_1) + \sum_{j=2}^{n}((j-1)!)^{-1} P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}, \\ |P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}|_p & \ll & H^{-s_j+(n-j)\epsilon_1} H^{-\theta_0(j-1)}. \end{array}$$

As $\theta_0 \geq \theta_j$, this implies that

$$|P^{(j)}(\gamma_1)| \ll H^{-s_j+(n-j)\epsilon_1+\frac{j-1}{j}(-w_2+s_j)} \leq H^{-w_2/2+(n-2)\epsilon_1} \quad \text{for } 2\leq j\leq n.$$

Thus,

$$|P'(\gamma_1)|_p \leq \max_{2\leq j\leq n}\{|P'(w)|_p, |P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}|_p\} \ll H^{-w_2/2+(n-2)\epsilon_1}$$

for $H > H_0(n)$.

Expressing the discriminant $D(P)$ in the form

$$|D(P)|_p = |a_n|_p^{2n-2} \prod_{1\leq i<j\leq n} |\gamma_i-\gamma_j|_p^2 = |a_n|_p^{2n-4} |P'(\gamma_1)|_p^2 \prod_{2\leq i<j\leq n} |\gamma_i-\gamma_j|_p^2$$

and using the facts that $|\gamma_i|_p \ll 1$ and $|a_n|_p \leq 1$, we obtain

$$|D(P)|_p \ll |P'(\gamma_1)|_p^2.$$

This contradicts (4.40) for $w_2 > 2n-2+2(n-2)\epsilon_1$ and sufficiently large H.

Therefore, $|P'(w)|_p^2 > |P(w)|_p$ holds for $w_2 > 2n-2+2(n-2)\epsilon_1$, and case I follows immediately from Hensel's Lemma 1.3. Hence, there exists a root $\gamma_1 \in \mathbb{Q}_p$ of P such that $|w-\gamma_1|_p \leq |P(w)|_p/|P'(w)|_p^2 < 1$.

Case II. Consider the polynomial P satisfying $D(P) = 0$. First, P is decomposed into irreducible polynomials $T_i(w) \in \mathbb{Z}[w]$, i.e.

$$P(w) = \prod_{i=1}^{k} T_i^{s_i}(w).$$

It will be shown that for some index j, $1 \leq j \leq k$,

$$|T_j(w)|_p < 2^{nw_2/2} H^{-w_2}(T_j). \tag{4.42}$$

Assume the opposite, so that

$$|T_j(w)|_p \geq 2^{nw_2/2} H^{-w_2}(T_j) \text{ for all } j,\ 1 \leq j \leq k.$$

Then, by Lemma 1.3,

$$|P(w)|_p \geq \prod_{j=1}^{k} (2^{nw_2/2} H^{-w_2}(T_j))^{s_j} \geq 2^{nw_2(\sum_{j=1}^{k} s_j/2-1)} H(P)^{-w_2} \geq H(P)^{-w_2}$$

which contradicts (4.37). Thus (4.42) holds.

Hence, applying the same method as in Case I for T_j, $D(T_j) \neq 0$, which satisfies (4.42), it follows that there exists a p-adic number γ_1 such that $|w-\gamma_1| < 1$ and $T_j(\gamma_1) = 0$. This implies $P(\gamma_1) = 0$. □

4.4 A result in the real and p-adic metrics

In this section a generalisation of the previous results is considered. The problem of Nesterenko in $\mathbb{R} \times \mathbb{Z}_p$ is investigated. The approach uses bounds on the derivatives of the polynomial when expanded as a Taylor series, and some of the ideas in Sprindžuk's book [85].

4.4.1 Statement of the Theorem

A polynomial is *leading* and *p-adic leading* if both

$$|a_n| > c_1 H(P), \quad |a_n|_p \gg 1 \tag{4.43}$$

hold. The following result is proved:

Theorem 4.5. *Let $P \in \mathbb{Z}[x]$ be leading and p–adic leading, of degree n and let the discriminant $D(P) \neq 0$. If at some point $(x, w) \in \mathbb{R} \times \mathbb{Q}_p^*$ the inequalities*

$$|P(x)| < H(P)^{-w_1}, \quad |P(w)|_p < H(P)^{-w_2} \tag{4.44}$$

and $x \in S_P(\alpha_1)$ or $w \in S_P(\gamma_1)$, hold for

$$w_1 + w_2 > 2n - 3, \tag{4.45}$$

$w_1 > 0$, $w_2 > 0$, and sufficiently large $H > H_0(n)$, then the root γ_1 of P closest to w belongs to $\mathbb{Q}_p$ and

$$|w - \gamma_1|_p < 1 \tag{4.46}$$

or the root α_1 of P closest to x belongs to $\mathbb{R}$ and

$$|x - \alpha_1| \ll H(P)^{-w_1 - w_2/2 + n - 2}. \tag{4.47}$$

It should be noted that this result is consistent with Theorem 4.1 when the result is restricted to the real case.

4.4.2 Preliminary setup and auxiliary Lemmas

Let $\mathcal{P}_n(H)$ be the set of polynomials $P \in \mathcal{P}_n$ satisfying (4.43) for which $H(P) = H$. Let $P \in \mathcal{P}_n(H)$ have roots $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{C}$ and roots $\gamma_1, \gamma_2, \dots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. Then, from (4.43) and [85], it follows that

$$|\alpha_i| \ll 1, \quad |\gamma_i|_p \ll 1, \quad i = 1, \dots, n;$$

i.e. the roots are bounded. Recall the following definitions:

$$\begin{aligned} S_P(\alpha_j) &= \{x \in \mathbb{R} : |x - \alpha_j| = \min_{1\le i\le n} |x - \alpha_i|\}, \\ T_P(\gamma_k) &= \{w \in \mathbb{Q}_p : |w - \gamma_k|_p = \min_{1\le i\le n} |w - \gamma_i|_p\}. \end{aligned}$$

We consider the sets $S_P(\alpha_j)$, $T_P(\gamma_k)$ for a fixed set j, k and for simplicity we will assume that $j = k = 1$. Reorder the other roots of P so that

$$\begin{aligned} |\alpha_1 - \alpha_2| &\le |\alpha_1 - \alpha_3| \le \dots \le |\alpha_1 - \alpha_n|, \\ |\gamma_1 - \gamma_2|_p &\le |\gamma_1 - \gamma_3|_p \le \dots \le |\gamma_1 - \gamma_n|_p. \end{aligned}$$

From now on it will be assumed without loss of generality that $x \in S_P(\alpha_1)$ and $w \in T_p(\gamma_1)$. In many places in the proof of the Theorem the values of the polynomials will be estimated by expanding the polynomial as a Taylor series. To obtain an upper bound on the terms in the Taylor series (and for other purposes) the following Lemma (proved in [19] and [68]) will be used.

Lemma 4.2. *If $P \in \mathcal{P}_n$ then*

$$\begin{aligned} |x - \alpha_1| &\le 2^n |P(x)||P'(\alpha_1)|^{-1}; \\ |w - \gamma_1|_p &\le |P(w)|_p |P'(\gamma_1)|_p^{-1}; \\ |x - \alpha_1| &\le \min_{2\le j\le n} \left(2^{n-j} |P(x)||P'(\alpha_1)|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k| \right)^{\frac{1}{j}}; \end{aligned}$$

and

$$|w-\gamma_1|_p \le \min_{2\le j\le n}\left(|P(w)|_p |P'(\gamma_1)|_p^{-1}\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p\right)^{\frac{1}{j}}.$$

4.4.3 Proof of Theorem 4.5.

Firstly, assume that $|P'(w)|_p^2 > |P(w)|_p$. In this case Hensel's Lemma (1.3) can be applied to obtain a p-adic root γ_1 of P such that $|w-\gamma_1|_p < |P(w)|_p/|P'(w)|_p^2 < 1$.

Secondly, assume that $|P'(w)|_p^2 \le |P(w)|_p$. From (4.44) it follows that $|P'(w)|_p < H^{-w_2/2}$. Then Lemma 4.2, is used together with $|a_n|_p \gg 1$, (4.44) and (4.43) to obtain

$$\begin{aligned}
|w-\gamma_1|_p &\le \min\nolimits_{1\le j\le n}(|P(w)|_p|P'(\gamma_1)|_p^{-1}\textstyle\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p)^{1/j}\\
&< \min\nolimits_{1\le j\le n}(H^{-w_2}|a_n|_p^{-1}\textstyle\prod_{k=j+1}^{n}|\gamma_1-\gamma_k|_p^{-1})^{1/j}\\
&\le \min\nolimits_{1\le j\le n}(H^{-w_2}|a_n|_p^{-1}H^{s_j})^{1/j}\\
&\ll \min\nolimits_{1\le j\le n} H^{\frac{-w_2+s_j}{j}}
\end{aligned}$$

where s_j is defined in (4.41). Let $\sigma(P)$ be the cylinder defined by this system. For convenience label

$$\theta_0 = \max_{1\le j\le n}\frac{w_2-s_j}{j}.$$

Expand P into its Taylor series and estimate each term at $w\in\sigma(P)$. This gives

$$P'(w) = P'(\gamma_1)+\sum_{j=2}^{n}((j-1)!)^{-1}P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}.$$

But

$$\begin{aligned}
|P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}|_p &\ll H^{-s_j+(n-j)\varepsilon}H^{-\theta_0(j-1)}\\
&\le H^{-s_j+(n-j)\varepsilon+\frac{j-1}{j}(-w_2+s_j)} \qquad (4.48)\\
&\le H^{-w_2/2+(n-2)\varepsilon}
\end{aligned}$$

for $2 \leq j \leq n$. Thus,

$$|P'(\gamma_1)|_p \leq \max_{2\leq j\leq n}\{|P'(w)|_p, |P^{(j)}(\gamma_1)(w-\gamma_1)^{j-1}|_p\} \ll H^{-w_2/2+(n-2)\varepsilon}$$

for $H > H_0(n)$.

From this, and the facts that $|\alpha_j| \ll 1$ and $|\gamma_j|_p \ll 1$, it follows that

$$1 \leq |D(P)||D(P)|_p \ll |a_n|^{2n-4}|a_n|_p^{2n-4}|P'(\alpha_1)|^2|P'(\gamma_1)|_p^2 \ll H^{2n-4}|P'(\alpha_1)|^2|P'(\gamma_1)|_p^2.$$

This further implies that

$$|P'(\alpha_1)| \gg H^{-n+2+w_2/2-(n-2)\varepsilon} \tag{4.49}$$

using the previous bounds on $|P'(\gamma_1)|_p$. Therefore, since

$$P'(\alpha_1) = a_n(\alpha_1-\alpha_2)\dots(\alpha_1-\alpha_n),$$

and $|\alpha_i| \ll 1$, we have

$$|\alpha_1-\alpha_2| \gg H^{-n+1+w_2/2-(n-2)\varepsilon}. \tag{4.50}$$

By (4.44), (4.49) and Lemma 4.2,

$$|x-\alpha_1| \ll |P(x)|/|P'(\alpha_1)| \ll H^{-w_1-w_2/2+n-2+(n-2)\varepsilon}. \tag{4.51}$$

Now, if α_1 is a complex root of P, then its conjugate is also a root of P. For simplicity, let $\alpha_2 = \bar{\alpha_1}$. Hence,

$$|\alpha_1-\alpha_2| \leq |x-\alpha_1| + |x-\alpha_2| \ll H^{-w_1-w_2/2+n-2+(n-2)\varepsilon}. \tag{4.52}$$

For $w_1 + w_2 > 2n-3+2(n-2)\varepsilon$ and sufficiently large $H > H_0(n)$ this contradicts (4.50). Hence, α_1 is a real root of P satisfying (4.51). □

Chapter 5

On regular systems of real algebraic numbers in small intervals

5.1 Introduction

Regular systems as defined by Baker and Schmidt [3] are defined in definition (1.10). In the results of Baker and Schmidt [3], Bernik [20] and Beresnevich [6] which were described in chapter 2, it was shown that the set of real algebraic numbers α of degree at most n together with the function $N(\alpha) = H(\alpha)^{n+1} \log^{-v} H(\alpha)$, forms a regular system when $v = 3n(n+1)$, 2 and 0 respectively. Here $H(\alpha)$ is the height of the algebraic number α, defined as the maximum of the absolute values of the coefficients of the minimal polynomial of α.

In [6] the constant c_1 is calculated, but T_0 is not (both of these are in the definition of a regular system). In [35] it is shown that for a given finite interval I in $[-1/2, 1/2]$ the value of $T_0(\Gamma, N(\alpha), I)$ is equal to

$$T_0(\mathbb{Q}, N(\alpha), I) = 10^4|I|^{-2}\log^2 100|I|^{-1}$$

for $n = 1$, and in [6] that

$$T_0(A_2, N(\alpha), I) = 72^3|I|^{-3}\log^3 72|I|^{-1}$$

for $n = 2$, where A_2 is the set of real algebraic numbers of degree two. In [35] (Section 6.1) Bugeaud notes that for $n \geq 3$ the relationship between $|I|$ and T_0 is not presently known.

In this chapter for $n = 3$ the relationship between $|I|$ and T_0 is examined, and it is shown that $T_0 = c_2|I|^{-4}$ for a constant c_2. Let A_n be the set of real algebraic numbers of degree n, and c_2, c_3 are positive constants.

For a positive integer Q define the set of polynomials

$$\mathcal{P}'_3(Q) = \{P \in \mathbb{Z}[x] : \deg P = 3, \quad H(P) \leq Q\}. \tag{5.1}$$

5.2 Statement of results

Theorem 5.1. *Let I be a finite interval contained in $[-1/2, 1/2]$. Then there exist positive constants c_1, c_2 and a positive number $T_0 = c_2|I|^{-4}$ such that for any $T \geq T_0$ there exist numbers $\alpha_1, \ldots, \alpha_t \in A_3 \cap I$ such that*

$$\begin{gathered} H(\alpha_i) \leq T^{1/4} \quad (1 \leq i \leq t), \\ |\alpha_i - \alpha_j| \geq T^{-1} \quad (1 \leq i < j \leq t), \\ t \geq c_1 T|I|. \end{gathered} \tag{5.2}$$

Note that from Theorem 5.1 it follows that the set of real algebraic numbers α of degree 3, together with the function $N(\alpha) = H^4(\alpha)$ form a regular system on $[-1/2, 1/2]$.

Let $\delta_0 \in \mathbb{R}^+$. Denote by $\bar{\mathcal{L}}_3 = \bar{\mathcal{L}}_3(Q, \delta_0, I)$ the set of $x \in I$, for which the system of inequalities

$$|P(x)| < Q^{-3}, \quad |P'(x)| < \delta_0 Q \tag{5.3}$$

are satisfied for some $P \in \mathcal{P}'_3(Q)$. The proof of Theorem 5.1 is based on the following metric result.

Theorem 5.2. *For any real number s, where $0 < s < 1$, there exists a constant δ_0, which satisfies the following property. For any interval $I \subset [-1/2, 1/2]$ there exists a sufficiently large number $Q_0 = Q_0(I)$ and a constant c_5 independent of Q_0 such that*

$$|I| > c_5 Q_0^{-1},$$

and for all $Q > Q_0$

$$|\bar{\mathcal{L}}_3(Q, \delta_0, I)| < s|I|. \tag{5.4}$$

5.3 Proof of Theorem 5.1

Let c_5 be a constant such that $c_5 \geq \frac{2.3^5}{(1-s)\delta_0}$ and for which Theorem (5.2) is valid. Denote by $\mathcal{L}_0(Q, I)$ the set of $x \in I$, for which $|P(x)| < Q^{-3}$ is satisfied for some $P \in P'_3(Q)$. It can be readily verified using Dirichlet's Box Principle that $\mathcal{L}_0(Q, I) = I$.

By Theorem 5.2 there exists a set $\mathcal{L}_3(Q, \delta_0, I) = I \setminus \bar{\mathcal{L}}_3(Q, \delta_0, I) \subset I$ such that $|\mathcal{L}_3(Q, \delta_0, I)| \geq (1 - s)|I|$ for all $Q > Q_0$, where $Q_0 > c_5|I|^{-1}$. Denote by $\mathcal{L}_{\leq 2}(Q, \delta_0, I)$ the union of the intervals

$$\sigma(\alpha) = \{x \in I : |x - \alpha| < 3\delta_0^{-1} Q^{-4}\}$$

over all real algebraic numbers of degree at most 2 and height at most Q. The number of different intervals in this union is at most $(2Q+1)^3$ and every interval has a length at most $6\delta_0^{-1}q^{-4}$, therefore it follows that

$$|\mathcal{L}_{\leq 2}(Q,\delta_0,I)| \leq \frac{(1-s)|I|}{2}$$

for $c_5 \geq \frac{2.3^5}{(1-s)\delta_0}$. Define

$$\mathcal{L}'_3(Q,\delta_0,I) = \mathcal{L}_3(Q,\delta_0,I) \setminus \mathcal{L}_{\leq 2}(Q,\delta_0,I).$$

Let $x \in \mathcal{L}'_3(Q,\delta_0,I)$. Then there exists a non-zero polynomial $P \in \mathcal{P}'_3(Q)$, satisfying

$$|P(x)| < Q^{-3}, \quad |P'(x)| \geq \delta_0 Q. \tag{5.5}$$

It will be shown that there exists a root α of P close to x. Let $y \in \mathbb{R}$, be such that $|y-x| = 3\delta_0^{-1}Q^{-4}$. By Taylor's formula

$$P(y) = \sum_{i=0}^{3} \frac{1}{i!} P^{(i)}(x)(y-x)^i.$$

As $x \in [-\frac{1}{2}, \frac{1}{2}]$, $|P^{(i)}(x)| \ll Q$ for $i = 1,2,3$. It is readily verified that

$$\left| P^{(i)}(x)(y-x)^i \right| \ll Q^{-7} \qquad \text{for } i \geq 2\,.$$

Also, by (5.5), $|P(x)| < Q^{-3}$. Thus,

$$\sum_{i=0,2,3} \left| \tfrac{1}{i!} P^{(i)}(x)(y-x)^i \right| < Q^{-3} + \sum_{i=2}^{3} 3^2(7\delta_0^{-2}Q^{-7}) < 2Q^{-3}. \tag{5.6}$$

On the other hand, by (5.5)

$$|P'(x)(y-x)| \geq 3Q^{-3}. \tag{5.7}$$

By (5.6) and (5.7) the behaviour of $P(y)$ is dominated by the behaviour of $P'(x)(y-x)$. It also follows from (5.6) and (5.7) that $P(y)$ has different signs at the endpoints of the interval $(x - 3\delta_0^{-1}Q^{-4}, x + 3\delta_0^{-1}Q^{-4})$. Thus, by the continuity of P, there exists a root α of P in this interval, and

$$|x - \alpha| < 3\delta_0^{-1}Q^{-4}\,. \tag{5.8}$$

Since $x \notin \mathcal{L}_{\leq 2}(Q, \delta_0, I)$, it follows that the degree of α is exactly 3. Choose a maximal collection of real algebraic numbers $\{\alpha_1, \dots, \alpha_t\} \subset I$, with degree $\deg\alpha_i = 3$ satisfying

$$H(\alpha_i) \leq Q, \quad |\alpha_i - \alpha_j| \geq 3\delta_0^{-1}Q^{-4}, \;\; 1 \leq i < j \leq t.$$

As has been shown, for any $x \in \mathcal{L}'_3(Q, \delta_0, I)$ there exists α, satisfying (5.8) with $H(\alpha) \leq Q$. Since the collection $\{\alpha_1, \dots, \alpha_t\}$ is maximal, there exists α_i in this collection such that $|\alpha - \alpha_i| \leq 3\delta_0^{-1}Q^{-4}$. From this and (5.8), by the triangle inequality it follows that $|x - \alpha_i| < 6\delta_0^{-1}Q^{-4}$. Then

$$\mathcal{L}'_3(Q, \delta_0, I) \subset \bigcup_{i=1}^{t}\{x \in I : \, |x - \alpha_i| < 6\delta_0^{-1}Q^{-4}\}.$$

Using $|\mathcal{L}'_3(Q, \delta_0, I)| \geq \frac{(1-s)|I|}{2}$, this gives

$$t \geq 2^{-3}3^{-1}\delta_0(1 - s)Q^4|I|.$$

Let $T_0 = Q_0^4$, then for any $T \geq T_0$, where

$$T_0 = (c_5 + 1)^4|I|^{-4},$$

there exists a collection $\alpha_1, \dots, \alpha_t \in I \cap A_3$ satisfying (5.2) which completes the proof of the Theorem. $\square$

5.4 Proof of Theorem 5.2

The proof uses the concept of essential and inessential domains extensively. This concept was first introduced by Sprindžuk in [85] and is described here.

Definition 5.1. *Let $\mathcal{P}$ be a set of polynomials satisfying certain conditions and $\sigma(P)$ be a set of points (defined for each $P \in \mathcal{P}$) which meet certain conditions. A domain $\sigma(P)$ is called* essential *if*

$$\left|\sigma(P) \bigcap \bigcup_{Q \in \mathcal{P}} \sigma(Q)\right| < \frac{1}{2}|\sigma(P)|$$

and is called inessential *otherwise.*

The proof of the Theorem is in two parts: the first when $|P'(x)| \gg Q$ and the second when $|P'(x)| \ll Q$. The first case has five sub-cases.

Case 1: Define $\tilde{\mathcal{L}}_3$ as the subset of $\bar{\mathcal{L}}_3$ containing the set of points $x \in I$, for which there exists a polynomial $P \in \mathcal{P}'_3(Q)$ such that the system

$$|P(x)| < Q^{-3}, \quad 2^6 Q^{-1} < |P'(x)| < \delta_0 Q \tag{5.9}$$

holds.

Denote by $\sigma_0(P)$ the set of solutions x of (5.9) for a fixed polynomial $P \in \mathcal{P}'_3(Q)$. Then can be written as $\tilde{\mathcal{L}}_3 = \cup_{P \in \mathcal{P}'_3(Q)} \sigma_0(P)$. Let α_1, α_2 and α_3 be the roots of $P \in \mathcal{P}'_3(Q)$ in $\mathbb{C}$. For simplicity only $S_P(\alpha_1) \cap I$, is considered as the arguments are the same for the other two $S_P(\alpha_i) \cap I$. Let $x \in \sigma_0(P) \cap S_P(\alpha_1)$. By the Mean Value Theorem

$$P'(x) = P'(\alpha_1) + P''(\theta_1)(x - \alpha_1), \quad \theta_1 \in (\alpha_1, x). \tag{5.10}$$

Estimating the second term by using Lemma 1.2 gives:

$$|P''(\theta_1)(x - \alpha_1)| \le 6Q3Q^{-3}|P'(x)|^{-1} < 1/2Q^{-1}. \tag{5.11}$$

Since $|P'(x)| > 2^6 Q^{-1}$, it follows from (5.10) and (5.11) that

$$1/2|P'(x)| < |P'(\alpha_1)| < 2|P'(x)|. \tag{5.12}$$

Now from (5.12), (5.3) and (5.9) it follows that

$$2^5 Q^{-1} < 1/2|P'(x)| < |P'(\alpha_1)| < 2|P'(x)| < 2\delta_0 Q.$$

Therefore the interval $\sigma_0(P) \cap S_P(\alpha_1)$ is contained in $\sigma(P) \cap S_P(\alpha_1)$, which is the set of all points in $S_P(\alpha_1)$ satisfying

$$|x - \alpha_1| < 6Q^{-3}|P'(\alpha_1)|^{-1}. \tag{5.13}$$

To obtain the measure of $\tilde{\mathcal{L}}_3$ it is necessary to consider five different subcases depending on the value of $|P'(\alpha_1)|$ lying in the interval $(2^5 Q^{-1}, 2\delta_0 Q)$. Throughout the proof let $v = \frac{5}{8}$.

Subcase A: Define the subset $\mathcal{L}_{31}$ of the set $\tilde{\mathcal{L}}_3$, as the set of points $x \in I$, for which there exists at least one polynomial $P \in \mathcal{P}_3'(Q)$, satisfying (5.9) and the inequality

$$Q^v < |P'(\alpha_1)| < 2\delta_0 Q \tag{5.14}$$

where $x \in S_P(\alpha_1)$.

Proposition 5.1. *For sufficiently small δ_0 and sufficiently large Q,*

$$|\mathcal{L}_{31}| < 2^{-4} s|I|.$$

Proof. For a polynomial $P \in \mathcal{P}_3'(Q)$ define the interval

$$\sigma_1(P) := \{x \in S_P(\alpha_1) \cap I : \ |x - \alpha_1| < c_6 Q^{-1}|P'(\alpha_1)|^{-1}, \ c_6 > 1\} \tag{5.15}$$

for a constant c_6 to be chosen later.

Using (5.13) and (5.15), it follows that

$$|\sigma(P)| < 6c_6^{-1} Q^{-2}|\sigma_1(P)|. \tag{5.16}$$

Note that from (5.14) it follows that $|\sigma_1(P)| < 2c_6Q^{-1-v}$, and for $Q > Q_0$ the interval $\sigma_1(P)$ is contained in $|I|$.

Now, write P as a Taylor series on the interval $\sigma_1(P)$ so that:

$$P(x) = P'(\alpha_1)(x-\alpha_1) + 1/2P''(\alpha_1)(x-\alpha_1)^2 + 1/6P'''(\alpha_1)(x-\alpha_1)^3.$$

Using (5.14) and (5.15) each term is estimated to obtain

$$|P(x)| < 2c_6Q^{-1}, \tag{5.17}$$

for $x \in \sigma_1(P)$, and $Q > Q_0$.

The polynomials in $P_3'(Q)$ are now partitioned into sets which have the same coefficients of x^2 and x^3. For integers $a_i, i = 2,3$ let b_1 be the pairs (a_3, a_2) and let $P_3(Q, b_1)$ be the set of polynomials in $P_3'(Q)$ for which the coefficient of x^i is a_i for $i = 2,3$. The intervals $\sigma_1(P)$ with $P \in P_3(Q, b_1)$ are now divided into two classes using Sprindžuk's method of essential and inessential domains. [85]. First the essential intervals $\sigma_1(P)$ are investigated. By definition

$$\sum_{\substack{P\in\mathcal{P}_3(Q,b_1)\\ \sigma_1(P)\ essential}} |\sigma_1(P)| \le 2|I|.$$

Using this, (5.16) and the fact that the number of different vectors b_1 does not exceed $(2Q+1)^2$, it follows that

$$\sum_{b_1}\sum_{\substack{P\in\mathcal{P}_3(Q,b_1)\\ \sigma_1(P)\ essential}} |\sigma(P)| < 2^7c_6^{-1}Q^2Q^{-2}|I| = 2^7c_6^{-1}|I|. \tag{5.18}$$

Next, consider the inessential intervals $\sigma_1(P)$. For polynomials P and $\bar{P}$ such that $P \neq \bar{P}$, and $P, \bar{P} \in \mathcal{P}_3(Q, b_1)$, the measure of the intersection $\sigma_1(P) \cap \sigma_1(\bar{P}) = \sigma_1(P, \bar{P})$, exceeds $\frac{|\sigma_1(P)|}{2}$. Hence, the inequalities (5.17) hold. As the coefficients a_3 and a_2 of the polynomials P and $\bar{P}$ are the same,

$R(x) = P(x) - \bar{P}(x)$ is linear and satisfies

$$|R(x)| = |ax - b| < 4c_6Q^{-1}, \quad \max(|a|, |b|) < 2Q, \;\; x \in \sigma_1(P, \bar{P}). \qquad (5.19)$$

Assume that $a > 0$. The values of a and $|b|$ are now estimated more precisely than in (5.19). From the Mean Value Theorem

$$P'(x) = P'(\alpha_1) + P''(\theta_2)(x - \alpha_1), \quad \theta_2 \in (\alpha_1, x),$$

and using (5.14) and $|P''(\theta_2)(x-\alpha_1)| < 5c_6Q^{-v}$, it follows that $|P'(x)| < 4\delta_0 Q$ for $Q > Q_0$. Therefore $|a| = |P'(x) - \bar{P}'(x)| < 8\delta_0 Q$, and using (5.19) it follows that $|b| < 16\delta_0 Q$. Thus, (5.19) can be rewritten as

$$|R(x)| = |ax - b| < 4c_6Q^{-1}, \quad \max(a, |b|) < 2^4\delta_0 Q, \;\; x \in \sigma_1(P, \bar{P}). \quad (5.20)$$

Now the measure of $x \in I$, for which (5.20) holds is estimated. For fixed a and b the first inequality in (5.20) holds for points $x \in I$ satisfying

$$|x - b/a| < 2^4c_6a^{-1}Q^{-1}. \qquad (5.21)$$

Denote this interval by $J(R)$, so that

$$|J(R)| = 2^5c_6a^{-1}Q^{-1}. \qquad (5.22)$$

We now wish to estimate $\sum |J(R)|$ where the sum is over a and b, such that $\frac{b}{a} \in I$ and $a, |b| < 2^4\delta_0 Q$. For fixed a denote by $M_I(a)$ the number of points b such that these conditions hold. Then,

$$M_I(a) \le \begin{cases} a|I| + 1 \le 2a|I|, & \text{if} \quad a \ge |I|^{-1}, \\ \gamma, & \text{if} \quad a < |I|^{-1}, \end{cases} \qquad (5.23)$$

where γ equals 1 or 0. First, let $a \ge |I|^{-1}$, then from (5.22), (5.23) it follows that

$$\sum_a \sum_{b:b/a\in I} |J(R)| < \sum_a 2^5c_6a^{-1}Q^{-1}2|I|a \le 2^{10}c_6\delta_0|I|. \qquad (5.24)$$

Next, consider $a < |I|^{-1}$ and use the second bound in (5.23) to find a constant $2^{-s}\delta_0 > c_7 \geq 2^4 c_6$, for which the intervals

$$J_1(R_1) := \{x \in I : \ |x - b_1/a_1| < c_7 a_1^{-1} Q^{-1}\},$$
$$J_1(R_2) := \{x \in I : \ |x - b_2/a_2| < c_7 a_2^{-1} Q^{-1}\},$$

where $J(R_i) \subseteq J_1(R_i), i = 1, 2$ do not intersect for $b_1/a_1 \neq b_2/a_2$. To see this is possible, suppose $J_1(R_1)$ and $J_1(R_2)$ intersect at x, then,

$$\frac{1}{a_1 a_2} \leq \frac{|b_1 a_2 - b_2 a_1|}{a_1 a_2} = |b_1/a_1 - b_2/a_2| \leq |x - b_1/a_1| + |x - b_2/a_2| \leq c_7 Q^{-1}(1/a_1 + 1/a_2).$$

Assuming WLOG that $a_2 > a_1$, this gives

$$1 \leq c_7 Q^{-1}(a_1 + a_2) < 2c_7 a_2 Q^{-1} < 2^5 c_7 \delta_0 \tag{5.25}$$

which is a contradiction. Thus,

$$\sum_R |J_1(R)| = \sum_{a \leq 4\delta_0 Q} 2c_7^{-1} a^{-1} Q^{-1} \gamma \leq |I|.$$

From this it follows that

$$\sum_{a \leq 4\delta_0 Q} \gamma a^{-1} \leq 2^{-1} c_7^{-1} Q |I|. \tag{5.26}$$

For fixed a and b the measure of the set $x \in I$, satisfying (5.21), does not exceed $2^5 c_6 a^{-1} Q^{-1}$. Hence, summing over b, from the second inequality in (5.23) it follows that $\sum_{b: b/a \in I} 2^5 c + 6a^{-1} Q{-}1 \leq 2^5 c_6 a^{-1} Q^{-1} \gamma$. Using (5.26), it follows that

$$\sum_{1 \leq a \leq 4\delta_0 Q} \sum_{b: b/a \in I} 2^5 c_6 a^{-1} Q^{-1} \leq 2^5 c_6 Q^{-1} \sum_{1 \leq a \leq 4\delta_0 Q} \gamma a^{-1} \leq 2^4 c_6 c_7^{-1} |I| \leq 2^{-6} s |I| \tag{5.27}$$

if $c_7 \geq 2^{10} c_6 s^{-1}$. Therefore

$$|\mathcal{L}_{31}| < (2^7 c_6^{-1} + 2^{10} c_6 \delta_0 + 2^{-6} s)|I|. \tag{5.28}$$

Choosing $c_6 = 2^{12}s^{-1}$, $\delta_0 = 2^{-28}s^2$ and $c_7 = 2^{22}s^{-2}$ completes the proof. $\square$

Subcase B: Define the subset $\mathcal{L}_{32}$ of the set $\tilde{\mathcal{L}}_3$, as the set of points $x \in I$, for which there exists at least one polynomial $P \in \mathcal{P}'_3(Q)$, satisfying (5.9) and the inequality

$$2^8 s^{-1/2} < |P'(\alpha_1)| \leq Q^{5/8}$$

with $x \in S_P(\alpha_1)$.

Proposition 5.2. *For sufficiently large Q*

$$|\mathcal{L}_{32}| < 2^{-4}s|I|.$$

Proof. The proof of proposition 5.2 closely follows that of proposition 5.1, so some details will be omitted. As before, for $P \in \mathcal{P}'_3(Q)$ and some positive constant $c_9 > 1$ (which will be specified later) consider the interval $\sigma(P)$ and define the interval

$$\sigma_2(P) := \{x \in S_P(\alpha_1) \cap I : \ |x - \alpha_1| < c_9 Q^{-1}|P'(\alpha_1)|^{-1}\}.$$

It is clear that

$$|\sigma(P)| < 6c_9^{-1}Q^{-2}|\sigma_2(P)|. \tag{5.29}$$

The definition of $|\mathcal{L}_{32}|$ implies that $|\sigma_2(P)| < |I|$. Expand P and P' as Taylor series on $\sigma_2(P)$, to obtain

$$|P(x)| < 2c_9 Q^{-1}, \tag{5.30}$$

and

$$|P'(x)| < 2|P'(\alpha_1)|. \tag{5.31}$$

Now consider the essential and inessential domains $\sigma_2(P)$, with $P \in \mathcal{P}_3(Q, b_1)$ where b_1 is as in proposition 5.1.

Following the proof of proposition 5.1 we obtain

$$\sum_{b_1} \sum_{\substack{P \in \mathcal{P}_3(Q,b_1) \\ \sigma_2(P)\ essential}} |\sigma(P)| < 2^7 c_9^{-1} |I|. \tag{5.32}$$

Now the inessential domains are considered. Assume $\sigma_2(P), P \in P_3(Q, b_1)$ is inessential. Thus there exists $\bar{P} \in \mathcal{P}_3(Q, b_1)$ with $P \neq \bar{P}$ such that

$$|\sigma_2(P, \bar{P})| = |\sigma_2(P) \cap \sigma_2(\bar{P})| \geq \frac{1}{2} |\sigma_2(P)|.$$

Let $T(x) = P(x) - \bar{P}(x) = gx - d$, then from (5.30) and (5.31)

$$|gx - d| < 4c_9 Q^{-1}. \tag{5.33}$$

The inequality (5.33) holds on an interval $J_2(T)$ with centre d/g and length $8c_9 g^{-1} Q^{-1}$. Fix g and denote by $M'_I(g)$ the number of points d/g, belonging to I. As in (5.23),

$$M'_I(g) \leq \begin{cases} 2g|I|, & \text{if } \ g \geq |I|^{-1}, \\ \gamma, & \text{if } \ g < |I|^{-1}, \end{cases}$$

where γ equals 1 or 0.

Again, first consider $g \geq |I|^{-1}$. Then, for $Q > Q_0$,

$$\sum_{1 \leq g \leq 2Q^{5/8}} \sum_{d: d/g \in I} |J_2(T)| \leq 2^5 c_9 Q^{-3/8} |I| \leq 2^{-6} s |I|. \tag{5.34}$$

Now consider $g < |I|^{-1}$. To show the sets $J_2(T)$ do not intersect, larger super-sets $J_3(T)$ defined below are shown not to intersect. Assume that for $c_{10} > 4c_9$ the intervals

$$J_3(T_1) := \{x \in I : \ |x - d_1/g_1| < c_{10} g_1^{-1} Q^{-1}\},$$
$$J_3(T_2) := \{x \in I : \ |x - d_2/g_2| < c_{10} g_2^{-1} Q^{-1}\},$$

intersect for $d_1/g_1 \neq d_2/g_2$. Then, as in (5.25):

$$1 \leq 2c_{10}(g_1+g_2)Q^{-1} \leq 8c_{10}Q^{-3/8}, \tag{5.35}$$

which is a contradiction for $Q > Q_0(c_{10})$. As in (5.26), it therefore follows that

$$\sum_{1\leq g\leq 2Q^{5/8}} \gamma g^{-1} \leq 2^{-1}c_{10}^{-1}Q|I|. \tag{5.36}$$

For fixed g the measure of the set $x \in I$, satisfying (5.33), does not exceed $8c_9g^{-1}Q^{-1}$. From (5.36), it follows that

$$\sum_{1\leq g\leq 2Q^{5/8}} \sum_{d,d/g\in I} 8c_9g^{-1}Q^{-1} \leq 8c_9Q^{-1} \sum_{1\leq g\leq 2Q^{5/8}} \gamma g^{-1} \leq 4c_9c_{10}^{-1}|I| \leq 2^{-6}s|I|$$

for $c_{10} \geq 2^8c_9s^{-1}$. From this, (5.32), and (5.34) it follows that

$$|\mathcal{L}_{32}| \leq (2^7c_9^{-1} + 2^{-6}s + 2^{-6}s)|I|.$$

Hence, choosing $c_9 = 2^{12}s^{-1}$ and $c_{10} = 2^{20}s^{-2}$, this completes the proof of proposition 5.2. □

Subcase C. Denote by $\mathcal{L}_{33} \subset \tilde{\mathcal{L}}_3$ the set of $x \in I$, for which there exists a polynomial $P \in \mathcal{P}_3'(Q)$, satisfying (5.9) and

$$2^{-3} < |P'(\alpha_1)| \leq 2^8s^{-1/2}$$

with $x \in S_P(\alpha_1)$.

Proposition 5.3. *For sufficiently large Q,*

$$|\mathcal{L}_{33}| < 2^{-4}s|I|.$$

Proof. For $P \in \mathcal{P}_3'(Q)$ and some $c_{11} > 1$ chosen later, define the interval

$$\sigma_3(P) := \{x \in S_P(\alpha_1) \cap I : \ |x-\alpha_1| < c_{11}Q^{-1}|P'(\alpha_1)|^{-1}\}.$$

Develop P and P' as a Taylor series on $\sigma_3(P)$, to obtain

$$|P(x)| < 2^9 c_{11}^2 Q^{-1}, \quad |P'(x)| < \max(2^9 s^{-1/2}, 2^6 c_{11})$$

for $Q > Q_0$. Consider again the essential and inessential domains $\sigma_3(P)$, $P \in \mathcal{P}_3(Q, b_1)$ defined as in proposition 5.1. As the approach is similar to previous propositions, the calculations are omitted. In the case of the essential domains the measure is at most $2^7 c_{11}^{-1}|I|$, and choosing $c_{11} > 2^{12}s^{-1}$ gives the measure of the points lying in essential domains as $2^{-5}s|I|$.

For the inessential domains, it is necessary to estimate the measure of $x \in I$, satisfying

$$|ax - b| < 2^{10} c_{11}^2 Q^{-1}, \ \max(a, |b|) < 2\max(2^9 s^{-1/2}, 2^6 c_{11}). \tag{5.37}$$

Direct calculations show that (5.37) holds on a set of $x \in I$, with measure at most $c_{13}Q^{-1}$ for some constant $c_{13} > 0$. Choosing $c_5 \geq 2^5 c_{13} s^{-1}$ in Theorem 5.2, the measure of the inessential domains is at most $2^{-5}s|I|$. So $|\mathcal{L}_{34}| \leq 2^{-4}s|I|$ as required. □

Subcase D: For some constant $c_{14} > 0$ chosen later, denote by $\mathcal{L}_{34} \subset \tilde{\mathcal{L}}_3$ the set of $x \in I$, for which there exists a polynomial $P \in \mathcal{P}_3'(Q)$, satisfying (5.9) and

$$c_{14}Q^{-1/2} < |P'(\alpha_1)| \leq 2^{-3}$$

where $x \in S_P(\alpha_1)$.

Proposition 5.4. *For sufficiently large Q*

$$|\mathcal{L}_{34}| < 2^{-4}s|I|.$$

Proof. For $P \in \mathcal{P}_3'(Q)$ and some $c_{15} > 1$ define the interval

$$\sigma_4(P) := \{x \in S_P(\alpha_1) \cap I : \ |x - \alpha_1| < c_{15}Q^{-2}|P'(\alpha_1)|^{-1}\}.$$

Clearly

$$|\sigma(P)| < 6c_{15}^{-1}Q^{-1}|\sigma_4(P)|. \tag{5.38}$$

Fix a_3. Let the subclass of polynomials $P \in \mathcal{P}_3'(Q)$ with the same leading coefficients be denoted by $\mathcal{P}_3(Q, a_3)$. Consider again the essential and inessential domains $\sigma_4(P)$, $P \in \mathcal{P}_3(Q, a_3)$.

From the definition of essential domains, it follows that

$$\sum_{\substack{P\in\mathcal{P}_3(Q,a_3)\\ \sigma_4(P)\ essential}} |\sigma_4(P)| \leq 2|I|.$$

Since the number of a_3 does not exceed $(2Q+1)$, summing over all a_3 and using (5.38), gives

$$\sum_{-Q\leq a_3\leq Q} \sum_{\substack{P\in\mathcal{P}_3(Q,a_3)\\ \sigma_4(P)\ essential}} |\sigma(P)| < 2^7 c_{15}^{-1}|I| = 2^{-5}s|I| \tag{5.39}$$

for $c_{15} = 2^{12}s^{-1}$.

Now consider the inessential domains. From the Taylor series expansions of $P_i(x)$ and $P_i'(x)$ on $\sigma_4(P_{i_1}, P_{i_2}) = \sigma_4(P_{i_1}) \cap \sigma_4(P_{i_2})$, $P_{i_1}, P_{i_2} \in \mathcal{P}_3(Q, a_3)$, $P_{i_1} \neq P_{i_2}$, the upper bounds of $|P_i(x)|$ and $|P_i'(x)|$, are

$$|P_i(x)| < 2c_{15}Q^{-2}, \tag{5.40}$$

and

$$|P_i'(x)| < 2|P'(\alpha_1)| \tag{5.41}$$

for $c_{14} \geq 2^2 c_{15}^{1/2}$. Since the leading coefficients of P_{i_1} and P_{i_2} are equal, then the polynomial

$$S(x) = P_{i_1}(x) - P_{i_2}(x) = f_2x^2 + f_1x + f_0$$

and by (5.40),

$$|S(x)| < 4c_{15}Q^{-2},\ \ |S'(x)| < 4|P'(\alpha_1)|,\ \ \ |f_i| \leq 2Q,\ \ 0 \leq i \leq 2.$$

Let β_1, $\beta_2 \in \mathbb{C}$ be the roots of S. Since the discrimimant $D(S)$ of S satisfies

$$|D(S)| = |S'(\beta_1)|^2 < 16|P'(\alpha_1)|^2 \leq 2^{-2},$$

this implies that $D(S) = 0$ and that S has a repeated root. Hence S has the form

$$S(x) = S_0^2(x) = (s_1 x - s_0)^2, \ \ |s_1| < 2Q^{1/2}.$$

Thus, we need to find the measure of $x \in I$, satisfying

$$|s_1 x - s_0| < 2c_{15}^{1/2} Q^{-1}, \ \ |s_1| < 2Q^{1/2}. \tag{5.42}$$

Hence, $|x - s_0/s_1| < 2c_{15}^{1/2}|s_1|^{-1}Q^{-1}$, which defines an interval $J_4(S_0)$ with centre at s_0/s_1 and length $4c_{15}^{1/2}|s_1|^{-1}Q^{-1}$. Fix s_1 and denote by $M_I''(s_1)$ the number of points s_0/s_1, belonging to I. As in (5.23), the following bounds

$$M_I''(s_1) \leq \begin{cases} 2s_1|I|, & \text{if } \ s_1 \geq |I|^{-1}, \\ \gamma, & \text{if } \ s_1 < |I|^{-1}, \end{cases}$$

are obtained, where γ equals 1 or 0.

Consider $s_1 \geq |I|^{-1}$. For $Q > Q_0$,

$$\sum_{1 \leq s_1 \leq 2Q^{1/2}} \ \sum_{s_0 : s_0/s_1 \in I} |J_4(S_0)| < 2^3 c_{15}^{1/2} Q^{-1/2}|I| \leq 2^{-6} s|I|. \tag{5.43}$$

Next let $s_1 < |I|^{-1}$. If for $c_{16} > 2c_{15}^{1/2}$ the intervals

$$J_5(S_1) := \{x \in I : \ |x - s_{0,1}/s_{1,1}| < c_{16} s_{1,1}^{-1} Q^{-1}\},$$
$$J_5(S_2) := \{x \in I : \ |x - s_{0,2}/s_{1,2}| < c_{16} s_{1,2}^{-1} Q^{-1}\},$$

intersect for $s_{0,1}/s_{1,1} \neq s_{0,2}/s_{1,2}$, then as in (5.25):

$$1 \leq 2c_{16}(s_{1,1} + s_{1,2})Q^{-1} < 2^3 c_{16} Q^{-1/2}, \tag{5.44}$$

which is a contradiction for $Q > Q_0(c_{16})$. Thus, using the same arguments as in proposition 5.1, as in (5.26)

$$\sum_{1 \le s_1 < 2Q^{1/2}} \gamma s_1^{-1} \le 2^{-1} c_{16}^{-1} Q|I|. \tag{5.45}$$

Since for fixed s_1 and s_0 the measure of the set $x \in I$, satisfying (5.42), is at most $2c_{15}^{1/2}|s_1|^{-1}Q^{-1}$, using (5.45), we get

$$\sum_{i=1,2} |J_5(s_i)| = \sum_{\substack{1 \le s_1 < 2Q^{1/2} \\ S_1 \subset |I|^{-1}}} \sum_{s_0 : s_0/s_1 \in I} 4c_{15}^{1/2}|s_1|^{-1}Q^{-1} \le 4c_{15}^{1/2}Q^{-1} \tag{5.46}$$

and

$$\sum_{1 \le s_1 < 2Q^{1/2}} \gamma |s_1|^{-1} \le 2c_{15}^{1/2} c_{16}^{-1}|I| \le 2^{-6}s|I| \tag{5.47}$$

for $c_{16} \ge 2^7 c_{15}^{1/2} s^{-1}$. Choose $c_{14} = 2^8 s^{-1/2}$. Summing the estimates (5.39), (5.43), (5.46) and (5.47), for the measures in the essential and inessential cases, it follows that $|\mathcal{L}_{34}| < 2^{-4}s|I|$. This concludes the proof of proposition 5.4. □

Subcase E: Denote by $\mathcal{L}_{35} \subset \tilde{\mathcal{L}}_3$ the set of $x \in I$, for which there exists $P \in \mathcal{P}'_3(Q)$, satisfying (5.9) and the following condition

$$2^5 Q^{-1} < |P'(\alpha_1)| \le 2^8 s^{-1/2} Q^{-1/2} \tag{5.48}$$

with $x \in S_P(\alpha_1)$.

Proposition 5.5. *For sufficiently large Q,*

$$|\mathcal{L}_{35}| < 2^{-2}s|I|.$$

Proof. Divide the interval I into smaller intervals J_i, where $|J_i| = Q^{-u}$ and $u > 1$. We say the polynomial P belongs to the interval J_i if there exists $x \in J_i$ such that (5.3) and (5.48) hold.

There is at most one irreducible polynomial belonging to an interval J_i. To see this, suppose the opposite. Assume there exists a point $x \in J_i$, for which (5.3) and (5.48) hold for two polynomials P_1 and P_2 from $\mathcal{P}_3'(Q)$. By the Mean Value Theorem, it follows that

$$P_1(x) = P_1'(\alpha_1)(x - \alpha_1) + P_1''(\theta_3)(x - \alpha_1)^2, \quad \theta_3 \in (\alpha_1, x),\ x \in J_i.$$

Using the estimate for $|P'(\alpha)|$ from (5.48), and the trivial bound $|P''(\theta_3)| \ll Q$, for $x \in J_i$

$$|P_1(x)| < 2^9 s^{-1/2} Q^{-1/2-u},\ x \in J_i,$$

for $u > 3/2$. Obviously the same estimate holds for P_2 in J_i.

We use the following Lemmas proved in [20].

Lemma 5.1. *Let $\delta > 0$ and $Q > Q_0(\delta)$. Further, let P_1 and P_2 be two integer polynomials of degree at most n with no common roots, and $\max(H(P_1), H(P_2)) \leq Q$. Let $J \subset \mathbb{R}$ be an interval of length $|J| = Q^{-\eta}$, $\eta > 0$. If there exists $\tau > 0$, such that for all $x \in J$*

$$|P_j(x)| < Q^{-\tau},$$

for $j = 1, 2$, then

$$\tau + 1 + 2\max(\tau + 1 - \eta, 0) < 2n + \delta. \tag{5.49}$$

Applying Lemma 5.1 with $\tau = 1/2 + u - \epsilon$, $\epsilon > 0$, and $\eta = u$, leads to a contradiction in (5.49) if $u > 3/2 + \delta + 3\epsilon$. Choose u satisfying $3/2 + \delta + 3\epsilon < u < 2$ and it follows that there is at most one irreducible polynomial on any J_i. As there is at most one polynomial $P \in \mathcal{P}_3'(Q)$ that belongs to any J_i then by Lemma 1.2, the measure of those x, satisfying (5.3) and (5.48), does not exceed

$$2^{-3} Q^{-2+u} |I| < 2^{-4} s |I|$$

for $u < 2$ and $Q > Q_0$.

If $P \in \mathcal{P}'_3(Q)$ is a reducible polynomial belonging to J_i then $P(x) = t_1(x)t_2(x)$, where t_1 is a first degree polynomial and t_2 is either a second degree polynomial or the product of two linear polynomials. Let $t_1(x) = ax + b$ and $t_2(x) = b_2x^2 + b_1x + b_0$. Assume that $a > 0$. By Gelfond's Lemma (1.1),

$$aH(t_2) \leq H(t_1)H(t_2) \leq 2^3H(P) \leq 2^3Q. \tag{5.50}$$

Denote by $\mathcal{L}_{351}$ the set of $x \in I$, for which the system

$$|t_1(x)| < Q^{-1}, \quad a < \delta_0 Q \tag{5.51}$$

is satisfied by some polynomial t_1. The system (5.51) is similar to (5.20). Hence, using the same arguments, it can be shown that $|\mathcal{L}_{351}| < 2^{-5}s|I|$. If (5.51) does not hold, there are three possibilities:

$$|t_1(x)| < Q^{-1}, \quad \delta_0 Q \leq a < 2^3Q, \tag{5.52}$$

$$|t_1(x)| \geq Q^{-1}, \quad a < \delta_0 Q, \tag{5.53}$$

or

$$|t_1(x)| \geq Q^{-1}, \quad \delta_0 Q \leq a < 2^3Q. \tag{5.54}$$

For (5.52) there are two further possibilities; namely,

$$\delta_0 Q^{-1} < |t_1(x)| < Q^{-1}, \quad \delta_0 Q \leq a < 2^3Q, \tag{5.55}$$

and

$$|t_1(x)| \leq \delta_0 Q^{-1}, \quad \delta_0 Q \leq a < 2^3Q. \tag{5.56}$$

Each of these will be considered in turn. Denote by $\mathcal{L}_{352}$ and by $\mathcal{L}_{353}$ the sets of $x \in I$, for which (5.55) and (5.56) are satisfied for polynomials t_1 respectively. The system (5.56) is similar to (5.20), and it is not difficult to show that $|\mathcal{L}_{353}| < 2^{-5}s|I|$.

Turning to $\mathcal{L}_{352}$ and using (5.50), (5.55) and $|P(x)| < Q^{-3}$ it follows that

$$|t_2(x)| \le \delta_0^{-1} Q^{-2}, \quad H(t_2) < 2^3 \delta_0^{-1}. \tag{5.57}$$

The number of polynomials that satisfy the second inequality in (5.57) does not exceed a constant depending on δ_0, say $c(\delta_0)$, therefore we conclude that $|\mathcal{L}_{352}| < 2^{-5} s|I|$.

Now we consider (5.53). Using (5.53) and (5.50)

$$|t_2(x)| < Q^{-2}, \quad H(t_2) < 2^3 Q. \tag{5.58}$$

First, t_2' is estimated from above on J_i. Using the equations

$$P'(x) = t_1'(x) t_2(x) + t_1(x) t_2'(x), \tag{5.59}$$

and

$$P'(x) = P'(\alpha_1) + P''(\alpha_1)(x - \alpha_1) + P'''(\alpha_1)(x - \alpha_1)^2/2,$$

the estimates (5.53), (5.58) and

$$|P'(x)| < 2^9 s^{-1/2} Q^{-1/2}, \;\; Q^{-1} \le |t_1(x)| \ll Q,$$

gives a contradiction for $|t_2'(x)| > Q^{5/8}$ and sufficiently large Q. Thus, $|t_2'(x)| \le Q^{5/8}$, $x \in J_i$. Then from (5.58),

$$|t_2(x)| < Q^{-2}, \quad |t_2'(x)| \le Q^{5/8} \tag{5.60}$$

hold. Denote by $\mathcal{L}_{354}$ the set of $x \in I$, for which (5.60) is satisfied for a polynomial t_2. The measure of $\mathcal{L}_{354}$ is estimated in a manner similar to proposition 5.2, giving $|\mathcal{L}_{354}| < 2^{-4} s|I|$.

Finally, we consider (5.54). Using (5.50), (5.54) and $|P(x)| < Q^{-3}$, it follows that

$$|t_2(x)| < Q^{-2}, \quad H(t_2) < 2^3 \delta_0^{-1}. \tag{5.61}$$

Denote by $\mathcal{L}_{355}$ the set of $x \in I$, for which (5.61) is satisfied for some t_2. The number of polynomials which satisfy the second inequality in (5.61) is bounded by a constant $c_2(\delta_0)$, therefore we conclude that $|\mathcal{L}_{355}| < 2^{-5}s|I|$.

Thus, $|\mathcal{L}_{35}| < 2^{-4}s|I| + \sum_{i=1}^{5} |\mathcal{L}_{35i}| < 2^{-2}s|I|$, which completes the proof of the proposition. □

Case 2: Define the subset $\check{\mathcal{L}}_3$ of the set $\bar{\mathcal{L}}_3$, as the set of points $x \in I$, for which there exists at least one polynomial $P \in \mathcal{P}_3'(Q)$ such that

$$|P(x)| < Q^{-3}, \quad |P'(x)| \leq 2^6 Q^{-1}. \tag{5.62}$$

Define by $\sigma_*(P)$ the set of solutions to (5.62) for a fixed polynomial $P \in \mathcal{P}_3'(Q)$. Let $x \in \sigma_*(P) \cap S_P(\alpha_1)$. First, it is shown that the value of the derivative of P at α_1, $P(\alpha_1) = 0$, satisfies

$$|P'(\alpha_1)| \leq 2^8 Q^{-1}. \tag{5.63}$$

To show this, assume the opposite holds. Develop P' as a Taylor series in the neighborhood of α_1, to obtain

$$P'(x) = P'(\alpha_1) + P''(\alpha_1)(x - \alpha_1) + 1/2P'''(\alpha_1)(x - \alpha_1)^2,$$

where $|x - \alpha_1| < 2^{-6}Q^{-2}$ by Lemma 1.2. Since $|P^k(\alpha_1)| \ll Q$ for $x, \alpha_1 \in [-\frac{1}{2}, \frac{1}{2}]$, it follows that

$$\max(|P''(\alpha_1)(x - \alpha_1)|, |1/2P'''(\alpha_1)(x - \alpha_1)^2|) < 2^{-3}Q^{-1},$$

and hence $|P'(x)| > 2^7 Q^{-1}$, which contradicts the condition that $|P'(x)| \leq 2^6 Q^{-1}$.

To estimate the measure of $\check{\mathcal{L}}_3$ two cases, depending on the value of $|P'(\alpha_1)|$, need to be considered. For some constant $c_{17} > 0$ denote by $\mathcal{L}_{36} \subset$

$\check{\mathcal{L}}_3$ the set of $x \in I$, for which there exists a polynomial $P \in \mathcal{P}'_3(Q)$, satisfying (5.62) and

$$|P'(\alpha_1)| < c_{17}Q^{-1}$$

with $x \in S_P(\alpha_1)$.

Proposition 5.6. *For sufficiently large Q*

$$|\mathcal{L}_{36}| < 2^{-2}s|I|.$$

Proof. First note that

$$1 \leq |D(P)| = |a_3^4(\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2| = |P'(\alpha_1)|^2 a_3^2 |\alpha_2 - \alpha_3|^2.$$

Also, $|\alpha_2 - \alpha_3| < 1$ is bounded as $\alpha_i \in [-\frac{1}{2}, \frac{1}{2}]$. Finally, as $|P'(\alpha_1)| < c_{17}Q^{-1}$, this gives

$$1 \leq D(P) \leq c_{17}^2 Q^{-2} Q^2 = c_{17}^2.$$

which does not hold when $c_{17} < 1$ thus, the discriminant of P satisfies $D(P) = 0$, which implies that P has a repeated root. Following the same approach as in proposition 5.4, it follows that $|\mathcal{L}_{36}| < 2^{-2}s|I|$. □

The second sub-case of case 2 is now considered. Denote by $\mathcal{L}_{37} \subset \check{\mathcal{L}}_3$ the set of $x \in I$, for which there exists some polynomial $P \in \mathcal{P}'_3(Q)$, satisfying (5.62) and

$$c_{17}Q^{-1} \leq |P'(\alpha_1)| \leq 2^8 Q^{-1} \tag{5.64}$$

with $x \in S_P(\alpha_1)$.

Proposition 5.7. *For Q sufficiently large,*

$$|\mathcal{L}_{37}| < 2^{-2}s|I|.$$

Proof. Divide the interval I into smaller intervals J'_i with $|J'_i| = Q^{-u'}$, where $u' > \frac{3}{2}$. The assumption that at least two irreducible polynomials belong to the interval J'_i will lead to a contradiction. To show this, suppose that P_1 and P_2 both belong to J'_i. By the Mean Value Theorem

$$P_1(x) = P'_1(\alpha_1)(x - \alpha_1) + P''_1(\theta_4)(x - \alpha_1)^2, \quad \theta_4 \in (\alpha_1, x), \;\; x \in J'_i.$$

Estimating each term gives

$$|P_1(x)| < 2^5 Q^{1-2u'}, \;\; x \in J'_i,$$

for $u' < 2$. Obviously the same estimate holds for P_2 on J'_i. Applying Lemma 5.1 with $\tau = -1+2u'-\epsilon'$, $\epsilon' > 0$, and $\eta = u'$, leads to a contradiction in (5.49) for $u' > 3/2 + \delta/4 + 3\epsilon'/4$. Thus, choose u', satisfying $3/2 + \delta/4 + 3\epsilon'/4 < u' < 2$.

Hence there is at most one polynomial $P \in \mathcal{P}'_3(Q)$ belonging to each J'_i. Therefore, by Lemma 1.2 the measure of those x, satisfying the first inequality of (5.3) and (5.64), is at most

$$4c_{17}^{-1} Q^{-2+u'} |I| < 2^{-4} s |I|$$

for $u' < 2$ and $Q > Q_0$.

When the polynomials are reducible, the proof exactly follows proposition 5.5. $\square$

Adding the measures calculated in propositions 5.1 to 5.7, it follows that the measure of $\bar{\mathcal{L}}_3$ satisfies (5.4).

Chapter 6

Conclusions and further questions

In this document problems have been investigated that fall into two distinct areas. First, in the p-adic norm, the Hausdorff dimension of the set of points that are well approximable by integer polynomials was investigated. The second area investigated is the nature of the roots of integer polynomials under the Archimedian and p-adic norms.

However, there are still many important problems that are not solved, which are interesting from the standpoint of both pure mathematics and applications. It appears that the approaches used in this thesis could usefully be used to investigate these problems. Some of the questions are listed below.

1. To study the distribution of algebraic numbers, metric Theorems on approximation in small intervals are needed. Presently, there are results for linear and quadratic polynomials. In this thesis third degree polynomials are studied, and results that improve on those proved by Bugeaud [35] were obtained. The obvious question is can the same be shown for polynomials of

any degree greater than three.

2. Best possible, or close to best possible, results have been obtained in Chapter 4 in the real case only for irreducible polynomials. The Theorem in the case of reducible polynomials is much weaker. This could present an opportunity to research methods suitable for use with reducible polynomials.

3. The problems investigated in chapter 4 were restricted to monotonic error functions. Can a Khintchine type Theorem for simultaneous Diophantine approximation in different metrics with non-monotonic error function be proved.

4. The work has been entirely theoretical. There has been no computational or computer based investigation of the problems considered. Additional computational investigation, which together with the theoretical results could help provide ideas as to what the true situation is. For example, Bugeaud, Mignotte and Schönhage, obtained results on the distribution of algebraic numbers using computational algorithms. To find the distribution of algebraic numbers with increasing height it is necessary to calculate numerically not only polynomials but also their derivatives. This has not yet taken place, and there is further opportunity here for further computational investigations which could help in the development of the theory.

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