Constructing Hecke-type Structures, their Representations and Applications

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B.Sc.

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Ollscoil na hÉireann Má Nuad

Thesis presented for the degree of

Doctor of Philosophy

to the

National University of Ireland Maynooth

Department of Mathematical Physics

February 2013

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To my parents and grandparents

“ petit à petit l’oiseau fait son nid ”

Declaration

This thesis has not been submitted in whole, or in part, to this or any other University for any other degree and is, except where otherwise stated, the original work of the author.

Signed: ____________________________

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Abstract

This thesis is primarily concerned with the construction of a large Hecke-type structure called the double affine Q-dependent braid group. The significance of this structure is that it is located at the top level of the hierarchy of all other structures that are known to be related to the braid group. In particular, as specialisations we obtain the Hecke algebra, in addition to the affine Hecke algebra, even the double affine Hecke algebra and also the elliptic braid group. To render the algebraic description of this group more accessible, we present an intuitive graphical representation that we have specifically developed to fully capture all of its structure. Contained within this representation are representations of all of the afore mentioned algebras which all contain the braid group as primary element. We also present finite dimensional matrix representations of affine Hecke algebras, emerging from tangles. Using these tangles we also obtain representations of the Temperley-Lieb algebra and the affine braid group. We conclude this thesis with our interpretation of the central role of the Hecke algebra in the development of knot theory. More specifically we explicitly derive the HOMFLY and Jones polynomials.

Contents

Chapter 1

Introduction

The following is a quote I came across while reading a textbook on differential equations.[1]

"We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things."

I find that this quote aptly describes most of the time I have spent researching, culminating with the presentation of this thesis. In my goal of understanding and developing Hecke algebras, much has been achieved, but as a result a whole new set of questions need to be answered.

This introduction, serves as motivation to highlight the physical relevance of the work contained in this thesis.

1.1 Hecke-type Structures

Representation theory is an essential tool in mathematical and physical research: it can reduce difficult problems in abstract algebra to more tractable problems in linear algebra for example. To this end, the theory of special functions, arithmetic and related combinatorics are the usual objectives of representation theory. A particularly potent example illustrating the power of representation theory may be offered in the context of Hecke-type algebras [2]. A significant portion of this thesis is dedicated to our construction and representation of a Hecke-type structure called the double affine Q-dependent braid group which we denote by $\mathcal{D}_N\{Q\}$. As the rest of the thesis follows on from the definition of $\mathcal{D}_N\{Q\}$, in the following figure, Figure 1.1 we specify its position in relation to other well known algebraic structures.

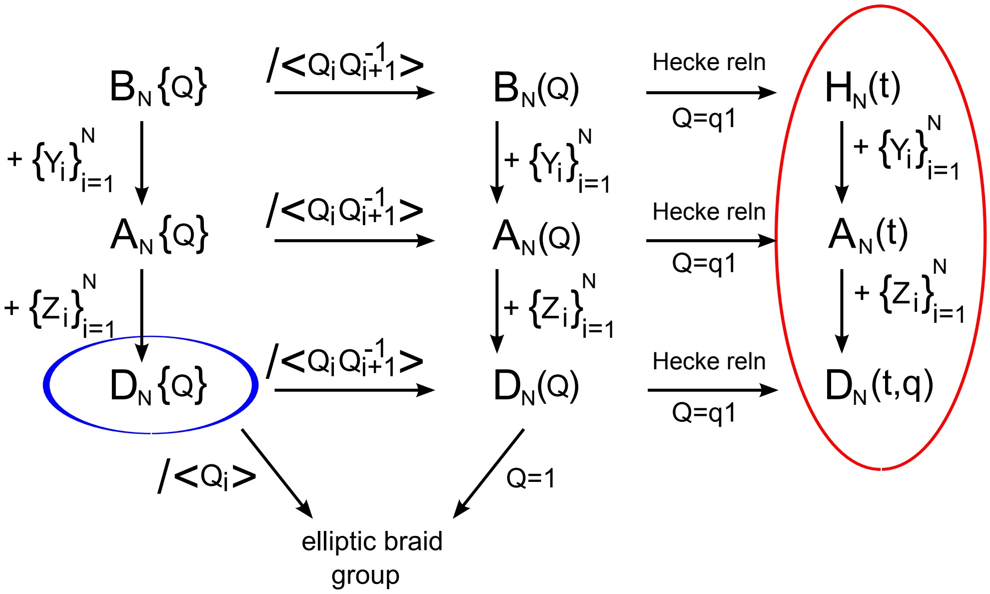


Figure 1.1: Diagram describing the relations of the double affine Q-dependent braid group, $\mathcal{D}_N\{Q\}$ (encircled in blue) with other well known algebraic structures. $\mathcal{D}_N\{Q\}$ depends on a set of N commuting deforming operators $\{Q_i\}$. Taking the quotient group of $\mathcal{D}_N\{Q\}$ by the normal freely generated group $\langle Q_i Q_{i+1}^{-1}\rangle$ yields a deformed double affine braid group $\mathcal{D}_N(Q)$, given in the centre column. In this quotient group there is only one deforming operator Q. Restricting the action of this single operator to a scalar factor, that is $Q = q\mathbb{1}$, and imposing the Hecke relation to the braid group generators, one recovers a presentation of the double affine Hecke algebra $\mathcal{D}_N(t,q)$. Encircled in red is an alternative way of constructing $\mathcal{D}_N(t,q)$, by appending to the Hecke algebra $\mathcal{H}_N(t)$ and affine Hecke algebra $\mathcal{A}_N(t)$ the generators $\{Y_i\}$ and $\{Z_i\}$.

Let us describe all of the relations included in Figure 1.1. Encircled in red on the right hand side are known algebraic structures which readers may be familiar with. To be more specific, located at the bottom of this column is $\mathcal{D}_N(t,q)$, the double affine Hecke algebra (DAHA), first introduced by Cherednik in [3]. In this thesis we explicitly describe its construction starting with its core element, the braid group $\mathcal{B}_N$ which we subsequently extend to the Hecke algebra $\mathcal{H}_N(t)$. By appending to the Hecke algebra a set of N generators $\{Y_i\} = \{Y_1, \ldots, Y_N\}$ we obtain the affine Hecke algebra $\mathcal{A}_N(t)$, which we then further extend to the DAHA by introducing another set of N operators $\{Z_i\} = \{Z_1, \ldots, Z_N\}$. This describes the rightmost column flowing from $\mathcal{H}_N(t)$ to $\mathcal{D}_N(t,q)$.

The rest of Figure 1.1 describes all of the various quotient groups and specialisations of the double affine Q-dependent braid group, $\mathcal{D}_N\{Q\}$, which we have encircled in blue. We now explain how we constructed this group in two straightforward steps.

Firstly we greatly generalised the DAHA, $\mathcal{D}_N(t,q)$, by considering the deformation parameter q, upon which it is characterised, as the resultant action of a single extra generator Q. In doing so we move to the centre column of Figure 1.1, where we obtained $\mathcal{D}_N(Q)$. We draw attention to the fact that $\mathcal{D}_N(Q)$ is a group structure and not an algebra, as in its construction we did not impose the Hecke relation to the braid group generators. To complete the explanation of this column, we show that $\mathcal{D}_N(Q)$ can also be constructed in a somewhat analogous fashion to the DAHA described above. That is, by extending the deformed braid group $\mathcal{B}_N(Q)$ to a deformed affine braid group $\mathcal{A}_N(Q)$ and then finally to $\mathcal{D}_N(Q)$. We highlight that both $\mathcal{B}_N(Q)$ and $\mathcal{A}_N(Q)$ depend on the extra generator Q and hence can be thought of as Q-deformed versions of the original braid group and affine Hecke algebra.

Secondly, to obtain the double affine Q-dependent braid group, $\mathcal{D}_N\{Q\}$ (encircled in blue in Figure 1.1) we then generalised $\mathcal{D}_N(Q)$. We did this by appending to its underlying braid group structure a set of N commuting operators $\{Q_i\} = \{Q_1, \ldots, Q_N\}$. Each operator Q_i acts solely on the i^{th} braid group generator without intertwining them. Extension of the resulting Q-dependent braid group $\mathcal{B}_N\{Q\}$, to the affine Q-dependent braid group $\mathcal{A}_N\{Q\}$, and finally to the double affine Q-dependent braid group $\mathcal{D}_N\{Q\}$ is achieved by the introduction of two sets of N generators, the Y_i and Z_i respectively. This completes the description of the left most column and also of all of the relations included in Figure 1.1. We will go into much greater detail in Chapter 3 where we present the majority of this work and also give the implications of the commutative diagram Figure 1.1.

As it is through representations that abstract concepts are rendered more accessible, in Chapter 3 we present a pictorial representation of $\mathcal{D}_N\{Q\}$ to complement its algebraic description. In the representation that we develop to fully incorporate all of the properties of $\mathcal{D}_N\{Q\}$, the braid group strands are turned into ribbons with the inclusion of 2π twists. Hence one may think of each ribbon as carrying extra information encoded by the operators $\{Q_i\}$. A particularly nice feature of this graphical representation is that it fully describes any double affine Hecke algebra, for all values of the deformation parameter q, upon which it is characterised. Therefore it is not restricted to $q = \mathbb{1}$, as has been the case until now.

1.2 Double Affine Hecke Algebras

In Chapter 2 we study the properties of Hecke algebras. Of particular interest to us, are DAHAs which we construct in several different ways. Each new approach that we outline, offers a unique perspective of its structure. Furthermore they enable the description of isomorphisms and involutions inherent to this algebra.

Algebraically the structure of a double affine Hecke algebra is very rich, and as a result, via representations offers significant physical relevance. Recently, polynomial representations [3] of DAHAs have become more familiar. Most significantly, their close connections to Macdonald polynomials, and therefore Jack polynomials [4], have played a huge part in this.

In this thesis we pay particular attention to Macdonald polynomials, obtained by simultaneously diagonalising the affine Hecke algebra generators. We give explicit calculations of their evaluation up to three dimensions and also define operators that generate all Macdonald polynomials of arbitrary dimension. These two variable polynomials are widely used to describe many existing physical models. For example in [5, 6] it is shown how, when subject to special wheel conditions, they yield interesting q-deformed Laughlin and Haldane-Rezayi wave functions [5, 7]. These are believed to be excellent candidates for describing quantum Hall effect ground states. By adjusting the wheel condition parameters, one may even fix the filling fraction of these wavefunctions.

In [7], Kasatani and Pasquier indicate how other polynomials directly obtained from the DAHA can, in a similar fashion, be used to describe the ground states of $O(n)$ models.

1.3 Topological Quantum Computation

Further motivation for the work contained in this thesis is in the area of quantum computation. Though we do not focus on this subject, it is nonetheless worthwhile to stress the importance of seemingly abstract mathematical results in the development of algorithms. In particular we give the following examples to highlight the central role of the Hecke algebra. In [8], the authors construct a polynomial quantum algorithm for approximating the Jones polynomial. This algorithm is based on the fact that the decomposition of N-strand braids can be assigned an algebra called the Temperley-Lieb algebra. The uniqueness of the Markov trace for this algebra, in addition to its path model representation which induces a unitary representation of the braid group, means that the algorithm can be applied efficiently by a quantum computer. It solves a bounded quantum polynomial (BQP) complete problem.

Much of this thesis deals with Hecke algebras which can be simply mapped to the Temperley-Lieb algebra as we show in the next chapter. The Markov trace due to Oceanu in [9] which is central to the quantum algorithm of [8] was initially defined on the Hecke algebra. We show explicitly how using this particular trace a two variable knot invariant is constructed. This knot invariant is the HOMFLY polynomial, of which the Jones polynomial is a specialisation.

Several attempts have been made to generalise the quantum algorithm described above. Most notably [10] provides polynomial quantum algorithms for additive approximations of the Tutte polynomial. The Tutte polynomial is a two variable polynomial defined for finite graphs with weighted edges and a scalar q. It has important implications in statistical mechanics due to its close connections to the Potts model. Also by constructing a medial graph, which translates a planar graph into a knot, for a particular choice of weights and q there is a simple connection between the Tutte polynomial of the original graph and the Jones polynomial of the knot. The quantum algorithms of [10] are largely based on generalising the Temperley-Lieb algebra to an infinite algebra of pictures [11], where the number of strands is not fixed. The allowance of creation and annihilation operators means that not only graphs originating from braids can be evaluated. In such a way an algorithm that calculates Kauffman brackets of a given medial graph is constructed.

In a somewhat similar way we will design a tangle representation of affine Hecke algebras. The basis of this representation is formed by elementary patterns with non-matching numbers of in and out going strands. Using these patterns we construct finite dimensional matrix representations of the affine Hecke algebra ($\mathcal{A}_N(t)$) and hence of the Temperley-Lieb algebra. We present explicit matrices corresponding to the action of the $\mathcal{A}_N(t)$ generators on the pattern basis for the $N = 2$, 3 and 4 cases.

1.4 Thesis Outline

Chapter 2 describes the structure of Hecke algebras, with emphasis on new approaches to constructing double affine Hecke algebras. In Chapter 3 the double affine Q-dependent braid group is introduced. Presented is an intuitive graphical representation to complement its algebraic definition. The polynomial representation of a DAHA is described in Chapter 4, showing precisely how to obtain Macdonald polynomials. Chapters 5 and 6 are closely connected. The first of these describes finite dimensional matrix representations of the affine Hecke algebra based on tangle diagrams. Tangle diagrams are decomposed into elementary patterns which act as matrix basis using moves associated to knot theory. In Chapter 6 the knot theory connection is further explored, resulting in the interpretation of the HOMFLY polynomial as a trace invariant on the Hecke algebra. Finally Chapter 7 discusses conclusions and future work.

This thesis is largely self contained. Proofs, derivations and calculations deemed too long to include in the main text can be found in the appendices located at the end of each chapter. Otherwise, where necessary, references to external sources are given. Several definitions which are frequently referred to throughout the text have been compiled into a glossary located at the back of this thesis. Though readers may be familiar with many of these, they serve to clarify what is meant when employed.

Chapter 2

The Braid Group and Hecke type algebras

In this chapter we focus on the right hand column of Figure 1.1 that is encircled in red. Our goal is to explicitly construct a double affine Hecke algebra, known more simply as a DAHA. The double affine Hecke algebra, which was first introduced by Cherednik in [3], has as its very underlying structure, the braid group. For this reason it is at the foundation of our construction, which we now proceed to describe.

We begin by introducing the well known N-strand braid group $\mathcal{B}_N$, due to Artin [12]. We then extend it to the Hecke algebra $\mathcal{H}_N(t)$. This is achieved by requiring that all of the braid group generators satisfy a particular quadratic relation, called the Hecke relation. The Hecke algebra is key to our construction of a DAHA. Furthermore it is very closely connected to other algebraic structures that readers may be familiar with. For instance, we show how it is related to the Symmetric Group, which is composed of $N-1$ elementary permutations. We also define a map from $\mathcal{H}_N(t)$ to $TL_N(d)$, the Temperley-Lieb algebra which first appeared in [13].

Having defined the Hecke algebra, the next step in constructing a DAHA, is to extend $\mathcal{H}_N(t)$ to an affine Hecke algebra $\mathcal{A}_N(t)$. It is known by [14] that any Hecke algebra can be extended to an affine Hecke algebra by simply appending to it a set of N invertible operators. However a more interesting presentation of $\mathcal{A}_N(t)$ purely in terms of the braid group generators and an element denoted by σ is outlined by Kasatani and Pasquier in [7]. We describe this approach in detail and explicitly derive all of the defining relations of $\mathcal{A}_N(t)$ in terms of the element σ.

Finally, we extend the affine Hecke algebra to a double affine Hecke algebra by introducing a further N generators. This completes our construction of a DAHA composed precisely of all of the relations first introduced by Cherednik.

It is important to highlight the fact that the construction of a DAHA is not unique. In Section 2.4 we present two new alternative constructions that offer the reader a deeper insight into this algebraic structure. The advantage of these alternative methods are that the defining relations of a DAHA become less complicated and easier to work with. Furthermore they enable the simple definition of automorphisms and involutions of this algebra.

The remainder of this chapter describes isomorphisms and automorphisms of the double affine Hecke algebra, resulting in the explicit construction of the modular group within this DAHA. The benefits of the alternative definitions are apparent in this section, particularly when describing involutions.

To conclude we present a very special case DAHA, the "one dimensional DAHA" put forward by Cherednik in [2]. We investigate in detail many of its interesting properties. In particular we describe automorphisms and involutions, as well as the presence of the modular group in this special DAHA.

2.1 The Braid Group and the Hecke Algebra

2.1.1 The Braid Group $\mathcal{B}_N$

The braid group is at the basis of all Hecke type algebras. The N-strand braid group $\mathcal{B}_N$ is defined as follows: $\mathcal{B}_N$ is the group generated by the $N-1$ invertible elements $\{T_i | i = 1, .., N-1\}$ satisfying the relations

$$\begin{aligned} T_iT_j &= T_jT_i \text{ for } |i-j| \geq 2, & (2.1)\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \text{ otherwise }. & (2.2) \end{aligned}$$

The second of the above is commonly referred to as the *braid relation* or the Yang-Baxter equation.

2.1.2 The Hecke Algebra $\mathcal{H}_N(t)$

The braid group is central to the Hecke algebra. In fact the extension of the braid group to an algebra by requiring that the T_i generators satisfy a particular equation defines the Hecke algebra.

We associate with $\mathcal{B}_N$ the Hecke algebra $\mathcal{H}_N(t)$. This is the group algebra of $\mathcal{B}_N$ over a field k parametrised by $t \in k$ such that each generator T_i satisfies the *Hecke relation*

$$\left(T_i - t^{1/2}\mathbb{1}\right)\left(T_i + t^{-1/2}\mathbb{1}\right) \;=\; 0. \tag{2.3}$$

We highlight that even though T_i^{-1} was assumed to exist in $\mathcal{B}_N$, this relation gives its form explicitly:

$$T_i^{-1} \;=\; T_i - \left(t^{1/2} - t^{-1/2}\right)\mathbb{1}.$$

It is often convenient to rewrite the Hecke relation (2.3) in the following way

$$T_i^2 - \left(t^{1/2} - t^{-1/2}\right) T_i \;=\; \mathbb{1}. \tag{2.4}$$

We note that the Hecke relation, $\mathcal{H}_N(t)$ depends on the parameter t. Varying the value of this parameter yields different group and algebraic structures. For example when $t = 1$ the Hecke algebra maps onto the Symmetric Group $\mathcal{S}_N$, whose generators S_i satisfy

$$\begin{aligned} S_i S_j &= S_j S_i \text{ for } |i-j| \geq 2, \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1}, \\ S_i^2 &= 1. \end{aligned}$$

By the above relations one may view the Hecke algebra as a "deformation" of the symmetric group.

2.1.3 The Temperley-Lieb Algebra $TL_N(d)$

The Hecke algebra is closely connected to the Temperley-Lieb algebra. We define the map from the $\mathcal{H}_N(t)$ generators T_i, to the $TL_N(d)$ generators e_i, as

$$T_i \;\longmapsto\; e_i + t^{1/2}\mathbb{1}.$$

Using this map the Hecke relation (2.4) gives one of the defining relations of $TL_N(d)$

$$\begin{aligned} e_i^2 &= (-t^{1/2} - t^{-1/2})e_i \\ &= de_i. \end{aligned}$$

The two other relations that define $\mathcal{H}_N(t)$, namely (2.1) and (2.2) give the remaining Temperley-Lieb algebra relations

$$e_i e_j = e_j e_i \text{ for } |i-j| \geq 2, \tag{2.5}$$

$$e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}. \tag{2.6}$$

2.2 The Affine Hecke Algebra $\mathcal{A}_N(t)$

Any Hecke algebra $\mathcal{H}_N(t)$ can be extended to an affine Hecke algebra (AHA) $\mathcal{A}_N(t)$ [14] by appending to it N invertible operators Y_i. These satisfy the relations

$$Y_i Y_j = Y_j Y_i \text{ for all } i, j, \tag{2.7}$$

$$T_i Y_j = Y_j T_i \text{ for } j \neq i, i+1, \tag{2.8}$$

$$T_i Y_{i+1} T_i = Y_i \text{ for } i = 1, \ldots, N-1. \tag{2.9}$$

Repeated applications of the last of these relations implies that we need only one of the Y_i (and all of the T_i) to generate the others. For example, (2.9) can be used to rewrite Y_i for $i = 2, \ldots, N$ as

$$Y_i = T_{i-1}^{-1} T_{i-2}^{-1} \ldots T_1^{-1} Y_1 T_1^{-1} \ldots T_{i-2}^{-1} T_{i-1}^{-1}. \tag{2.10}$$

It is perhaps worth pointing out that even though the above definitions involve only multiplication, we need the full Hecke algebraic structure in order to consistently order the operators. For example, T_1 and Y_3 can be reordered as we like, however this is not true for T_1 and Y_2. In this case we must use the Hecke relation:

$$\begin{aligned} T_1 Y_2 &= Y_2 T_1^{-1} \\ &= Y_2 \left[T_1 - \left(t^{1/2} - t^{-1/2}\right) \mathbb{1}\right] \\ &= Y_2 T_1 - \left(t^{1/2} - t^{-1/2}\right) Y_2. \end{aligned}$$

$\mathcal{A}_N(t)$ is thus fully generated by Y_1 and the T_i, and we can reorder them as necessary.

2.2.1 The AHA $\mathcal{A}_N(t)$ in terms of σ

A more elementary presentation of $\mathcal{A}_N(t)$ is to write all the Y_i in terms of T_i and an element σ defined as

$$\sigma \quad := \quad T_{N-1}^{-1}T_{N-2}^{-1}\dots T_1^{-1}Y_1. \tag{2.11}$$

Using this presentation outlined by Kasatani and Pasquier in [7], we show exactly how all of the Y_i can now be written in terms of σ and the T_i using (2.9):
When $i = 1$

$$\begin{aligned} \sigma &= T_{N-1}^{-1}T_{N-2}^{-1}\dots T_1^{-1}Y_1 \\ \Rightarrow Y_1 &= T_1T_2\dots T_{N-1}\sigma. \end{aligned}$$

When $i = 2, \dots, N-1$

$$\begin{aligned} Y_2 &= T_1^{-1}Y_1T_1^{-1} = T_2\dots T_{N-1}\sigma T_1^{-1} \\ \Rightarrow Y_3 &= T_2^{-1}Y_2T_2^{-1} = T_3\dots T_{N-1}\sigma T_1^{-1}T_2^{-1}. \end{aligned}$$

By repeated iteration we find

$$Y_i = T_i\dots T_{N-1}\sigma T_1^{-1}\dots T_{i-1}^{-1} \text{ for all } i = 2, \dots, N-1.$$

Finally when $i = N$

$$Y_i = T_i\dots T_{N-1}\sigma T_1^{-1}\dots T_{i-1}^{-1} \quad \Rightarrow \quad Y_N = \sigma T_1^{-1}\dots T_{N-1}^{-1}.$$

Therefore in terms of the element σ all of the Y_i are given by:

$$Y_i \quad = \quad \begin{cases} T_1T_2\dots T_{N-1}\sigma & i = 1\,, \\ T_i\dots T_{N-1}\sigma T_1^{-1}\dots T_{i-1}^{-1} & i = 2, \dots, N-1, \\ \sigma T_1^{-1}\dots T_{N-1}^{-1} & i = N. \end{cases}$$

In Appendix 2A.1 at the end of this chapter we explicitly derive the other defining relations for an AHA in terms of the Hecke algebra generators and σ. They provide the reader with a good insight into working with the many commutation relations. We simply give the results here; namely that (2.7) and (2.8), may also be rewritten in terms of σ as

$$T_{i-1}\sigma = \sigma T_i, \quad i = 2, \ldots, N-1, \tag{2.12}$$

$$T_{N-1}\sigma^2 = \sigma^2 T_1. \tag{2.13}$$

If we define $T_N = \sigma T_1 \sigma^{-1}$, then we get that $T_{i-1}\sigma = \sigma T_i$ for all $i = 2, \ldots, N$.

In Appendix 2A.1 we derive that the above relations imply $\sigma^N T_i = T_i \sigma^N$. This tells us that σ^N commutes with all the T_i, and therefore with all the Y_i too. In fact it is the product of the Y_i:

$$\sigma^N = \prod_{i=1}^{N} Y_i. \tag{2.14}$$

Though a known result [2], due to its importance we offer a proof in Appendix 2A.2. Its significance implies σ^N is central in $\mathcal{A}_N(t)$. We could therefore if necessary label irreducible representations of this AHA with the eigenvalues of σ^N.

Before concluding this section, it is essential to point out that, given a Hecke algebra, an AHA *always* exists. This is because all of the Y_i are defined recursively (2.9), hence setting the value of any Y_i is sufficient to construct an AHA. To illustrate this point we see that the choice $Y_1 = 1$ fulfills all the necessary criteria. By (2.10) the elements

$$Y_i = T_{i-1}^{-1} \ldots T_2^{-1} T_1^{-2} T_2^{-1} \ldots T_{i-1}^{-1}$$

give all the Y_i for $i = 2, \ldots, N$ and

$$\sigma = T_{N-1}^{-1} T_{N-2}^{-1} \ldots T_1^{-1}.$$

As a further example one may take $Y_N = 1$ which by (2.10) means

$$Y_1 = T_1 \ldots T_{N-2} T_{N-1}^2 T_{N-2} \ldots T_1.$$

Repeated applications of the recursive relation (2.9) yields

$$Y_i = T_i \ldots T_{N-2} T_{N-1}^2 T_{N-2} \ldots T_i$$

which gives all of the Y_i for $i = 1, \dots, N-1$ and

$$\sigma \;=\; T_{N-1}T_{N-2}\dots T_1.$$

Therefore it is clear that any Hecke algebra $\mathcal{H}_N(t)$ already contains an AHA; it's just that the AHA generators are not independent of each other.

2.3 The Double Affine Hecke Algebra $\mathcal{D}_N(t, q)$

To complete the construction of a DAHA, we extend the affine Hecke algebraic structure to a double affine Hecke algebra of type A, [2, 15] by introducing a further N invertible generators Z_i. This particular DAHA denoted by $\mathcal{D}_N(t, q)$, satisfies the relations

$$Z_iZ_j \;=\; Z_jZ_i \text{ for all } i, j, \tag{2.15}$$
$$T_iZ_j \;=\; Z_jT_i \text{ for } j \neq i, i+1, \tag{2.16}$$
$$T_iZ_{i+1}T_i \;=\; Z_i \text{ for } i = 1, \dots, N-1, \tag{2.17}$$

together with a new parameter $q \in k$ which appears explicitly in relations intertwining the Y_i and the Z_i [2]:

$$Y_1Z_2Y_1^{-1}Z_2^{-1} \;=\; T_1^2, \tag{2.18}$$
$$Y_i\left(\prod_{j=1}^{N} Z_j\right) \;=\; q\left(\prod_{j=1}^{N} Z_j\right)Y_i, \tag{2.19}$$
$$Z_i\left(\prod_{j=1}^{N} Y_j\right) \;=\; q^{-1}\left(\prod_{j=1}^{N} Y_j\right)Z_i. \tag{2.20}$$

This double affine Hecke algebra follows Cherednik's original definition. It is the most widely used presentation of a DAHA but not the only one. In the following sections we introduce alternative DAHA presentations.

2.3.1 The DAHA $\mathcal{D}_N(t, q)$ in terms of σ

Here we present another way of defining the DAHA described in the previous section. Recall that in Subsection 2.2.1, we derived the defining relations of $\mathcal{A}_N(t)$ purely in terms of the Hecke algebra generators T_i and the operator σ, which replaced all of the Y_i. In a similar fashion we can define a DAHA in terms of the T_i, Z_i and σ by choosing to eliminate the Y_i in favour of the operator σ. To accomplish this we rewrite the intertwining relations (2.18) and (2.19).

In terms of σ, we show in detail in Appendix 2A.3 that (2.18) and (2.19) can be rewritten as

$$\begin{aligned} Z_{i-1}\sigma &= \sigma Z_i \text{ for } i = 2, \ldots, N, \\ Z_N\sigma &= q^{-1}\sigma Z_1, \end{aligned}$$

or, if we define $Z_0 = qZ_N$, then $Z_{i-1}\sigma = \sigma Z_i$ for all i.

We will now use the above equations to learn more about the $\sigma - Z$ interaction. Of interest is σ^N which we saw is central to $\mathcal{A}_N(t)$. We investigate if this is also the case for $\mathcal{D}_N(t,q)$.

Firstly we know that $Z_{i-1}\sigma = \sigma Z_i$ for $i = 1, \ldots, N$ when $Z_0 = qZ_N$.
Therefore this implies that $\sigma^{-1}Z_{i-1}\sigma = Z_i$ for all i. Replacing Z_{i-1} with $\sigma^{-1}Z_{i-2}\sigma$ we obtain

$$\sigma^{-2}Z_{i-2}\sigma^2 = Z_i.$$

After i iterations and using $Z_0 = qZ_N$ gives

$$\begin{aligned} \sigma^{-i}qZ_N\sigma^i &= Z_i \\ \Rightarrow Z_N\sigma^i &= q^{-1}\sigma^i Z_i \\ \Rightarrow \sigma^{N-i}Z_N\sigma^i &= q^{-1}\sigma^N Z_i. \end{aligned}$$

Ideally we would like the same index of Z on both sides. For this to be the case we can decrease Z_N to Z_i by pushing σs through using $\sigma Z_i = Z_{i-1}\sigma$. After $N - i$ iterations

$$\begin{aligned} \sigma^{N-1-i}\sigma Z_N\sigma^i &= q^{-1}\sigma^N Z_i \\ \Rightarrow \sigma^{N-1-i}Z_{N-1}\sigma^{i+1} &= q^{-1}\sigma^N Z_i \\ \Rightarrow \sigma^{N-(N-i)-i}Z_i\sigma^{i+N-i} &= q^{-1}\sigma^N Z_i \\ \Rightarrow Z_i\sigma^N &= q^{-1}\sigma^N Z_i. \end{aligned}$$

We can now use the identity $\prod_{j=1}^{N} Y_j = \sigma^N$, which gives us precisely (2.20); that is

$$Z_i\left(\prod_{j=1}^{N} Y_j\right) = q^{-1}\left(\prod_{j=1}^{N} Y_j\right) Z_i.$$

Therefore, in our definition of $\mathcal{D}_N(t,q)$ in terms of the T_i, the Z_i and σ, it is clear that (2.20) is not independent of the other relations intertwining the Y_i and Z_i. Although it is often included in the literature as part of the definition of a DAHA we have just shown how it can be derived using equations (2.18) and (2.19).

Hence, to summarise, from now on we can take a DAHA to be the algebra generated by T_i, Y_i and Z_i which satisfy equations (2.1)-(2.3), (2.7)-(2.9) and (2.15)-(2.19).

2.4 Equivalent DAHA constructions

The purpose of this section is to introduce new methods we developed of constructing a double affine Hecke algebra. As we have already seen, our definition of $\mathcal{D}_N(t,q)$ in terms of σ revealed several interesting relations but more importantly helped us gain a deeper understanding of the structure of a DAHA. Similarly, in the following novel approaches, where we treat a DAHA as the combination of two separate AHAs, we offer the reader a completely different perspective of the double affine Hecke algebra.

Recall that in the definition of an affine Hecke algebra, we introduced an element σ (2.11), and then proceeded in writing the defining relations of this AHA purely in terms of σ and the T_i. In a similar fashion within the double affine Hecke algebra one could define an element analogous to σ since, by (2.15)-(2.17), the T_i and Z_i form an AHA by themselves. The relations intertwining both AHAs, (2.18)-(2.20), therefore determine the relationship between σ and its analogue.

Defining an element analogous to σ can be done in one of two ways. We now present these in what follows.

2.4.1 Method 1 - Defining ζ_1

The first approach is straightforward. We follow the construction of σ (2.11) exactly by defining ζ_1 as

$$\zeta_1 \quad = \quad T_{N-1}^{-1} \dots T_1^{-1} Z_1. \tag{2.21}$$

Using this definition we can now describe the AHA formed by the Z_i and the T_i solely in terms of the T_i and ζ_1. Unsurprisingly, since the Z_i and Y_i obey the same relations with the T_i, ζ_1 has very similar properties to σ; that is (2.15)-(2.17) become

$$\begin{aligned} Z_i &= T_i \dots T_{N-1}\zeta_1 T_1^{-1} \dots T_{i-1}^{-1}, \\ T_{i-1}\zeta_1 &= \zeta_1 T_i \text{ for } i = 2, \dots, N-1, \\ T_{N-1}\zeta_1^2 &= \zeta_1^2 T_1. \end{aligned} \tag{2.22}$$

As in Subsection 2.2.1, defining $T_N = \zeta_1 T_1 \zeta_1^{-1}$, implies that $T_{i-1}\zeta_1 = \zeta_1 T_i$ for all $i = 2, \dots, N$. In addition to this the above equations also imply that $\zeta_1^N T_i = T_i \zeta_1^N$; its derivation can be found in Appendix 2A.4.

Analogous to the construction of σ, ζ_1^N is the product of the Z_i, $\left(\zeta_1^N = \prod_{i=1}^N Z_i\right)$, and therefore commutes with all the Z_i and T_i.

So we now know how the two separate AHAs behave independently. Recall that the first AHA where the Y_i were eliminated in favour of σ is defined by the equations (2.11)-(2.13). However to construct a valid DAHA both of these AHAs must be combined in a specific way. It is the intertwining relations (2.18) and (2.19) that govern the relationship between these two AHAs, the first in terms of the T_i and σ, and the second in terms of the T_i and ζ_1. Hence to complete the construction of this DAHA we must rewrite the intertwining relations in terms of the elements σ and ζ_1.
We begin with the first intertwining relation (2.18).

1. We have $Y_1 Z_2 Y_1^{-1} Z_2^{-1} = T_1^2 \quad \Rightarrow Y_1 Z_2 = T_1^2 Z_2 Y_1$. We can write the lefthand side of this expression using the definitions of Y_1 and Z_2 in terms of σ and ζ_1 respectively to get

$$Y_1 Z_2 = (T_1 \dots T_{N-1}\sigma)\left(T_2 \dots T_{N-1}\zeta_1 T_1^{-1}\right).$$

Using $T_{i-1}\sigma = \sigma T_i$ for $i = 2, \dots, N-1$ repeatedly allows us to move σ to the right:

$$\begin{aligned} Y_1 Z_2 &= (T_1 \dots T_{N-1})\left(T_1 \sigma T_3 \dots T_{N-1}\zeta_1 T_1^{-1}\right) \\ &= (T_1 \dots T_{N-1})(T_1 \dots T_{N-2})\,\sigma\zeta_1 T_1^{-1}. \end{aligned}$$

By definition of Y_1 we can write $T_1^2 Z_2 Y_1$ as follows

$$T_1^2 Z_2 Y_1 = T_1 T_1 Z_2 T_1 \dots T_{N-1}\sigma.$$

Firstly using (2.17) and then (2.16) repeatedly we can push Z_1 to the right of the

T_i:

$$\begin{aligned}
T_1^2 Z_2 Y_1 &= T_1 Z_1 T_2 \ldots T_{N-1}\sigma \\
&= T_1 T_2 Z_1 T_3 \ldots T_{N-1}\sigma \\
&= (T_1 \ldots T_{N-1})\, Z_1 \sigma \\
&= (T_1 \ldots T_{N-1})\,(T_1 \ldots T_{N-1})\,\zeta_1 \sigma.
\end{aligned}$$

Using $Y_1 Z_2 = T_1^2 Z_2 Y_1$ we obtain the final expression

$$\begin{aligned}
(T_1 \ldots T_{N-1})\,(T_1 \ldots T_{N-2})\,\sigma\zeta_1 T_1^{-1} &= (T_1 \ldots T_{N-1})\,(T_1 \ldots T_{N-1})\,\zeta_1\sigma \\
\Rightarrow \sigma\zeta_1 T_1^{-1} &= T_{N-1}\zeta_1\sigma \\
\Rightarrow \sigma\zeta_1 &= T_{N-1}\zeta_1\sigma T_1.
\end{aligned}$$

So rewriting (2.18) in terms of σ and ζ_1 gives the first relation describing the interaction between σ and ζ_1. The second independent relation is obtained in a similar fashion by rewriting (2.19) in terms of σ and ζ_1.

2. By (2.19), $Y_i \left(\prod_{j=1}^{N} Z_j\right) = q \left(\prod_{j=1}^{N} Z_j\right) Y_i$.
   It is shown in Appendix 2A.3 that in terms of σ this relation becomes

   $$\sigma Z_1 = q Z_N \sigma.$$

   Using the definitions of Z_1 and Z_N in terms of ζ_1, (2.21) and (2.22), gives the final expression

   $$\sigma\,(T_1 \ldots T_{N-1})\,\zeta_1 = q\zeta_1 \left(T_1^{-1} \ldots T_{N-1}^{-1}\right) \sigma.$$

As a consequence of these two relations between σ and ζ_1 we also obtain relations describing the interaction between the product of all the σ, that is, σ^N with ζ_1 and vice versa, which we now derive. Recall that σ^N is central to the AHA in the T_i and σ and that ζ_1^N is central to the AHA formed by the T_i and ζ_1.

3. In Subsection 2.3.1 we showed that $Z_i \sigma^N = q^{-1} \sigma^N Z_i$. Using the definition of Z_i in terms of ζ_1, (2.22), this relation becomes

$$T_i \dots T_{N-1} \zeta_1 T_1^{-1} \dots T_{i-1}^{-1} \sigma^N = q^{-1} \sigma^N T_i \dots T_{N-1} \zeta_1 T_1^{-1} \dots T_{i-1}^{-1}.$$

But we know that σ^N commutes with all the Ts since $\sigma^N T_i = T_i \sigma^N$, therefore we get the expression

$$\begin{aligned} T_i \dots T_{N-1} \zeta_1 \sigma^N T_1^{-1} \dots T_{i-1}^{-1} &= q^{-1} T_i \dots T_{N-1} \sigma^N \zeta_1 T_1^{-1} \dots T_{i-1}^{-1} \\ \Rightarrow \zeta_1 \sigma^N &= q^{-1} \sigma^N \zeta_1. \end{aligned}$$

4. Since ζ_1^N is the product of the Z_i, $\left(\zeta_1^N = \prod_{i=1}^N Z_i \right)$, we can rewrite equation (2.19) as

$$Y_i \zeta_1^N = q \zeta_1^N Y_i.$$

Using the definition of Y_i in terms of σ, this relation becomes

$$T_i \dots T_{N-1} \sigma T_1^{-1} \dots T_{i-1}^{-1} \zeta_1^N = q \zeta_1^N T_i \dots T_{N-1} \sigma T_1^{-1} \dots T_{i-1}^{-1}.$$

However since $\zeta_1^N T_i = T_i \zeta_1^N$, then ζ_1^N commutes with all the Ts and we obtain the final expression

$$\begin{aligned} T_i \dots T_{N-1} \sigma \zeta_1^N T_1^{-1} \dots T_{i-1}^{-1} &= q T_i \dots T_{N-1} \zeta_1^N \sigma T_1^{-1} \dots T_{i-1}^{-1} \\ \Rightarrow \sigma \zeta_1^N &= q \zeta_1^N \sigma. \end{aligned}$$

This completes the description of $\mathcal{D}_N(t,q)$ in terms of the T_i, σ and ζ_1.

To summarise this new method of constructing a DAHA, we have included the complete set of relations which define $\mathcal{D}_N(t,q)$ in terms of the T_i, σ and ζ_1 in Table 2.3 located at the end of Appendix 2.

2.4.2 Method 2 - Defining ζ_2

In this section we present the second approach we developed for defining an element analogous to σ. In a similar fashion to the method just described we will again split the original definition of a DAHA into two separate AHAs. The intertwining relations of the original DAHA, between the Y_i and Z_i, now in terms of the element σ and its alternative analogue which we denote by ζ_2, give the relations that the two AHAs must satisfy in order to come together to form a DAHA.

We define the element ζ_2 in a somewhat reversed way to ζ_1

$$\zeta_2 = Z_1 T_1^{-1} \dots T_{N-1}^{-1}. \tag{2.23}$$

Note that by writing Z_1 in terms of ζ_1 by (2.21), the definition of ζ_2 above tells us precisely how ζ_1 and ζ_2 are related; $\zeta_2 \left(T_{N-1} \dots T_1\right) = \left(T_1 \dots T_{N-1}\right) \zeta_1$.

Predictably, the properties of ζ_2 are somewhat reversed compared to ζ_1. This is clearly seen as in terms of ζ_2, the AHA formed by the T_i and ζ_2 that is (2.15)-(2.17) is given by

$$\begin{aligned} Z_i &= T_{i-1}^{-1} \dots T_1^{-1} \zeta_2 T_{N-1} \dots T_i, \\ T_{i+1}\zeta_2 &= \zeta_2 T_i \text{ for } i = 1, \dots, N-2, \\ T_1 \zeta_2^2 &= \zeta_2^2 T_{N-1}. \end{aligned} \tag{2.24}$$

In a similar (yet reversed) fashion to ζ_1, defining $T_N = \zeta_2^{-1} T_1 \zeta_2$ implies that $T_{i+1}\zeta_2 = \zeta_2 T_i$ for $i = 1, \dots, N-1$. In Appendix 2A.4 we prove that the relations above also imply that $T_i \zeta_2^N = \zeta_2^N T_i$. Therefore ζ_2^N commutes with all the Ts and is central to the AHA formed by the T_i and ζ_2.

In addition to the above relations, ζ_2^N is the product of the Z_i, $\left(\zeta_2^N = \prod_{i=1}^N Z_i\right)$, making it commute with all the Z_i and T_i.

At this point we have all of the relations satisfied by both independent AHAs. The first in terms of the T_i and σ is defined by (2.11)-(2.13). The second AHA which we have just derived is given by (2.23) and (2.24). We can now rewrite the intertwining relations (2.18) and (2.19) in terms of ζ_2 and σ. These two relations are of great importance as they determine the necessary conditions for the two separate AHAs to fuse and form a double affine Hecke algebra. As before, we begin by examining the first intertwining relation.

1. We have by (2.18) that $Y_1 Z_2 Y_1^{-1} Z_2^{-1} = T_1^2 \quad \Rightarrow Y_1 Z_2 = T_1^2 Z_2 Y_1$. Writing the lefthand side of this expression using the definitions of Y_1 and Z_2 in terms of σ and ζ_2 gives

$$Y_1 Z_2 = \left(T_1 \dots T_{N-1}\right) \sigma T_1^{-1} \zeta_2 \left(T_{N-1} \dots T_2\right).$$

Similarly we can write $T_1^2 Z_2 Y_1$ as follows

$$\begin{aligned} T_1^2 Z_2 Y_1 &= T_1^2 T_1^{-1} \zeta_2 \left(T_{N-1} \dots T_2\right) \left(T_1 \dots T_{N-1}\right) \sigma \\ &= T_1 \zeta_2 \left(T_{N-1} \dots T_2\right) \left(T_1 \dots T_{N-1}\right) \sigma. \end{aligned}$$

Using the braid relation (2.2) firstly and then by (2.1) we get that

$$\begin{aligned} T_1^2 Z_2 Y_1 &= T_1 \zeta_2 \left(T_{N-1} \dots T_3 T_1 T_2 T_1 T_3 \dots T_{N-1}\right) \sigma \\ &= T_1 \zeta_2 \left(T_{N-1} \dots T_1 T_3 T_2 T_3 T_1 \dots T_{N-1}\right) \sigma. \end{aligned}$$

We repeat the last step until eventually

$$T_1^2 Z_2 Y_1 = T_1 \zeta_2 \left(T_1 \dots T_{N-2} T_{N-1} T_{N-2} \dots T_1\right) \sigma.$$

Now we can push ζ_2 through the T_i by (2.24) and move σ to the left of the expression using $T_{i-1}\sigma = \sigma T_i$ for $i = 2, \dots, N-1$ repeatedly

$$T_1^2 Z_2 Y_1 = T_1 T_2 \dots T_{N-1} \zeta_2 T_{N-1} \sigma T_{N-1} \dots T_2.$$

Setting $Y_1 Z_2 = T_1^2 Z_2 Y_1$ we obtain the final expression

$$\begin{aligned} \left(T_1 \dots T_{N-1}\right) \sigma T_1^{-1} \zeta_2 \left(T_{N-1} \dots T_2\right) &= \left(T_1 T_2 \dots T_{N-1}\right) \zeta_2 T_{N-1} \sigma \left(T_{N-1} \dots T_2\right) \\ \Rightarrow \sigma T_1^{-1} \zeta_2 &= \zeta_2 T_{N-1} \sigma. \end{aligned}$$

Equation (2.18) in terms of σ and ζ_2 gave the first relation between σ and ζ_2. In a similar fashion, (2.19) in terms of σ and ζ_2, gives the second independent relation governing the interaction of both separate AHAs.

2. Again we refer to Appendix 2A.3 for the explicit derivation and just state the result here; that is, in terms of σ (2.19) is given by

$$\sigma Z_1 = q Z_N \sigma.$$

Substituting into this relation the definitions of Z_1 and Z_N in terms of ζ_2, (2.23) and (2.24), gives the final expression

$$\sigma \zeta_2 \left(T_{N-1} \dots T_1\right) = q \left(T_{N-1}^{-1} \dots T_1^{-1}\right) \zeta_2 \sigma.$$

As in our construction of $\mathcal{D}_N(t,q)$ in terms of the T_i, σ and ζ_1, we obtain equations describing the interaction between the product of all the σ with ζ_2^N and vice versa. However, even though ζ_1 and ζ_2 are defined differently, the product ζ_2^N is equal to ζ_1^N and so shares its properties; namely we obtain

$$\zeta_2\sigma^N = q^{-1}\sigma^N\zeta_2 \quad \text{and} \quad \sigma\zeta_2^N = q\zeta_2^N\sigma.$$

These equations complete the description of $\mathcal{D}_N(t,q)$ in terms of the T_i, σ and ζ_2.

To summarise the structure of double affine Hecke algebras, we reiterate that the construction of a DAHA is not unique. Initially we introduced Cherednik's original definition of a DAHA denoted $\mathcal{D}_N(t,q)$, defined in terms of the braid group operators T_i in addition to the affine Hecke algebra generators, the Y_i and the double affine generators the Z_i. Then we eliminated the Y_i in favour of σ and proceeded in describing $\mathcal{D}_N(t,q)$ in terms of the T_i, the Z_i and σ. Finally treating $\mathcal{D}_N(t,q)$ as the specific combination of two separate AHAs, allowed us to construct two new presentations of a DAHA.

In the first of these we opted to describe the second independent AHA in terms of the element ζ_1, before forming $\mathcal{D}_N(t,q)$ in terms of the T_i, σ and ζ_1 by combining it with the first independent AHA which was given by the T_i and σ.
The second new DAHA in terms of the T_i, σ and ζ_2 was formed by combining the first AHA in T_i and σ, with another AHA initially formed by the T_i and Z_i, now in terms of the T_i and the element ζ_2.

A complete summary of all of the equations used to define all of the DAHAs we have discussed is included in Table 2.3 at the end of Appendix 2.

As a final comment before concluding this section, recall that at the end of Subsection 2.2.1 we pointed out that given a Hecke algebra, an AHA *always* exists. However it is now clear that the AHA construction which takes either Y_1 or Y_N equal to 1 will not work in a DAHA where $q \neq 1$. In fact using an AHA where *any* of the Y_i are set to 1 can only be extended to a DAHA with $q = 1$. Since we know $Y_i\zeta_{1,2}^N = q\zeta_{1,2}^N Y_i$, whichever Y_i is set to 1 will simply commute with $\zeta_{1,2}^N$.

Therefore, it is not apparent that, given either $\mathcal{H}_N(t)$ or $\mathcal{A}_N(t)$, we can extend it to a DAHA with $q \neq 1$.

2.5 Automorphisms and Involutions on $\mathcal{D}_N(t,q)$

In the remainder of this chapter we will investigate many of the properties of the double affine Hecke algebra, $\mathcal{D}_N(t,q)$. Recall that in the previous sections we constructed $\mathcal{D}_N(t,q)$ and showed that it is generated by the T_i, Y_i and Z_i satisfying the relations (2.1)-(2.3), (2.7)-(2.9) and (2.15)-(2.19).

Initially we will begin by defining automorphisms of $\mathcal{D}_N(t,q)$ which have appeared in [2]. We are particularly interested in applying these definitions; in doing so we successfully derive a representation of the modular group within $\mathcal{D}_N(t,q)$. Finally we expand the known domain by presenting the simple form of the action of involutions on $\mathcal{D}_N(t,q)$ in terms of our previously described second alternative presentation of a DAHA in the T_i, σ and ζ_2.

2.5.1 Automorphisms on $\mathcal{D}_N(t,q)$

The following are all automorphisms of $\mathcal{D}_N(t,q)$; each takes T_i to itself and their actions on Y_1 and Z_1 are given in Table 2.1.

map	$Y_1 \mapsto$	$Z_1 \mapsto$
τ_+	$Y_1 Z_1$	Z_1
τ_-	Y_1	$Y_1 Z_1$
τ_-^{-1}	Y_1	$Y_1^{-1} Z_1$
λ	Z_1	$Z_1^{-1} Y_1^{-1} Z_1$
λ^{-1}	$Y_1 Z_1^{-1} Y_1^{-1}$	Y_1

Table 2.1: Automorphisms of $\mathcal{D}_N(t,q)$.

Note that Table 2.1 is not a complete list of all the automorphisms of $\mathcal{D}_N(t,q)$; it merely gives the ones of most interest to us for future calculations. For example, since τ_+ is an automorphisms of $\mathcal{D}_N(t,q)$, then τ_+^{-1} is also an automorphism, yet we have not included it in Table 2.1.

The actions on Y_i and Z_i are found via their definitions in terms of Y_1 and Z_1; for example the map λ^{-1} defined in Table 2.1 acts on the Y_i as

$$\begin{aligned}
\lambda^{-1}(Y_i) &= \lambda^{-1}\left(T_{i-1}^{-1}\right)\ldots\lambda^{-1}\left(T_1^{-1}\right)\lambda^{-1}(Y_1)\,\lambda^{-1}\left(T_1^{-1}\right)\ldots\lambda^{-1}\left(T_{i-1}^{-1}\right)\\
&= T_{i-1}^{-1}\ldots T_1^{-1} Y_1 Z_1^{-1} Y_1^{-1} T_1^{-1}\ldots T_{i-1}^{-1}.
\end{aligned}$$

Using these definitions it is straightforward to prove the validity of the automorphisms defined in Table 2.1.

We would like to find a relationship between these automorphisms . We achieve this by defining an isomorphism ϵ from $\mathcal{D}_N(t,q) \to \mathcal{D}_N(t^{-1},q^{-1})$. ϵ is an isomorphism satisfying $\epsilon^{-1} = \epsilon$ whose action is

$$\epsilon\left(Y_i\right) = Z_i^{-1}, \quad \epsilon\left(Z_i\right) = Y_i^{-1}, \quad \epsilon\left(T_i\right) = T_i^{-1}.$$

Since $\epsilon^2 = 1$, it is trivial to show that the automorphisms described in Table 2.1 are related via

$$\begin{aligned} \tau_- &= \epsilon\tau_+\epsilon^{-1}, \\ \lambda = \tau_+\tau_-^{-1}\tau_+ &= \tau_-^{-1}\tau_+\tau_-^{-1} = \epsilon\lambda^{-1}\epsilon^{-1}. \end{aligned}$$

The latter is of particular interest to us; upon closer examination we see that it is the braid relation, meaning the braid group $\mathcal{B}_3$ is a subgroup of $\mathrm{Aut}(\mathcal{D}_N(t,q))$. In $\mathcal{B}_3$ there are only two generators: letting them be τ_+ and τ_-^{-1}, we can say

$$\mathcal{B}_3 = \left\langle \tau_+, \tau_- | \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1} \right\rangle.$$

The appearance of $\mathcal{B}_3$ as a subgroup of $\mathrm{Aut}(\mathcal{D}_N(t,q))$ suggests that if there exists a normal subgroup C, generated by a and b with $a^2 = b^3 = 1$, then there is a representation of the modular group within $\mathcal{D}_N(t,q)$. This exciting prospect is a direct consequence of the following property of the modular group $PSL(2,\mathbb{Z})$:

$$PSL(2,\mathbb{Z}) = \mathcal{B}_3 / \left\langle C \right\rangle.$$

We outline our construction of such a subgroup in the following section.

2.5.2 The Modular Group in $\mathcal{D}_N(t,q)$

Our goal is to find a normal subgroup of $\mathcal{B}_3 = \left\langle \tau_+, \tau_- | \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1} \right\rangle$. Define C to be λ^2, that is:

$$\begin{aligned} C &= \left(\tau_+\tau_-^{-1}\tau_+\right)^2 = \left(\tau_-^{-1}\tau_+\tau_-^{-1}\right)^2 \\ &= \left(\tau_+\tau_-^{-1}\tau_+\right)\left(\tau_-^{-1}\tau_+\tau_-^{-1}\right) \\ &= \left(\tau_+\tau_-^{-1}\right)^3. \end{aligned}$$

Clearly C was carefully chosen such that it is generated by two elements, saying S and U, that are related via $S^2 = U^3$. We now verify that C is a normal subgroup of $\mathcal{B}_3$.

$$\begin{aligned} C\tau_+ &= \left(\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\right)\tau_+ & C\tau_-^{-1} &= \left(\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\right)\tau_-^{-1} \\ &=\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\tau_-^{-1}\tau_+\tau_-^{-1} & &=\tau_+\tau_-^{-1}\tau_+\tau_+\tau_-^{-1}\tau_+\tau_-^{-1} \\ &=\tau_+\left(\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\right) & &=\tau_-^{-1}\left(\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\tau_+\tau_-^{-1}\right) \\ &=\tau_+ C & &=\tau_-^{-1}C \end{aligned}$$

So clearly C commutes with τ_+ and τ_-^{-1}, therefore the group freely generated by C, $\langle C\rangle = \langle\left(\tau_+\tau_-^{-1}\tau_+\right)^2\rangle$, is a normal subgroup of $\mathcal{B}_3$. This means we can construct the quotient $\mathcal{B}_3/\langle C\rangle$.

Setting $1 = [C]$, $S = [\tau_+\tau_-^{-1}\tau_+]$ and $U = [\tau_+\tau_-^{-1}]$, we have the group

$$\left\langle S, U | S^2 = U^3 = 1\right\rangle,$$

which is precisely the presentation of the modular group $PSL(2,\mathbb{Z})$.

Having derived a representation of the modular group within $\mathcal{D}_N(t,q)$ via the subgroup C, we now deduce the action of C on the DAHA generators. Using Table 2.1, we find that

1. $C(T_i) = \lambda^2(T_i) = T_i$,
2. $C(Y_1) = \lambda^2(Y_1) = \lambda(Z_1) = Z_1^{-1}Y_1^{-1}Z_1$,
3. $C(Z_1) = \lambda^2(Z_1) = \lambda(Z_1^{-1}Y_1^{-1}Z_1) = Z_1^{-1}Y_1Z_1^{-1}Y_1^{-1}Z_1$.

Writing these as $C(Y_1) = \left(Z_1^{-1}Y_1\right)Y_1^{-1}\left(Y_1^{-1}Z_1\right)$ and $C(Z_1) = \left(Z_1^{-1}Y_1\right)Z_1^{-1}\left(Y_1^{-1}Z_1\right)$, we can see that the action of C on Y_1 and Z_1 is – up to conjugation by $Z_1^{-1}Y_1$ – inversion. Therefore taking $[C] = 1$ is thus akin to identifying these generators with their inverses, which is perhaps too trivial.

It is then more interesting to look at C^2. $C^2 = \lambda^4$ acts on the DAHA generators as follows:

1. $C^2(T_i) = \lambda^2(T_i) = T_i$,
2. $C^2(Y_1) = \lambda^2(Z_1^{-1}Y_1^{-1}Z_1) = \lambda(Z_1^{-1}Y_1Z_1^{-1}Y_1^{-1}Z_1) = (Z_1^{-1}Y_1Z_1Y_1^{-1})Y_1(Y_1Z_1^{-1}Y_1^{-1}Z_1)$,
3. $C^2(Z_1) = \lambda^2(Z_1^{-1}Y_1^{-1}Z_1) = \lambda(Z_1^{-1}Y_1Z_1Y_1Z_1^{-1}Y_1^{-1}Z_1) = (Z_1^{-1}Y_1Z_1Y_1^{-1})Z_1(Y_1Z_1^{-1}Y_1^{-1}Z_1)$.

We write these in a simpler form. Letting $u = Y_1 Z_1^{-1} Y_1^{-1} Z_1$, C^2 acts on the generators as

$$C^2\left(T_i\right) = T_i, \quad C^2\left(Y_1\right) = u^{-1} Y_1 u, \quad C^2\left(Z_1\right) = u^{-1} Z_1 u.$$

In terms of the Hecke algebra generators T_i, using (2.9), u is defined as

$$\begin{aligned} u &= Y_1 Z_1^{-1} Y_1^{-1} Z_1 \\ &= T_1 \dots T_{N-1} \sigma Z_1^{-1} \sigma^{-1} T_{N-1}^{-1} \dots T_1^{-1} Z_1. \end{aligned}$$

We can pull the Z_1 back as far as σ^{-1} using $\sigma Z_i = Z_{i-1}\sigma$ for $i = 2, \dots, N$.

$$u = T_1 \dots T_{N-1} \sigma Z_1^{-1} \sigma^{-1} Z_N T_{N-1} \dots T_1.$$

Using $Z_N \sigma = q^{-1} \sigma Z_1$ to write $\sigma^{-1} Z_N$ as $q^{-1} Z_1 \sigma^{-1}$ yields the final expression

$$u = q^{-1} T_1 \dots T_{N-2} T_{N-1}^2 T_{N-2} \dots T_1.$$

This remarkable result means that taking the quotient of DAHA by the free group generated by C^2 does not place any restrictions on Y_1 and Z_1 (although it does so on the Ts). Therefore constructing the quotient group $\mathcal{B}_3/\langle C^2\rangle$ gives $SL(2,\mathbb{Z})$. This is not the modular group, but still a very nice group nonetheless, with the generators represented by the 2×2 matrices

$$[\tau_+] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \left[\tau_-^{-1}\right] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

2.5.3 Involutions on $\mathcal{D}_N(t,q)$

We define two antiisomorphisms θ_1 and θ_2 which act on $\mathcal{D}_N(t,q)$, conjugating all numbers in the process. The first antiisomorphism θ_1 is the map from $\mathcal{D}_N(t,q) \longrightarrow \mathcal{D}_N(t^*,q^*)$ defined as

$$\theta_1(T_i) = T_i, \quad \theta_1(Y_i) = Z_i, \quad \theta_1(Z_i) = Y_i.$$

It is straightforward, given the actions above, to verify that θ_1 is an antiisomorphism on $\mathcal{D}_N(t,q)$. The only relation which requires some work is the action of θ_1 on the intertwining relation (2.18); as such we derive it below.

Applying θ_1 to (2.18) yields

$$\theta_1 \left(Y_1 Z_2 Y_1^{-1} Z_2^{-1} \; = \; T_1^2 \right) \; \longrightarrow \; Y_2^{-1} Z_1^{-1} Y_2 Z_1 \; = \; T_1^2.$$

Equation (2.17) gives $Z_1 = T_1 Z_2 T_1$ and (2.9) gives $Y_2^{-1} = T_1 Y_1^{-1} T_1$, therefore after direct substitution we now have

$$T_1 Y_1^{-1} T_1 Z_1^{-1} Y_2 T_1 Z_2 T_1 \; = \; T_1^2.$$

But $Y_2 T_1 = T_1^{-1} Y_1$ by (2.9) and $T_1 Z_1^{-1} = Z_2^{-1} T_1$ by (2.17), which gives

$$\begin{aligned} Y_1^{-1} Z_2^{-1} T_1^{-2} Y_1 Z_2 &= 1 \\ \Rightarrow Y_1^{-1} Z_2^{-1} &= Z_2^{-1} Y_1^{-1} T_1^2 \\ \Rightarrow Y_1 Z_2 Y_1^{-1} Z_2^{-1} &= T_1^2 \end{aligned}$$

as required.

The second antiisomorphism θ_2 on $\mathcal{D}_N(t,q)$ is the map from $\mathcal{D}_N(t,q) \longrightarrow \mathcal{D}_N((t^*)^{-1},(q^*)^{-1})$ defined as

$$\theta_2(T_i) \; = \; T_i^{-1}, \quad \theta_2(Y_i) \; = \; Y_i^{-1}, \quad \theta_1(Z_i) \; = \; Z_i^{-1}.$$

Verifying the action of θ_2 on the defining relations of $\mathcal{D}_N(t,q)$ follows in a similar way to the last proof.

A particularly nice form of the two antiisomorphisms θ_1 and θ_2 comes from their action on $\mathcal{D}_N(t,q)$ in terms of T_i, σ and ζ_2, which we introduced in Subsection 2.4.2. Eliminating the Y_i and Z_i in favour of σ and ζ_2 means

$$\begin{aligned} \theta_1(\sigma) &= \theta(T_{N-1}^{-1} \dots T_1^{-1} Y_1) & \qquad \theta_1(\zeta_2) &= \theta_1(Z_1 T_1^{-1} \dots T_{N-1}^{-1}) \\ &= Z_1 T_1^{-1} \dots T_{N-1}^{-1} & &= T_{N-1}^{-1} \dots T_1^{-1} Y_1 \\ &= \zeta_2 & &= \sigma \end{aligned}$$

We can similarly show that $\qquad \theta_2(\sigma) \; = \; \sigma^{-1} \qquad$ and $\qquad \theta_2(\zeta_2) \; = \; \zeta_2^{-1}$.

Both of these maps square to the identity in $\mathrm{Aut}(\mathcal{D}_N(t,q))$; as a result θ_1 is an involution on $\mathcal{D}_N(t,q)$ if $t^* = t$ and $q^* = q$, resulting in a real representation of $\mathcal{D}_N(t,q)$. θ_2 is an involution on $\mathcal{D}_N(t,q)$ if $t^* = t^{-1}$ and $q^* = q^{-1}$ which is useful if one requires a unitary representation of $\mathcal{D}_N(t,q)$.

2.6 The DAHA $\mathcal{D}_1(t, q)$

In the concluding section of this chapter we look at a very special case DAHA, the case when $N = 1$ which is due to [2]. Since this DAHA is very a little known algebra we will describe it in great detail. Though of little physical relevance in terms of the braid group, its algebraic structure displays very interesting properties which we now proceed to investigate.

Among the defining relations of the double affine Hecke algebra $\mathcal{D}_N(t, q)$ constructed in Section 2.3 were the braid group relations, (2.1) and (2.2), which govern the N-strand braid group $\mathcal{B}_N$ generated by the $N - 1$ invertible elements $\{T_i | i = 1, .., N - 1\}$.

It is clear that when $N = 1$ we can't use these definitions for a DAHA, because the braid group is trivial in this case; $N - 1$ generators equals zero generators, and therefore seemingly no Ts.

However we can get around this by defining $\mathcal{D}_1(t, q)$ with no mention of the braid group. We do this by assuming there is a *single* element T which satisfies the Hecke relation (2.3), that is, $\left(T - t^{1/2}\mathbb{1}\right)\left(T + t^{-1/2}\mathbb{1}\right) = 0$.

Completing its definition there are, in addition, two invertible elements X and Y satisfying

$$TXT = X^{-1}, \qquad TY^{-1}T = Y,$$
$$XY = q^{1/2}YXT^2.$$

These relations appear quite similar to the definitions for $N \geq 2$ DAHAs as in Section 2.3. However, they are completely independent from all of the relations for $N \geq 2$ DAHAs and cannot be obtained from them by simply setting N equal to 1.

As in the general case, we can also find automorphisms on $\mathcal{D}_1(t, q)$. The maps τ_+, τ_- and λ are all automorphisms on $\mathcal{D}_1(t, q)$ which leave T unchanged and act on X and Y as given in Table 2.2.

In addition to the automorphisms, we define an isomorphism ϵ from $\mathcal{D}_1(t, q)$ to $\mathcal{D}_1(t^{-1}, q^{-1})$, satisfying $\epsilon^{-1} = \epsilon$, whose action on the generators T, Y and X is

$$\epsilon(T) = T^{-1}, \quad \epsilon(Y) = X, \quad \epsilon(X) = Y.$$

map	$X \mapsto$	$Y \mapsto$
τ_+	X	$q^{-1/4}XY$
τ_-	$q^{1/4}YX$	Y
τ_-^{-1}	$q^{-1/4}Y^{-1}X$	Y
λ	Y^{-1}	$q^{-1/2}Y^{-1}XY$
λ^{-1}	$q^{1/2}X^{-1}YX$	X^{-1}

Table 2.2: Automorphisms of $\mathcal{D}_1(t,q)$.

Using this definition of ϵ, the automorphisms described in Table 2.2 are related via

$$\begin{aligned} \tau_- &= \epsilon\tau_+\epsilon^{-1}, \\ \lambda = \tau_+\tau_-^{-1}\tau_+ &= \tau_-^{-1}\tau_+\tau_-^{-1} = \epsilon\lambda^{-1}\epsilon^{-1}. \end{aligned}$$

Once again the last relation implies the appearance of $\mathcal{B}_3$ as a subgroup of $\mathrm{Aut}(\mathcal{D}_1(t,q))$. Following Subsection 2.5.2 we show that there exists a representation of the modular group within $\mathcal{D}_1(t,q)$.

Define C to be λ^2. It is easy to see that $C = (\tau_+\tau_-^{-1})^3$, and also that C commutes with both τ_+ and τ_-. The group freely generated by C, $\langle C\rangle = \langle \left(\tau_+\tau_-^{-1}\tau_+\right)^2\rangle$, is therefore a normal subgroup of $\mathcal{B}_3$, and we can construct the quotient $\mathcal{B}_3/\langle C\rangle$.

With $1 = [C]$, $S = [\tau_+\tau_-^{-1}\tau_+]$ and $U = [\tau_+\tau_-^{-1}]$, we have precisely the presentation of the modular group $PSL(2,\mathbb{Z})$,

$$\left\langle S, U | S^2 = U^3 = 1\right\rangle.$$

We would now like to see if the action of the subgroup C on the $\mathcal{D}_1(t,q))$ generators is somewhat trivial, as was the case for the N dimensional DAHA. Using Table 2.2, the action of C on the $\mathcal{D}_1(t,q))$ generators is given by

1. $C(T) = \lambda^2(T) = T$,
2. $C(Y) = \lambda^2(Y) = q^{-1/2}Y^{-1}X^{-1}Y^{-1}XY$,
3. $C(X) = \lambda^2(X) = q^{1/2}Y^{-1}X^{-1}Y$.

Using the defining relations of $\mathcal{D}_1(t,q)$), we can rewrite these as follows:

$$C(T) = T, \quad C(Y) = T^{-1}YT, \quad C(X) = T^{-1}XT.$$

From the above it is clear that $C = \lambda^2$ acts on the generators via conjugation by T. Therefore, in this case the representation of the modular group $PSL(2,\mathbb{Z})$, obtained via the quotient $\mathcal{B}_3/\langle C\rangle$, is more meaningfull than for the $N > 1$ case where the the generators were just identified with their inverses.

2.6.1 Involutions on $\mathcal{D}_1(t,q)$

As in $\mathcal{D}_N(t,q)$ we also define two antiisomorphisms θ_1 and θ_2 which act on $\mathcal{D}_1(t,q)$.

The first map θ_1: $\mathcal{D}_1(t,q) \longrightarrow \mathcal{D}_1(t^*,q^*)$ is the antiisomorphism

$$\theta_1(T) = T, \quad \theta_1(Y) = X^{-1}, \quad \theta_1(X) = Y^{-1}.$$

If t and q are real then θ_1 defines a Hermitian adjoint on $\mathcal{D}_1(t,q)$.

The second antiisomorphism θ_2: $\mathcal{D}_1(t,q) \longrightarrow \mathcal{D}_1((t^*)^{-1},(q^*)^{-1})$ is given by

$$\theta_2(T) = T^{-1}, \quad \theta_2(Y) = Y^{-1}, \quad \theta_2(X) = X^{-1}.$$

θ_2 can be used as a Hermitian adjoint if both t and q have modulus 1.

To conclude this chapter, which focused mainly on Hecke algebras, or more specifically double affine Hecke algebras, we highlight some of the main points. Starting with the definition of the braid group and the Hecke algebra, we explicitly constructed a DAHA in several different ways. We pointed out that given a Hecke algebra, an AHA always exists; yet further extension to a DAHA with $q \neq 1$ is not apparent.

Furthermore we investigated some interesting properties of a DAHA, particularly automorphisms and the presence of a representation of the modular group within. We also looked at a special case DAHA, that of $\mathcal{D}_1(t,q)$.

In the next chapter we will introduce an even richer structure which we created, called the double affine Q-dependent braid group ($\mathcal{D}_N\{Q\}$). It's structure resembles in many ways that of a DAHA, but is much more general. In fact we will show that the double affine Hecke algebra is a small quotient group of the double affine Q-dependent braid group, hence all of its properties which we have just discussed are contained within $\mathcal{D}_N\{Q\}$.

Appendix 2

2A.1 The AHA $\mathcal{A}_N(t)$ in terms of σ

In terms of σ all of the Y_i are given by:

$$Y_i = \begin{cases} T_1T_2\dots T_{N-1}\sigma & i=1\,, \\ T_i\dots T_{N-1}\sigma T_1^{-1}\dots T_{i-1}^{-1} & i=2,\dots,N-1, \\ \sigma T_1^{-1}\dots T_{N-1}^{-1} & i=N. \end{cases}$$

We want to rewrite the other defining relations of $\mathcal{A}_N(t)$ in terms of σ, namely

$$\begin{aligned} Y_iY_j &= Y_jY_i \ \text{ for all } \ i,j, \\ T_iY_j &= Y_jT_i \ \text{ for } \ j\neq i,i+1. \end{aligned}$$

1. We have $T_iY_j = Y_jT_i$ for $j \neq i, i+1$. So for $j=1$ and $i=2,\dots,N-1$:

$$\begin{aligned} T_iY_1 &= T_iT_1T_2\dots T_{N-1}\sigma \\ &= T_iT_1\dots T_{i-2}T_{i-1}T_iT_{i+1}\dots T_{N-1}\sigma. \end{aligned}$$

Can pull the first T_i through since it commutes with $T_1,\dots,T_{i-2}$ and then use the braid relation

$$\begin{aligned} T_iY_1 &= T_1\dots T_{i-2}\left(T_iT_{i-1}T_i\right)T_{i+1}\dots T_{N-1}\sigma \\ &= T_1\dots T_{i-2}\left(T_{i-1}T_iT_{i-1}\right)T_{i+1}\dots T_{N-1}\sigma. \end{aligned}$$

Second T_{i-1} can be pushed all the way to the right since it commutes with $T_{i+1},\dots,T_{N-1}$

$$T_iY_1 = T_1\dots T_{N-1}T_{i-1}\sigma.$$

Since $T_iY_j = Y_jT_i$ for $j\neq i, i+1$, and using the definition of Y_1, we must have that:

$$\begin{aligned} T_1\dots T_{N-1}T_{i-1}\sigma &= T_1\dots T_{N-1}\sigma T_i \\ \Rightarrow T_{i-1}\sigma &= \sigma T_i \ \text{ for } \ i=2,\dots,N-1. \end{aligned}$$

This relation between the T_i and σ is valid for $i=2,\dots,N-1$ only, yet in $\mathcal{A}_N(t)$

there are $(N-1)\,T_i$ generators. So can we find a similar relation when $i = N$?

2. We know that $Y_iY_j = Y_jY_i$ for all i, j. So for $j = 1$ and $i = N$, we rewrite $Y_1Y_N = Y_NY_1$ in terms of σ.

   By definition of Y_1 and Y_N

$$\begin{aligned} Y_1Y_N &= T_1 \dots T_{N-1}\sigma^2 T_1^{-1} \dots T_{N-1}^{-1}, \\ Y_NY_1 &= \sigma T_1^{-1}T_2^{-1} \dots T_{N-1}^{-1} T_1 \dots T_{N-1}\sigma. \end{aligned}$$

   Can pull T_1 all the way back to T_2^{-1} since $\left(T_3^{-1} \dots T_{N-1}^{-1}\right)$ commutes with T_1 and then use the braid relation

$$\begin{aligned} Y_NY_1 &= \sigma T_1^{-1}T_2^{-1}T_1 \dots T_{N-1}^{-1}T_2 \dots T_{N-1}\sigma \\ &= \sigma T_2 T_1^{-1}T_2^{-1}T_3^{-1} \dots T_{N-1}^{-1}T_2T_3 \dots T_{N-1}\sigma. \end{aligned}$$

   Repeat the previous step until all the T_i on the right are moved to the left

$$\begin{aligned} Y_NY_1 &= \sigma T_2 \dots T_{N-1}T_1^{-1} \dots T_{N-1}^{-1}T_{N-1}\sigma \\ &= \sigma T_2 \dots T_{N-1}T_1^{-1} \dots T_{N-2}^{-1}\sigma. \end{aligned}$$

   We previously found that $T_{i-1}\sigma = \sigma T_i$ for $i = 2, \dots, N-1 \;\Rightarrow\; \sigma T_i^{-1} = T_{i-1}^{-1}\sigma$. We use this to move the sigmas until they meet in the middle

$$\begin{aligned} Y_NY_1 &= T_1\sigma T_3 \dots T_{N-1}T_1^{-1}T_2^{-1} \dots \sigma T_{N-1}^{-1} \\ &= T_1T_2\sigma \dots T_{N-1}T_1^{-1}T_2^{-1} \dots \sigma T_{N-2}^{-1}T_{N-1}^{-1} \\ &= T_1 \dots T_{N-2}\sigma^2 T_2^{-1} \dots T_{N-1}^{-1}. \end{aligned}$$

   Setting expressions equal we obtain the final expression:

$$\begin{aligned} Y_1Y_N &= Y_NY_1 \\ \Rightarrow T_1 \dots T_{N-1}\sigma^2 T_1^{-1} \dots T_{N-1}^{-1} &= T_1 \dots T_{N-2}\sigma^2 T_2^{-1} \dots T_{N-1}^{-1} \\ \Rightarrow T_{N-1}\sigma^2 T_1^{-1} &= \sigma^2 \\ \Rightarrow T_{N-1}\sigma^2 &= \sigma^2 T_1. \end{aligned}$$

   Having obtained the expression $T_{N-1}\sigma^2 = \sigma^2 T_1$, one questions what happens after repeated applications of σ; that is, is there a relation involving σ^N?

3. $T_{i-1}\sigma = \sigma T_i$ for $i = 2, \ldots, N-1 \Rightarrow T_i = \sigma^{-1}T_{i-1}\sigma$ for $i = 2, \ldots, N-1$. Therefore we can replace T_{i-1} with $T_{i-1} = \sigma^{-1}T_{i-2}\sigma$ to get:

$$T_i = \sigma^{-2}T_{i-2}\sigma^2.$$

Now replace T_{i-2} with $T_{i-2} = \sigma^{-1}T_{i-3}\sigma$ and repeat this procedure until finally

$$\begin{aligned} T_i &= \sigma^{-3}T_{i-3}\sigma^3 \\ &= \sigma^{-(i-1)}T_1\sigma^{i-1}. \end{aligned}$$

When $i = N-1$

$$\begin{aligned} T_{N-1} &= \sigma^{-(N-2)}T_1\sigma^{N-2} \\ \Rightarrow T_{N-1}\sigma^2 &= \sigma^{-(N-2)}T_1\sigma^{N}. \end{aligned}$$

But $T_{N-1}\sigma^2 = \sigma^2 T_1$ so we substitute to get:

$$\begin{aligned} \sigma^2 T_1 &= \sigma^{-(N-2)}T_1\sigma^N \\ \Rightarrow \sigma^N T_1 &= T_1\sigma^N \\ \Rightarrow \sigma^N T_i &= T_i\sigma^N \quad \text{for} \quad i = 1, \ldots, N-1. \end{aligned}$$

2A.2 $\prod_{j=1}^{N} Y_j = \sigma^N$

We show that $\prod_{j=1}^{N} Y_j = \sigma^N$. Although this identity is already well-known [2], we present our own proof for the interested reader.

Define the operator P_k by

$$P_k := \sigma^k (T_1 \ldots T_k)^{-1} (T_2 \ldots T_{k+1})^{-1} \ldots (T_{N-k} \ldots T_{N-1})^{-1}. \tag{2A.1}$$

We want to show by induction that this is equal to $P_k = \displaystyle\prod_{j=N-k+1}^{N} Y_j$.

1. For $k = 1$:

$$\begin{aligned} P_1 &:= \sigma^1 (T_1)^{-1} (T_2)^{-1} \ldots (T_{N-1})^{-1} \\ &= \sigma T_1^{-1} T_2^{-1} \ldots T_{N-1}^{-1} \\ &= Y_N, \end{aligned}$$

so P_1 is indeed equal to $\prod_{j=N-1+1}^{N} Y_j = Y_N$, and the assertion is true for $k = 1$.

2. Now assume that our assertion is true for some k, namely,

$$P_k = \sigma^k (T_1 \ldots T_k)^{-1} (T_2 \ldots T_{k+1})^{-1} \ldots (T_{N-k} \ldots T_{N-1})^{-1} = \prod_{j=N-k+1}^{N} Y_j.$$

If this holds, then $P_k Y_{N-k}$ is $\prod_{j=N-k}^{N} Y_j$ because all the Y_i commute. Using $Y_{N-k} = T_{N-k} \ldots T_{N-1} \sigma T_1^{-1} \ldots T_{N-k-1}^{-1}$, we can rewrite this same expression as

$$\begin{aligned} P_k Y_{N-k} &= \left[\sigma^k (T_1 \ldots T_k)^{-1} (T_2 \ldots T_{k+1})^{-1} \ldots (T_{N-k} \ldots T_{N-1})^{-1}\right] \\ &\quad \times \left[T_{N-k} \ldots T_{N-1} \sigma T_1^{-1} \ldots T_{N-k-1}^{-1}\right] \\ &= \left[\sigma^k (T_1 \ldots T_k)^{-1} (T_2 \ldots T_{k+1})^{-1} \ldots (T_{N-k-1} \ldots T_{N-2})^{-1}\right] \\ &\quad \times \left[\sigma T_1^{-1} \ldots T_{N-k-1}^{-1}\right]. \end{aligned}$$

Using $T_i^{-1} \sigma = \sigma T_{i+1}^{-1}$, all σs can be moved to the left:

$$\begin{aligned} P_k Y_{N-k} &= \left[\sigma^{k+1} (T_2 \ldots T_{k+1})^{-1} (T_3 \ldots T_{k+2})^{-1}\right] \ldots \\ &\quad \ldots \left[(T_{N-k} \ldots T_{N-1})^{-1} T_1^{-1} \ldots T_{N-k-1}^{-1}\right]. \end{aligned}$$

T_i commutes with all other Ts except T_{i+1} and T_{i-1}, so we may pull the rightmost operators T_1^{-1} to T_{N-k-1}^{-1} as far as possible to the left:

$$\begin{aligned} P_k Y_{N-k} &= \sigma^{k+1} \left[(T_2 \ldots T_{k+1})^{-1} T_1^{-1}\right] \left[(T_3 \ldots T_{k+2})^{-1} T_2^{-1}\right] \ldots \\ &\quad \ldots \left[(T_{N-k} \ldots T_{N-1})^{-1} T_{N-k-1}^{-1}\right] \\ &= \sigma^{k+1} (T_1 \ldots T_{k+1})^{-1} (T_2 \ldots T_{k+2})^{-1} \ldots (T_{N-k-1} \ldots T_{N-1})^{-1}. \end{aligned}$$

But (2A.1) tells us that this is precisely the definition of P_{k+1}. Thus, $P_k Y_{N-k} = P_{k+1}$, so $P_{k+1} = \prod_{j=N-k}^{N} Y_j$ and our assertion holds for $k + 1$ if it holds for k.

This therefore verifies that

$$\sigma^k (T_1 \dots T_k)^{-1} (T_2 \dots T_{k+1})^{-1} \dots (T_{N-k} \dots T_{N-1})^{-1} \quad = \quad \prod_{j=N-k+1}^{N} Y_j$$

for all $k = 1, 2, \dots, N-1$. For $k = N-1$, this gives

$$\sigma^{N-1} (T_1 \dots T_{N-1})^{-1} \quad = \quad \prod_{j=2}^{N} Y_j.$$

But $\sigma^{-1}(T_1 \dots T_{N-1})^{-1} = Y_1^{-1}$, so we find that

$$\prod_{j=1}^{N} Y_j = \sigma^N. \qquad \blacksquare$$

2A.3 The DAHA $\mathcal{D}_N(t, q)$ in terms of σ

We want to write the defining relations of a DAHA solely in terms of σ and the T_i; that is we must rewrite the following equations

$$\begin{aligned} Y_1 Z_2 Y_1^{-1} Z_2^{-1} &= T_1^2, \\ Y_i \left(\prod_{j=1}^{N} Z_j \right) &= q \left(\prod_{j=1}^{N} Z_j \right) Y_i. \end{aligned}$$

1. We have that $Y_1 Z_2 Y_1^{-1} Z_2^{-1} = T_1^2$ and by definition of Y_1

$$\begin{aligned} Y_1 Z_2 &= T_1^2 Z_2 Y_1 \\ &= T_1 T_1 Z_2 T_1 \dots T_{N-1} \sigma. \end{aligned}$$

   Now we can push Z_2 all the way to the right of the expression. First write Z_2 in terms of Z_1 by (2.17), then using (2.16) and since Z_1 commutes with $(T_3 \dots T_{N-1})$ we get

$$\begin{aligned} Y_1 Z_2 &= T_1 Z_1 T_2 \dots T_{N-1} \sigma \\ &= T_1 T_2 Z_1 T_3 \dots T_{N-1} \sigma \\ &= T_1 \dots T_{N-1} Z_1 \sigma. \end{aligned}$$

   But we already know that $Y_1 Z_2 = T_1 \dots T_{N-1} \sigma Z_2$ by definition of Y_1.

Therefore:

$$\begin{aligned} T_1 \dots T_{N-1} \sigma Z_2 &= T_1 \dots T_{N-1} Z_1 \sigma \\ \Rightarrow \sigma Z_2 &= Z_1 \sigma \\ \Rightarrow \sigma Z_i &= Z_{i-1} \sigma \text{ for } i = 2, \dots, N. \end{aligned}$$

This relation, $\sigma Z_i = Z_{i-1}\sigma$ is valid only for $i = 2, \dots, N$, so we must also consider the case $i = 1$.

2. We begin by examining the product of the Z_i by the product of the T_i. Using $T_i Z_j = Z_j T_i$ for $j \neq i, i+1$ this can be rewritten as follows

$$(Z_1 \dots Z_N)(T_1 \dots T_{N-1}) = Z_1 (Z_2 T_1)(Z_3 T_2) \dots (Z_N T_{N-1}).$$

By (2.17) we can push each Z_i to the right of each T_i

$$(Z_1 \dots Z_N)(T_1 \dots T_{N-1}) = Z_1 \left(T_1^{-1} Z_1\right) \left(T_2^{-2} Z_2\right) \dots \left(T_{N-1}^{-1} Z_{N-1}\right).$$

Now (2.16) means all the Z_i commute with the T_i to their right

$$(Z_1 \dots Z_N)(T_1 \dots T_{N-1}) = Z_1 T_1^{-1} \dots T_{N-1}^{-1} Z_1 \dots Z_{N-1}.$$

Use (2.17) repeatedly to push Z_1 to the right of the expression through the T_i^{-1}

$$\begin{aligned} (Z_1 \dots Z_N)(T_1 \dots T_{N-1}) &= T_1 Z_2 T_2^{-1} \dots T_{N-1}^{-1} Z_1 Z_2 \dots Z_{N-1} \\ &= T_1 \dots T_{N-1} Z_N Z_1 \dots Z_{N-1}. \end{aligned}$$

All Z_i commute with each other by (2.15) giving us the final expression

$$(Z_1 \dots Z_N)(T_1 \dots T_{N-1}) = (T_1 \dots T_{N-1})(Z_1 \dots Z_N).$$

Therefore we see that the product of the T_i, $(T_1 \dots T_{N-1})$, commutes with the product of the Z_i, $(Z_1 \dots Z_N)$.
We shall use this important result to find an expression for $\sigma Z_i = Z_{i-1}\sigma$ when $i = 1$.

3. Let us see how this commutation relation influences $Y_i\left(\prod_{j=1}^{N} Z_j\right) = q\left(\prod_{j=1}^{N} Z_j\right) Y_i$.

   For $i=1$, by the definition of Y_1 and since all the Z_i commute with each other

$$\begin{aligned} q\left(\prod_{j=1}^{N} Z_j\right) Y_1 &= q\left(Z_1 \ldots Z_N\right)\left(T_1 \ldots T_{N-1}\right)\sigma \\ &= q\left(T_1 \ldots T_{N-1}\right)\left(Z_N \ldots Z_1\right)\sigma. \end{aligned}$$

   Using $\sigma Z_i = Z_{i-1}\sigma$ for $i = 2, \ldots, N$ to move σ to the left of the Z_i

$$q\left(\prod_{j=1}^{N} Z_j\right) Y_1 = q\left(T_1 \ldots T_{N-1}\right) Z_N \sigma \left(Z_N \ldots Z_2\right).$$

   This must equal (2.19) with $i=1$

$$\begin{aligned} Y_1\left(\prod_{j=1}^{N} Z_j\right) &= \left(T_1 \ldots T_{N-1}\right)\sigma\left(Z_1 \ldots Z_N\right) \\ &= \left(T_1 \ldots T_{N-1}\right)\sigma\left(Z_N \ldots Z_1\right). \end{aligned}$$

   Setting both expressions equal yields:

$$\begin{aligned} \left(T_1 \ldots T_{N-1}\right)\sigma\left(Z_N \ldots Z_1\right) &= q\left(T_1 \ldots T_{N-1}\right) Z_N \sigma\left(Z_N \ldots Z_2\right) \\ \Rightarrow \sigma Z_1 &= q Z_N \sigma. \end{aligned}$$

   Since we already found that $\sigma Z_i = Z_{i-1}\sigma$ for $i = 2, \ldots, N$, defining $Z_0 = qZ_N$, completes the expression

$$\sigma Z_i = Z_{i-1}\sigma \text{ for } i = 1, \ldots, N.$$

2A.4 The derivations $\zeta_1^N T_i = T_i \zeta_1^N$ and $T_i \zeta_2^N = \zeta_2^N T_i$

We have the relations $T_{i-1}\zeta_1 = \zeta_1 T_i$ for $i = 2, \ldots, N-1$ and $T_{N-1}\zeta_1^2 = \zeta_1^2 T_1$.

1. $T_{i-1}\zeta_1 = \zeta_1 T_i$ for $i = 2, \ldots, N-1 \Rightarrow T_i = \zeta_1^{-1} T_{i-1}\zeta_1$ for $i = 2, \ldots, N-1$. Therefore we can replace T_{i-1} with $T_{i-1} = \zeta_1^{-1} T_{i-2}\zeta_1$ to get:

$$T_i = \zeta_1^{-2} T_{i-2}\zeta_1^2.$$

Now replace T_{i-2} with $T_{i-2} = \zeta_1^{-1} T_{i-3} \zeta_1$ and repeat this procedure until finally

$$\begin{aligned} T_i &= \zeta_1^{-3} T_{i-3} \zeta_1^3 \\ &= \zeta_1^{-(i-1)} T_1 \zeta_1^{i-1}. \end{aligned}$$

When $i = N-1$

$$\begin{aligned} T_{N-1} &= \zeta_1^{-(N-2)} T_1 \zeta_1^{N-2} \\ \Rightarrow T_{N-1}\zeta_1^2 &= \zeta_1^{-(N-2)} T_1 \zeta_1^{N}. \end{aligned}$$

But $T_{N-1}\zeta_1^2 = \zeta_1^2 T_1$ so we substitute to get:

$$\begin{aligned} \zeta_1^2 T_1 &= \zeta_1^{-(N-2)} T_1 \zeta_1^N \\ \Rightarrow \zeta_1^N T_1 &= T_1 \zeta_1^N \\ \Rightarrow \zeta_1^N T_i &= T_i \zeta_1^N \quad \text{for} \quad i = 1, \ldots, N-1. \end{aligned}$$

We have the two relations $T_{i+1}\zeta_2 = \zeta_2 T_i$ for $i = 1, \ldots, N-2$ and $\zeta_2^2 T_{N-1} = T_1 \zeta_2^2$.

2. $T_{i+1}\zeta_2 = \zeta_2 T_i$ for $i = 1, \ldots, N-2 \Rightarrow T_{i+1} = \zeta_2 T_i \zeta_2^{-1}$ for $i = 1, \ldots, N-2$. Therefore we can replace T_i with $T_i = \zeta_2 T_{i-1} \zeta_2^{-1}$ to get:

$$T_{i+1} = \zeta_2^2 T_{i-1} \zeta_2^{-2}.$$

Now replace T_{i-1} with $T_{i-1} = \zeta_2 T_{i-2} \zeta_2^{-1}$ and repeat this procedure until finally

$$\begin{aligned} T_{i+1} &= \zeta_2^3 T_{i-2} \zeta_2^{-3} \\ &= \zeta_2^i T_1 \zeta_2^{-i}. \end{aligned}$$

When $i = N-2$

$$\begin{aligned} T_{N-1} &= \zeta_2^{N-2} T_1 \zeta_2^{-(N-2)} \\ \Rightarrow \zeta_2^2 T_{N-1} &= \zeta_2^N T_1 \zeta_2^{-(N-2)}. \end{aligned}$$

But $\zeta_2^2 T_{N-1} = T_1 \zeta_2^2$ so we substitute to get:

$$\begin{aligned} T_1\zeta_2^2 &= \zeta_2^N T_1 \zeta_2^{-(N-2)} \\ \Rightarrow T_1\zeta_2^N &= \zeta_2^N T_1 \\ \Rightarrow T_i\zeta_2^N &= \zeta_2^N T_i \quad \text{for} \quad i = 1, \ldots, N-1. \end{aligned}$$

Chapter 3

Graphical Calculus for the Double Affine Q-Dependent Braid Group

In this chapter our primary objective is to give readers a clear picture of the structure of a large group we created called a double affine Q-dependent braid group ($\mathcal{D}_N\{Q\}$). We initially developed this group as a generalisation of the double affine Hecke algebra. As such it is constructed by appending to the braid group a set of N commuting operators $\{Q_i\} = \{Q_1 \ldots, Q_N\}$, before extending it to an affine Q-dependent braid group. In addition we establish its position in relation to other well known abstract algebraic structures, in particular, the double affine Hecke algebra, $\mathcal{D}_N(t, q)$ which we defined in the previous chapter. As it is through representations that we learn most about abstract mathematical concepts, we also present a novel intuitive graphical calculus to complement the original algebraic description.

Our interest in $\mathcal{D}_N\{Q\}$ stems from its pole position with respect to other algebraic structures whose primary element is a braid group. In fact, appending to the double affine braid group a set of operators $\{Q_i\}$ generalises, without affecting, the structure of the underlying braid group. It does so by turning braid group strands into ribbons and permitting 2π twists. Since the operators $\{Q_i\}$ do not intertwine the braid group generators, then the original braid group corresponds to $\mathcal{B}_N\{Q\}/\langle Q_i\rangle$, where $\langle Q_i\rangle$ is the normal group freely generated by the operators Q_i. Thus the original braid group is in other words equivalent to $\mathcal{B}_N\{Q\}$ where $Q = \mathbb{1}$. Similarly the affine braid group corresponds to $\mathcal{A}_N\{Q\}/\langle Q_i\rangle$. Naturally the elliptic braid group [16, 17] is obtained from $\mathcal{D}_N\{Q\}$ by ignoring the twists or equivalently by contracting ribbons to strands, i.e. $\mathcal{D}_N\{Q\}/\langle Q_i\rangle$. In addition, taking the quotient $\mathcal{D}_N\{Q\}/\langle Q_i Q_{i+1}^{-1}\rangle$ is equivalent to considering twists on different ribbons as identical. Furthermore imposing the Hecke relation and setting $Q_i = q\mathbb{1}$, where $q \in \mathbb{C}$, we obtain the double affine Hecke algebra (of type A) [2, 15]. We illustrate all of these relations in Figure 3.1.

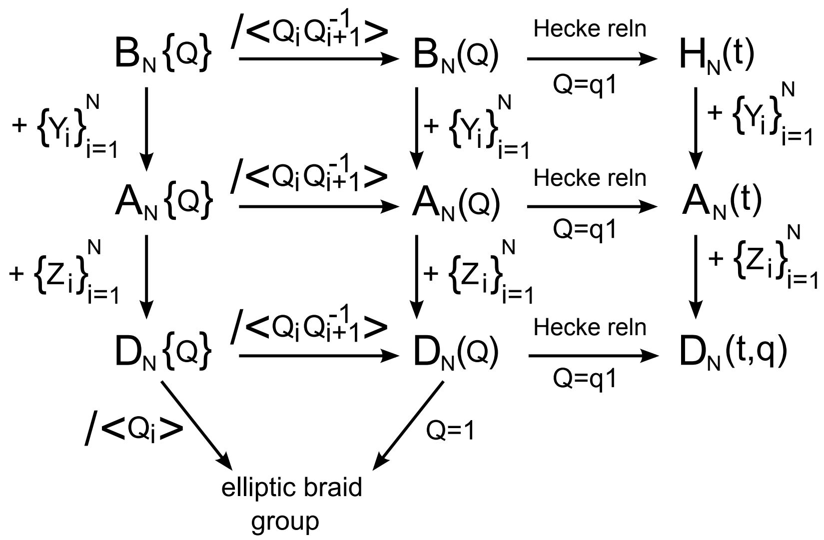


Figure 3.1: Diagram describing the relations of $\mathcal{D}_N\{Q\}$ with other algebraic structures whose primary element is a braid group. To comment on the notation we introduce in this figure and will adopt in this chapter, note that $\mathcal{B}_N\{Q\} = \mathcal{B}_N(Q_1, Q_2, \dots, Q_N)$ and $\mathcal{B}_N(Q) = \mathcal{B}_N\{Q\}/\langle Q_i Q_{i+1}^{-1}\rangle \simeq \mathcal{B}_N(Q, Q, \dots, Q)$.

To further clarify the position of $\mathcal{D}_N\{Q\}$ in relation to other known algebraic structures we describe Figure 3.1 in greater detail. The rightmost column of this figure corresponds to the construction of the double affine Hecke algebra $\mathcal{D}_N(t, q)$ which we described in the previous chapter. As illustrated, and as we previously showed, $\mathcal{D}_N(t, q)$ can be obtained by firstly appending to the Hecke algebra $\mathcal{H}_N(t)$, a set of operators Y_i, to define the affine Hecke algebra $\mathcal{A}_N(t)$. Appending to $\mathcal{A}_N(t)$ a further N generators denoted by Z_i, then completes the description of $\mathcal{D}_N(t, q)$.

In the middle column of Figure 3.1 we depict a Q-dependent generalisation of the construction of $\mathcal{D}_N(t, q)$. We develop this group by considering the parameter q, which as we saw in the previous chapter is included in the definition of a DAHA, as a special value corresponding to the action of an extra generator Q. Therefore the middle column describes the construction of the double affine Q dependent braid group $\mathcal{D}_N(Q)$ from which $\mathcal{D}_N(t, q)$ is obtained by setting the single generator $Q = q\mathbb{1}$. Note also that the braid group generators T_i do not satisfy the Hecke relation and for this reason the middle column describes a group structure and not an algebra.

Finally, the complete description of the construction of $\mathcal{D}_N\{Q\}$ is given in the first column. $\mathcal{D}_N\{Q\}$ is the double affine Q-dependent braid group which has an additional set of N commuting generators Q_i with respect to $\mathcal{D}_N(Q)$. In this case all of the Q_i are unique and it is only when we consider them as identical via the quotient $\langle Q_i Q_{i+1}^{-1}\rangle$ that we recover $\mathcal{D}_N(Q)$.

To complement the algebraic description of the double affine Q-dependent braid group, we also provide a pictorial representation that fully captures all of its complex structure. The graphical calculus we develop is based on ribbons within cubes, where opposite vertical faces of the cube are identified; a topologically equivalent presentation is given in terms of ribbons living inside a toroid. We clearly illustrate all of the defining relations of $\mathcal{D}_N\{Q\}$ in our new cube-ribbon representation. It provides a concrete visual description of its structure, in particular we obtain a very straightforward interpretation of the action of the generators Q_i which create 2π twists in the ribbons. In the quotient group $\mathcal{D}_N\{Q\}/\langle Q_i Q_{i+1}^{-1}\rangle$, where we obtain the double affine Hecke algebra, we show that q corresponds to the factor when replacing a ribbon with a twist by one with no twist at all. Hence a major achievement of our cube-ribbon representation is that it describes double affine Hecke algebras for all values of q, something which has not been accomplished until now. In $\mathcal{D}_N\{Q\}/\langle Q_i\rangle$ the ribbons are reduced to strands and twists are no longer possible, therefore our pictorial representation gives a toroidal description of the elliptic braid group.

The first step in describing our construction of the double affine Q-dependent braid group, $\mathcal{D}_N\{Q\}$, is to define its underlying Q-dependent braid group $\mathcal{B}_N\{Q\}$.

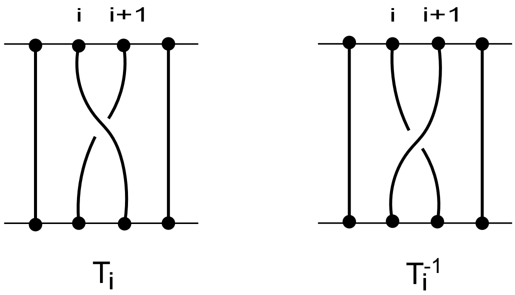
3.1 The Q-Dependent Braid Group $\mathcal{B}_N\{Q\}$

We begin by defining the braid group and its Q-dependent extension. These are essential to our construction of $\mathcal{D}_N\{Q\}$. Similarly its well-established pictorial representation serves as a starting point for our cube-ribbon representation.

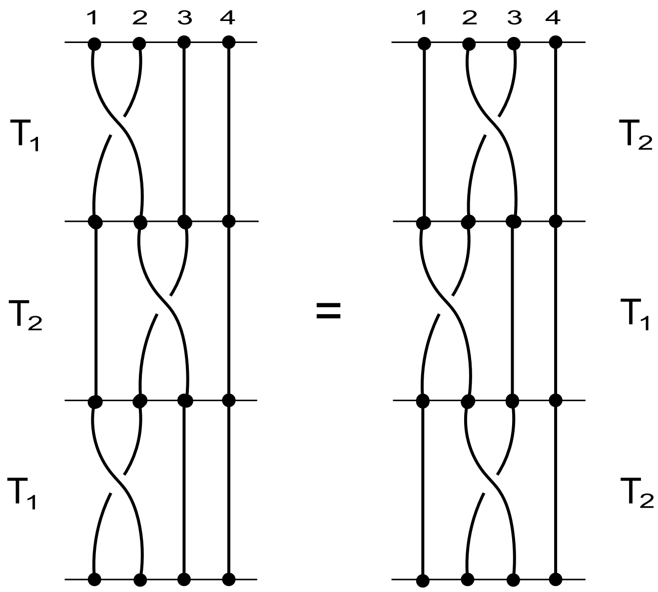
As previously presented in Section 2.1, the N-strand braid group $\mathcal{B}_N$ is the group generated by the $N-1$ invertible elements $\{T_i | i = 1, .., N-1\}$ satisfying the relations

$$\begin{aligned} T_i T_j &= T_j T_i \text{ for } |i-j| \geq 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \text{ otherwise .} \end{aligned}$$

It is indeed well known that this algebraic description can be incorporated into a pictorial one by defining T_i and its inverse T_i^{-1} to correspond to the exchange of the i^{th} and $(i+1)^{\text{th}}$ strands as illustrated below:



Multiplication is then defined by stacking: AB is the braid obtained by stacking A on top of B and gluing the bottom ends of the strands in A to the top ends of those in B. To illustrate, we demonstrate the braid relation in $\mathcal{B}_4$ for $i = 1$, i.e. $T_1T_2T_1 = T_2T_1T_2$:



Pulling all strands tight one can see that the relation is satisfied. All other braid group relations follow similarly.

We now generalise the braid group in a specific way, by defining the N-strand Q-dependent braid group, $\mathcal{B}_N\{Q\}$, as follows: $\mathcal{B}_N\{Q\}$ is the group generated by the invertible elements $\{T_i|i = 1, .., N-1\}$ satisfying (2.1) and (2.2), in addition to a set of commuting elements $\{Q_i|i = 1, .., N\}$ satisfying the relations

$$Q_iQ_j = Q_jQ_i \text{ for all } i, j, \tag{3.1}$$

$$T_iQ_j = Q_jT_i \text{ for } j \neq i, i+1, \tag{3.2}$$

$$T_iQ_i = Q_{i+1}T_i \text{ for } i = 1, \dots, N-1, \tag{3.3}$$

$$T_iQ_{i+1} = Q_iT_i \text{ for } i = 1, \dots, N-1. \tag{3.4}$$

Note that the Q_i do not intertwine the T_i. Furthermore the latter two relations give $Q_{i+1} = T_i Q_i T_i^{-1}$ and $Q_{i+1} = T_i^{-1} Q_i T_i$ for all $i = 1, \ldots, N-1$. Therefore

$$\begin{aligned} T_i Q_i T_i^{-1} &= T_i^{-1} Q_i T_i \\ \Rightarrow T_i^2 Q_i &= Q_i T_i^2 \text{ for } i = 1, \ldots, N-1. \end{aligned}$$

Since $T_i^2 Q_j = Q_j T_i^2$ for all i, j, then the Qs commute with all even powers of the Ts but not with odd powers.

As it stands, only the trivial braids – those whose strands go straight from top to bottom without crossing – can represent the Qs in a way consistent with (3.1)-(3.4). We shall see later how to introduce nontrivial graphical representations for the Qs.

3.2 Affine Braid Groups

3.2.1 The Affine Braid Group $\mathcal{A}_N$

In Section 2.2 we described how the braid group can be extended to an affine Hecke algebra $\mathcal{A}_N(t)$. In a similar fashion to describe our definition of an affine Q-dependent braid group, we must necessarily firstly extend the Q-dependent braid group $\mathcal{B}_N\{Q\}$ to an affine braid group $\mathcal{A}_N$. We accomplish this by appending to it N invertible operators Y_i which satisfy the relations

$$\begin{aligned} Y_i Y_j &= Y_j Y_i \text{ for all } i, j, && (3.5) \\ T_i Y_j &= Y_j T_i \text{ for } j \neq i, i+1, && (3.6) \\ T_i Y_{i+1} T_i &= Y_i \text{ for } i = 1, \ldots, N-1. && (3.7) \end{aligned}$$

Again we see that $\mathcal{A}_N$ is fully generated by Y_1 and the T_i, since we need only one of the Y_i (and all of the T_i) to generate the others. Furthermore, equation (3.7) can be used to rewrite Y_i for $i = 2, \ldots, N$ as

$$Y_i = T_{i-1}^{-1} T_{i-2}^{-1} \ldots T_1^{-1} Y_1 T_1^{-1} \ldots T_{i-2}^{-1} T_{i-1}^{-1}.$$

As in Subsection 2.2.1, introducing the element σ defined as:

$$\sigma := T_{N-1}^{-1} T_{N-2}^{-1} \ldots T_1^{-1} Y_1, \tag{3.8}$$

and following the derivations of Appendix 2A.1, the defining relations for $\mathcal{A}_N$ may be rewritten in terms of σ and the T_i as

$$Y_i = \begin{cases} T_1T_2\dots T_{N-1}\sigma & i=1\,, \\ T_i\dots T_{N-1}\sigma T_1^{-1}\dots T_{i-1}^{-1} & i=2,\dots,N-1, \\ \sigma T_1^{-1}\dots T_{N-1}^{-1} & i=N, \end{cases}$$

$$\begin{aligned} T_{i-1}\sigma &= \sigma T_i, \qquad i=2,\dots,N-1, \\ T_{N-1}\sigma^2 &= \sigma^2 T_1. \end{aligned}$$

Recall also that the above relations imply that $\sigma^N T_i = T_i\sigma^N$ which tells us that σ^N commutes with all the Y_i, and is thus central in $\mathcal{A}_N$.

Before defining our generalisation of $\mathcal{A}_N$ to the affine Q-dependent braid group, it is perhaps worth pointing out that unlike the definition of the affine Hecke algebra in Section 2.2 where the braid group generators are required to satisfy the Hecke relation, in the affine braid group $\mathcal{A}_N$ there is no such restriction on the Q-dependent braid group generators. They are part of a group and *not* an algebra.

3.2.2 The Affine Q-Dependent Braid Group $\mathcal{A}_N\{Q\}$

In a similar fashion to $\mathcal{B}_N\{Q\}$, we extend $\mathcal{A}_N$ to an affine Q-dependent braid group, $\mathcal{A}_N\{Q\}$, by defining how the set of elements $\{Q_i|i=1,..,N\}$ interact with the affine generators Y_i.

Therefore in addition to all of the defining relations of $\mathcal{A}_N$, the generators of $\mathcal{A}_N\{Q\}$ must also satisfy

$$Y_iQ_j = Q_jY_i \text{ for all } i,j. \tag{3.9}$$

Using the definition of σ, (3.8), one can rewrite (3.9), to obtain $\mathcal{A}_N\{Q\}$ purely in terms of the T_i, σ and Q_i:

$$\begin{aligned} \sigma Q_i &= Q_{i-1}\sigma \text{ for } i=2,\dots,N, \\ \sigma Q_1 &= Q_N\sigma. \end{aligned}$$

It is straightforward to show that these relations also imply that $\sigma^N Q_i = Q_i\sigma^N$.

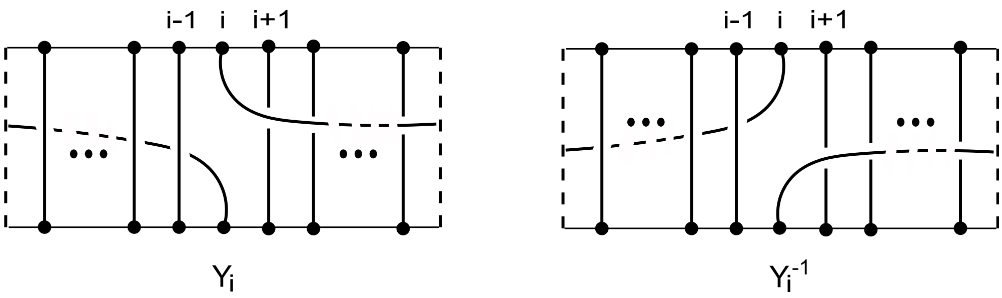
Having fully described our definition of an affine Q-dependent braid group, $\mathcal{A}_N\{Q\}$, we now incorporate its algebraic structure into an intuitive graphical one.

3.2.3 Pictorially Representing $\mathcal{A}_N\{Q\}$

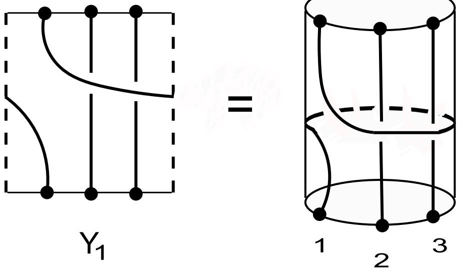
We have already seen that in the pictorial representation of the braid group $\mathcal{B}_N$, the braiding of the strands takes place in the strip in a strict top-to-bottom direction. To incorporate the extra set of generators, we develop the idea of [7] where the Y_i were thought of as braiding on the surface of cylinders.

In order for us to do this, we turn the braid group strip into a cylinder by identifying the left and right edges; to highlight this point, we represent these edges with dashed lines. This means that we can now braid in a left-to-right (or vice versa) fashion by wrapping strands around the cylinder. This application of periodic boundary conditions is what gives us a pictorial representation for the affine Q-dependent braid group $\mathcal{A}_N\{Q\}$. (The braid group generators T_i still braid top-to-bottom as they did before we identified the sides.)

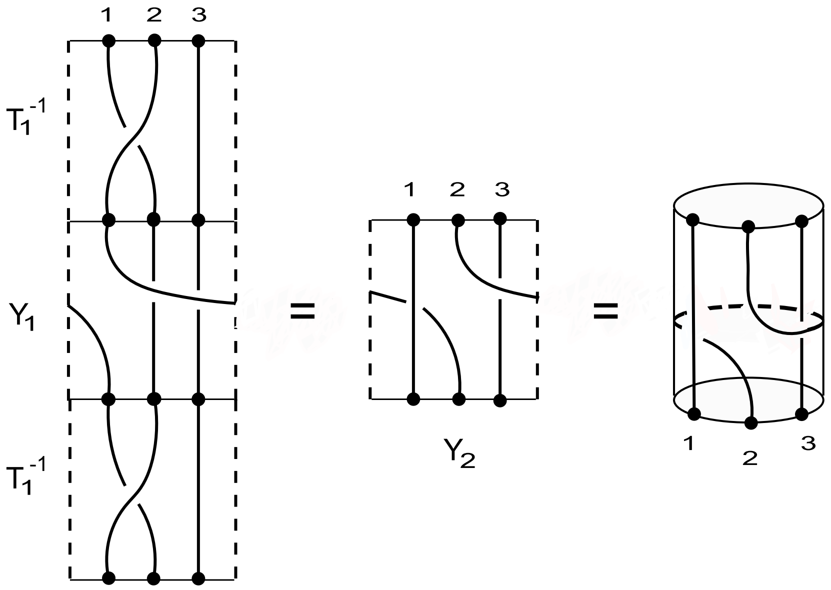
To illustrate this, we define the pictorial representations of the $\mathcal{A}_N\{Q\}$ generator Y_i and its inverse Y_i^{-1} as follows:



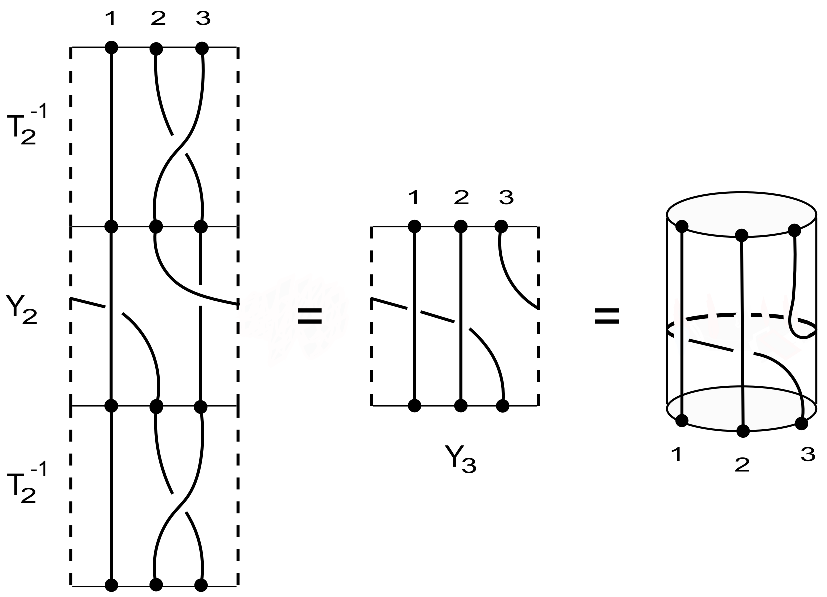
So we see that Y_i takes the strand starting at point i on the top edge and takes it to the same point on the bottom edge and leaves all other strands untouched, and does so such that it goes over all strands to the right $(i+1, \dots, N)$ and under all strands to the left $(1, \dots, i-1)$. For example, in the $N = 3$ case, Y_1 is given by either of the two pictures below:



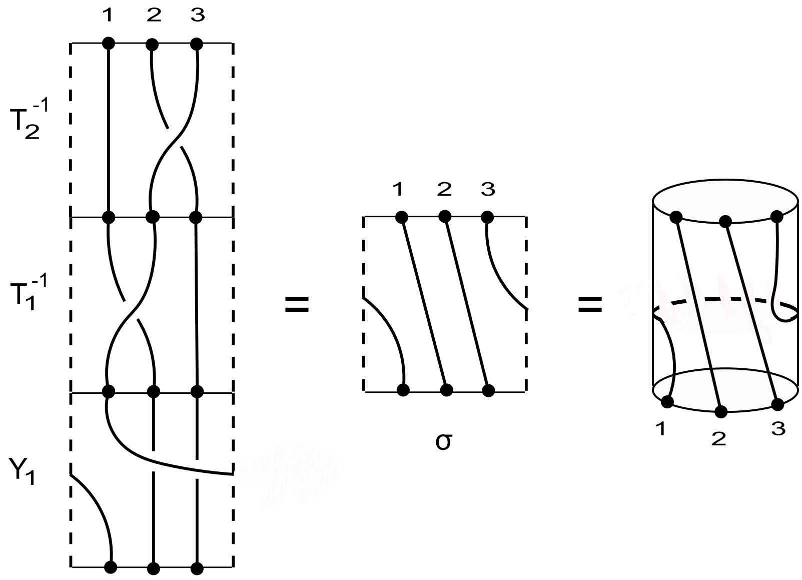
Multiplication is now defined by stacking cylinders on top of one another. We explicitly demonstrate this for Y_2. Recall that by (3.7) all of the Y_i are defined recursively so given Y_1 and the T_i we can construct all other Y_i. Algebraically in the $N = 3$ case $Y_2 = T_1^{-1} Y_1 T_1^{-1}$ and hence to demonstrate the multiplication of braiding on cylinders and to show that our pictorial representation is consistent we have that:



To complete the pictorial representation of the Y_i in the $N = 3$ strand case, we show that $Y_3 = T_2^{-1} Y_2 T_2^{-1}$.



Recall, from (3.8), that σ was defined in terms of Y_1: $\sigma = T_{N-1}^{-1} T_{N-2}^{-1} .... T_1^{-1} Y_1$. Since we now have a pictorial representation for all of the Ts and all of the Ys, we can illustrate the operator σ. In the $N = 3$ strand case, we have $\sigma = T_2^{-1} T_1^{-1} Y_1$, which looks like



From the illustration it is clear that σ has the same general form for all N, namely, it acts as a kind of raising operator on the indices by taking point i on the top to point $i+1$ on the bottom (with the cylindrical topology identifying point $N+1$ with 1). Therefore, we take this to be the pictorial definition of σ, and so together with the cylinders representing the T_i, all of the defining relations of the $\mathcal{A}_N\{Q\}$ follow suit.

At this point we have a complete pictorial representation for the Ys. However, the Qs are still only representable by trivial braids. Despite this we can extend $\mathcal{A}_N\{Q\}$ to a double affine Q-dependent braid group by incorporating a whole new set of generators and their graphical representations, as we will now show.

3.3 Double Affine Braid Groups

3.3.1 The Double Affine Q-Dependent Braid Group $\mathcal{D}_N\{Q\}$

We can extend $\mathcal{A}_N\{Q\}$ to a double affine Q-dependent braid group $\mathcal{D}_N\{Q\}$ by introducing a further N invertible generators Z_i satisfying the relations

$$
\begin{aligned}
Z_i Z_j &= Z_j Z_i \text{ for all } i, j, && (3.10)\\
T_i Z_j &= Z_j T_i \text{ for } j \neq i, i+1, && (3.11)\\
T_i Z_{i+1} T_i &= Z_i \text{ for } i = 1, \ldots, N-1, && (3.12)
\end{aligned}
$$

together with the set of elements $\{Q_i | i = 1, .., N\}$ which commute with all the Z_i and appear explicitly in relations intertwining the Y_i and the Z_i:

$$
\begin{aligned}
Z_i Q_j &= Q_j Z_i \text{ for all } i, j, && (3.13)\\
Y_1 Z_2 Y_1^{-1} Z_2^{-1} &= T_1^2, && (3.14)\\
Y_i \left(\prod_{j=1}^{N} Z_j \right) &= Q_i \left(\prod_{j=1}^{N} Z_j \right) Y_i, && (3.15)\\
Z_i \left(\prod_{j=1}^{N} Y_j \right) &= Q_i^{-1} \left(\prod_{j=1}^{N} Y_j \right) Z_i. && (3.16)
\end{aligned}
$$

As in the construction of the double affine Hecke algebra in Section 2.3, we can choose to eliminate the Y_i in favour of the cyclic operator σ. Then (3.14) and (3.15) can be rewritten as

$$
\begin{aligned}
Z_{i-1}\sigma &= \sigma Z_i, \text{ for } i = 2, \ldots, N,\\
Z_N \sigma &= Q_N^{-1} \sigma Z_1.
\end{aligned}
$$

Furthermore using the above relations, one can quickly see that

$$
Z_i \sigma^N = Q_i^{-1} \sigma^N Z_i, \qquad (3.17)
$$

and this, in addition to the identity $\prod_{j=1}^{N} Y_j = \sigma^N$ gives us (3.16). Therefore it is not independent of the other relations.

To summarise, we define a double affine Q-dependent braid group $\mathcal{D}_N\{Q\}$ to be the group generated by the T_i, Y_i, Z_i and Q_i which satisfy equations (2.1) and (2.2), (3.1)-(3.4), (3.5)-(3.7) alongside (3.9) and (3.10)-(3.15). We shall see shortly that the appearance of the operators Q_i in the last of these defining relations will strongly influence our choice of pictorial representation for $\mathcal{D}_N\{Q\}$.

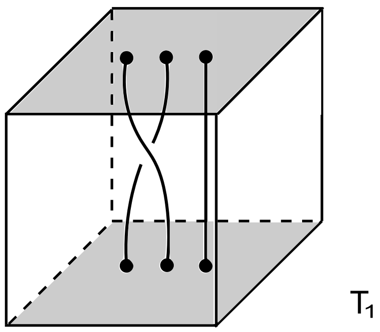
3.3.2 Graphical Representation of $\mathcal{D}_N\{Q\}$

Previously we extended the well known pictorial representation of the braid group to that of an $\mathcal{A}_N\{Q\}$ by identifying the two vertical edges and defining the action of the Y_i generators on the strands as wrapping around the resulting cylinder. We would now like to extend this $\mathcal{A}_N\{Q\}$ representation to one for a $\mathcal{D}_N\{Q\}$ by somehow incorporating the new generators Z_i into the picture.

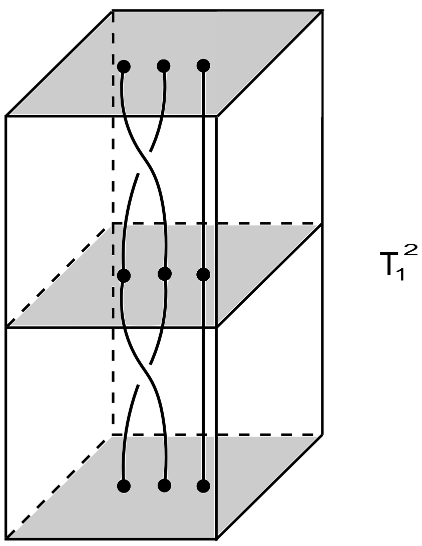
Our method for doing so is motivated by the $\mathcal{A}_N\{Q\}$ construction: the braid group generators do not wind strands at all; they simply connect points on the top edge to ones on the bottom. The Y_i generators, however, do wind the strands "perpendicular" to the T_i, namely, left-to-right (or vice versa) instead of top-to-bottom.

As the new Z_i generators have exactly the same relations between themselves and the Ts as the Ys do (both form independent affine braid groups), this suggests that we need a *third* direction. Therefore instead of a strip whose two vertical sides are identified, we now use a *cube* whose opposite vertical faces are identified. So the left and right faces of the cube are identified with the Y_i operators taking strands through them, while the front and back faces are identified with the Z_i generators taking strands through them.

To see this, first consider drawing each braid group generator T_i in a cube. The braiding now takes place within the cube from top to bottom:

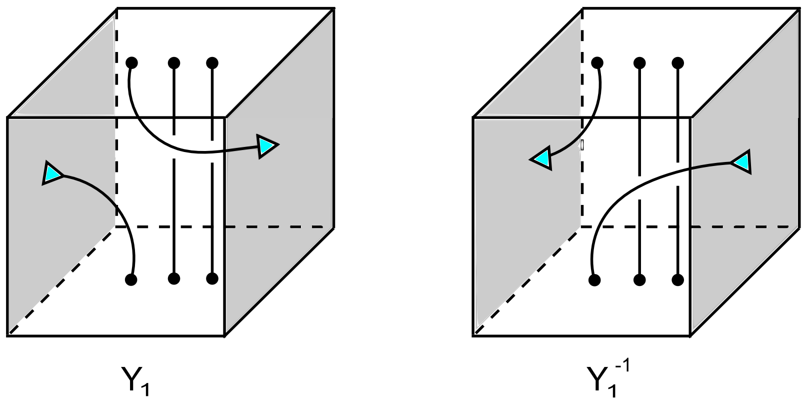


Multiplication is defined in the usual way, by stacking one cube onto another. So for example the element T_1^2 is represented as follows:

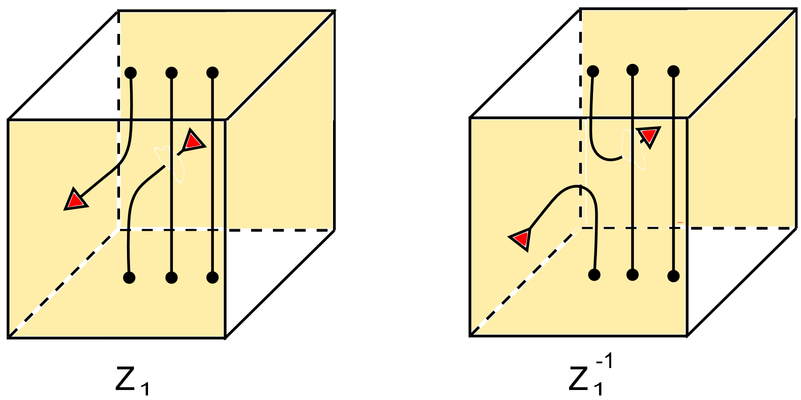


This representation is essentially the same as that for the elliptic braid group on a torus [16, 17], which is generated by T_i, Y_i and Z_i but requires all the Q_i to be unity. In Subsection 3.4.1, we show that the Q_i are indeed $\mathbb{1}$ for our representation, as expected. This is not a surprising result though as the elliptic braid group is simply $\mathcal{D}_N\{Q\}/\langle Q_i\rangle$. However, using three-dimensional cubes rather than a two-dimensional torus will allow us to generalise to values of Q_i other than unity, as we illustrate in Subsection 3.4.2.

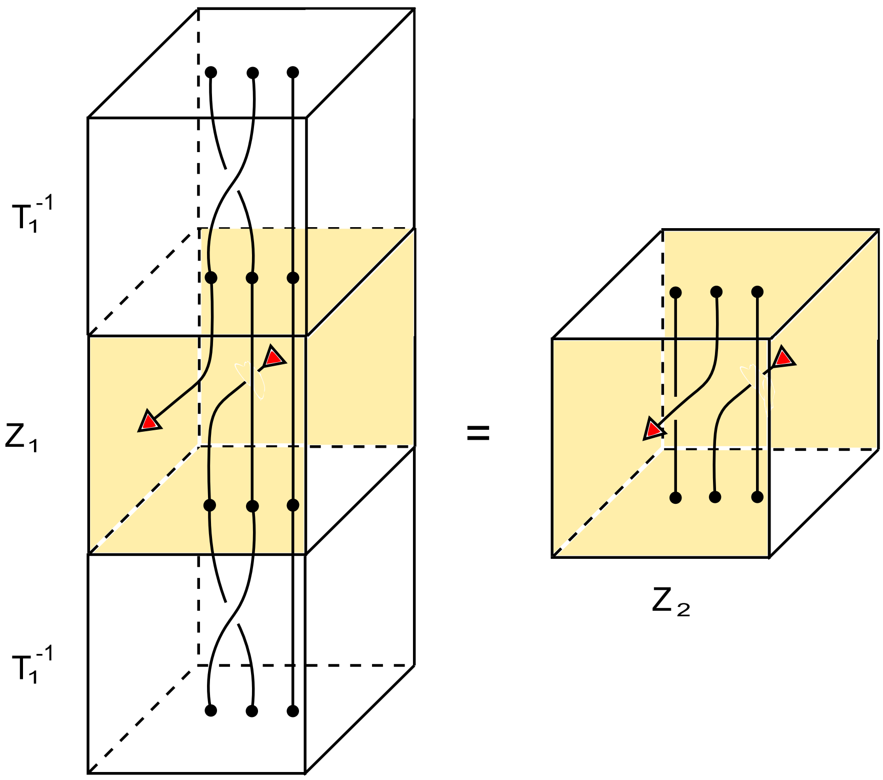
Recall that the affine Q-dependent braid group generators Y_i identified the left and right sides with each other to give braiding on a cylinder. In the cube representation, we identify the left and right faces of the cube with each other. In the following figure, the turquoise arrows traverse the coloured blue planes and wrap the strand around the cube from one to the other.



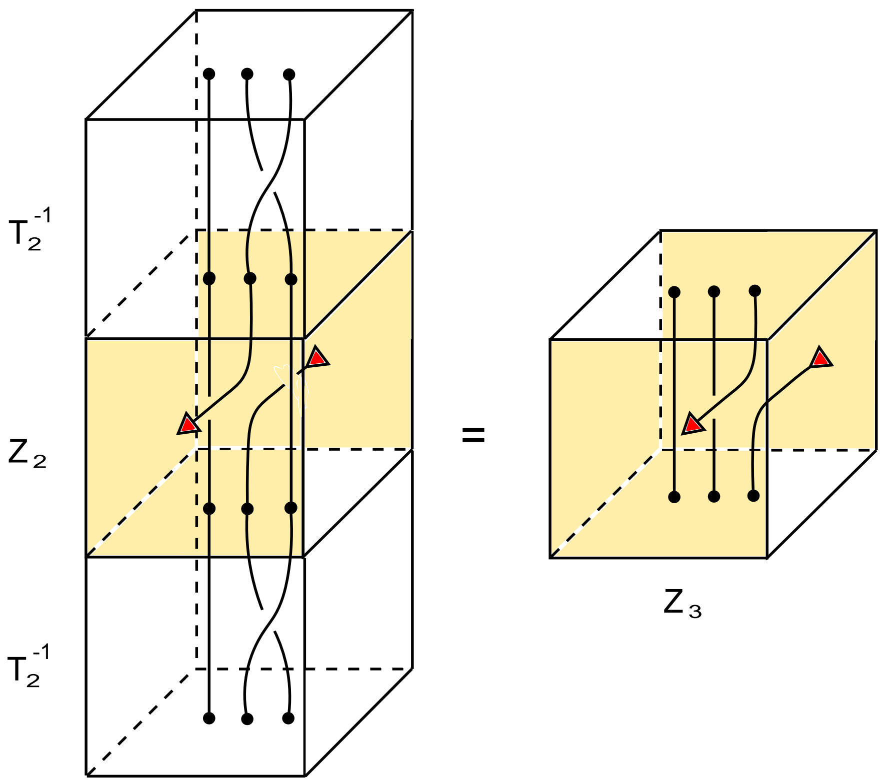
The additional $\mathcal{D}_N\{Q\}$ generators Z_i identify the front face of the cube with its back face. In the figure below, we use red arrows to indicate that the strand passes out through the coloured front face of the cube, then wraps around until it meets the strand that passes out the back face. More specifically, for the $N = 3$ case we define Z_1 (and its inverse) as



Having defined Z_1, we can now obtain all of the other Z_i for $i = 2, \ldots, N$ using $T_i Z_{i+1} T_i = Z_i$. So, for example, $Z_2 = T_1^{-1} Z_1 T_1^{-1}$:



For completeness we also give the pictorial representation of Z_3, and so all of the Z_i in the $N = 3$ case are depicted.

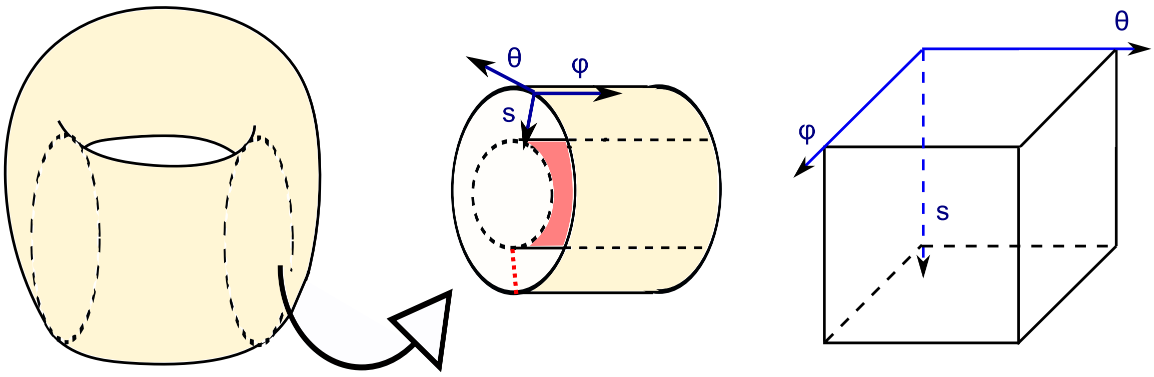


For the general N strand case one may proceed in this manner to construct Z_i for any i.

We see that its action is to take the i^{th} point on the top face, bring it out the front face of the cube, wrap around to come in the back face, and connect to the i^{th} point on the bottom, with all other strands simply going straight from top to bottom.

At this point, we highlight the fact that our cube is topologically equivalent to a hollowed-out toroid: identification of the opposing sides of any horizontal slice of the cube gives a 2-torus, and the region between the top and bottom faces – a time interval I if we view our strands as worldlines – gives the thickness. Thus, each of our generators is represented as N strands within the toroid $S^1 \times S^1 \times I$.

To illustrate this further, define two angles, θ and φ. We let θ be the direction in which the Y_i generators wrap around the toroid and φ is the direction the Z_i wrap around the toroid. So, in effect, the $\mathcal{A}_N\{Q\}$ generators Y_i encircle the torus within the toroid whereas the additional $\mathcal{D}_N\{Q\}$ generators Z_i encircle the empty space bounded by the toroid, as illustrated below:



where $s \in I$ is the time parameter.

A particularly nice feature of this toroidal representation, is that one can now clearly see the distinct directions in which the different generators wrap. In Figure 3.2 particular cross sections of the torus are illustrated, each one indicating precisely how the different generators are represented. For completeness we show all three types of generators, the T_i, Y_i and Z_i. Firstly we illustrate the generator T_2 which braids strands in the region between both torus surfaces. Then we show the generator Y_1 which encircles the inner torus and the generator Z_1 which braids strands around the empty space bounded by the toroid.

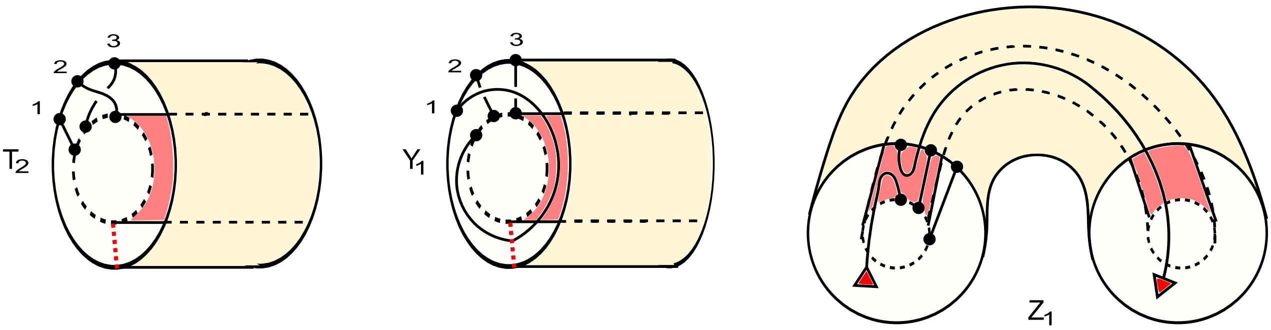


Figure 3.2: Toroidal representation of the action of the generators T_2, Y_1 and Z_1 in the $N = 3$ strand case.

We define multiplication by stuffing toroids inside each other: this is done such that the points on the inner boundary of the first (in order of multiplication) generator correspond to the points on the outer boundary of the second generator. As an example in Figure 3.3 we illustrate the product T_2Y_1: that is,we stuff Y_1 into T_2 such that the numbered points on the outer boundary of Y_1 correspond to the points on the inner boundary of T_2.

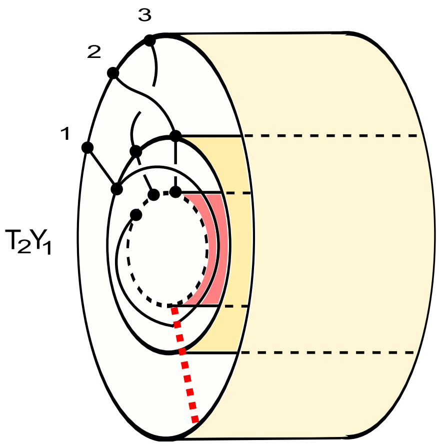


Figure 3.3: Toroidal representation of the product T_2Y_1 where only a small cross section of the overall toroid is shown.

3.4 Graphical Representation of the action of Q_i

3.4.1 The case $Q_i =$identity

We must confirm that our cubic/toroidal representation works for all the $\mathcal{D}_N\{Q\}$ axioms. We start by verifying (3.14), i.e. $Y_1Z_2Y_1^{-1}Z_2^{-1} = T_1^2$. From Figure 3.4, we see that this is satisfied by our cube representation.

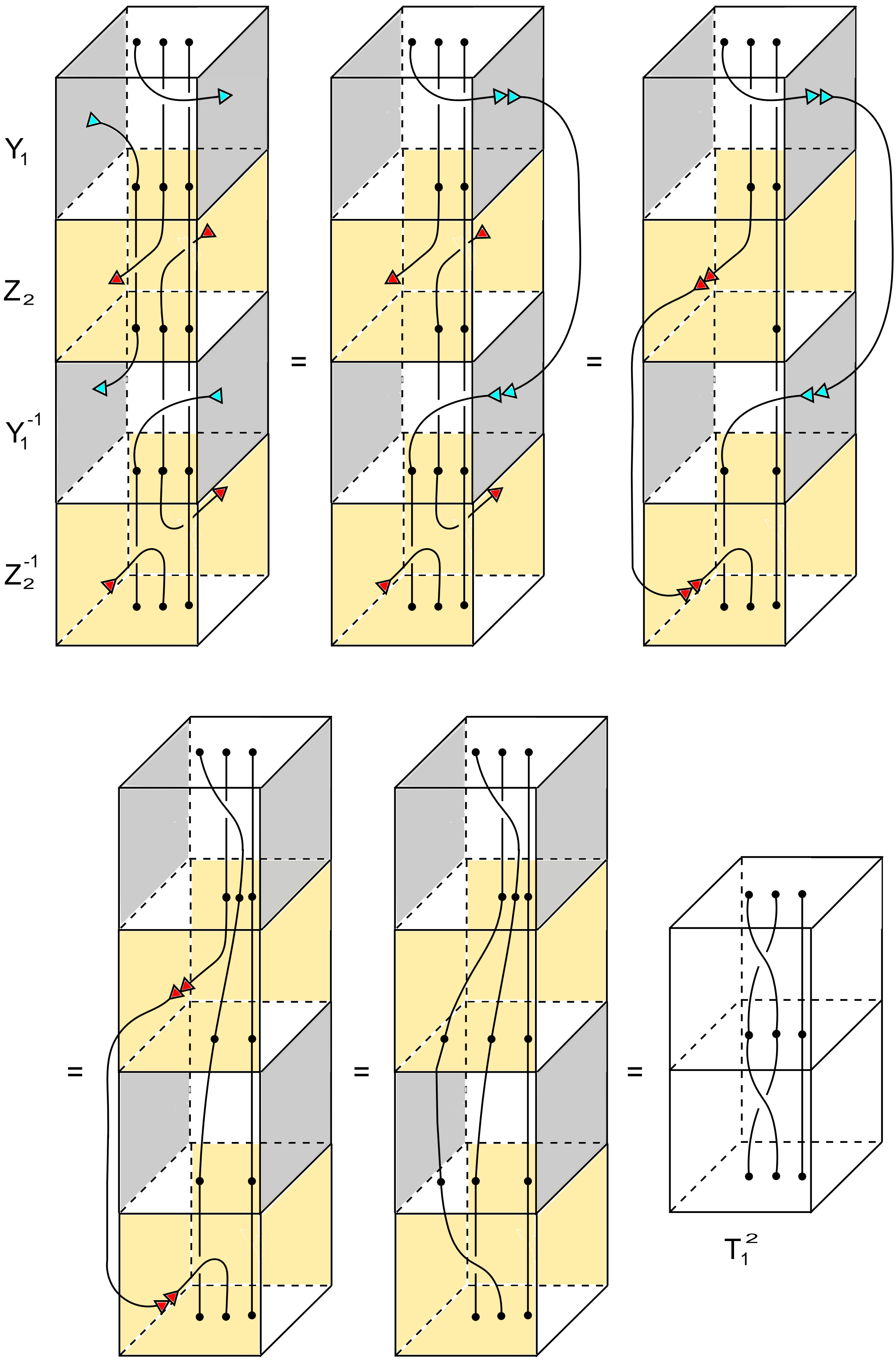


Figure 3.4: Step-by-step verification of the relation $Y_1 Z_2 Y_1^{-1} Z_2^{-1} = T_1^2$ in the cube representation.

Equations (3.15) and (3.16) must also hold in our representation, of course. These are the relations that depend explicitly on the elements Q_i. In fact, they give us various ways of writing the Q_i; for example, in the $N = 3$ case, we find $Q_3 = \sigma Z_1 \sigma^{-1} Z_3^{-1}$. We have pictorial representations for all the generators on the right-hand side of this relation, so we may explicitly find the pictorial representation of Q_3. From Figure 3.5, we see that Q_3 acts only on the third strand while leaving the other two untouched. For clarity, we have indicated the twisting using arrows; one must start form the top of the third strand and follow the arrows around all faces of the cube.

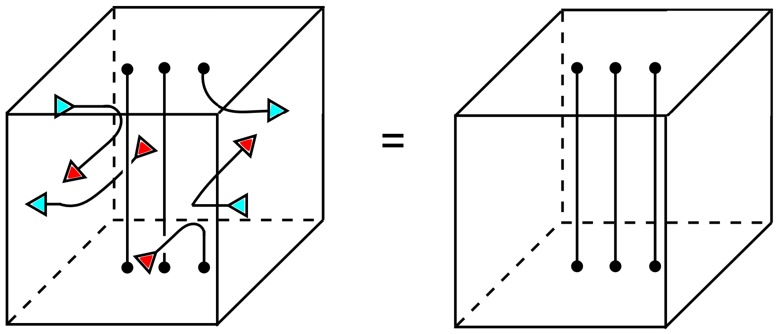


Figure 3.5: Pictorial representation of $Q_3 = \sigma Z_1 \sigma^{-1} Z_3^{-1}$. Pulling all strands tight yields the identity.

This is the pictorial representation of Q_3. By pulling the strands tight, we find that this is precisely the operator which leaves the strands entirely alone: the identity $\mathbb{1}$, namely, the trivial braid. This result is not unique to Q_3; we find that the graphical representation for each of the Qs is simply the identity.

Although this cube representation is successful in describing the T_i, Y_i and Z_i generators of $\mathcal{D}_N\{Q\}$, it still only allows the Q_i to be represented by trivial braids, and so is really only valid when $Q_i = \mathbb{1}$. Therefore, this is simply a representation of $\mathcal{D}_N\{Q\}/\langle Q_i \rangle$, i.e. the elliptic braid group [16, 17] (see Figure 3.1). However, if we wish to allow for values of Q_i *other* than unity, our cube representation needs to be modified. We now proceed to describe the modification that is required.

3.4.2 The General Case $Q_i \neq$ identity: Introducing Ribbons

To obtain a nontrivial pictorial representation which accommodates $Q_i \neq \mathbb{1}$, we modify our cube representation by replacing the strands by ribbons. This modification is not unmotivated: in order to extend the $\mathcal{A}_N\{Q\}$ representation to one for a $\mathcal{D}_N\{Q\}$, we increased the dimension of our space from two to three, and so it is reasonable to increase the dimension of our strands.

Doing so is precisely what we need in order for our representation to work for *all* $\mathcal{D}_N\{Q\}$s, not just those where the $Q_i = \mathbb{1}$. We therefore no longer braid one-dimensional strands, but do so instead with two-dimensional *ribbons*. This extra degree of freedom will enable us to completely describe a double affine Q-dependent braid group for any Q_i.

However, before we revisit the elements Q_i, we must verify that all of the previous $\mathcal{D}_N\{Q\}$ axioms still hold when using ribbons within our cube representation. It is straightforward to show that they do; to illustrate this point, we explicitly show (3.14), as this relation contains all three types of generators, the T_i, Y_i and Z_i. (For clarity, we have coloured the front and back of each ribbon with black and green respectively.) This example, illustrated in Figure 3.6, also allows us to clearly lay out the braiding conventions that we use.

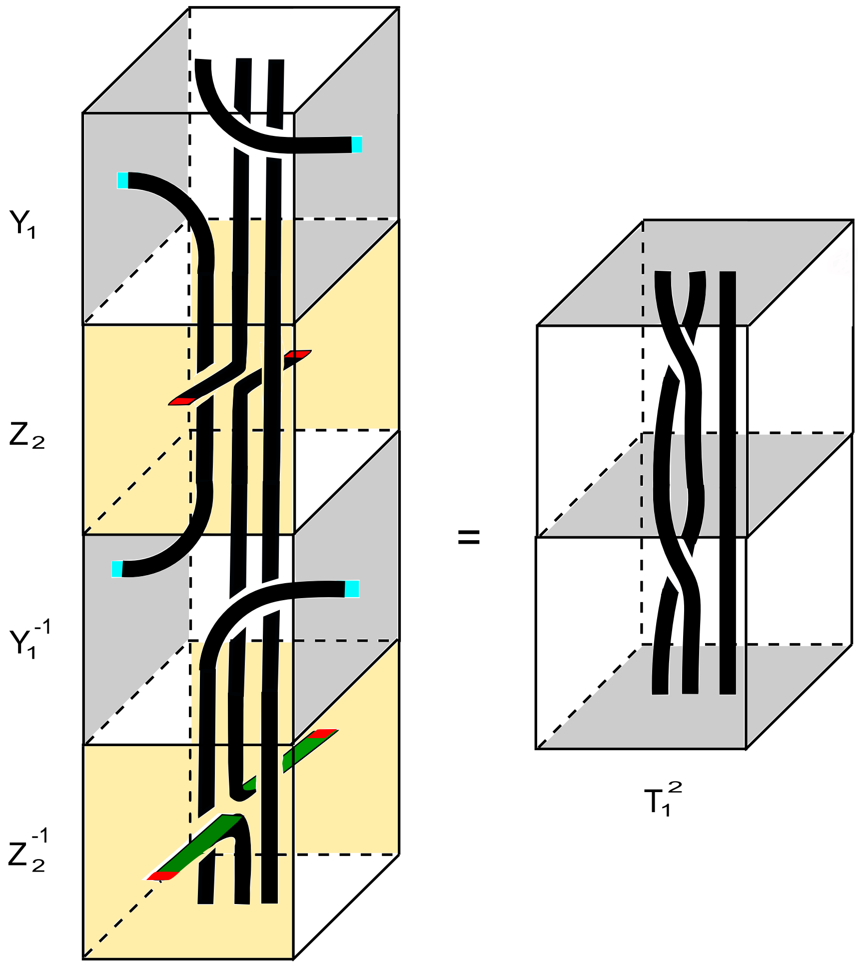


Figure 3.6: Cube ribbon representation of the intertwining relation $T_1^2 = Y_1 Z_2 Y_1^{-1} Z_2^{-1}$.

When the ribbon wraps in a left/right direction – representing a Y_i operator – we use turquoise for the tips that are identified with each other. It is vital to stress that these link the left and right faces of the cube in a very particular fashion: the ribbon must pass through a left or right face of the cube oriented *vertically*. This condition ensures that the ribbon doesn't twist while wrapping around the cube.

In a similar fashion, the ribbons representing the Z_i generators are coloured so that when a red tip is visible, this implies that the ribbon passes through either the back or front face of the cube. We require that whenever such a ribbon intersects the front or back face of the cube, it does so oriented *horizontally*.

We now revisit the relation $Q_3 = \sigma Z_1 \sigma^{-1} Z_3^{-1}$ which, when represented by 1-dimensional strands, was equivalent to the identity element. Now using ribbons instead of strands, we construct the pictorial representation of Q_3. (For clarity, we show only the third ribbon, as this is the only one which behaves nontrivially.) Keeping with the colour convention defined earlier, we obtain Q_3, and, by pulling the ribbons tight, yields the key result we require: a twist in the ribbon is created! This important result is illustrated in Figure 3.7.

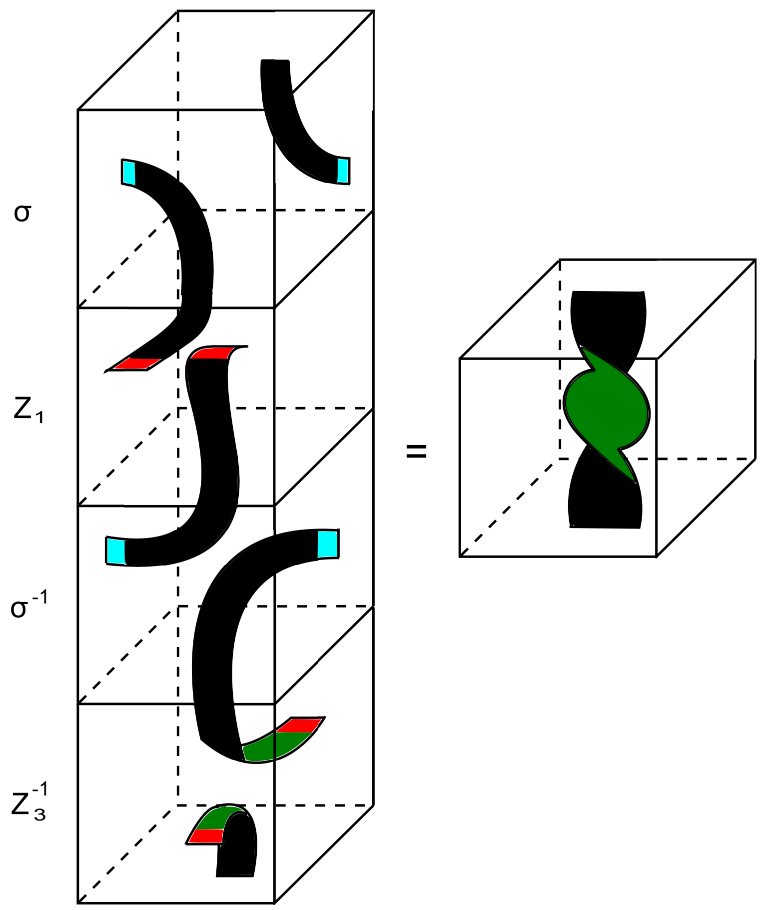
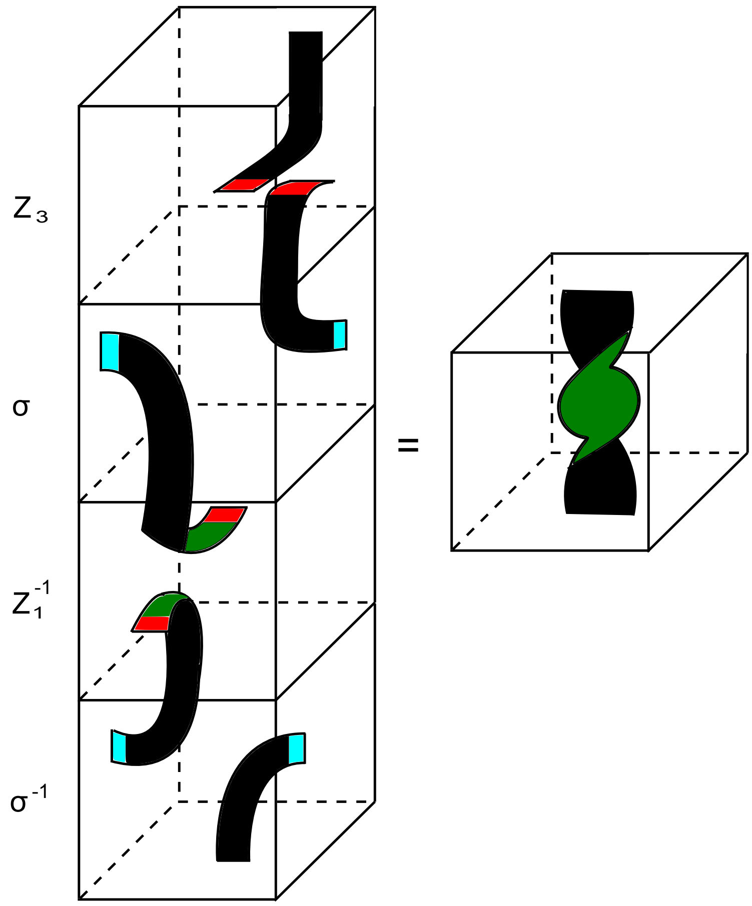


Figure 3.7: Q_3, the creation of a twist in the third ribbon.

As this is a significant feature of our ribbon representation, let us explain in detail how this comes about: in constructing $\sigma Z_1 \sigma^{-1} Z_3^{-1}$, both the black and green faces of the ribbon are clearly visible. Upon closer inspection, we see that the ribbon undergoes a *full anticlockwise twist* in going from the top face to the bottom one. First, the front black face of the ribbon is visible. Then, having undergone half an anticlockwise twist, the back green face becomes visible until finally the full anticlockwise twist leaves the black face facing forwards.

This significant result can be generalised. We have just shown that in our cube-ribbon representation Q_3 creates a twist in the third ribbon. It is easily shown, following the construction of Q_3, that in our particular representation the action of Q_i is to create a single full anticlockwise twist in the i^{th} ribbon.

We can also verify that an expression like $Z_3 \sigma Z_1^{-1} \sigma^{-1}$, which the $\mathcal{D}_N\{Q\}$ axioms require to be Q_3^{-1} for $N = 3$, is indeed a full *clockwise* twist in the third ribbon, again totally consistent with our interpretation of Q_i. We illustrate this relation below, where for clarity only the third ribbon is shown.



Other expressions could be used to determine Q_i; for example, (3.15) gives

$$Q_i \;=\; Y_i \left(\prod_{j=1}^{N} Z_j\right) Y_i^{-1} \left(\prod_{j=1}^{N} Z_j^{-1}\right).$$

Or we could use (3.17): $Q_i = \sigma^N Z_i \sigma^{-N} Z_i^{-1}$. For these and any other representation for Q_i the result is the same. Therefore the interpretation of Q_i is now clear: it is the generator that creates a full anticlockwise twist in the i^{th} ribbon. Similarly Q_i^{-1} creates a full clockwise twist in the i^{th} ribbon. As these are no longer trivial actions on the ribbons, we have a pictorial representation for $Q_i \neq \mathbb{1}$, and a complete description of the structure of $\mathcal{D}_N\{Q\}$.

As the creation of a full anticlockwise twist in the ribbon may be somewhat difficult to visualise we have included a more mathematically rigorous argument to convince the reader in Appendix 3A.1.

3.5 The Double Affine Hecke Algebra within $\mathcal{D}_N\{Q\}$

In the previous section we highlighted the fact that the elliptic braid group is given by $\mathcal{D}_N\{Q\}/\langle Q_i\rangle$. Similarly our definition of a double affine Q-dependent braid group closely resembles that of a double affine Hecke algebra (DAHA), which we described in Section 2.3. One of the main differences is that a $\mathcal{D}_N\{Q\}$ is a group structure and not an algebra. Hence the braid group generators T_i are not constrained to obey the Hecke relation, as in the DAHA. However this doesn't mean that a DAHA isn't contained within a $\mathcal{D}_N\{Q\}$; we will now show precisely how to obtain a DAHA given our construction of a double affine Q-dependent braid group.

3.5.1 The Double Affine Hecke Algebra within $\mathcal{D}_N\{Q\}$

Consider the subgroup $\mathcal{C}$ of the Q-dependent braid group $\mathcal{B}_N\{Q\}$ defined as

$$\mathcal{C} \;=\; \langle Q_i Q_{i+1}^{-1}, i = 1, \ldots, N-1\rangle.$$

It is easily shown that $\mathcal{C}$ is a normal subgroup of $\mathcal{B}_N\{Q\}$ (see Appendix 3A.2), and so we can construct the quotient $\mathcal{G} = \mathcal{B}_N\{Q\}/\langle\mathcal{C}\rangle$, which is precisely the group we require to define a DAHA. Within $\mathcal{G}$, the Q_i are indistinguishable from one another; therefore, we refer to each of their cosets $[Q_i]$ as Q. Most importantly, using (3.1)-(3.4), we see that the Q now commute with not only the squares of the braid group generators T_i^2, but also with the T_i themselves. We are now in a position to extend the quotient group $\mathcal{G}$ to a Hecke algebra.

3.5.2 The Hecke Algebra $\mathcal{H}_N(t)$

Before defining a DAHA, we must extend our quotient group $\mathcal{G}$ to an algebra in which the T_i generators satisfy a particular relation; this defines the Hecke algebra.

Associate with $\mathcal{G}$ the Hecke algebra $\mathcal{H}_N(t)$. This is the group algebra of $\mathcal{G}$ over a field k parametrised by $t \in k$ such that each generator T_i satisfies the *Hecke relation* (2.3)

$$\left(T_i - t^{1/2}\mathbb{1}\right)\left(T_i + t^{-1/2}\mathbb{1}\right) \;=\; 0.$$

3.5.3 The Double Affine Hecke Algebra $\mathcal{D}_N(t, q)$

To complete the DAHA construction we must firstly extend the Hecke algebra $\mathcal{H}_N(t)$ to an affine Hecke algebra $\mathcal{A}_N(t)$. This is achieved with the introduction of N invertible operators Y_i which satisfy (2.7)-(2.9).

Recall that the $\mathcal{A}_N$ was fully generated by Y_1 and the T_i. It is perhaps worth pointing out that the affine Hecke algebra is also fully generated by Y_1 and the T_i, and we can reorder them as necessary. This was not true for the $\mathcal{A}_N\{Q\}$ as we needed the full Hecke algebraic structure in order to consistently order the operators.

Following Subsection 2.3.1 we take a DAHA $\mathcal{D}_N(t, q)$ of type A to be the algebra generated by the T_i, Y_i and Z_i which satisfy equations (2.1)-(2.2), the Hecke relation (2.3) along with (2.7)-(2.9) and (2.15)-(2.17).

In addition to these the Y_i and Z_i obey the intertwining relations [2]

$$Y_1 Z_2 Y_1^{-1} Z_2^{-1} \;=\; T_1^2, \tag{3.18}$$

$$Y_i \left(\prod_{j=1}^{N} Z_j\right) \;=\; q\left(\prod_{j=1}^{N} Z_j\right) Y_i, \tag{3.19}$$

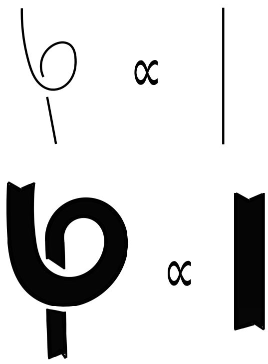
$$Z_i \left(\prod_{j=1}^{N} Y_j\right) \;=\; q^{-1}\left(\prod_{j=1}^{N} Y_j\right) Z_i, \tag{3.20}$$

where $q \in k$.

(As in the $\mathcal{D}_N\{Q\}$ (3.20) is not independent of the other relations, although it is often included in the literature as part of the definition of a DAHA.)

One must note that unlike our definition of the $\mathcal{D}_N\{Q\}$ where we have a set of elements Q_i, in the DAHA q is simply a parameter. So a DAHA $\mathcal{D}_N(t,q)$ depends on the two variables t and q. This is entirely consistent with our construction of a DAHA from $\mathcal{D}_N\{Q\}$ via the quotient group $\mathcal{G}$ if we set $Q = q\mathbb{1}$. We therefore have a representation of a DAHA in $\mathcal{B}_N\{Q\}/\langle\mathcal{C}\rangle$ when we impose $Q = q\mathbb{1}$.

In terms of our cube representation we can replace a ribbon with a full anticlockwise twist by one with no twist at all, only if we multiply the resulting cube by a factor of q. As a result, one may view this twist as the first Reidemeister move on a ribbon:



Therefore the interpretation of q is clear: it is the multiplicative factor in front of a DAHA element whenever we replace a ribbon with a full anticlockwise twist by one with no twist at all. Furthermore since q does not describe the actual position of the twist in the ribbon, one can have a factor of q^n in front of a DAHA element corresponding to n anticlockwise twists occurring *anywhere* in the cube. As there is no restriction on what value q can take we have a pictorial representation that fully describes any DAHA.

This has not been the case until now. In previous works, for example in [2], DAHA representations were limited to having $q = 1$. Our cube-ribbon representation does not place any restrictions on the value of q, as such we have significantly expanded the representation theory of double affine Hecke algebras.

3.6 $\mathcal{D}_N\{Q\}$ Summary

In this chapter we developed a large Hecke type structure called the double affine Q-dependent braid group, $\mathcal{D}_N\{Q\}$. We showed that this group, primarily created by generalising the braid group, is related to many well known algebraic and group structures such as the elliptic braid group and the double affine Hecke algebra. In fact from Figure

3.1 we indicated how all of the known Hecke algebra extensions of type A can be obtained from $\mathcal{D}_N\{Q\}$ via specific quotient groups and by restricting the action of the operators Q_i.

Complementing the algebraic description, and to give readers a deeper understanding of the structure of $\mathcal{D}_N\{Q\}$, we also presented a graphical representation of the double affine Q-dependent braid group. Following the method of extending the pictorial representation of the Q-dependent braid group to one for an $\mathcal{A}_N\{Q\}$, we found that all of the relations not explicitly involving the operators Q_i could be satisfied by a $\mathcal{D}_N\{Q\}$ depicted using 1-dimensional strands embedded in a cube whose opposing vertical sides were identified, i.e. a hollowed-out toroid.

This representation was consistent only for a $\mathcal{D}_N\{Q\}$ where all the $Q_i = \mathbb{1}$; that is, the elliptic braid group. However, by replacing the strands with ribbons, our cube representation allowed us to capture all aspects of a $\mathcal{D}_N\{Q\}$ and gave us a nice interpretation of the action of any Q_i as a single full anticlockwise twist in the i^{th} ribbon. We thus obtain an intuitive pictorial representation which clearly incorporates all of the structure of the more abstract $\mathcal{D}_N\{Q\}$.

We showed that our new graphical representation is also valid for all DAHAs. Our definition of a $\mathcal{D}_N\{Q\}$ reduced to one of a double affine Hecke algebra simply by attaching the Hecke algebra to one of its quotient groups. The DAHA depends on two parameters t and q. We found that graphically, in our cube-ribbon representation, the parameter q corresponds to a full anticlockwise twist in the ribbon. Hence within this representation we can fully capture all of the structure of any DAHA for all values of the parameter q. In the following chapter we will describe another infinite dimensional representation of the DAHA, that is its polynomial representation.

By construction, our representation is related to tangles and knot theory. Using elementary tangles via Reidemeister moves to describe this algebra appears quite possible; in fact, the replacement of a full twist by a factor of q is very much a Reidemeister-like move. This indicates a relation between our cube-ribbon representation and elementary tangle representations of affine Hecke algebras. In Chapter 5 we will look further into this suspected relationship and use our new pictorial representation to transform this cube-ribbon representation into an equivalent matrix one.

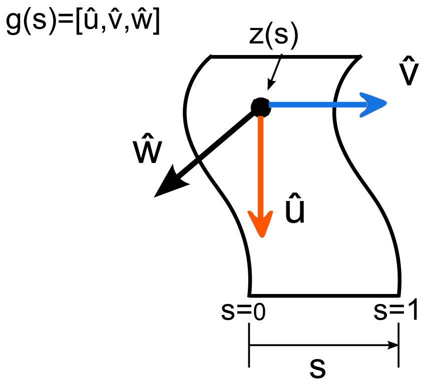
* The author recognises that some of the results obtained in this chapter are in collaboration with Dr. Vincent Pasquier.

Appendix 3

3A.1 Q: The Twist Operator

Here we show that the twist in the ribbon generated by Q_3 is precisely 2π. We demonstrate this specifically for the case of $Q_3 = \sigma Z_1 \sigma^{-1} Z_3^{-1}$ as in Figure 3.6 where, from top to bottom, a full anticlockwise twist in the third ribbon is obtained. For clarity we illustrate only the third ribbon as it is the only one that behaves non-trivially.

Firstly let $z(s), (0 \leq s \leq 1)$ denote the position of a point on the ribbon. Then $\hat{v}$ is the unit vector indicating the ribbon orientation and always lies on the surface of the ribbon. The direction of motion is given by the unit vector $\hat{u}$, where at all times $\hat{u}.\hat{v} = 0$. The vector $\hat{w} = \hat{u} \times \hat{v}$ defines the normal to the ribbon.
So there is an orthogonal frame $g(s) = [\hat{u}, \hat{v}, \hat{w}]$ attached to each point on the ribbon as indicated in the diagram below.



We now follow a point as it travels down the ribbon. Attached to this point is the orthogonal frame $g(s)$. We impose that the ribbon cannot twist around the direction of motion, that is; $\omega.\hat{u} = 0$ where ω is the angular velocity of the frame $g(s)$. We measure the degree of rotation of $g(s)$, between the top and bottom of the ribbon, relative to a fixed frame. This yields the size of the twist in the ribbon.

Figure 3.8 (a) shows the frame $g(s)$ at various points along the ribbon, from the top of the ribbon labelled point (A), to the bottom of the ribbon; point (B). Between these points we show that the moving frame $g(s)$ undergoes a full 2π rotation relative to the inertial reference frame $(\hat{x}, \hat{y}, \hat{z})$.

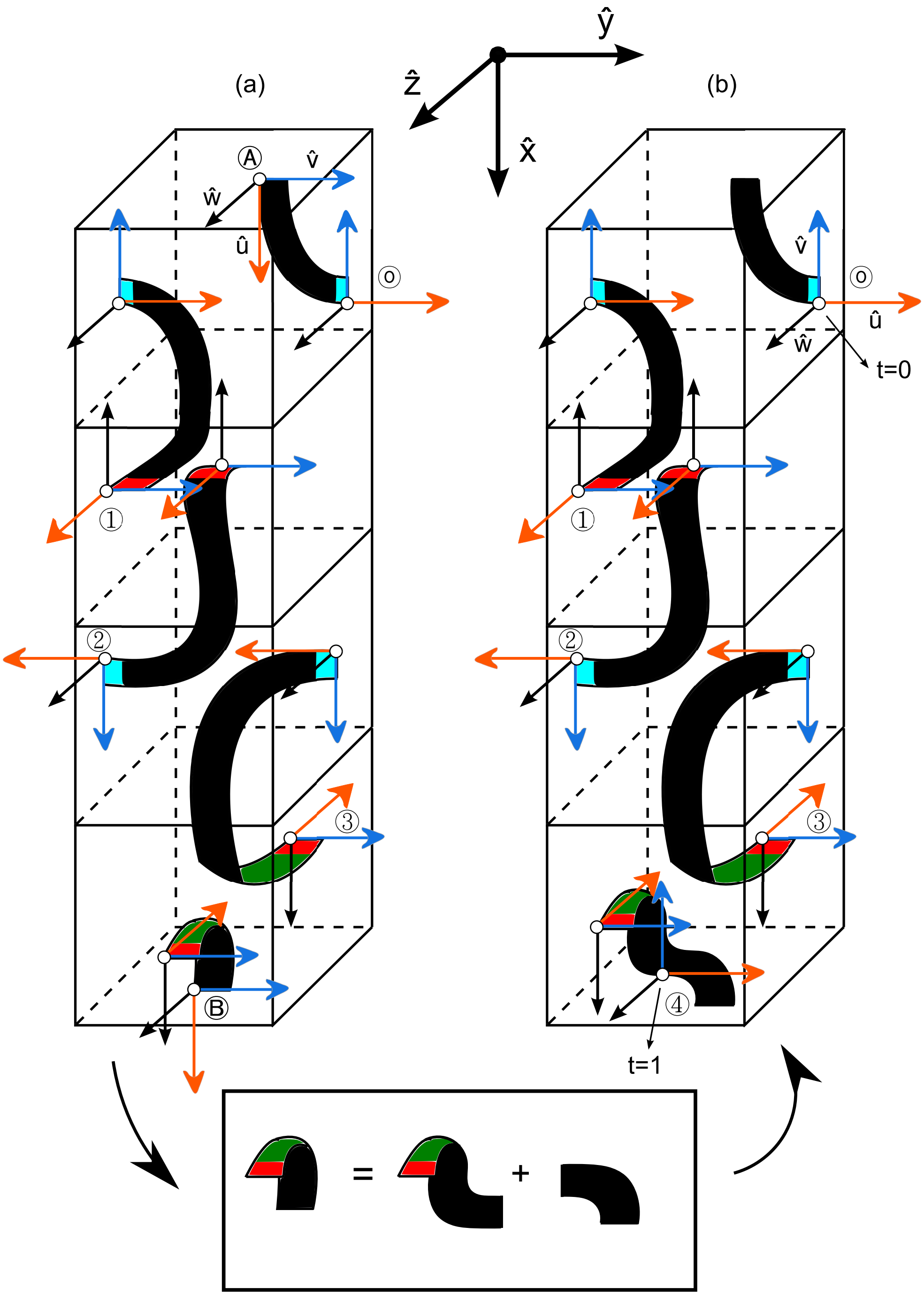


Figure 3.8: Figure (a) shows $g(s)$ at various points along the ribbon $Q_3 = \sigma Z_1 \sigma^{-1} Z_3^{-1}$. In Figure (b) we redraw the relation such that between times $t = 0$ and $t = 1$ one can see $\hat{u}$ rotating by 2π in the $\hat{y} - \hat{z}$ plane.

Notice that between points (A) and (0), the ribbon itself does not undergo any rotation. Therefore without losing any information we can measure the twist starting from point (0), which we now call time $t = 0$, as in Figure 3.8 (b).

Furthermore in Figure 3.8 (b), the bottom of the ribbon is redrawn in such a way that the extra turns do not contribute to the overall twist. Then following $g(s)$ from $t=0$ to $t=1$, one can immediately see that $\hat{u}$ rotates *only* in the $\hat{y}-\hat{z}$ plane. In fact it does exactly a 2π clockwise rotation. So at any time t, $\hat{u}$ can be written as follows:

$$\hat{u}(t)=\cos(2\pi t)\hat{y}+\sin(2\pi t)\hat{z}.$$

One can easily check this holds. For example at time $t=1/2$, $\hat{u}(1/2)=-\hat{y}$. This is verified upon inspection of point ② in the diagram.

Further inspection reveals that as $\hat{u}$ rotates in the $\hat{y}-\hat{z}$ plane, the vectors $\hat{v}$ and $\hat{w}$ rotate in a clockwise fashion around $\hat{u}$.
We introduce a frame $[\hat{e}_1,\hat{e}_2,\hat{e}_3]$, where $\hat{e}_1=\hat{u}$ and $\hat{e}_2$, $\hat{e}_3$ are functions of $\hat{v}$ and $\hat{w}$, to measure the rotation of $\hat{v}$ and $\hat{w}$ around $\hat{u}$. Impose that at $t=0$, $\hat{e}_1=\hat{u},\hat{e}_2=\hat{v}$ and $\hat{e}_3=\hat{w}$. It is important to note that $\hat{e}_1=\hat{u}$ at all times; that is we have $\hat{u}(t)=\hat{e}_1$.
Therefore in terms of this frame $[\hat{e}_1,\hat{e}_2,\hat{e}_3]$ we can write:

$$\hat{v}(t)=\cos(2\pi t)\hat{e}_2-\sin(2\pi t)\hat{e}_3,$$
$$\hat{w}(t)=\sin(2\pi t)\hat{e}_2+\cos(2\pi t)\hat{e}_3.$$

Again these can easily be verified through simple substitution and by referring to the above diagram.
The vector $\hat{u}$ was fixed to $\hat{e}_1$ so in terms of the inertial reference frame we have:

$$\hat{e}_1(t)=\cos(2\pi t)\hat{y}+\sin(2\pi t)\hat{z}.$$

Following the vector $\hat{e}_2$ between $t=0$ and $t=1$ we see that it always points in the negative $\hat{x}$ direction. This implies that:

$$\hat{e}_2(t)=-\hat{x}.$$

Since $[\hat{e}_1,\hat{e}_2,\hat{e}_3]$ form an orthogonal frame we must have that:

$$\hat{e}_3(t)=-\sin(2\pi t)\hat{y}+\cos(2\pi t)\hat{z}.$$

Finally in terms of the fixed frame $(\hat{x},\hat{y},\hat{z})$;

$$\begin{aligned}\hat{u}(t)&=\cos(2\pi t)\hat{y}+\sin(2\pi t)\hat{z},\\ \hat{v}(t)&=-\cos(2\pi t)\hat{x}+\sin^2(2\pi t)\hat{y}-\sin(2\pi t)\cos(2\pi t)\hat{z},\\ \hat{w}(t)&=-\sin(2\pi t)\hat{x}-\sin(2\pi t)\cos(2\pi t)\hat{y}+\cos^2(2\pi t)\hat{z}.\end{aligned}$$

One can clearly see that $\hat{v}$ undergoes a full 2π clockwise rotation from $t=0$ to $t=1$. $\hat{v}$ lies on the ribbon surface at all times, therefore requiring the ribbon to undergo the same rotation. This yields precisely the required result; Q_3 creates a full anticlockwise twist in the third ribbon.

3A.2 $\mathcal{C}$ is a Normal Subgroup of $\mathcal{B}_N\{Q\}$

We would like to construct the quotient $\mathcal{G} = \mathcal{B}_N\{Q\}/\langle\mathcal{C}\rangle$. Recall that $\mathcal{C}$ is a subgroup of the Q-dependent braid group $\mathcal{B}_N\{Q\}$ defined as

$$\mathcal{C} \;=\; \langle Q_iQ_{i+1}^{-1}, i = 1, \ldots, N-1\rangle.$$

In order to construct $\mathcal{G}$, we prove that $\mathcal{C}$ is a normal subgroup of $\mathcal{B}_N\{Q\}$, by showing that for each element $Q_iQ_{i+1}^{-1} \in \mathcal{C}$ and each $T_j \in \mathcal{B}_N\{Q\}$ the element $T_j(Q_iQ_{i+1}^{-1})T_j^{-1} \in \mathcal{C}$. We look at all possible values of j, hence all $T_j \in \mathcal{B}_N\{Q\}$.

1. When $j = i$ we have that

$$T_j(Q_iQ_{i+1}^{-1})T_j^{-1} \;=\; T_i(Q_iQ_{i+1}^{-1})T_i^{-1}.$$

But by (3.3), $T_iQ_i = Q_{i+1}T_i$ which implies that

$$T_j(Q_iQ_{i+1}^{-1})T_j^{-1} \;=\; Q_{i+1}T_iQ_{i+1}^{-1}T_i^{-1}.$$

Using (3.4), $T_iQ_{i+1} = Q_iT_i \quad\Rightarrow\quad Q_{i+1}^{-1}T_i^{-1} = T_i^{-1}Q_i^{-1}$, so

$$\begin{aligned} T_j(Q_iQ_{i+1}^{-1})T_j^{-1} &= Q_{i+1}T_iT_i^{-1}Q_i^{-1} \\ &= Q_{i+1}Q_i^{-1}. \end{aligned}$$

Since $\mathcal{C}$ is generated by all elements $Q_iQ_{i+1}^{-1}$, we see that $Q_{i+1}Q_i^{-1}$ is merely the inverse of this element and hence $\in \mathcal{C}$.
We now examine another value of j.

2. When $j = i+1$ we have that

$$T_j(Q_iQ_{i+1}^{-1})T_j^{-1} \;=\; T_{i+1}(Q_iQ_{i+1}^{-1})T_{i+1}^{-1}.$$

But by (3.3), $T_iQ_i = Q_{i+1}T_i \quad\Rightarrow\quad Q_{i+1}^{-1}T_{i+1}^{-1} = T_{i+1}^{-1}Q_{i+2}^{-1}$, so

$$T_j(Q_iQ_{i+1}^{-1})T_j^{-1} \;=\; T_{i+1}Q_iT_{i+1}^{-1}Q_{i+2}^{-1}.$$

By (3.2), Q_i commutes with T_{i+1} so we get the final expression

$$\begin{aligned} T_j(Q_iQ_{i+1}^{-1})T_j^{-1} &= Q_iT_{i+1}T_{i+1}^{-1}Q_{i+2}^{-1} \\ &= Q_iQ_{i+2}^{-1}. \end{aligned}$$

Since $\mathcal{C}$ is generated by all elements $Q_i Q_{i+1}^{-1}$, we see that $Q_i Q_{i+2}^{-1}$ is just the product of two such elements. This is particularly clear if we write $Q_i Q_{i+2}^{-1}$ as $\left(Q_i Q_{i+1}^{-1}\right)\left(Q_{i+1} Q_{i+2}^{-1}\right)$. Therefore $Q_i Q_{i+2}^{-1} \in \mathcal{C}$. We now examine the last case.

3. When $j > i+1$ and when $j < i$ then by (3.2) we have that

$$\begin{aligned} T_j(Q_i Q_{i+1}^{-1})T_j^{-1} &= Q_i T_j Q_{i+1}^{-1} T_j^{-1} \\ &= Q_i T_j T_j^{-1} Q_{i+1}^{-1} \\ &= Q_i Q_{i+1}^{-1}. \end{aligned}$$

As in the first case, $Q_{i+1} Q_i^{-1}$ is simply the inverse of one of the elements, namely $Q_i Q_{i+1}^{-1}$, which generates $\mathcal{C}$. Therefore $Q_{i+1} Q_i^{-1}$ is also $\in \mathcal{C}$.

This completes our proof that $\mathcal{C}$ is a normal subgroup of $\mathcal{B}_N\{Q\}$ as we have shown that for all values of j, the following holds true

$$T_j(Q_i Q_{i+1}^{-1})T_j^{-1} \in \mathcal{C}.$$

Chapter 4

The Double Affine Hecke Algebra Polynomial Representation

As we have already seen, algebraically the structure of a double affine Hecke algebra is very rich. Furthermore via representations, as in our cube-ribbon representation, a DAHA offers much physical relevance. For example, in terms of the ribbons, one could interpret the factor of q as corresponding to a phase factor resulting from the interchange of the worldlines of two particles. Similarly, in this chapter we present another infinite dimensional representation of a double affine Hecke algebra that offers significant physical relevance. In particular we introduce the polynomial representation of a DAHA.

We present the polynomial representation of the double affine Hecke algebra from the point of view of Kasatani and Pasquier in [7]. However we also go beyond [7] and derive the explicit polynomials resulting from the action of all of the generators at each point of the construction. In addition to this we pay particular attention to Macdonald polynomials [18], which are two variable polynomials obtained by simultaneously diagonalising the affine Hecke algebra generators. As highlighted in the introduction to this thesis, Macdonald polynomials are widely used to describe many existing physical models. Due to their importance, and to present readers with clear examples of these polynomials, we explicitly evaluate all Macdonald polynomials up to three dimensions. We show exactly how we obtained them, giving all of the detailed calculations.

Furthermore we also define intertwining operators, due to Cherednik [19], which given any Macdonald polynomial, can be used to generate all other Macdonald polynomials of arbitrary dimension. With specific examples we will describe in detail how to employ these operators, resulting in the obtainment of all Macdonald polynomials up to three dimensions.

4.1 The Polynomial Representation

Following our presentation of a DAHA $\mathcal{D}_N(t,q)$ in Chapter 2, we now describe its polynomial representation, which appeared originally in [19]. $\mathcal{D}_N(t,q)$ has a simple irreducible representation U on the ring of Laurent polynomials in N variables $x_i^{\pm 1}$. Unsurprisingly this representation depends on two parameters t and q.

It is at this point perhaps useful to outline our choice of notation. Throughout this chapter we will follow the notation introduced in [7]; that is, to avoid confusion between the abstract algebraic generators and these same generators acting on polynomials, we adopt the following convention: all generators when acting on polynomials are denoted with a bar, that is $\bar{T}_{i,j}$ denotes the action of the Hecke algebra generator. In contrast the abstract Hecke algebra generator is just denoted by T_i. We also impose that all generators with a bar act on polynomials from the right.

4.1.1 Representing the T_i

The first step in defining the DAHA polynomial representation is to describe the action of the Hecke generators T_i on polynomials. Inspired by [7], we begin by introducing permutation operators s_i, which permute the variables x_i and x_{i+1}

$$\begin{aligned} x_i s_i &= s_i x_{i+1}, \\ x_{i+1} s_i &= s_i x_i, \\ x_\ell s_i &= s_i x_\ell \text{ for } \ell \neq i, i+1. \end{aligned}$$

Using the Hecke relation (2.3) a solution to these equations is given by

$$\begin{aligned} s_i &= x_{i+1}\bar{T}_i - x_i \bar{T}_i^{-1} \\ &= (x_{i+1} - x_i)\bar{T}_i + x_i(t^{1/2} - t^{-1/2}). \end{aligned}$$

However the s_i are permutation operators, hence we must introduce the normalisation factor $(t^{1/2}x_i - t^{-1/2}x_{i+1})^{-1}$ to ensure that $s_i^2 = 1$

$$s_i = (x_{i+1} - x_i)\bar{T}_i(t^{1/2}x_i - t^{-1/2}x_{i+1})^{-1} + x_i(t^{1/2} - t^{-1/2})(t^{1/2}x_i - t^{-1/2}x_{i+1})^{-1}.$$

Rearranging the above equation yields an expression describing the action of the Hecke algebra $\mathcal{H}_N(t)$ generators $\bar{T}_i$ on polynomials. We find that

$$\begin{aligned}\bar{T}_i &= (x_i - x_{i+1})^{-1}x_i(t^{1/2} - t^{-1/2}) - (x_i - x_{i+1})^{-1}s_i(t^{1/2}x_i - t^{-1/2}x_{i+1}) \\ &= -t^{-1/2}s_i + (t^{1/2} - t^{-1/2})(1 - s_i)\frac{x_i}{x_i - x_{i+1}}.\end{aligned}$$

Clearly, from the above definition we see that the operator $\bar{T}_i$ depends only on the permutation operators s_i. As the s_i act only on the variables x_i and x_{i+1}, it is useful to define the operator $\bar{T}_{i,j} = \bar{T}_{i,i+1}$ as follows;

$$\bar{T}_{i,j} = -t^{-1/2}s_{ij} + (t^{1/2} - t^{-1/2})(1 - s_{ij})\frac{x_i}{x_i - x_j}, \tag{4.1}$$

where s_{ij} permutes the variables x_i and x_j.

By (4.1) we know exactly how the $\mathcal{H}_N(t)$ generators act on polynomials. For the $N = 3$ case, using the explicit calculations included in Appendix 4A.1 we show that the matrices corresponding to the action of $\bar{T}_{1,2}$ and $\bar{T}_{2,3}$ on degree one monomials using $\{1, x_1, x_2, x_3\}$ as basis are given by:

$$\bar{T}_{1,2} = \begin{bmatrix} -t^{-1/2} & 0 & 0 & 0 \\ 0 & (t^{1/2} - t^{-1/2}) & -t^{1/2} & 0 \\ 0 & -t^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & -t^{-1/2} \end{bmatrix},$$

$$\bar{T}_{2,3} = \begin{bmatrix} -t^{-1/2} & 0 & 0 & 0 \\ 0 & -t^{-1/2} & 0 & 0 \\ 0 & 0 & (t^{1/2} - t^{-1/2}) & -t^{1/2} \\ 0 & 0 & -t^{-1/2} & 0 \end{bmatrix}.$$

These matrices are easily inverted giving us a complete description of all of the $\mathcal{H}_3(t)$ generators on degree one monomials. Similarly the $\mathcal{H}_3(t)$ generators on degree two monomials, using the basis $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}$, give rise to the following matrices:

$$\bar{T}_{1,2} = \begin{bmatrix} (t^{1/2} - t^{-1/2}) & -t^{1/2} & 0 & 0 & 0 & 0 \\ -t^{-1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t^{-1/2} & 0 & 0 & 0 \\ (t^{1/2} - t^{-1/2}) & -(t^{1/2} - t^{-1/2}) & 0 & -t^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & (t^{1/2} - t^{-1/2}) & -t^{1/2} \\ 0 & 0 & 0 & 0 & -t^{-1/2} & 0 \end{bmatrix},$$

$$\bar{T}_{2,3} = \begin{bmatrix} -t^{-1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & (t^{1/2}-t^{-1/2}) & -t^{1/2} & 0 & 0 & 0 \\ 0 & -t^{-1/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (t^{1/2}-t^{-1/2}) & -t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{-1/2} & 0 & 0 \\ 0 & (t^{1/2}-t^{-1/2}) & -(t^{1/2}-t^{-1/2}) & 0 & 0 & -t^{-1/2} \end{bmatrix}.$$

Again inverting these matrices yields $\bar{T}_{1,2}^{-1}$ and $\bar{T}_{2,3}^{-1}$. We have included all of the calculations surrounding the formulation of these matrices in Appendix 4A.1. These results are easily verified. One only needs to apply (4.1) to all degree one and two monomials.

Having presented the polynomial representation of the Hecke algebra, we now do likewise with the affine Hecke algebra $\mathcal{A}_N(t)$. In a way we are following the order of construction of the algebraic description of $\mathcal{D}_N(t,q)$, which we presented in Chapter 2.

4.1.2 Representing the Y_i

Describing the action of the affine Hecke algebra generators on polynomials is relatively straightforward as we already have the action of the Hecke algebra generators, $\bar{T}_{i,j}$. In Subsection 2.2.1 we derived that the $\mathcal{A}_N(t)$ generators Y_i, in terms of the element σ (2.11), are given by

$$Y_i = T_i \dots T_{N-1} \sigma T_1^{-1} \dots T_{i-1}^{-1}.$$

Therefore acting on a polynomial we have

$$\bar{Y}_i = \bar{T}_i \dots \bar{T}_{N-1} \bar{\sigma} \bar{T}_1^{-1} \dots \bar{T}_{i-1}^{-1}. \tag{4.2}$$

As we have already described $\bar{T}_{i,j}$, that leaves us to introduce $\bar{\sigma}$.

We need a representation of σ that has all of the same characteristics as the σ we previously defined in Subsection 2.2.1. Furthermore in our cylindrical representation, we showed how σ acted as a kind of raising operator on the indices. Hence we decompose the operator $\bar{\sigma}$ into a product of elementary permutations s_i and a diagonal operator $\hat{q}_1$

$$\bar{\sigma} = s_{N-1} \dots s_2 s_1 \hat{q}_1, \tag{4.3}$$

where $f(x_1,\ldots,x_i,\ldots,x_N)\hat{q}_i = f(x_1,\ldots q^{-1}x_i,\ldots,x_N)$.

Using (4.3) and imposing that $1\bar{\sigma} = 1$, we look at the action of $\bar{\sigma}$ on all x_i. Its action on x_i is

$$\begin{aligned} x_i\bar{\sigma} &= x_i s_{N-1}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}\ldots x_i s_i s_{i-1}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}\ldots s_i x_{i+1} s_{i-1}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}\ldots s_2 s_1\hat{q}_1 x_{i+1} \\ &= \bar{\sigma}x_{i+1}. \end{aligned}$$

The case $i = N$ gives

$$\begin{aligned} x_N\bar{\sigma} &= x_N s_{N-1}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}x_{N-1}s_{N-2}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}s_{N-2}x_{N-2}\ldots s_2 s_1\hat{q}_1 \\ &= s_{N-1}\ldots s_2 s_1 x_1\hat{q}_1 \\ &= s_{N-1}\ldots s_2 s_1\hat{q}_1 q^{-1}x_1. \end{aligned}$$

We already derived that $x_i\bar{\sigma} = \bar{\sigma}x_{i+1}$ so this implies that $x_N\bar{\sigma} = \bar{\sigma}x_{N+1}$, giving the final expression

$$\begin{aligned} s_{N-1}\ldots s_2 s_1\hat{q}_1 x_{N+1} &= s_{N-1}\ldots s_2 s_1\hat{q}_1 q^{-1}x_1 \\ \Rightarrow \hat{q}_1 q^{-1}x_1 &= \hat{q}_1 x_{N+1} \\ \Rightarrow q^{-1}x_1 &= x_{N+1}. \end{aligned}$$

If we define $x_{N+1} = q^{-1}x_1$, then we get that $x_i\bar{\sigma} = \bar{\sigma}x_{i+1}$ for all $i = 1,\ldots,N$.

We have showed that the action of $\bar{\sigma}$ is straightforward. For example on degree one monomials in the $N = 3$ case, $\bar{\sigma}$ acts as follows: $\bar{\sigma}1 = 1$, $\bar{\sigma}x_1 = x_2$, $\bar{\sigma}x_2 = x_3$ and $\bar{\sigma}x_3 = q^{-1}x_1$. Compiling these results into a matrix with basis given by $\{1, x_1, x_2, x_3\}$, we have that the action of $\bar{\sigma}$ on degree one monomials is described by

$$\bar{\sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

A similar calculation shows that the action of $\bar{\sigma}$ on degree two monomials in the $N = 3$ case is described by the matrix with basis $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}$ given by

$$\bar{\sigma} = \begin{bmatrix} 0 & 0 & q^{-2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The expression for $\bar{\sigma}$ now completes the action of $\bar{Y}_i$ on polynomials. Substituting (4.3) into equation (4.2) gives the following expression for $\bar{Y}_i$

$$\bar{Y}_i = \bar{T}_i \ldots \bar{T}_{N-1} s_{N-1} \ldots s_2 s_1 \hat{q}_1 \bar{T}_1^{-1} \ldots \bar{T}_{i-1}^{-1}.$$

We will rewrite this expression for $\bar{Y}_i$ in terms of the $\bar{T}_{i,j}$ previously defined. For consistency we must also write the permutations s_i as s_{ij}. However before doing so we note that since s_i is a member of the permutation group, then $s_i = s_i^{-1}$, and we can write $\bar{\sigma}$ in a more symmetric way as

$$\begin{aligned} \bar{\sigma} &= s_{N-1} \ldots . s_2 s_1 \hat{q}_1 \\ &= s_{N-1} \ldots . s_{i-1} s_i \hat{q}_i s_{i-1}^{-1} \ldots . s_1^{-1}. \end{aligned}$$

Using this expression for $\bar{\sigma}$, $\bar{Y}_i$ in terms of $\bar{T}_{i,j}$ becomes

$$\bar{Y}_i = \bar{T}_{i,i+1} \bar{T}_{i+1,i+2} \ldots \bar{T}_{N-1,N} s_{N-1,N} \ldots s_{i,i+1} \hat{q}_i s_{i-1,i}^{-1} \ldots s_{1,2}^{-1} \bar{T}_{1,2}^{-1} \ldots \bar{T}_{i-1,i}^{-1}, \qquad (4.4)$$

where $\bar{T}_{i,j} = \bar{T}_{i,i+1}$.

Expression (4.4), though perfectly reasonable is quite inelegant and not practical to work with. A much nicer way is to express (4.4) in terms of triangular operators [7] $\bar{X}_{i,j}$, defined as $\bar{X}_{i,j} = \bar{T}_{i,j} s_{ij}$.

We will now rewrite equation (4.4) in terms of these triangular operators. As it is quite a long expression, we break it into two parts. Firstly, the left-hand side of (4.4) as far as $\hat{q}_i$, that is $\bar{T}_{i,i+1}\bar{T}_{i+1,i+2}\ldots\bar{T}_{N-1,N}s_{N-1,N}\ldots s_{i,i+1}\hat{q}_i$, is given in terms of $\bar{X}_{i,j}$ as

$$\begin{aligned} & \bar{T}_{i,i+1}\bar{T}_{i+1,i+2}\ldots\bar{T}_{N-1,N}s_{N-1,N}s_{N-2,N-1}\ldots s_{i,i+1}\hat{q}_i \\ = \; & \bar{T}_{i,i+1}\bar{T}_{i+1,i+2}\ldots\bar{T}_{N-2,N-1}\bar{X}_{N-1,N}s_{N-2,N-1}\ldots s_{i,i+1}\hat{q}_i\,, \end{aligned}$$

since by definition $\bar{X}_{N-1,N} = \bar{T}_{N-1,N}s_{N-1,N}$.
The operator $\bar{X}_{N-1,N}$ can be pulled through $s_{N-2,N-1}$ to give

$$\bar{T}_{i,i+1}\bar{T}_{i+1,i+2}\ldots\bar{T}_{N-2,N-1}s_{N-2,N-1}\bar{X}_{N-2,N}s_{N-3,N-2}\ldots s_{i,i+1}\hat{q}_i.$$

Now we replace $\bar{T}_{N-2,N-1}s_{N-2,N-1}$ with $\bar{X}_{N-2,N-1}$, and repeating the previous two steps until all of the $s_{i,j}$ on the right meet with the $\bar{T}_{i,j}$, we get

$$\bar{X}_{i,i+1}\bar{X}_{i,i+2}\ldots\bar{X}_{i,N}\hat{q}_i.$$

We use the same technique to express the right-hand side, from $\hat{q}_i$ onwards, of (4.4) in terms of $\bar{X}_{i,j}$.

$$\begin{aligned} & \hat{q}_i s_{i-1,i}^{-1}\ldots s_{1,2}^{-1}\bar{T}_{1,2}^{-1}\ldots\bar{T}_{i-1,i}^{-1} \\ = \; & \hat{q}_i s_{i-1,i}^{-1}\ldots s_{2,3}^{-1}\bar{X}_{1,2}^{-1}\bar{T}_{2,3}^{-1}\ldots\bar{T}_{i-1,i}^{-1} \\ = \; & \hat{q}_i s_{i-1,i}^{-1}\ldots s_{3,4}^{-1}\bar{X}_{1,3}^{-1}s_{2,3}^{-1}\ldots\bar{T}_{i-1,i}^{-1} \\ = \; & \hat{q}_i\bar{X}_{1,i}^{-1}\ldots\bar{X}_{i-1,i}^{-1}. \end{aligned}$$

Therefore a tidier expression of (4.4) in terms of the triangular operators $\bar{X}_{i,j}$ is given by

$$\bar{Y}_i = \bar{X}_{i,i+1}\bar{X}_{i,i+2}\ldots\bar{X}_{i,N}\hat{q}_i\bar{X}_{1,i}^{-1}\ldots\bar{X}_{i-1,i}^{-1}. \tag{4.5}$$

This last expression describes the complete action of the affine Hecke algebra generators Y_i on polynomials.

Since by definition $\bar{X}_{i,j} = \bar{T}_{i,j}s_{ij}$, its explicit action in terms of the permutation operators s_{ij} is simply

$$\bar{X}_{i,j} = -t^{-1/2} + (t^{1/2} - t^{-1/2})(1 - s_{ij})\frac{x_i}{x_i - x_j}. \tag{4.6}$$

In Appendix 4A.2 at the end of this chapter we use equations (4.5) and (4.6) to explicitly calculate the action of $\bar{Y}_i$ on degree one and two monomials. We just present the results here. That is, the action of the $\mathcal{A}_3(t)$ generators, on degree one monomials, in matrix format with basis $\{1, x_1, x_2, x_3\}$ are given by

$$\bar{Y}_1 = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & q^{-1}t^{-1} & 0 & 0 \\ 0 & t^{-1}-1 & 1 & 0 \\ 0 & t^{-1}-1 & 0 & 1 \end{bmatrix},$$

$$\bar{Y}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-t^{-1} & q^{-1}t^{-1} & 0 \\ 0 & 0 & 1-t & t \end{bmatrix},$$

$$\bar{Y}_3 = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 1-t^{-1} & t-1 & q^{-1}t^{-1} \end{bmatrix}.$$

We highlight the fact that one can alternatively obtain these matrices using the algebraic relations describing the Y_i. In Section 2.2, where we derived all of the defining relations of the affine Hecke algebra in terms of the T_i and σ, we obtained $Y_i = T_i \dots T_{N-1}\sigma T_1^{-1} \dots T_{i-1}^{-1}$ and $T_i Y_{i+1} T_i = Y_i$.

Therefore for the $N = 3$ case we have $Y_1 = T_1 T_2 \sigma$ which on polynomials implies that $\bar{Y}_1 = \bar{\sigma}\bar{T}_{2,3}\bar{T}_{1,2}$. So by simply multiplying the matrices $\bar{T}_{i,j}$ and $\bar{\sigma}$, which we have already obtained, one can verify that the matrices given above for $\bar{Y}_i$ on degree one monomials are recovered. Using the recursive relation $T_i Y_{i+1} T_i = Y_i$, we can now generate all $\bar{Y}_i$. For example $\bar{Y}_2 = \bar{T}_{1,2}^{-1}\bar{Y}_1\bar{T}_{1,2}^{-1}$ and $\bar{Y}_3 = \bar{T}_{2,3}^{-1}\bar{Y}_2\bar{T}_{2,3}^{-1}$.

4.1.3 The complete DAHA polynomial representation

To complete the polynomial representation of the double affine Hecke algebra $\mathcal{D}_N(t, q)$ we need to describe the action of the $\mathcal{D}_N(t, q)$ generators Z_i.

The action of the double affine Hecke algebra generators Z_i on polynomials is particularly simple and given by

$$f(x_1, \ldots, x_i, \ldots, x_N)\bar{Z}_i \quad = \quad x_i f(x_1, \ldots, x_i, \ldots, x_N). \tag{4.7}$$

This concludes the presentation of the irreducible polynomial representation U of $\mathcal{D}_N(t,q)$. To summarise, the defining relations required to fully describe U are given by (4.1), (4.3) and (4.5) - (4.7).

In the following section we will present an interesting property of the irreducible polynomial representation U. More specifically we will describe how to obtain Macdonald polynomials and even construct several examples in detail.

4.2 Macdonald Polynomials

Non symmetric Macdonald polynomials are monic simultaneous eigenvectors of $\bar{Y}_i$. They form a basis of the polynomial representation U. Therefore in order to obtain these non symmetric Macdonald polynomials we must ensure the $\bar{Y}_i$ are simultaneously diagonalisable on U. In order to achieve this we introduce a particular ordering of the monomial basis which induces U to be $\bar{Y}$-semisimple, that is, simultaneously diagonalisable with respect to the generators Y_i.

4.2.1 Ordering the $\bar{Y}_i$

Following the particular ordering that was introduced by Kasatani in [5], we define the ordering $\succ$ such that U is $\bar{Y}$-semisimple.

$$\textbf{The ordering } \succ : \qquad \lambda \succ \mu \quad \Leftrightarrow \quad (\lambda^+ > \mu^+) \;\text{ or }\; (\lambda^+ = \mu^+ \;\text{ and }\; \lambda > \mu) \tag{4.8}$$

where $>$ is the dominance ordering:

$$\lambda \geq \mu \quad \Leftrightarrow \quad \sum_{j=1}^{\ell} \lambda_j \geq \sum_{j=1}^{\ell} \mu_j\,, \qquad \text{for } 1 \leq \ell \leq N,$$

and λ^+ is the partition $(\lambda_1^+ \geq \lambda_2^+ \ldots \geq \lambda_N^+)$.
Therefore if $\lambda \succ \mu$, under this ordering $x^\lambda = x_1^{\lambda_1} \ldots x_N^{\lambda_N}$ is greater than $x^\mu = x_1^{\mu_1} \ldots x_N^{\mu_N}$.

Under the ordering $\succ$ the action of $\bar{Y}_i$ on any polynomial can be written in the form [7]

$$x^{\lambda_\pi}\bar{Y}_j = (-t^{-1/2})^{N-1}q^{-\lambda_{\pi j}}t^{\pi_j - 2}x^{\lambda_\pi} + \sum_{\mu \prec \lambda} c_{\lambda,\mu}x^{\mu}\,, \tag{4.9}$$

where λ_π is the dominant degree term and π is the shortest permutation of λ^+ such that $(\lambda_\pi)_i = \lambda^+_\pi$.
The $\bar{Y}_i$ operators are now realised as triangular operators in the monomial basis subject to the ordering $\succ$. This makes finding their eigenvalues trivial, as the eigenvalues of a triangular matrix is just their diagonal entries.

As the definition of $\succ$ may look rather complicated, we present several examples to illustrate its action.

<u>**Example**</u> For $N = 3$ and given $\lambda = (0, 0, 2) \Rightarrow x_1^0 x_2^0 x_3^2$, we find all μ such that $\lambda \succ \mu$.

Firstly we obtain λ^+ by rearranging λ_i in strictly decreasing order.

$$\lambda_i = (0, 0, 2) \quad \Rightarrow \quad \lambda^+ = (2, 0, 0)$$

Using the partial sum $\sum_{j=1}^{\ell}\lambda_j \geq \sum_{j=1}^{\ell}\mu_j$ we then find all μ^+; that is all μ in strictly decreasing order such that:

$$\mu_1^+ \leq 2, \quad \mu_1^+ + \mu_2^+ \leq 2, \quad \mu_1^+ + \mu_2^+ + \mu_3^+ \leq 2.$$

In this example we see that the only permitted μ^+ are $\mu^+ = (2, 0, 0)$ and $\mu^+ = (1, 1, 0)$.

Using μ^+ we now deduce all possible μ by cyclic permutation, noting that the μ don't have to be in decreasing order.

$$\mu^+ = (2, 0, 0) \;\;\Rightarrow\;\; \mu = (2, 0, 0)\,,\ (0, 2, 0)\,,\ (0, 0, 2).$$
$$\mu^+ = (1, 1, 0) \;\;\Rightarrow\;\; \mu = (1, 1, 0)\,,\ (0, 1, 1)\,,\ (1, 0, 1).$$

We now have all μ, however not all are permitted since by definition if $\lambda^+ = \mu^+$, then we must also have $\lambda > \mu$.
In this example $\lambda^+ = \mu^+ = (0, 0, 2)$ so using our original $\lambda = (0, 0, 2)$ we must have:

$$\mu_1 \leq 0, \quad \mu_1 + \mu_2 \leq 0, \quad \mu_1 + \mu_2 + \mu_3 \leq 2.$$

This implies that $\mu = (0,0,2)$ is the only permissible μ from the set $\{(2,0,0),(0,2,0),(0,0,2)\}$. Therefore all μ satisfying $\lambda \succ \mu$ for $\lambda = (0,0,2)$ are

$$\mu = (0,0,2)\,,\ (1,1,0)\,,\ (0,1,1)\,,\ (1,0,1).$$

These values of μ correspond to x_3^2, x_1x_2, x_2x_3 and x_1x_3.

So whenever $\bar{Y}_i$ acts on $x^\lambda = x^{(0,0,2)} = x_3^2$, by (4.9) only the terms $x^\mu = x_1x_2$, x_2x_3 and x_1x_3 should appear, preceded by the dominant term x_3^2.
This is easily verified; (4.5) gives the action $\bar{Y}_3$ on x_3^2 as

$$x_3^2\bar{Y}_3 \;\; = \;\; q^{-2}t^{-1}x_3^2 + q^{-2}(t^{-1}-1)x_1x_3 + q^{-2}(t^{-1}-1)x_2x_3,$$

which can be written in the form

$$x_3^2\bar{Y}_3 \;\; = \;\; q^{-2}t^{-1}x_3^2 + \sum_{\mu\prec\lambda} c_{\lambda,\mu}x^\mu.$$

One can also use (4.9) to construct the action of $\bar{Y}_i$ on any polynomial as is illustrated in the following example.

Example We want to find $x_3^2\bar{Y}_3$ for the $N=3$ case.

By (4.9) we know it is of the form

$$x^{\lambda_\pi}\bar{Y}_j \;\; = \;\; q^{-\lambda_{\pi j}}t^{\pi_j - 2}x^{\lambda_\pi} + \sum_{\mu\prec\lambda} c_{\lambda,\mu}x^\mu.$$

We want $x_3^2\bar{Y}_3 \; = \; x^{\lambda_\pi}\bar{Y}_j \quad \Rightarrow \quad \lambda_\pi = (0,0,2)$ and $j = 3 \quad \Rightarrow \quad \lambda_{\pi j} = 2.$

Now $\lambda = (0,0,2)$ which implies the dominant term $\lambda^+ = (2,0,0)$.
By definition π is the shortest permutation of λ^+ such that $(\lambda_\pi)_i = \lambda^+_\pi$, so with the least number of permutations we obtain $\lambda_\pi = (0,0,2)$ from $\lambda^+ = (2,0,0)$.

$$\begin{aligned} \lambda^+ &= (2,0,0) \to (0,2,0) \to (0,0,2) \; = \; \lambda_\pi \\ \pi &= (1,2,3) \to (2,1,3) \to (2,3,1) \end{aligned}$$

We see that $\pi_{j=3} = 1$ and hence $\pi_j - 2 = -1$.

Therefore by (4.9) we can write

$$x_3^2\bar{Y}_3 \;=\; q^{-2}t^{-1}x_3^2 \;+\; \text{lower terms},$$

which we saw in the previous example is precisely the desired result.

It is worth stressing the importance of the permutation patterns π_j. In Table 4.1 we give the value of π_j, for all degree one and two monomials in the $N = 3$ case.

λ_{π_j}	$\lambda^+_{\pi_j}$	π_j
(2,0,0)	(2,0,0)	(1,2,3)
(0,2,0)	(2,0,0)	(2,1,3)
(0,0,2)	(2,0,0)	(2,3,1)
(1,1,0)	(1,1,0)	(1,2,3)
(1,0,1)	(1,1,0)	(1,3,2)
(0,1,1)	(1,1,0)	(3,1,2)

Table 4.1: Permutation patterns for $N = 3$ degree one and two monomials.

Consider for example if we wanted to find $x_2x_3\bar{Y}_3$ for the $N = 3$ case. We have $x_1^0x_2^1x_3^1\bar{Y}_{j=3}$ which implies that $\lambda_{\pi_j} = (0, 1, 1)$ and $j = 3$. The dominant term $\lambda^+_{\pi_j} = (1, 1, 0)$ so we need just find $\pi_{j=3}$ corresponding to $\lambda^+_{\pi_j} = (1, 1, 0)$. The last row of Table 4.1 tells us immediately that the appropriate π_j is (3,1,2). The third element is 2, so $\pi_{j=3} = 2$. Then by (4.9) we have

$$x_2x_3\bar{Y}_3 \;=\; q^{-1}x_2x_3 \;+\; \text{lower terms}.$$

4.2.2 Non symmetric Macdonald polynomials

The monomial basis with respect to the ordering $\succ$ induces U to be $\bar{Y}$-semisimple. Furthermore since the $\bar{Y}_i$ are now triangular operators, finding their eigenvalues and hence their eigenvectors is much simplified. The non symmetric Macdonald polynomials are the monic simultaneous eigenvectors of $\bar{Y}_i$.

Under the ordering $\succ$ we present the matrices with basis $\{1, x_1, x_2, x_3\}$, corresponding to the action of $\bar{Y}_i$ on degree zero and degree one monomials in the $N = 3$ case below.

$$\bar{Y}_1 = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & q^{-1}t^{-1} & 0 & 0 \\ 0 & t^{-1}-1 & 1 & 0 \\ 0 & t^{-1}-1 & 0 & 1 \end{bmatrix}$$

$\bar{Y}_1$ is clearly lower triangular so its eigenvalues are $\eta_1 = t^{-1}, q^{-1}t^{-1}, 1, 1$.

$$\bar{Y}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-t^{-1} & q^{-1}t^{-1} & 0 \\ 0 & 0 & 1-t & t \end{bmatrix}$$

Eigenvalues of $\bar{Y}_2$ are $\eta_2 = 1, 1, q^{-1}t^{-1}, t$.

$$\bar{Y}_3 = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 1-t^{-1} & t-1 & q^{-1}t^{-1} \end{bmatrix}$$

Eigenvalues of $\bar{Y}_3$ are $\eta_3 = t, t, t, q^{-1}t^{-1}$.

Having found all of the eigenvalues we now find the monic $\bar{Y}$-eigenvectors satisfying the above matrices. As we have explicitly calculated these eigenvectors by hand, the calculations are quite long and for this reason we have not included them in this text. Instead we simply present the results here, that is, the non symmetric Macdonald polynomials E_{λ_π} of degree zero and one are given by

$$\begin{aligned} E_{\lambda_\pi} &= E_{(0,0,0)} = 1 & \eta_1 &= t^{-1}, \eta_2 = 1, \eta_3 = t, \\ E_{\lambda_\pi} &= E_{(1,0,0)} = x_1 + \frac{t^{-1}-1}{q^{-1}t^{-1}-1}x_2 + \frac{t^{-1}-1}{q^{-1}t^{-1}-1}x_3 & \eta_1 &= q^{-1}t^{-1}, \eta_2 = 1, \eta_3 = t, \\ E_{\lambda_\pi} &= E_{(0,1,0)} = x_2 + \frac{1-t}{q^{-1}t^{-1}-t}x_3 & \eta_1 &= 1, \eta_2 = q^{-1}t^{-1}, \eta_3 = t, \\ E_{\lambda_\pi} &= E_{(0,0,1)} = x_3 & \eta_1 &= 1, \eta_2 = t, \eta_3 = q^{-1}t^{-1}. \end{aligned}$$

Similarly for $N = 3$, the action of $\bar{Y}_i$ under the ordering $\succ$, on degree two monomials with the basis $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}$ is given by the following matrices. Under the ordering $\succ$,the $\bar{Y}_i$ are lower triangular, hence their eigenvalues are simply the diagonal entries.

$$\bar{Y}_1 = \begin{bmatrix} q^{-2}t^{-1} & 0 & 0 & 0 & 0 & 0 \\ t^{-1}-1 & 1 & 0 & 0 & 0 & 0 \\ t^{-1}-1 & 0 & 1 & 0 & 0 & 0 \\ q^{-1}(t^{-1}-1) & q^{-1}(1-t^{-1}) & 0 & q^{-1}t^{-1} & 0 & 0 \\ q^{-1}(t^{-1}-1) & 0 & q^{-1}(1-t^{-1}) & 0 & q^{-1}t^{-1} & 0 \\ -(2-t-t^{-1}) & 2-t-t^{-1} & 0 & t^{-1}-1 & 1-t & t \end{bmatrix}$$

Eigenvalues of $\bar{Y}_1$ are $\eta_1 = q^{-2}t^{-1}, 1, 1, q^{-1}t^{-1}, q^{-1}t^{-1}, t$.

$$\bar{Y}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1-t^{-1} & q^{-2}t^{-1} & 0 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 & 0 & 0 \\ 1-t^{-1} & q^{-2}(t^{-1}-1) & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1-t & t & 0 \\ 0 & q^{-1}(t^{-1}-1) & q^{-1}(1-t^{-1}) & 2-t-t^{-1} & t-1 & q^{-1}t^{-1} \end{bmatrix}$$

Eigenvalues of $\bar{Y}_2$ are $\eta_2 = 1, q^{-2}t^{-1}, t, q^{-1}, t, q^{-1}t^{-1}$.

$$\bar{Y}_3 = \begin{bmatrix} t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 \\ 1-t^{-1} & t-1 & q^{-2}t^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 \\ 1-t^{-1} & 0 & q^{-2}(t^{-1}-1) & t-1 & q^{-1} & 0 \\ 2-t-t^{-1} & t-1 & q^{-2}(t^{-1}-1) & t-1 & 0 & q^{-1} \end{bmatrix}$$

Eigenvalues of $\bar{Y}_3$ are $\eta_3 = t, t, q^{-2}t^{-1}, t, q^{-1}, q^{-1}$.

As before, we have found the non symmetric Macdonald polynomials by simultaneously diagonalising these matrices. We give their explicit form here, a result of lengthy calculations. The non symmetric Macdonald polynomials of degree two are:

$$E_{\lambda_\pi} = E_{(0,1,1)} = x_2x_3 \qquad \eta_1 = t, \eta_2 = q^{-1}t^{-1}, \eta_3 = q^{-1},$$

$$E_{\lambda_\pi} = E_{(1,0,1)} = x_1x_3 + \frac{1-t}{q^{-1}t^{-1}-t}x_2x_3 \qquad \eta_1 = q^{-1}t^{-1}, \eta_2 = t, \eta_3 = q^{-1},$$

$$E_{\lambda_\pi} = E_{(1,1,0)} = x_1x_2 + \frac{t-1}{t-q^{-1}}x_1x_3 + \frac{t-1}{t-q^{-1}}x_2x_3 \qquad \eta_1 = q^{-1}t^{-1}, \eta_2 = q^{-1}, \eta_3 = t,$$

$$E_{\lambda_\pi} = E_{(0,0,2)} = x_3^2 + \frac{q^{-1}-q^{-1}t^{-1}}{1-q^{-1}t^{-1}}x_1x_3 + \frac{q^{-1}-q^{-1}t^{-1}}{1-q^{-1}t^{-1}}x_2x_3 \quad \eta_1 = 1, \eta_2 = t, \eta_3 = q^{-2}t^{-1},$$

$$\begin{aligned} E_{\lambda_\pi} = E_{(0,2,0)} = x_2^2 &+ \frac{1-t}{q^{-2}t^{-1}-t}x_3^2 + \frac{q^{-1}(1-t^{-1})}{1-q^{-1}t^{-1}}x_1x_2, \\ &+ \frac{q^{-1}(1-t^{-1})(1-t)}{(1-q^{-1}t^{-1})(q^{-2}t^{-1}-t)}x_1x_3, \\ &+ \frac{(1-t^{-1})(q^{-2}t^{-1}+q^{-1}-q^{-1}t-t)}{(1-q^{-1}t^{-1})(q^{-2}t^{-1}-t)}x_2x_3 \qquad \eta_1 = 1, \eta_2 = q^{-2}t^{-1}, \eta_3 = t, \end{aligned}$$

$$\begin{aligned} E_{\lambda_\pi} = E_{(2,0,0)} = x_1^2 &+ \frac{t^{-1}-1}{q^{-2}t^{-1}-1}x_2^2 + \frac{t^{-1}-1}{q^{-2}t^{-1}-1}x_3^2 + \frac{(t^{-1}-1)(1-q^{-2})}{(q^{-2}t^{-1}-1)(1-q^{-1})}x_1x_2, \\ &+ \frac{(t^{-1}-1)(1-q^{-2})}{(q^{-2}t^{-1}-1)(1-q^{-1})}x_1x_3, \\ &+ \frac{(t^{-1}-1)(1-q^{-2})(2-t^{-1}-1)}{(q^{-2}t^{-1}-1)(1-q^{-1})(1-q^{-1}t^{-1})}x_2x_3 \qquad \eta_1 = 1, \eta_2 = t, \eta_3 = q^{-1}t^{-1}. \end{aligned}$$

As can be seen from above, even for low dimensions the Macdonald polynomials can be quite long. In the next section we will describe a much simpler method for obtaining Macdonald polynomials.

4.3 Intertwining Operators

As the Macdonald polynomials are found by simultaneously diagonalising the $\bar{Y}_i$, they are not always easily obtained, especially if calculated by hand as we have done so far. This is particularly true the larger the dimension of the monomial basis becomes. Even for the $N = 4$ case, finding non symmetric Macdonald polynomials of degree two is extremely tedious and time consuming. It is therefore useful to define intertwining operators which enable us to find all non symmetric Macdonald polynomials, given any one non symmetric

Macdonald polynomial. Though these intertwining operators are due to Cherednik [19], we will define them in a similar fashion to Kasatani [5].

4.3.1 The Intertwining Operator $\bar{A}$

The intertwining operator $\bar{A}$ increases the degree of Y-eigenvectors. It is defined as follows:

$$\bar{A} \;=\; \bar{Z}_1\bar{\sigma}^{-1}, \tag{4.10}$$

and acts on eigenvectors as

$$E_{\lambda_\pi}\bar{A} \;=\; q^{\lambda_{\pi_1}+1}E_{\lambda_\pi\bar{\sigma}^{-1}},$$

where $\lambda_\pi\bar{\sigma}^{-1} \;=\; (\lambda_{\pi_2}, \ldots, \lambda_{\pi_N}, \lambda_{\pi_1}+1)$.

Since degree one monomials are just constants, their simultaneous Y-eigenvector is just 1. It therefore makes sense to start from here to construct all other Y-eigenvectors. We act with $\bar{A}$ on the eigenvector $E_{(0,0,0)} \;=\; 1$ to obtain another eigenvector of higher degree. In the $N = 3$ monomial basis we have

$$\begin{aligned} E_{(0,0,0)}\bar{A} \;=\; (1)\bar{Z}_1\bar{\sigma}^{-1} \;=\; x_1\bar{\sigma}^{-1} \;&=\; qx_3 \\ \Rightarrow q^{\lambda_{\pi_1}+1}E_{\lambda_\pi\bar{\sigma}^{-1}} \;=\; qE_{(0,0,1)} \;&=\; qx_3 \\ \Rightarrow E_{(0,0,1)} \;&=\; x_3. \end{aligned}$$

Therefore we have just obtained the first degree one Macdonald polynomial using the operator $\bar{A}$. Acting $\bar{A}$ again yields a degree two Macdonald polynomial.

$$\begin{aligned} E_{(0,0,1)}\bar{A} \;=\; (x_3)\bar{Z}_1\bar{\sigma}^{-1} \;=\; x_3x_1\bar{\sigma}^{-1} \;&=\; qx_2x_3 \\ \Rightarrow q^{\lambda_{\pi_1}+1}E_{\lambda_\pi\bar{\sigma}^{-1}} \;=\; qE_{(0,1,1)} \;&=\; qx_2x_3 \\ \Rightarrow E_{(0,1,1)} \;&=\; x_2x_3. \end{aligned}$$

We can continue applying the operator $\bar{A}$ to obtain Macdonald polynomials in such a way. We note that repeated applications of $\bar{A}$, starting at $E_{(0,0,0)} \;=\; 1$ only yields the lowest ordered Macdonald polynomial of each degree. However from the previous section we know that there are three simultaneous Y-eigenvectors of degree one. $\bar{A}$ only enabled us to find one of these, namely, $E_{(0,0,1)} = x_3$. To find the other two we introduce the intertwining operator $\bar{B}_i$.

4.3.2 The Intertwining Operator $\bar{B}$

Unlike the operator $\bar{A}$, which increased the degree of the Y_i by one, we want the intertwining operators $\bar{B}_i$ to maintain the same degree. They commute with the affine generators Y_i so

$$\begin{aligned} Y_j B_i &= B_i Y_j \text{ for } j \neq i, i+1, \\ Y_i B_i &= B_i Y_{i+1}, \\ Y_{i+1} B_i &= B_i Y_i. \end{aligned}$$

Using the definition of $\bar{Y}_i$, a solution to these equations is given by $\bar{B}_i = \bar{T}_i \bar{Y}_{i+1} - \bar{T}_i^{-1} \bar{Y}_i$. Since we want $\bar{B}_i$ to act on eigenvectors as $E_{\lambda_\pi} \bar{B}_i = \gamma E_{s_i \lambda_\pi} \bar{B}_i$, we must introduce a normalisation factor. The intertwining operator $\bar{B}_i$ is therefore defined as

$$\bar{B}_i = \frac{\bar{T}_i \bar{Y}_{i+1} - \bar{T}_i^{-1} \bar{Y}_i}{\bar{Y}_i - \bar{Y}_{i+1}}, \tag{4.11}$$

and acts on eigenvectors as

$$E_{\lambda_\pi} \bar{B}_i = t^{1/2} E_{s_i \lambda_\pi} \bar{B}_i,$$

where $s_i \lambda_\pi \succ \lambda_\pi$.

Having used the operator $\bar{A}$ to obtain the first eigenvector of degree one, $E_{(0,0,1)} = x_3$, let us now act on it with $\bar{B}_i$ to find the other two, $E_{(0,1,0)}$ and $E_{(1,0,0)}$.

Firstly we want to find $E_{(0,1,0)}$ given $E_{(0,0,1)}$. Therefore in order to obtain $(0,1,0)$ from $(0,0,1)$ we apply the permutation operator s_2, since $s_2(0,0,1) = (0,1,0)$.
By the ordering $\succ$, (4.8) we have $(0,1,0) \succ (0,0,1)$, so we must determine $E_{(0,0,1)} \bar{B}_2$.

$$\begin{aligned} E_{(0,0,1)} \bar{B}_2 &= x_3 \bar{B}_2 \\ &= x_3 \frac{\bar{T}_2 \bar{Y}_3 - \bar{T}_2^{-1} \bar{Y}_2}{\bar{Y}_2 - \bar{Y}_3} \\ &= t^{1/2} x_2 + \frac{t^{1/2}(1-t)}{q^{-1} t^{-1} - t} x_3. \end{aligned}$$

However by (4.11) we know that $E_{(0,0,1)}\bar{B}_2 = t^{1/2}E_{(0,1,0)}$, which gives us

$$E_{(0,1,0)} = t^{1/2}x_2 + \frac{1-t}{q^{-1}t^{-1}-t}x_3.$$

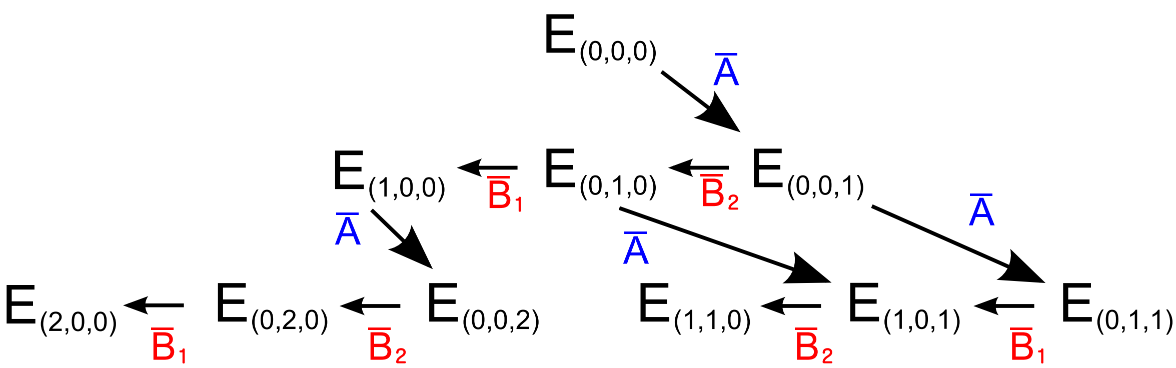
Similarly, to obtain $E_{(1,0,0)}$ from $E_{(0,1,0)}$ we calculate $E_{(0,1,0)}\bar{B}_1$, since $s_1(0,1,0) = (1,0,0)$ with $(1,0,0) \succ (0,1,0)$.

$$E_{(0,1,0)}\bar{B}_1 = t^{1/2}x_1 + t^{1/2}\left(\frac{t^{-1}-1}{q^{-1}t^{-1}-1}\right)x_2 + t^{1/2}\left(\frac{t^{-1}-1}{q^{-1}t^{-1}-1}\right)x_3.$$

This gives us the last eigenvector of degree one (in the $N = 3$ case)

$$E_{(1,0,0)} = x_1 + \frac{t^{-1}-1}{q^{-1}t^{-1}-1}x_2 + \frac{t^{-1}-1}{q^{-1}t^{-1}-1}x_3.$$

To summarise, using both intertwining operators $\bar{A}$ and $\bar{B}_i$, allows to obtain all simultaneous Y-eigenvectors of any degree. Starting with the eigenvector of degree zero $E_{(0,0,0)} = 1$, we apply $\bar{A}$ and obtain an eigenvector of degree one. Now $\bar{B}_i$ lets us find all other degree one eigenvectors. Repeated applications of both intertwining operators thus generates all Macdonald polynomials. For the $N = 3$ case we therefore have the following diagram:



This concludes the chapter on the polynomial representation of a double affine Hecke algebra. In the next chapter we shift our focus to affine Hecke algebra representations. We present a tangle representation which we developed to generate finite dimensional matrix representations of $\mathcal{A}_N(t)$. The construction of this tangle representation was motivated by the apparent close relationship between many knot theory ideas and our cube-ribbon construction.

Appendix 4

4A.1 $\bar{T}_{i,j}$ on degree zero, one and two monomials

We calculate the action of the Hecke algebra generators $\bar{T}_{i,j}$ on degree zero, one and two monomials. We look at the case $N = 3$, where there are two generators $\bar{T}_{1,2}$ and $\bar{T}_{2,3}$. Recall that (4.1) gave us

$$\bar{T}_{i,j} = -t^{-1/2}s_{ij} + (t^{1/2} - t^{-1/2})(1 - s_{ij})\frac{x_i}{x_i - x_j}.$$

On degree zero and one monomials $1, x_1, x_2, x_3$ this gives us the following results:

$$\begin{aligned}
1\bar{T}_{1,2} &= -t^{-1/2} \\
x_1\bar{T}_{1,2} &= (t^{1/2} - t^{-1/2})x_1 - t^{-1/2}x_2 \\
x_2\bar{T}_{1,2} &= -t^{1/2}x_1 \\
x_3\bar{T}_{1,2} &= -t^{-1/2}x_3
\end{aligned}$$

$$\begin{aligned}
1\bar{T}_{2,3} &= -t^{-1/2} \\
x_1\bar{T}_{2,3} &= -t^{-1/2}x_1 \\
x_2\bar{T}_{2,3} &= (t^{1/2} - t^{-1/2})x_2 - t^{-1/2}x_3 \\
x_3\bar{T}_{2,3} &= -t^{1/2}x_2
\end{aligned}$$

On degree two monomials $x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$ we get:

$$\begin{aligned}
x_1^2\bar{T}_{1,2} &= (t^{1/2} - t^{-1/2})x_1^2 - t^{1/2}x_2^2 + (t^{1/2} - t^{-1/2})x_1x_2 \\
x_2^2\bar{T}_{1,2} &= -t^{1/2}x_1^2 - (t^{1/2} - t^{-1/2})x_1x_2 \\
x_3^2\bar{T}_{1,2} &= -t^{-1/2}x_3^2 \\
x_1x_2\bar{T}_{1,2} &= -t^{-1/2}x_1x_2 \\
x_1x_3\bar{T}_{1,2} &= (t^{1/2} - t^{-1/2})x_1x_3 - t^{-1/2}x_2x_3 \\
x_2x_3\bar{T}_{1,2} &= -t^{1/2}x_1x_3 \\
x_1^2\bar{T}_{2,3} &= -t^{-1/2}x_1^2 \\
x_2^2\bar{T}_{2,3} &= (t^{1/2} - t^{-1/2})x_2^2 - t^{-1/2}x_3^2 + (t^{1/2} - t^{-1/2})x_2x_3 \\
x_3^2\bar{T}_{2,3} &= -t^{1/2}x_2^2 - (t^{1/2} - t^{-1/2})x_2x_3
\end{aligned}$$

$$
\begin{aligned}
x_1x_2\bar{T}_{2,3} &= (t^{1/2}-t^{-1/2})x_1x_2 - t^{-1/2}x_1x_3 \\
x_1x_3\bar{T}_{2,3} &= -t^{1/2}x_1x_2 \\
x_2x_3\bar{T}_{2,3} &= -t^{-1/2}x_2x_3
\end{aligned}
$$

4A.2 $\bar{Y}_i$ on degree zero, one and two monomials

We calculate the action of the affine Hecke algebra generators $\bar{Y}_i$ on degree zero, one and two monomials. We look at the case $N=3$, where there are three generators $\bar{Y}_1$, $\bar{Y}_2$ and $\bar{Y}_3$.

Recall that (4.5) gave us

$$
\bar{Y}_i = \bar{X}_{i,i+1}\bar{X}_{i,i+2}\ldots\bar{X}_{i,N}\hat{q}_i\bar{X}_{1,i}^{-1}\ldots\bar{X}_{i-1,i}^{-1},
$$

where the action of the $\bar{X}_{i,j}$ is given by

$$
\bar{X}_{i,j} = -t^{-1/2} + (t^{1/2}-t^{-1/2})(1-s_{ij})\frac{x_i}{x_i-x_j}.
$$

Therefore $\bar{Y}_1 = \bar{X}_{1,2}\bar{X}_{1,3}\hat{q}_1$, $\bar{Y}_2 = \bar{X}_{2,3}\hat{q}_2\bar{X}_{1,2}^{-1}$ and $\bar{Y}_3 = \hat{q}_3\bar{X}_{1,3}^{-1}\bar{X}_{2,3}^{-1}$. We firstly calculate $\bar{X}_{1,2}$, $\bar{X}_{1,3}$ and $\bar{X}_{2,3}$ along with their inverses on degree zero, one and two monomials.

$$
\begin{aligned}
1\bar{X}_{1,2} &= -t^{-1/2} \\
x_1\bar{X}_{1,2} &= -t^{-1/2}x_1 + (t^{1/2}-t^{-1/2})x_2 \\
x_2\bar{X}_{1,2} &= -t^{1/2}x_2 \\
x_3\bar{X}_{1,2} &= -t^{-1/2}x_3 \\
x_1^2\bar{X}_{1,2} &= -t^{-1/2}x_1^2 + (t^{1/2}-t^{-1/2})x_2^2 + (t^{1/2}-t^{-1/2})x_1x_2 \\
x_2^2\bar{X}_{1,2} &= -t^{1/2}x_2^2 - (t^{1/2}-t^{-1/2})x_1x_2 \\
x_3^2\bar{X}_{1,2} &= -t^{-1/2}x_3^2 \\
x_1x_2\bar{X}_{1,2} &= -t^{-1/2}x_1x_2 \\
x_1x_3\bar{X}_{1,2} &= -t^{-1/2}x_1x_3 + (t^{1/2}-t^{-1/2})x_2x_3 \\
x_2x_3\bar{X}_{1,2} &= -t^{1/2}x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
1\bar{X}_{1,2}^{-1} &= -t^{1/2} \\
x_1\bar{X}_{1,2}^{-1} &= -t^{1/2}x_1 - (t^{1/2} - t^{-1/2})x_2 \\
x_2\bar{X}_{1,2}^{-1} &= -t^{-1/2}x_2 \\
x_3\bar{X}_{1,2}^{-1} &= -t^{1/2}x_3 \\
x_1^2\bar{X}_{1,2}^{-1} &= -t^{1/2}x_1^2 - (t^{1/2} - t^{-1/2})x_2^2 - (t^{1/2} - t^{-1/2})x_1x_2 \\
x_2^2\bar{X}_{1,2}^{-1} &= -t^{-1/2}x_2^2 + (t^{1/2} - t^{-1/2})x_1x_2 \\
x_3^2\bar{X}_{1,2}^{-1} &= -t^{1/2}x_3^2 \\
x_1x_2\bar{X}_{1,2}^{-1} &= -t^{1/2}x_1x_2 \\
x_1x_3\bar{X}_{1,2}^{-1} &= -t^{1/2}x_1x_3 - (t^{1/2} - t^{-1/2})x_2x_3 \\
x_2x_3\bar{X}_{1,2}^{-1} &= -t^{-1/2}x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
1\bar{X}_{1,3} &= -t^{-1/2} \\
x_1\bar{X}_{1,3} &= -t^{-1/2}x_1 + (t^{1/2} - t^{-1/2})x_3 \\
x_2\bar{X}_{1,3} &= -t^{-1/2}x_2 \\
x_3\bar{X}_{1,3} &= -t^{1/2}x_3 \\
x_1^2\bar{X}_{1,3} &= -t^{-1/2}x_1^2 + (t^{1/2} - t^{-1/2})x_2^3 + (t^{1/2} - t^{-1/2})x_1x_3 \\
x_2^2\bar{X}_{1,3} &= -t^{-1/2}x_2^2 \\
x_3^2\bar{X}_{1,3} &= -t^{1/2}x_3^2 - (t^{1/2} - t^{-1/2})x_1x_3 \\
x_1x_2\bar{X}_{1,3} &= -t^{-1/2}x_1x_2 + (t^{1/2} - t^{-1/2})x_2x_3 \\
x_1x_3\bar{X}_{1,3} &= -t^{-1/2}x_1x_3 \\
x_2x_3\bar{X}_{1,3} &= -t^{1/2}x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
1\bar{X}_{1,3}^{-1} &= -t^{1/2} \\
x_1\bar{X}_{1,3}^{-1} &= -t^{1/2}x_1 - (t^{1/2} - t^{-1/2})x_3 \\
x_2\bar{X}_{1,3}^{-1} &= -t^{1/2}x_2 \\
x_3\bar{X}_{1,3}^{-1} &= -t^{-1/2}x_3 \\
x_1^2\bar{X}_{1,3}^{-1} &= -t^{1/2}x_1^2 - (t^{1/2} - t^{-1/2})x_2^3 - (t^{1/2} - t^{-1/2})x_1x_3 \\
x_2^2\bar{X}_{1,3}^{-1} &= -t^{1/2}x_2^2 \\
x_3^2\bar{X}_{1,3}^{-1} &= -t^{-1/2}x_3^2 + (t^{1/2} - t^{-1/2})x_1x_3 \\
x_1x_2\bar{X}_{1,3}^{-1} &= -t^{1/2}x_1x_2 - (t^{1/2} - t^{-1/2})x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
x_1x_3\bar{X}_{1,3}^{-1} &= -t^{1/2}x_1x_3 \\
x_2x_3\bar{X}_{1,3}^{-1} &= -t^{-1/2}x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
1\bar{X}_{2,3} &= -t^{-1/2} \\
x_1\bar{X}_{2,3} &= -t^{-1/2}x_1 \\
x_2\bar{X}_{2,3} &= -t^{-1/2}x_2 + (t^{1/2} - t^{-1/2})x_3 \\
x_3\bar{X}_{2,3} &= -t^{1/2}x_3 \\
x_1^2\bar{X}_{2,3} &= -t^{-1/2}x_1^2 \\
x_2^2\bar{X}_{2,3} &= -t^{-1/2}x_2^2 + (t^{1/2} - t^{-1/2})x_3^2 + (t^{1/2} - t^{-1/2})x_2x_3 \\
x_3^2\bar{X}_{2,3} &= -t^{1/2}x_3^2 - (t^{1/2} - t^{-1/2})x_2x_3 \\
x_1x_2\bar{X}_{2,3} &= -t^{-1/2}x_1x_2 + (t^{1/2} - t^{-1/2})x_1x_3 \\
x_1x_3\bar{X}_{2,3} &= -t^{1/2}x_1x_3 \\
x_2x_3\bar{X}_{2,3} &= -t^{-1/2}x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
1\bar{X}_{2,3}^{-1} &= -t^{1/2} \\
x_1\bar{X}_{2,3}^{-1} &= -t^{1/2}x_1 \\
x_2\bar{X}_{2,3}^{-1} &= -t^{1/2}x_2 - (t^{1/2} - t^{-1/2})x_3 \\
x_3\bar{X}_{2,3}^{-1} &= -t^{-1/2}x_3 \\
x_1^2\bar{X}_{2,3}^{-1} &= -t^{1/2}x_1^2 \\
x_2^2\bar{X}_{2,3}^{-1} &= -t^{1/2}x_2^2 - (t^{1/2} - t^{-1/2})x_3^2 - (t^{1/2} - t^{-1/2})x_2x_3 \\
x_3^2\bar{X}_{2,3}^{-1} &= -t^{-1/2}x_3^2 + (t^{1/2} - t^{-1/2})x_2x_3 \\
x_1x_2\bar{X}_{2,3}^{-1} &= -t^{1/2}x_1x_2 - (t^{1/2} - t^{-1/2})x_1x_3 \\
x_1x_3\bar{X}_{2,3}^{-1} &= -t^{-1/2}x_1x_3 \\
x_2x_3\bar{X}_{2,3}^{-1} &= -t^{1/2}x_2x_3
\end{aligned}
$$

We have that $\bar{Y}_1 = \bar{X}_{1,2}\bar{X}_{1,3}\hat{q}_1$, so we calculate the action of $\bar{Y}_1$ on degree zero, one and two monomials.

$$
\begin{aligned}
1\bar{Y}_1 &= t^{-1} \\
x_1\bar{Y}_1 &= q^{-1}t^{-1}x_1 + (t^{-1}-1)x_2 + (t^{-1}-1)x_3 \\
x_2\bar{Y}_1 &= x_2 \\
x_3\bar{Y}_1 &= x_3 \\
x_1^2\bar{Y}_1 &= q^{-2}t^{-1}x_1^2 + (t^{-1}-1)x_2^2 + (t^{-1}-1)x_3^2 + q^{-1}(t^{-1}-1)x_1x_2 \\
&+ q^{-1}(t^{-1}-1)x_1x_3 - (2-t-t^{-1})x_2x_3 \\
x_2^2\bar{Y}_1 &= x_2^2 + q^{-1}(1-t^{-1})x_1x_2 + (2-t-t^{-1})x_2x_3 \\
x_3^2\bar{Y}_1 &= x_3^2 + q^{-1}(1-t^{-1})x_1x_3 \\
x_1x_2\bar{Y}_1 &= q^{-1}t^{-1}x_1x_2 + (t^{-1}-1)x_2x_3 \\
x_1x_3\bar{Y}_1 &= q^{-1}t^{-1}x_1x_3 + (1-t)x_2x_3 \\
x_2x_3\bar{Y}_1 &= tx_2x_3
\end{aligned}
$$

Since $\bar{Y}_2 = \bar{X}_{2,3}\hat{q}_2\bar{X}_{1,2}^{-1}$ we can now find the action of $\bar{Y}_2$ on degree zero, one and two monomials.

$$
\begin{aligned}
1\bar{Y}_2 &= 1 \\
x_1\bar{Y}_2 &= x_1 + (1-t^{-1})x_2 \\
x_2\bar{Y}_2 &= q^{-1}t^{-1}x_2 + (1-t)x_3 \\
x_3\bar{Y}_2 &= tx_3 \\
x_1^2\bar{Y}_2 &= x_1^2 + (1-t^{-1})x_2^2 + (1-t^{-1})x_1x_2 \\
x_2^2\bar{Y}_2 &= q^{-2}t^{-1}x_2^2 + (1-t)x_3^2 + q^{-2}(t^{-1}-1)x_1x_2 + q^{-1}(t^{-1}-1)x_2x_3 \\
x_3^2\bar{Y}_2 &= tx_3^2 + q^{-1}(1-t^{-1})x_2x_3 \\
x_1x_2\bar{Y}_2 &= q^{-1}x_1x_2 + (1-t)x_1x_3 + (2-t-t^{-1})x_2x_3 \\
x_1x_3\bar{Y}_2 &= tx_1x_3 + (t-1)x_2x_3 \\
x_2x_3\bar{Y}_2 &= q^{-1}t^{-1}x_2x_3
\end{aligned}
$$

Finally $\bar{Y}_3 = \hat{q}_3 \bar{X}_{1,3}^{-1} \bar{X}_{2,3}^{-1}$, so we can calculate the action of $\bar{Y}_3$ on degree zero, one and two monomials.

$$\begin{array}{rcl}
1\bar{Y}_3 & = & t \\
x_1\bar{Y}_3 & = & tx_1 + (1-t^{-1})x_3 \\
x_2\bar{Y}_3 & = & tx_2 + (t-1)x_3 \\
x_3\bar{Y}_3 & = & q^{-1}t^{-1}x_3 \\
x_1^2\bar{Y}_3 & = & tx_1^2 + (1-t^{-1})x_3^2 + (1-t^{-1})x_1x_3 + (2-t-t^{-1})x_2x_3 \\
x_2^2\bar{Y}_3 & = & tx_2^2 + (t-1)x_3^2 + (t-1)x_2x_3 \\
x_3^2\bar{Y}_3 & = & q^{-2}t^{-1}x_3^2 + q^{-2}(t^{-1}-1)x_1x_3 + q^{-2}(t^{-1}-1)x_2x_3 \\
x_1x_2\bar{Y}_3 & = & tx_1x_2 + (t-1)x_1x_3 + (t-1)x_2x_3 \\
x_1x_3\bar{Y}_3 & = & q^{-1}x_1x_3 \\
x_2x_3\bar{Y}_3 & = & q^{-1}x_2x_3
\end{array}$$

Chapter 5

Tangle Representation of the Affine Hecke Algebra

In this chapter our principal goal is to investigate the possibility of finite dimensional DAHA representations. We aim to find such representations by firstly developing and then extending finite dimensional matrix representations of affine Hecke algebras. More specifically, we explicitly develop matrix representations of AHAs emerging from tangles and present all of the necessary conditions for their extension to matrix representations of DAHAs.

Our tangle representation of the affine Hecke algebra $\mathcal{A}_N(t)$, is motivated by the work of Kasatani and Pasquier in [7], where they define a pattern representation of the affine Temperley-Lieb algebra, which is a specific quotient group of $\mathcal{A}_N(t)$. In this chapter the tangle representation that we construct is much more general and provides finite dimensional matrix representations for all affine Hecke algebras, in addition to the Temperley-Lieb algebra.

For clarity we highlight the main points of its construction here. Firstly, we define exactly how to obtain planar tangles called elementary patterns which form a basis of our representation. We present combinatorial formulas describing the exact number of basis elements for all dimensions. Secondly we use the graphical representation of the $\mathcal{A}_N(t)$ generators, which we introduced in Chapter 3 in terms of braiding on cylinders, to describe the action of these generators on the pattern basis. Lastly, to obtain finite dimensional matrices, we use knot theory techniques. In particular we employ Reidemeister moves and moves associated to the evaluation of the Kauffman bracket, to decompose the tangle diagrams resulting from the action of the $\mathcal{A}_N(t)$ generators on the elementary patterns, into a linear combination of basis elements.

We begin this chapter by defining tangles and Reidemeister moves before explicitly describing the construction of the elementary pattern basis.

5.1 Tangles

A tangle is made by a set of open strings embedded into a three dimensional ball, such that their extremities are on the boundary [7]. The strings are not allowed to cross each other. In the special case that the ball is punctured, that is, pierced with a flux running through it, strings cannot cross the flux but can connect the flux to the boundary.
In either case the extremities of the strings on the boundary are labelled by N points ordered anticlockwise.

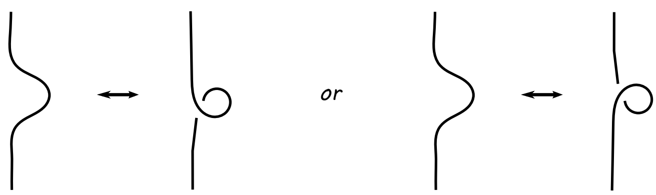
To obtain a tangle diagram complete with crossings, we project onto a flat disk and note the position of under and over crossings. In the punctured ball case, we project the flux onto the origin of the disk. Therefore strings can connect the origin to the boundary but cannot in any case cross the origin.

We highlight that even though tangle diagrams may appear completely unique, they are in fact equivalent if there exists an ambient isotopy taking one to the other using Reidemeister moves.

5.1.1 Reidemeister moves

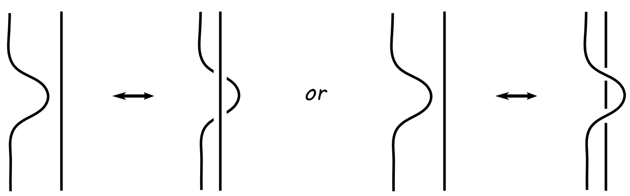
The Reidemeister moves [20] are a set of ambient isotopies that allow us to change the projection of a tangle diagram without changing the tangle diagram represented by the projection. They are defined as follows: [21]

1. The first Reidemeister move allows us to include or exclude a twist in any strand as described in the diagram below.



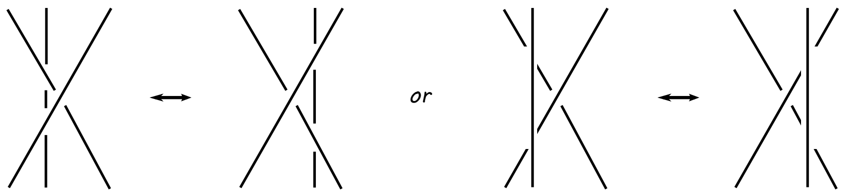
first Reidemeister move

2. The second Reidemeister move allows us to either add or remove two crossings to a tangle diagram. In the figure below we depict both cases.



second Reidemeister move

3. The third Reidemeister move allows us to move a strand from one side of a crossing to the other side of the crossing, as illustrated in the following diagram.



third Reidemeister move

Using these moves it is straightforward to identify equivalent tangle diagrams.

5.2 Patterns

Having gone from tangles to tangle diagrams, we now introduce patterns which will form the basis of our representation.

Patterns are planar tangles, therefore they do not contain any crossings. We are particularly interested in elementary patterns, as linear combinations of these give tangle diagrams. As such, elementary patterns are the basis elements of all tangles and also the basis of our finite dimensional representations of the affine Hecke algebra.

5.2.1 Encoding Patterns

We can encode a pattern π, [7, 22], with a string of letters α and β as follows:

(i) If point i on the boundary of the disk is connected to the origin, label it α.

(ii) If points i and j are connected with $i < j$, then label points i and j by α and β respectively.

Note that these rules imply that at any point of a pattern π there must be a greater or equal number of αs as βs. Also given any pattern π we can easily locate the position of the isolated α by successively erasing factors of $\alpha\beta$ corresponding to paired points. To illustrate these rules, we present the following examples.

Consider the pattern $\pi = \alpha\alpha\beta\alpha\beta\beta$. In this pattern all points are paired since $N = 6$ is even. One can easily see that there is no isolated α as π contains the same number of αs as βs. Starting from the centre of π we factorise points into pairs (factors of $\alpha\beta$). Using the second rule, points 2 and 3 are a pair; as are points $4, 5$ and $1, 6$. Therefore we can now draw π as in Figure 5.1.

As another example take the pattern $\pi = \beta\alpha\alpha$. In this case $N = 3$ is odd so there is an isolated α connecting the origin to the boundary. π begins with a β meaning the first element is connected to the last element of π, using the cyclic nature of the patterns. Therefore since points 3 and 1 are connected, point 2 connects the origin to the boundary. The pattern $\pi = \beta\alpha\alpha$ is illustrated in Figure 5.1.

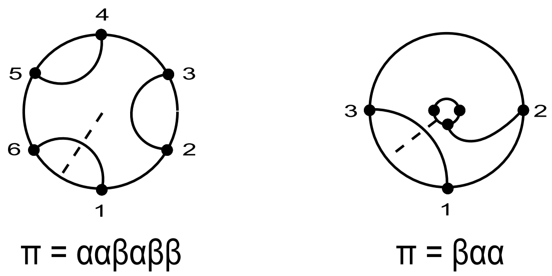


Figure 5.1: The patterns $\pi = \alpha\alpha\beta\alpha\beta\beta$ and $\pi = \beta\alpha\alpha$.

Let us now examine the three possible distinct types of patterns.

5.2.2 N Even Patterns

If we have an even number of ordered points N on the boundary of the disk, then all points are paired. This implies that every pattern must contain the same number of αs as βs. There is no isolated α and hence we have unpunctured disks. Each pattern must also begin by an α.

The number of basis elements (patterns π) for N even is given by

$$C_N = \frac{2^{N/2}(N-1)!!}{(N/2+1)!}, \tag{5.1}$$

where $(N-1)!!$ is the product of odd terms only. We see that the number of basis elements increases quite rapidly as the number of ordered points on the boundary increases. For the first three even N we find that:

$$\begin{aligned}
N = 4 \quad &\Rightarrow \quad C_4 = \frac{2^2(3)!!}{(3)!} = \frac{4.3.1}{6} = 2 \\
N = 6 \quad &\Rightarrow \quad C_6 = \frac{2^3(5)!!}{(4)!} = \frac{8.5.3.1}{24} = 5 \\
N = 8 \quad &\Rightarrow \quad C_8 = \frac{2^4(7)!!}{(5)!} = \frac{16.7.5.3.1}{120} = 14
\end{aligned}$$

It is worth pointing out that $N = 2$ is the first even case. However this is a special case as given two points there is only one way to connect them. This case is trivial, therefore the first non trivial even case is when $N = 4$. We can puncture the $N = 2$ case though, which yields two distinct basis elements. We will study this case more thoroughly in Section 5.4.

5.2.3 N Odd Patterns

Given an odd number of ordered points N on the boundary of the disk, there must be an isolated α in every pattern. The isolated α connects the boundary to the origin, meaning that all N odd patterns are punctured. This implies that every pattern must contain a greater number of αs than βs. It is now useful to view each pattern as an infinite periodic string with the identification $\pi_{i+N} = \pi_i$. In this way for N odd patterns, the first element doesn't necessarily have to be an α.

For each odd N the number of basis elements is given by

$$C_N = \frac{2^{(N-1)/2} N!!}{(\frac{N+1}{2})!}. \tag{5.2}$$

The dimension of the first four odd basis are:

$$\begin{aligned}
N = 1 \quad &\Rightarrow \quad C_1 = \text{this is trivial as we have only one point} \\
N = 3 \quad &\Rightarrow \quad C_3 = \frac{2^1(3)!!}{(2)!} = \frac{2.3.1}{2} = 3 \\
N = 5 \quad &\Rightarrow \quad C_5 = \frac{2^2(5)!!}{(3)!} = \frac{4.5.3.1}{6} = 10 \\
N = 7 \quad &\Rightarrow \quad C_7 = \frac{2^3(7)!!}{(4)!} = \frac{8.7.5.3.1}{24} = 35
\end{aligned}$$

5.2.4 N Even and Punctured

All odd N patterns are inherently punctured. However this is not true for even N patterns. Therefore we describe how to puncture patterns with an even number of points. In this situation then, all points remain paired and strings cannot cross the puncture located at the origin. In essence puncturing patterns increases the dimension of the basis as there are now more distinct ways in which one can connect the ordered points on the boundary. As an example in Section 5.4 we derive in detail the $N = 2$ punctured case.

5.3 Operators in the Tangle Representation

Before we explicitly derive tangle representations of the affine Hecke algebra it is necessary to describe the action of operators on the pattern basis. The operators of interest to us are the affine Hecke algebra generators. In Chapter 2, we defined $\mathcal{A}_N(t)$ in terms of the generators T_i and Y_i. We also derived all of the defining relations of $\mathcal{A}_N(t)$ in terms of the T_i and the operator σ (2.11). Graphically, in Chapter 3 we represented all of these generators and described their action as braiding on the surface of a cylinder, or equivalently, as braiding on an infinitely long strip with vertical edges identified with each other. Now we define how these generators act on the basis elements of our tangle representation, that is on elementary patterns.

5.3.1 Operators as Annuli

In the tangle representation operators are represented as annuli. Each operator or annulus has M ordered points on its inner boundary and $\bar{M}$ ordered points on its outer boundary. Strands may cross each other as they connect points on $\bar{M}$ to M. The identity operator contains no crossings as it is simply the annulus where all points $\bar{i}$ on the outer boundary are connected to corresponding ordered points i on the inner boundary.

When an operator acts on a pattern, we simply place the pattern inside the annulus such that all N points on the boundary of the disk containing the pattern are identified with the corresponding M points of the inner boundary of the annulus. To fully describe the action of an operator on a pattern, in Figure 5.2 we show the Hecke algebra generator T_1 acting on the pattern $\pi = \beta\alpha\alpha$.

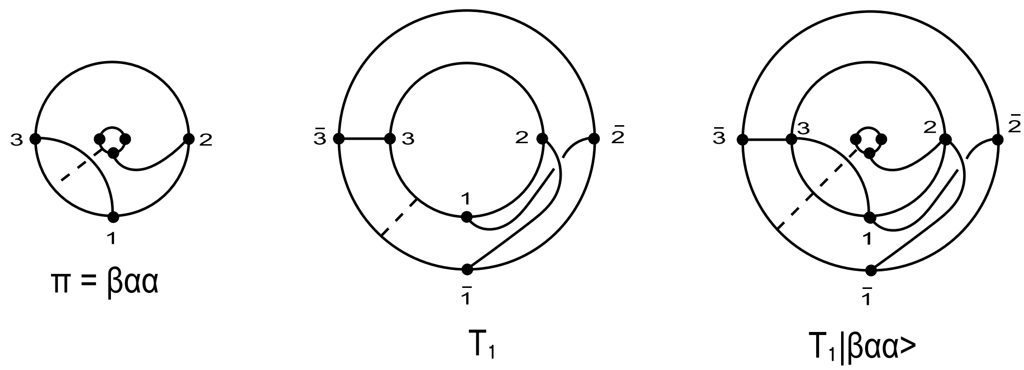
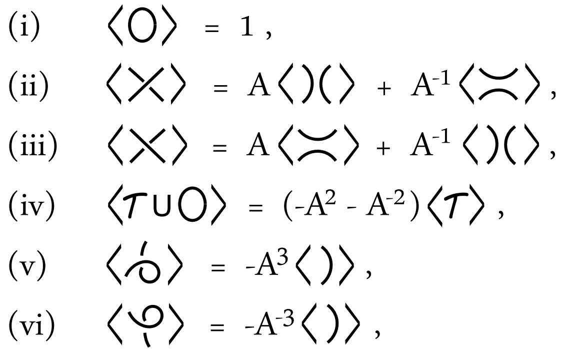


Figure 5.2: The operator T_1 acting on the pattern $\pi = \beta\alpha\alpha$. Note that π has points $N = 1, 2, 3$ on its boundary while T_1 has points $M = 1, 2, 3$ points on its inner boundary and points $\bar{M} = \bar{1}, \bar{2}, \bar{3}$ on its outer boundary.

5.3.2 Decomposing Tangle Diagrams

From Figure 5.2 it is clear that when an operator acts on a pattern we are left with a tangle diagram. As our goal is to obtain finite dimensional matrix representations of the affine Hecke algebra, we decompose tangle diagrams into linear combinations of elementary patterns. The elementary patterns thus form a basis of our tangle representation.

We use rules from knot theory to decompose tangle diagrams. Specifically, we apply the rules for calculating the Kauffman bracket [23] of link diagrams. These rules describe how to rewrite a link (tangle) diagram with crossings into linear combinations of diagrams without crossings. For a tangle diagram $\mathcal{T}$ we use the following rules:



where $A = t^{-1/4}$.
The first rule states that 1 is the value given to the particular projection of the unknot that has no crossing at all. Rules (ii) and (iii) describe how to rewrite positive and negative crossings, while rules (v) and (vi) get rid of twists. The fourth rule says that a disjoint loop in a tangle diagram can be replaced by a factor of $(-A^2 - A^{-2})$ times the tangle diagram without the loop. We derive these rules in detail in Chapter 6 in a knot theory context; for the purpose of the tangle representation the above is sufficient.

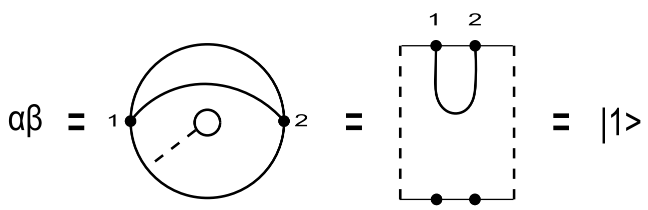
Using the Kauffman rules we can now decompose any tangle diagram into a linear combination of elementary patterns. This allows us to construct tangle representations of the affine Hecke algebra.

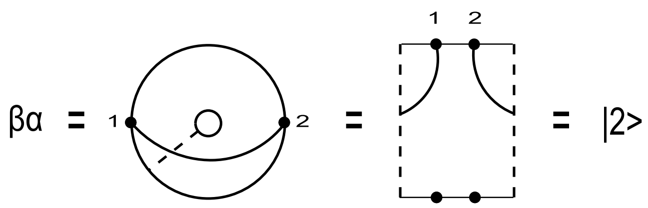
5.4 Tangle Representations of $\mathcal{A}_N(t)$

Starting with the first non trivial case, ($N = 2$ punctured), we construct finite dimensional matrix representations of the affine Hecke algebra $\mathcal{A}_N(t)$. The representations are obtained by examining the action of the $\mathcal{A}_N(t)$ generators T_i, Y_i and σ on the pattern basis elements. Using the pictorial representation of the affine Hecke algebra generators which we introduced in Chapter 3, the action of these generators on the pattern basis yields tangle diagrams. We then decompose the tangle diagrams into linear combinations of basis elements to obtain finite dimensional representations of $\mathcal{A}_N(t)$.

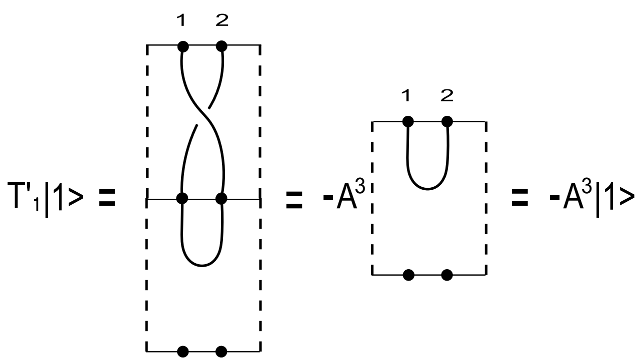
5.4.1 $N = 2$ Tangle Representation

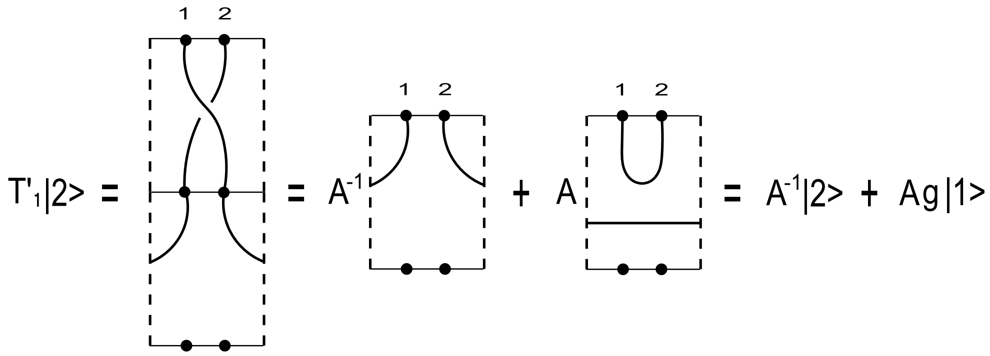
Recall that the non punctured $N = 2$ case is trivial, therefore to obtain a non trivial matrix representation of $\mathcal{A}_N(t)$ we puncture each disk containing the elementary patterns. Each pattern contains two points on the boundary of a disk with a puncture located at its origin. There are only two distinct ways of connecting both points together, giving a two dimensional pattern basis. Using the rules of Subsection 5.2.1, these patterns are $\alpha\beta$ and $\beta\alpha$ which we illustrate below.





Given our two basis elements we must now determine the action of the braid group generators $\{T'_i | i = 1, .., N-1\}$ on these elementary patterns. Since $N = 2$, we have only one generator T'_1. Using the Kauffman rules we get rid of crossings in the resulting tangle diagrams in favour of a linear combination of the two basis patterns $\alpha\beta$ and $\beta\alpha$ as we now explicitly show:





Therefore the matrix corresponding to the action of the braid group generator T_1' on the patterns $\alpha\beta$ and $\beta\alpha$ is given by:

$$T_1' = \begin{bmatrix} -A^3 & Ag \\ 0 & A^{-1} \end{bmatrix}.$$

Note that we have given the value "g" to a loop surrounding the puncture located at the origin. Furthermore we choose to denote braid group generators as T_i' to distinguish them from the Hecke algebra generators T_i. To obtain the Hecke algebra generators we normalise the braid group generators so that they obey the Hecke relation (2.3). As we have only one braid group generator T_1', we set

$$\gamma^{-1}T_1'^{-1} \quad = \quad \gamma T_1' - (t^{1/2} - t^{-1/2})\mathbb{1}, \tag{5.3}$$

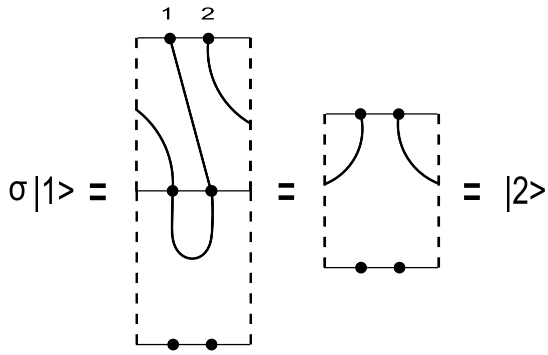
where γ is a normalisation factor.

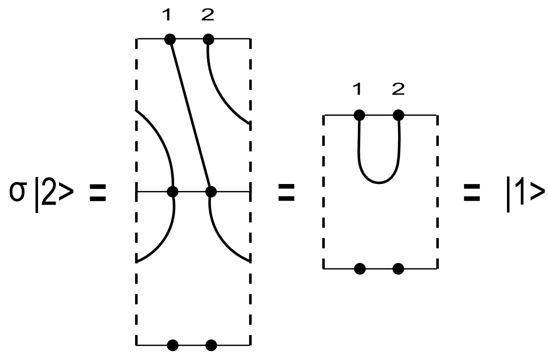
Solving (5.3) gives $\gamma = t^{1/4} = A^{-1}$. Thus for the $N = 2$ punctured disk the Hecke algebra generator T_1 and its inverse T_1^{-1} are given by the following matrices:

$$T_1 = \begin{bmatrix} -A^2 & g \\ 0 & A^{-2} \end{bmatrix} \quad , \quad T_1^{-1} = \begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix}.$$

In Subsection 2.2.1 we showed that the defining relations of $\mathcal{A}_N(t)$ can be written either in terms of the Hecke algebra generators T_i and the affine Hecke algebra generators Y_i, or purely in terms of the T_i and the cyclic operator σ. For completeness we obtain matrices for both sigma and the Y_i.

Using the graphical representation of σ that we constructed in Chapter 3, we see that its action on the two dimensional basis is as follows:



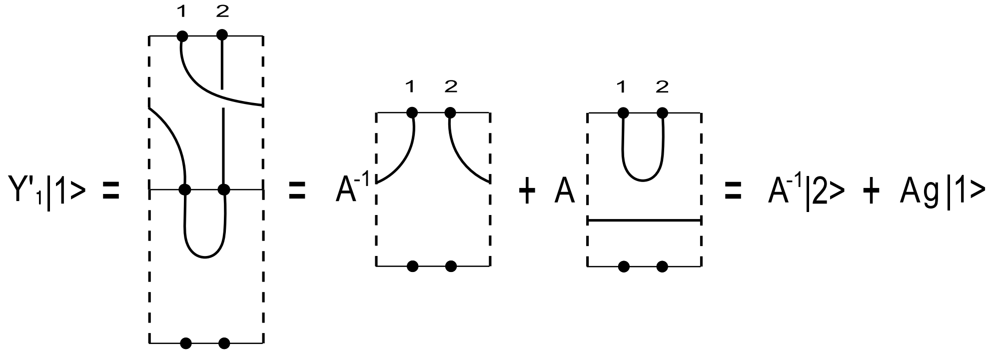


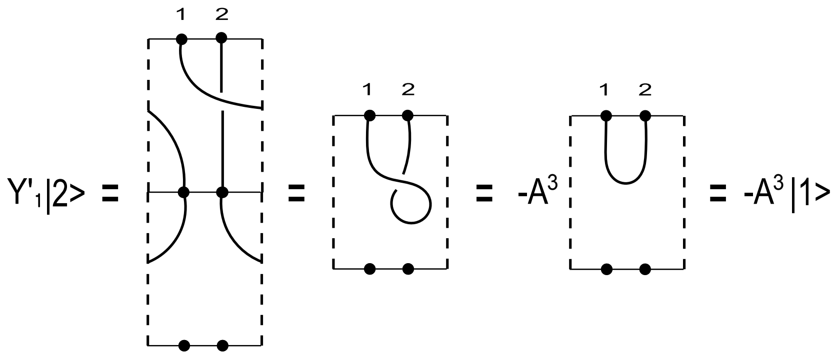
In terms of matrices this implies that σ has the following form:

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the matrices that we have just found for T_1 and σ, it is evident that they satisfy the relation (2.13), that is, $\sigma^2 T_1 = T_1 \sigma^2$ in the $N = 2$ case.

To complete the tangle representation of $\mathcal{A}_2(t)$ we obtain matrices corresponding to the action of the affine Hecke algebra generators $\{Y_i | i = 1, .., N\}$ on the patterns $\alpha\beta$, $\beta\alpha$. Since $N = 2$ there are just two generators Y_1 and Y_2. We denote the matrices obtained from the tangle diagrams as Y_i' as strictly speaking they are affine braid group generators and need to be normalised to become valid affine Hecke algebra generators.





This means that Y_1' is given by the matrix

$$Y_1' = \begin{bmatrix} Ag & -A^3 \\ A^{-1} & 0 \end{bmatrix}.$$

We can easily verify that this matrix is the unnormalised matrix corresponding to the affine Hecke algebra generator Y_1. Equation (2.11) becomes $Y_1 = T_1\sigma$ in the $N = 2$ case, so

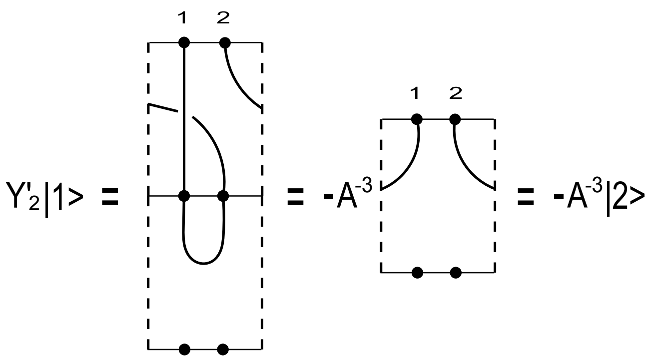
$$Y_1 = T_1\sigma = \begin{bmatrix} -A^2 & g \\ 0 & A^{-2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-1} \begin{bmatrix} Ag & -A^3 \\ A^{-1} & 0 \end{bmatrix} = A^{-1}Y_1'.$$

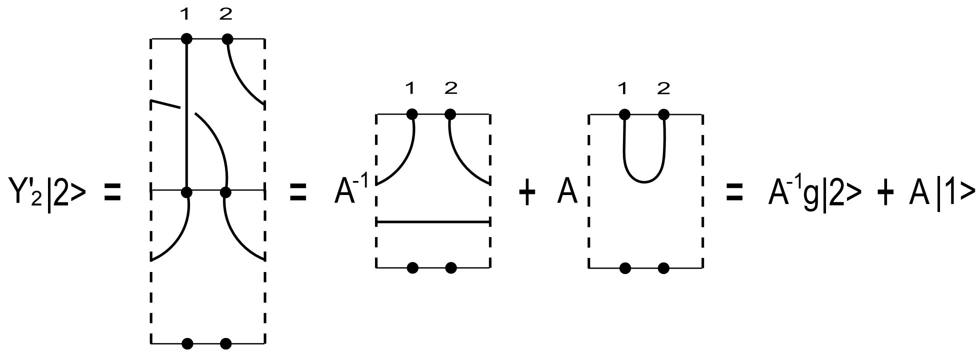
This is exactly as expected since $Y_1 = T_1\sigma = A^{-1}T_1'\sigma = A^{-1}Y_1'$, and verifies that the matrix

$$Y_1 = \begin{bmatrix} g & -A^2 \\ A^{-2} & 0 \end{bmatrix}$$

describes the action of the affine Hecke algebra Y_1 on the two dimensional pattern basis.

In a very similar fashion we obtain the matrix representation of the affine Hecke algebra generator Y_2.





This means that Y_2' is given by the matrix

$$Y_2' = \begin{bmatrix} 0 & A \\ -A^{-3} & A^{-1}g \end{bmatrix},$$

which we verify is correct. By (2.9) we know that $Y_2 = T_1^{-1} Y_1 T_1^{-1}$, therefore

$$Y_2 = \begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix} \begin{bmatrix} g & -A^2 \\ A^{-2} & 0 \end{bmatrix} \begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix} = A \begin{bmatrix} 0 & A \\ -A^{-3} & A^{-1}g \end{bmatrix} = AY_2'.$$

Accounting for the normalisation factors, this is precisely the required result as

$$Y_2 \;=\; T_1^{-1} Y_1 T_1^{-1} \;=\; AT'^{-1} A^{-1} Y_1' A T_1'^{-1} \;=\; AT_1'^{-1} Y_1' T_1'^{-1} \;=\; AY_2'.$$

Therefore the affine Hecke algebra generator Y_2 is given by

$$Y_2 = \begin{bmatrix} 0 & A^2 \\ -A^{-2} & g \end{bmatrix}.$$

To summarise our construction of the tangle representation of $\mathcal{A}_2(t)$ we reiterate the following points. Using the Kauffman bracket rules to write a tangle diagram as a linear combination of elementary planar tangles, we firstly found two dimensional square matrices for the braid group generators. Normalising these matrices to satisfy the Hecke relation yielded matrices describing the Hecke generators. Then using the graphical representation of the $\mathcal{A}_N(t)$ generators we found matrices for the affine braid group which when normalised gave finite dimensional matrices for the AHA generators Y_i. Finally, we showed that all of the matrices we obtained satisfy the defining relations of $\mathcal{A}_2(t)$ and hence form a two dimensional matrix representation of its algebraic description.

Below we give all of the matrices, along with their inverses, which describe the tangle representation of the two strand affine Hecke algebra $\mathcal{A}_2(t)$ in terms of the basis patterns

$\alpha\beta$, $\beta\alpha$.

$$T_1 = \begin{bmatrix} -A^2 & g \\ 0 & A^{-2} \end{bmatrix} \quad , \quad T_1^{-1} = \begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix},$$

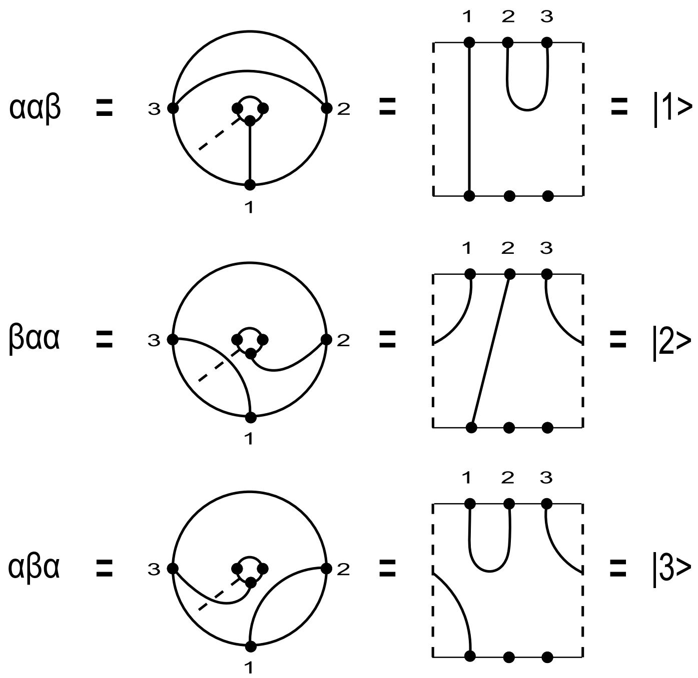
$$Y_1 = \begin{bmatrix} g & -A^2 \\ A^{-2} & 0 \end{bmatrix} \quad , \quad Y_1^{-1} = \begin{bmatrix} 0 & -A^2 \\ -A^{-2} & g \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0 & A^2 \\ -A^{-2} & g \end{bmatrix} \quad , \quad Y_2^{-1} = \begin{bmatrix} g & -A^2 \\ A^{-2} & 0 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \sigma^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

5.4.2 $N = 3$ Tangle Representation

In the $N = 3$ case, the basis elements are punctured since N is odd. Equation (5.2) says there are three elementary patterns that form the basis of the tangle representation. Using the rules for encoding patterns yields three elementary patterns $\alpha\alpha\beta$, $\beta\alpha\alpha$ and $\alpha\beta\alpha$, all of which we illustrate below.



In a similar fashion to the $N = 2$ case, here we obtain a 3 dimensional matrix representation of $\mathcal{A}_3(t)$ given by the pattern basis $\{\alpha\alpha\beta\,,\beta\alpha\alpha\,,\alpha\beta\alpha\}$. We have included the explicit construction in Appendix 5A.1, we simply present the results in this section.

As in the $N = 2$ case we begin by obtaining matrices corresponding to the Hecke algebra generators T_1 and T_2 by decomposing the tangle diagrams formed by the action of the generators on the three dimensional basis. The resulting matrices are given by:

$$T_1 = \begin{bmatrix} A^{-2} & 0 & 0 \\ 0 & A^{-2} & 0 \\ \alpha & 1 & -A^2 \end{bmatrix} \quad , \quad T_1^{-1} = \begin{bmatrix} A^{2} & 0 & 0 \\ 0 & A^{2} & 0 \\ \alpha & 1 & -A^{-2} \end{bmatrix} ,$$

$$T_2 = \begin{bmatrix} -A^{2} & \alpha & \alpha^{-1} \\ 0 & A^{-2} & 0 \\ 0 & 0 & A^{-2} \end{bmatrix} \quad , \quad T_2^{-1} = \begin{bmatrix} -A^{-2} & \alpha & \alpha^{-1} \\ 0 & A^{2} & 0 \\ 0 & 0 & A^{2} \end{bmatrix} .$$

It is also beneficial to obtain the matrix representation of the operator σ in the $N = 3$ case. Given σ and the Ts, we now have a complete representation of $\mathcal{A}_3(t)$.

$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad \sigma^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \alpha & 0 \end{bmatrix} .$$

Alternatively, $\mathcal{A}_3(t)$ is fully described in terms of the Ts and the affine Hecke algebra generators $\{Y_i | i = 1, .., 3\}$. Their matrix representations are given by:

$$Y_1 = \begin{bmatrix} A^{-2}\alpha^{-1} & -1 & A^{-2} \\ 0 & 0 & A^{-4}\alpha^{-1} \\ 0 & -A^2\alpha & A^{-2}\alpha^{-1} + \alpha \end{bmatrix} \quad , \quad Y_1^{-1} = \begin{bmatrix} A^{2}\alpha & A^2 & -1 \\ 0 & A^2\alpha + \alpha^{-1} & -A^{-2}\alpha^{-1} \\ 0 & A^4\alpha & 0 \end{bmatrix} ,$$

$$Y_2 = \begin{bmatrix} A^{2}\alpha^{-1} + \alpha & 1 - A^4 & -A^{-2} \\ A^{-2} & A^{-2}\alpha^{-1} & -A^{-4}\alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad Y_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -A^{-4} & A^{-2}\alpha & A^{-2}\alpha^{-1} \\ -A^{-2} & \alpha - A^4\alpha & A^2\alpha + \alpha^{-1} \end{bmatrix} ,$$

$$Y_3 = \begin{bmatrix} 0 & A^4 & 0 \\ -A^{-2} & A^2\alpha^{-1} + \alpha & 0 \\ -1 & A^2\alpha & A^2\alpha^{-1} \end{bmatrix} \quad , \quad Y_3^{-1} = \begin{bmatrix} A^{-2}\alpha + \alpha^{-1} & -A^2 & 0 \\ A^{-4} & 0 & 0 \\ A^{-2} & -\alpha & A^{-2}\alpha \end{bmatrix} .$$

We now have a complete description of all of the generators of $\mathcal{A}_3(t)$ in the tangle representation. We point out the dependence on the parameter α. This is the value we have given to the removal of a 2π clockwise twist of a strand around the origin. We have explicitly illustrated this twisting in Appendix 5A.1 where we have also included the complete detailed construction of the $N = 3$ tangle representation.

5.4.3 Larger N Tangle Representations

Following the rules we laid out in Subsection 5.2.1 for encoding patterns, we now know how to construct a tangle representation for all AHAs given any number of points on the boundary of the disk. Furthermore equations (5.1) and (5.2) tell us exactly the dimension of the tangle basis. In the appendices at the end of this chapter we give all of the elementary patterns needed to construct the tangle representation of the affine Hecke algebra for the cases when $N = 4$, $N = 5$ and $N = 6$. Furthermore we also include the specific matrices which define each AHA in the pattern basis for the above cases. All other cases follow similarly.

5.5 Tangle Representations of the Temperley-Lieb Algebra

In Subsection 2.1.3 we introduced the Temperley-Lieb algebra $TL_N(d)$. Recall that the map from the $\mathcal{H}_N(t)$ generators T_i, to the $TL_N(d)$ generators e_i, is

$$T_i \longmapsto e_i + t^{1/2}\mathbb{1},$$

and the defining relations of $TL_N(d)$ are:

$$\begin{aligned} e_i^2 &= de_i, && (5.4)\\ e_ie_j &= e_je_i \text{ for } |i-j| \geq 2, && (5.5)\\ e_ie_{i+1}e_i - e_i &= e_{i+1}e_ie_{i+1} - e_{i+1}, && (5.6) \end{aligned}$$

where $d = -t^{1/2} - t^{-1/2}$.
Using the established pictorial representation of the Temperley-Lieb generators, see [24] for example, where the generator e_i corresponds to a cup and cap between the i^{th} and $i+1^{th}$ points, we can construct the tangle representation of $TL_N(d)$.

Note that the parameter $d = -t^{1/2} - t^{-1/2}$ is the value given to the removal of a disjoint closed loop in tangle diagrams as in rule (iv) for calculating the Kauffman bracket. Keeping with common notation, within the affine Hecke algebra we will denote d by $\tau = -A^{-2} - A^2$ where $A = t^{-1/4}$.

5.5.1 Tangle Representation of $TL_2(d)$

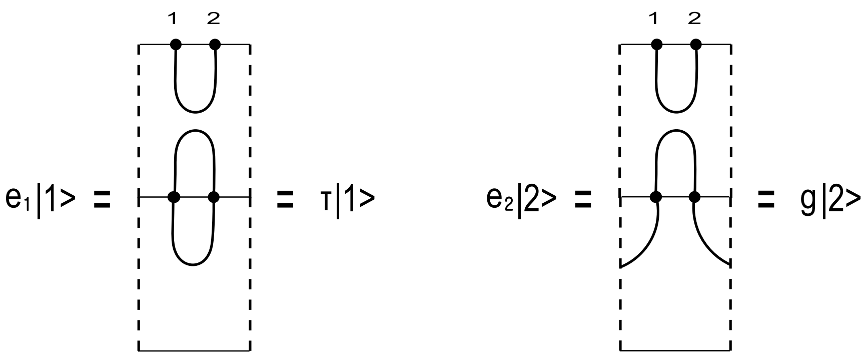
In the $N = 2$ case there is only one Hecke algebra generator T_1. We found that its tangle representation on the pattern basis $\alpha\beta, \beta\alpha$ is given by the matrix

$$T_1 = \begin{bmatrix} -A^2 & g \\ 0 & A^{-2} \end{bmatrix}.$$

Mapping to the Temperley-Lieb operator e_1, implies that in the tangle basis we obtain for e_1 the matrix

$$e_1 = \begin{bmatrix} \tau & g \\ 0 & 0 \end{bmatrix}.$$

We can explicitly verify this result by examining the action of the generator e_1 on the two elementary patterns.



The resulting matrix is

$$e_1 = \begin{bmatrix} \tau & g \\ 0 & 0 \end{bmatrix},$$

exactly as expected. Equation (5.4) is also clearly satisfied:

$$e_1^2 = \begin{bmatrix} \tau & g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau & g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tau^2 & \tau g \\ 0 & 0 \end{bmatrix} = \tau e_1 = d e_1.$$

There is only one generator e_1 in $TL_2(d)$ so this completes the construction of the tangle representation of the Temperley-Lieb algebra in the case of $N = 2$.

5.5.2 Tangle Representation of $TL_3(d)$

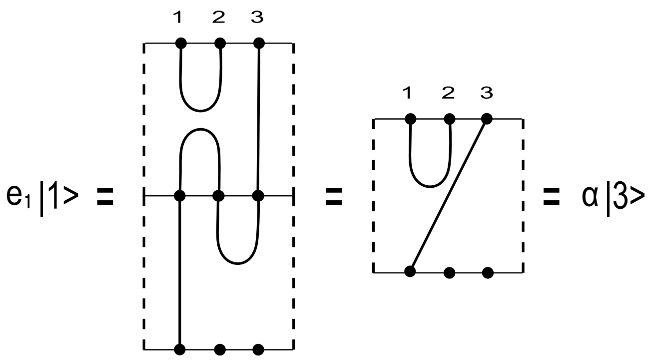
The $N = 3$ case follows similarly. From Subsection 5.4.2, matrices corresponding to the action of the Hecke Algebra generators T_1 and T_2 on the elementary basis elements $\alpha\alpha\beta, \beta\alpha\alpha, \alpha\beta\alpha$ are:

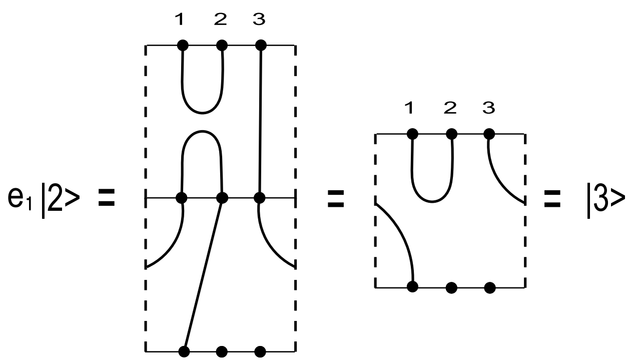
$$T_1 = \begin{bmatrix} A^{-2} & 0 & 0 \\ 0 & A^{-2} & 0 \\ \alpha & 1 & -A^2 \end{bmatrix} \quad , \quad T_2 = \begin{bmatrix} -A^2 & \alpha & \alpha^{-1} \\ 0 & A^{-2} & 0 \\ 0 & 0 & A^{-2} \end{bmatrix}.$$

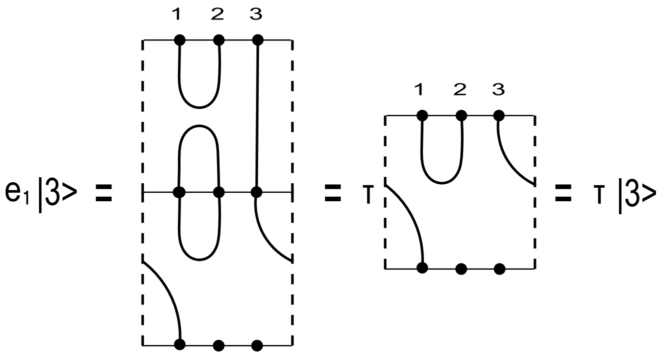
The map $T_i \longmapsto e_i + t^{1/2}\mathbb{1}$ implies that in the tangle representation, the Temperley-Lieb generators e_1 and e_2 are given by:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 1 & \tau \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} \tau & \alpha & \alpha^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These matrices are in perfect agreement with the matrices obtained from decomposing the tangle diagrams resulting from the action of the generators e_1 and e_2 on the elementary patterns. From the tangle diagrams we get:



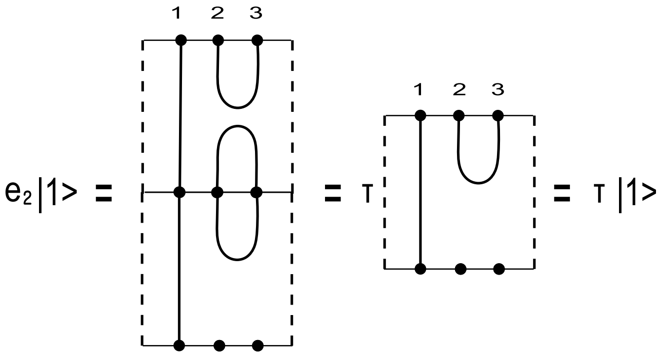


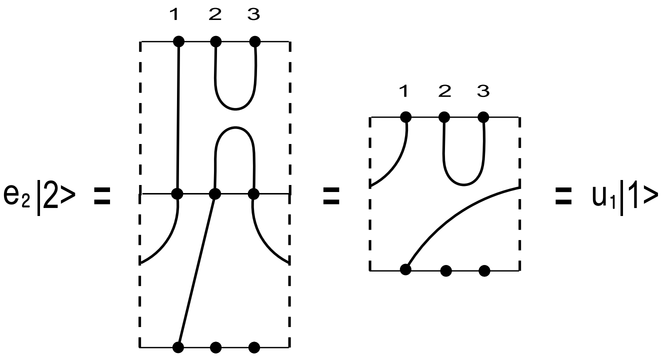


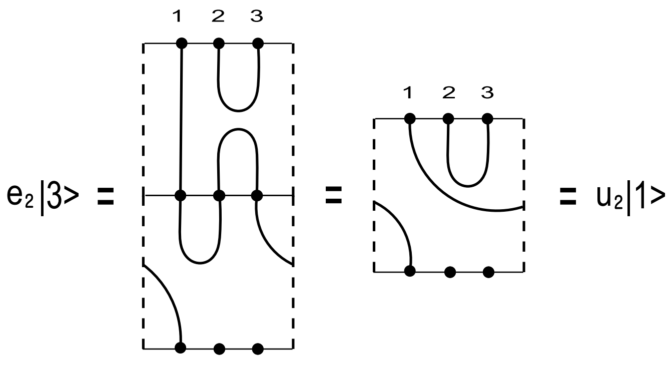
As predicted, the matrix corresponding to the action of e_1 on the pattern basis is

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 1 & \tau \end{bmatrix}.$$

Similarly e_2 acts on the basis elements as follows:







With the values previously obtained for u_1 and u_2, that is $u_1 = \alpha$ and $u_2 = \alpha^{-1}$, the matrix corresponding to the action of e_2 on the pattern basis is as expected:

$$e_2 = \begin{bmatrix} \tau & \alpha & \alpha^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is trivial to show that the matrices obtained for e_1 and e_2 satisfy the defining relations of the Temperley-Lieb algebra. In $N = 3$ they satisfy (5.4) and (5.6), notably $e_1^2 = \tau e_1$, $e_2^2 = \tau e_2$ and $e_1 e_2 e_1 - e_1 = e_2 e_1 e_2 - e_2$.

To summarise, we can construct the tangle representation of the Temperley-Lieb algebra by decomposing the tangle diagrams resulting from the action of the Temperley-Lieb generators e_i on the pattern basis, into a linear combination of elementary patterns. The finite dimensional matrices obtained are consistent with the algebraic description of $TL_N(d)$. Furthermore they are easily mapped to the matrices describing the action of the Hecke algebra generators on elementary patterns.

5.6 Extending the Tangle Representation

Having successfully obtained finite dimensional matrix representations of the affine Hecke algebra using tangles, the natural step is to extend the tangle representation to the double affine Hecke algebra $\mathcal{D}_N(t, q)$. In Chapter 3 we developed a pictorial representation of all the double affine Hecke algebra generators Z_i. Describing the action of these generators, in the cube representation, on elementary tangles therefore yields the tangle representation of $\mathcal{D}_N(t, q)$.

However limiting factors are encountered when trying to obtain finite dimensional matrix representations of $\mathcal{D}_N(t, q)$. Firstly we cannot use the Kauffman rules to decompose the tangle diagrams resulting from the action of the Z_i generators on the basis patterns. The reason is simple; in the cube representation we have "three dimensional crossings", that is we do not have simple over and under crossings as in the affine Hecke algebra. In the cube representation the Y_i generators braid strands through the left and right faces whereas the Z_i generators braid strands through the front and back faces. Therefore to find a tangle representation of $\mathcal{D}_N(t, q)$ we need to be able to decompose crossings that lie in two different perpendicular planes. To accomplish this a three dimensional analogue of the Reidemeister and Kauffman rules is required.

The difficulty in constructing tangle representations of $\mathcal{D}_N(t,q)$ emerging from tangle diagrams does not however rule out the existence of finite dimensional tangle representations of $\mathcal{D}_N(t,q)$. For example we can extend the $N=2$ tangle representation of the affine Hecke algebra of Section 5.4.1 to one for the double affine Hecke algebra.

5.6.1 Tangle Representation of $\mathcal{D}_2(t,q)$

We begin by noting that in $\mathcal{D}_2(t,q)$ there are two Z_i generators, Z_1 and Z_2. Also we found that in the tangle representation of $\mathcal{A}_2(t)$ the operator sigma is given by:

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and so $\sigma^2 = \mathbb{1}$. This immediately restricts the value of the parameter q, upon which $\mathcal{D}_2(t,q)$ depends on, to be one. By (2.14) and (2.20) we have that $Z_i\sigma^N = q^{-1}\sigma^N Z_i$, and with $\sigma^N = \mathbb{1}$ we must have $q=1$. However we can still try to find a finite dimensional matrix representation of $\mathcal{D}_2(t,q)$ with $q=1$. Firstly to check if there exists a two dimensional matrix representation of the double affine Hecke algebra generator Z_1 we impose all of the relations of $\mathcal{D}_2(t,q)$ to an arbitrary (2×2) matrix defined as:

$$Z_1 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Imposing all of the relations that Z_1 must satisfy (see Subsection 2.3.1) in addition to the matrices obtained for $\mathcal{A}_2(t)$ means that Z_1 must have the following form:

$$Z_1 = \begin{bmatrix} Z_{11} & Z_{12} \\ -Z_{12} & A^{-4}Z_{11} + gA^{-2}Z_{12} \end{bmatrix},$$

where as before g is the value given to a closed loop surrounding the origin and $A = t^{-1/4}$. Since all of the Z_i are defined recursively according to (2.17), Z_2 is simply given by $Z_2 = T_1^{-1}Z_1T_1^{-1}$, which in matrix form is

$$\begin{aligned} Z_2 &= \begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ -Z_{12} & A^{-4}Z_{11} + gA^{-2}Z_{12} \end{bmatrix}\begin{bmatrix} -A^{-2} & g \\ 0 & A^2 \end{bmatrix} \\ &= \begin{bmatrix} A^{-4}Z_{11} + gA^{-2}Z_{12} & -Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix}. \end{aligned}$$

Interestingly the product of the Z is proportional to the identity. We get that

$$Z_1Z_2 = \left(A^{-4}Z_{11}^2 + gA^{-2}Z_{11}Z_{12} + Z_{12}^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The existence of the two matrices corresponding to the action of the double affine Hecke algebra generators Z_1 and Z_2 on the pattern basis $\alpha\beta, \beta\alpha$ indicates that it may be possible to obtain the same matrices by decomposing their corresponding tangle diagrams. Z_1 and Z_2 depend on several free parameters, Z_{11}, Z_{12} and g. These parameters should be fixed by rules describing the decomposition of "three dimensional crossings" into linear combinations of planar patterns. In this manner one could obtain finite dimensional representations of $\mathcal{D}_N(t,q)$.

5.6.2 Tangle Representation of $\mathcal{D}_3(t,q)$

In the extension of the $N = 3$ affine Hecke algebra tangle representation to one for the double affine Hecke algebra, a different limiting factor is encountered. The problem of decomposing "three dimensional crossings" still remains but it is not the only one.

Following the method of the previous subsection we can impose all of the defining relations of $\mathcal{D}_3(t,q)$ to an arbitrary matrix Z_1 and hence define all of the other Z generators using (2.17). Using all of the matrices describing $\mathcal{A}_3(t)$, which we obtained in Subsection 5.4.2, the resulting (3×3) matrix for Z_1 is

$$Z_1 = \begin{bmatrix} A^{-2}\alpha^{-1} & A^{-2}\alpha^{-1} & -\alpha^{-1} \\ A^{-4}\alpha^{-2} & A^{-4}\alpha^{-2} & -A^{-2}\alpha^{-2} \\ 0 & 1 & -A^2 + A^2\alpha^{-1} \end{bmatrix}.$$

However this matrix is singular; that is, it is non-invertible. Hence it cannot represent the action of the double affine Hecke algebra generator Z_1 on the pattern basis. All Z_i generators are required to be invertible.

In an effort to obtain a finite dimensional representation of $\mathcal{D}_3(t,q)$ we take (6×6) matrices in block diagonal form composed of two (3×3) representations and impose all of the relations of $\mathcal{D}_3(t,q)$ to an arbitrary matrix Z of off diagonal form. So the Hecke algebra generators T_i and affine Hecke algebra generators Y_i have the following form:

$$T_i = \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array}\right] \quad , \quad Y_i = \left[\begin{array}{c|c} Y_1 & 0 \\ \hline 0 & Y_2 \end{array}\right],$$

where T_1 and Y_1 form a (3×3) representation of $\mathcal{A}_3(t)$ depending on the parameters A and α. T_2 and Y_2 also form a (3×3) representation of $\mathcal{A}_3(t)$ and depend on the parameters B and β.
We impose all of the relations of $\mathcal{D}_3(t,q)$ to the matrix Z which has an off diagonal form, that is

$$Z = \left[\begin{array}{c|c} Z_1 & 0 \\ \hline 0 & Z_2 \end{array}\right],$$

where Z_1 and Z_2 are both arbitrary three dimensional square matrices.

Imposing all of the conditions and solving for Z yields three distinct matrices. In each case though the resulting matrix is singular and hence cannot represent a double affine Hecke algebra generator. This result indicates that in order to obtain finite dimensional matrix representations of $\mathcal{D}_N(t,q)$ one needs to be able to systematically decompose three dimensional crossings into a linear combination of planar tangles.

This concludes the chapter on our construction of finite dimensional matrix representations of the affine Hecke algebra based on elementary patterns. Our tangle representation is not solely restricted to affine Hecke algebras; we also showed how we can obtain finite dimensional matrices that generate the Temperley-Lieb algebra.

Furthermore our tangle representation also provides explicit matrices for all the braid group and affine braid group generators. Recall that in order to obtain the Hecke algebra generators we needed to normalise the braid group matrices to satisfy the Hecke relation. Similar normalisation was required to obtain the AHA generators. Therefore without subjecting the matrices to this normalisation we effectively have a representation of $\mathcal{A}_N(Q)$ (see Figure 1.1) for $Q = \mathbb{1}$.

We also gave explicit conditions for the existence of finite dimensional matrix representations, emerging from tangles, of double affine Hecke algebras. For the two dimensional DAHA with $q = 1$, we found explicit (2×2) matrices corresponding to the generators Z_1 and Z_2. These matrices depended on several free parameters, which we indicated should be fixed by rules describing the decomposition of three dimensional crossings into linear combinations of planar patterns. Therefore, we have in effect presented the necessary conditions for these decomposition rules to satisfy in order to be able to construct finite dimensional DAHA representations via tangle diagrams.

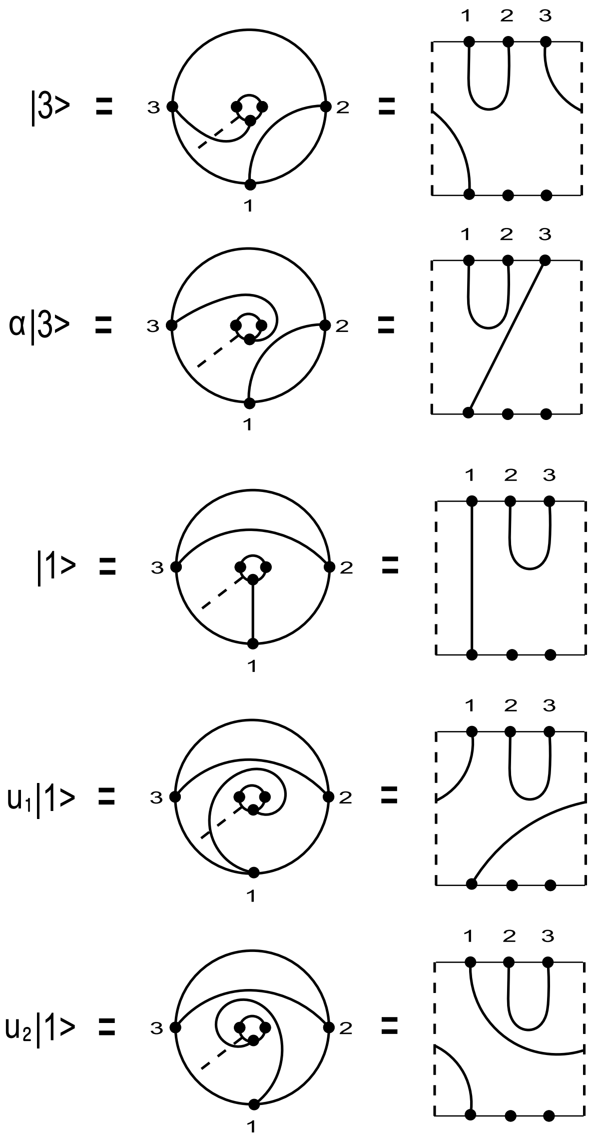
As we have described, our construction of tangle representations relies heavily on techniques that are largely associated to knot theory. As such, the affine Hecke algebra tangle representation also highlights the central role of Hecke algebras in knot theory; this important role is at the center of the discovery of both the Jones and HOMFLY polynomials, both knot invariants which we will introduce in the next chapter.

Appendix 5

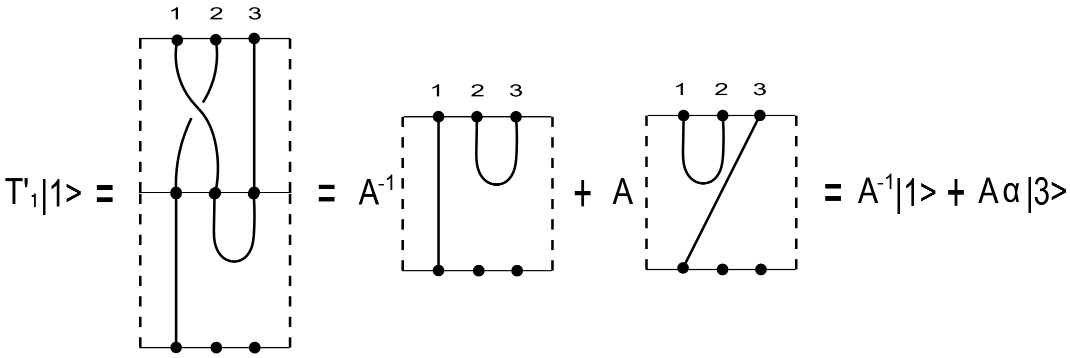
5A.1 $N = 3$ Tangle Representation

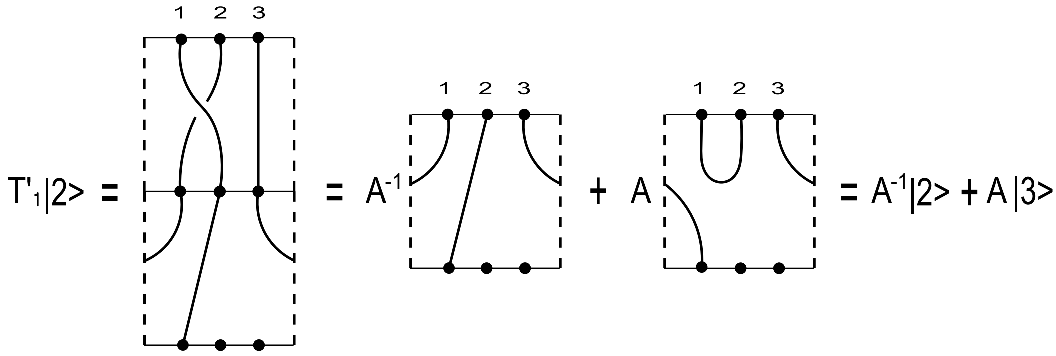
We explicitly construct the tangle representation for $\mathcal{A}_3(t)$, using the basis $\{\alpha\alpha\beta\,, \beta\alpha\alpha\,, \alpha\beta\alpha\}$.

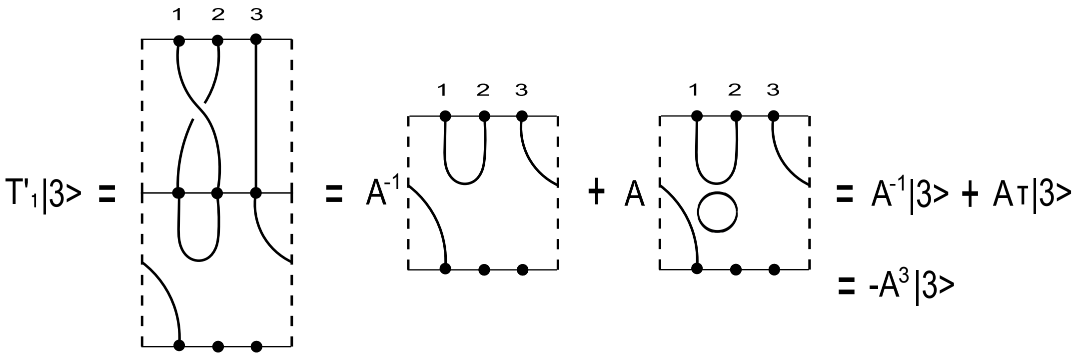
However before doing so it is necessary to introduce the following notation: we can remove a disjoint closed loop by multiplying the corresponding basis element without the loop by a factor of $\tau = -A^{-2} - A^2$. Similarly a closed loop around the origin is removed by introducing a factor of g. In addition to these, the factor α corresponds to a 2π twist, whereas u_1 and u_2 correspond to 2π clockwise and anticlockwise twists respectively. For clarity we illustrate these twists below.



Firstly we obtain the matrices for the braid group generators T'_1 and T'_2.



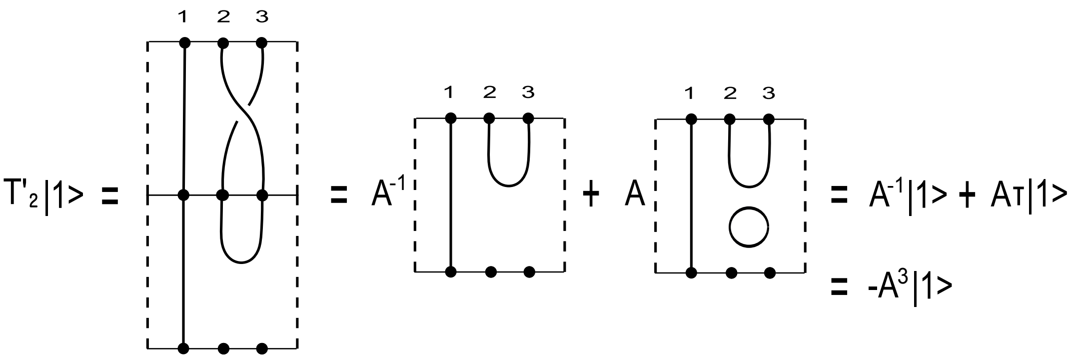


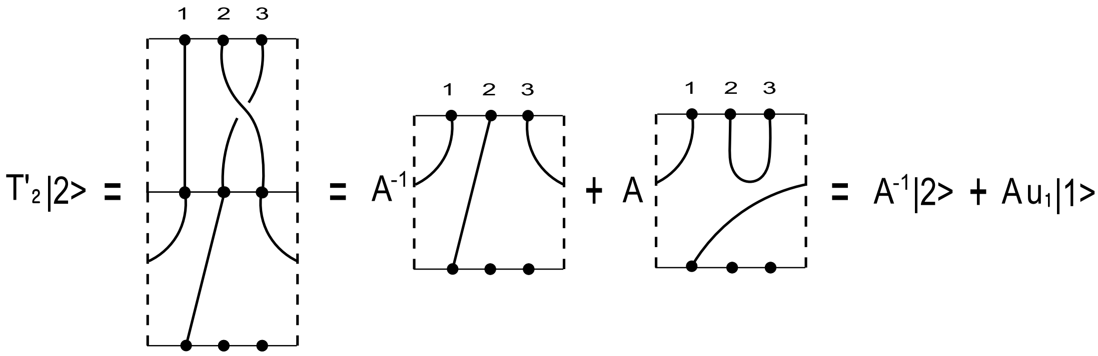


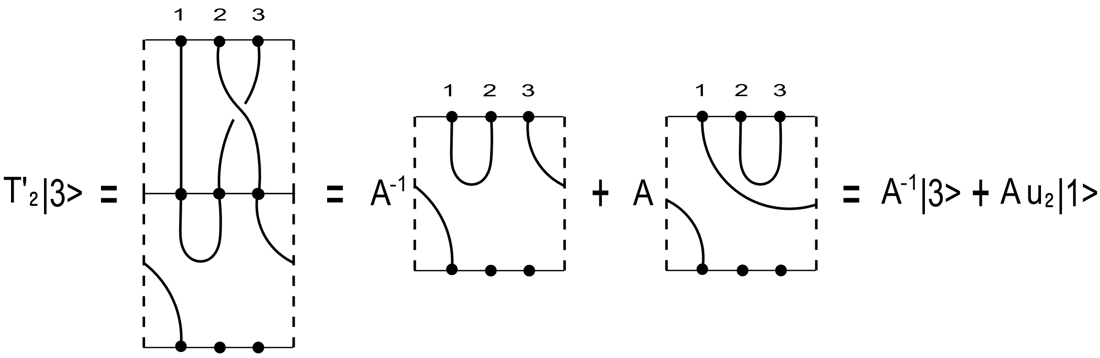
In matrix form we find:

$$T'_1 = \begin{bmatrix} A^{-1} & 0 & 0 \\ 0 & A^{-1} & 0 \\ A\alpha & A & -A^3 \end{bmatrix}.$$

Using the same rules we obtain T'_2.







Clearly the matrix T_2' depends on the two parameters u_1 and u_2, which have yet to be determined.

$$T_2' = \begin{bmatrix} -A^3 & Au_1 & Au_2 \\ 0 & A^{-1} & 0 \\ 0 & 0 & A^{-1} \end{bmatrix}.$$

To obtain the Hecke algebra generators T_1 and T_2, we normalise the braid group generators by a factor of γ in order for them to satisfy the Hecke relation

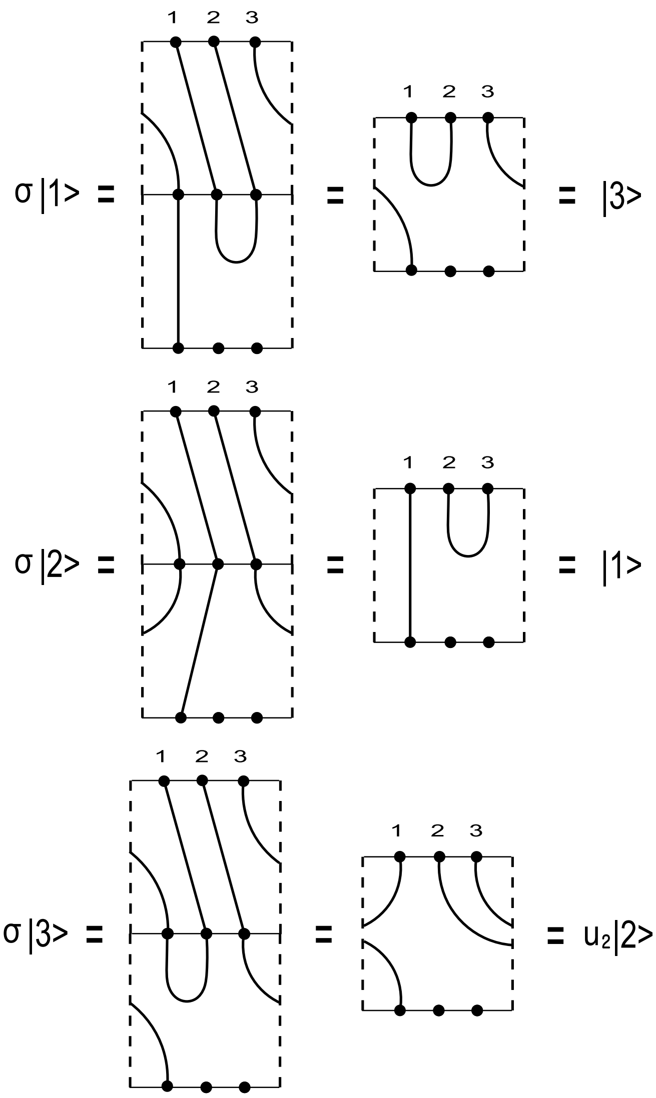
$$\gamma^{-1} T_i'^{-1} \;\; = \;\; \gamma T_i' - (t^{1/2} - t^{-1/2})\mathbb{1}.$$

The required normalisation factor is $\gamma = A^{-1}$ which implies the Hecke algebra generators are:

$$T_1 = \begin{bmatrix} A^{-2} & 0 & 0 \\ 0 & A^{-2} & 0 \\ \alpha & 1 & -A^2 \end{bmatrix} \quad , \quad T_1^{-1} = \begin{bmatrix} A^2 & 0 & 0 \\ 0 & A^2 & 0 \\ \alpha & 1 & -A^{-2} \end{bmatrix},$$

$$T_2 = \begin{bmatrix} -A^2 & u_1 & u_2 \\ 0 & A^{-2} & 0 \\ 0 & 0 & A^{-2} \end{bmatrix} \quad , \quad T_2^{-1} = \begin{bmatrix} -A^{-2} & u_1 & u_2 \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{bmatrix}.$$

The cyclic operator σ acts on the elementary patterns as follows:



Similar to T_2 the σ matrix also depends on the parameter u_2:

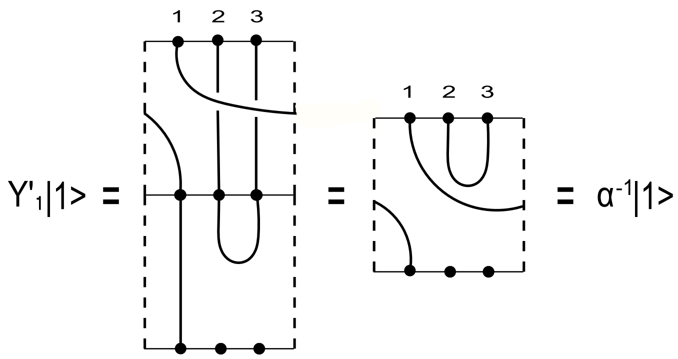
$$\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & u_2 \\ 1 & 0 & 0 \end{bmatrix}.$$

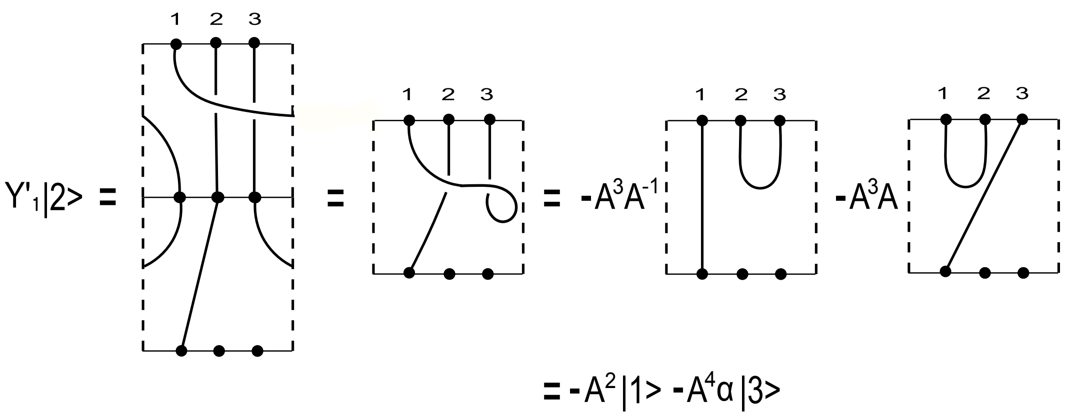
The value of the undetermined parameters u_1 and u_2 are easily found. The Hecke algebra generators must satisfy the braid relation, (2.2), which in $N = 3$ is simply $T_1T_2T_1 = T_1T_2T_1$. This relation implies that $u_2 = \alpha^{-1}$.

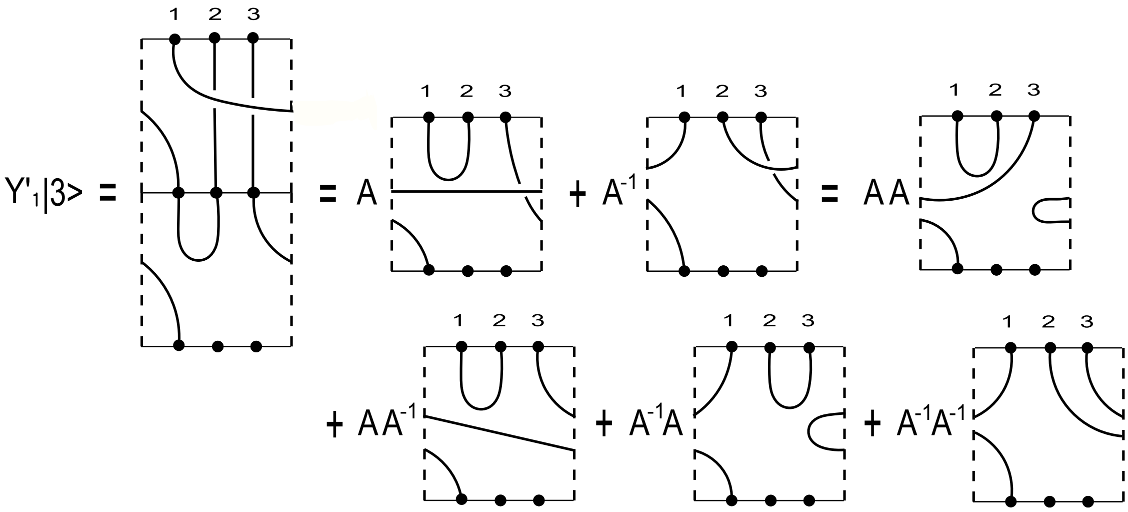
Furthermore, the Ts and σ must satisfy the relations (2.12) and (2.13), that is $T_1\sigma = \sigma T_2$ and $T_2\sigma^2 = \sigma^2 T_1$ when $N = 3$. Both of these relations imply that $u_1 = \alpha$. These two results are consistent with the fact that u_1 and u_2 correspond to 2π clockwise and anticlockwise twists around the origin. As we have already given the value α to a 2π clockwise twist, it is then unsurprising that calculations yield $u_1 = \alpha$ and $u_2 = \alpha^{-1}$. Therefore using these values we can now write the matrices for T_2 and σ as:

$$T_2 = \begin{bmatrix} -A^2 & \alpha & \alpha^{-1} \\ 0 & A^{-2} & 0 \\ 0 & 0 & A^{-2} \end{bmatrix} \quad , \quad T_2^{-1} = \begin{bmatrix} -A^{-2} & \alpha & \alpha^{-1} \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{bmatrix} \quad , \quad \sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix}.$$

To complete the matrix representation of $\mathcal{A}_3(t)$ in the tangle basis, we describe the action of the affine Hecke algebra generators Y_1, Y_2 and Y_3 on the elementary patterns. The matrices emerging from the tangle diagrams correspond to the affine braid group generators Y_i' as the T's aren't normalised to T yet. Thus we must normalise the Y_i' to obtain the affine Hecke algebra generators Y_i.







$$= A^2\alpha|3\rangle + g|3\rangle + |1\rangle + A^{-2}\alpha^{-1}|2\rangle$$

These yield the matrix:

$$Y_1' = \begin{bmatrix} \alpha^{-1} & -A^2 & 1 \\ 0 & 0 & A^{-2}\alpha^{-1} \\ 0 & -A^4\alpha & A^2\alpha + g \end{bmatrix}.$$

To obtain the affine Hecke algebra generator Y_1, we write it in terms of σ using (2.11). This implies that $Y_1 = T_1 T_2 \sigma$ and so

$$\begin{aligned} Y_1 &= \begin{bmatrix} A^{-2} & 0 & 0 \\ 0 & A^{-2} & 0 \\ \alpha & 1 & -A^2 \end{bmatrix} \begin{bmatrix} -A^2 & \alpha & \alpha^{-1} \\ 0 & A^{-2} & 0 \\ 0 & 0 & A^{-2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix} \\ &= A^{-2} \begin{bmatrix} \alpha^{-1} & -A^2 & 1 \\ 0 & 0 & A^{-2}\alpha^{-1} \\ 0 & -A^4\alpha & A^2\alpha + \alpha^{-1} \end{bmatrix} \\ &= A^{-2} Y_1' \quad \text{when} \quad g = \alpha^{-1}. \end{aligned}$$

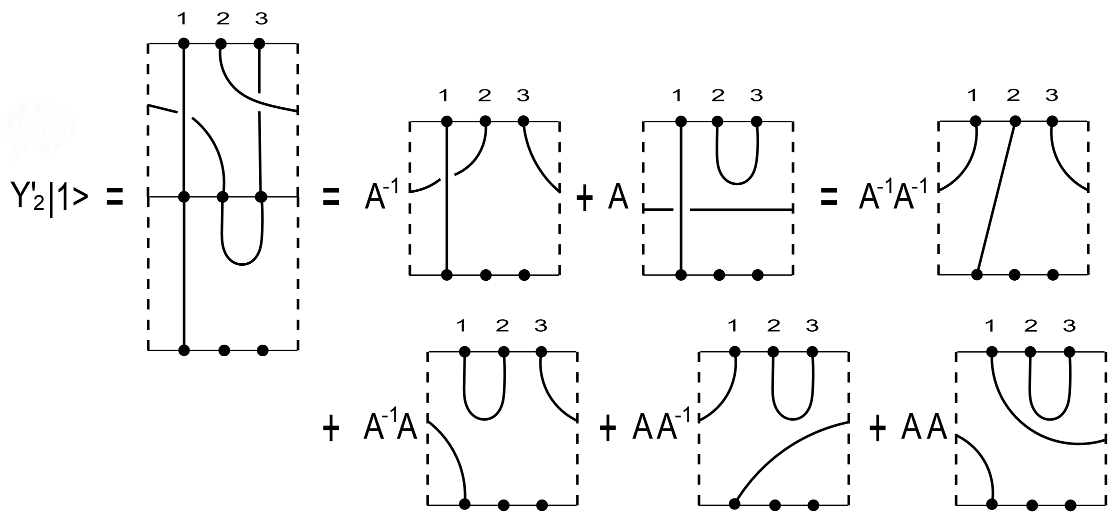
This result is exactly what we expect and agrees with our tangle construction, giving the correct normalisation factor as

$$Y_1 \;=\; T_1 T_2 \sigma \;=\; A^{-1} T_1' A^{-1} T_2' \sigma \;=\; A^{-2} T_1' T_2' \sigma \;=\; A^{-2} Y_1.$$

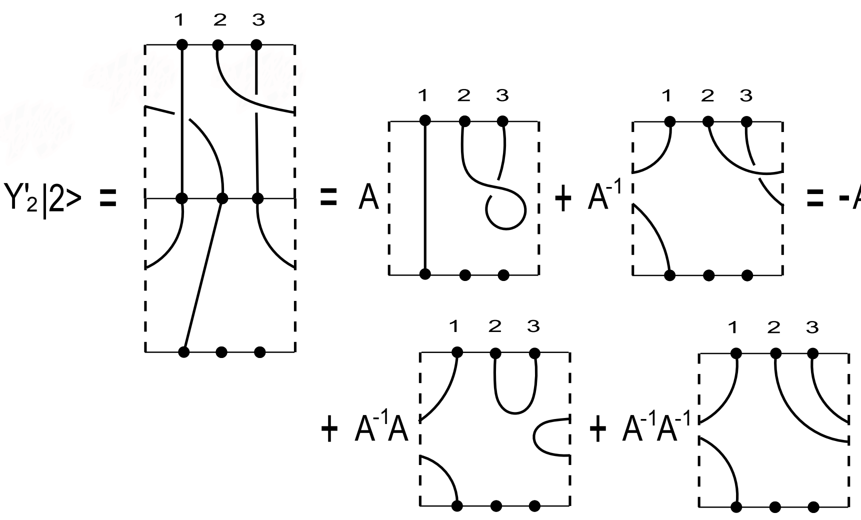
It also fixes the value of g to be $g = \alpha^{-1}$. So the affine Hecke algebra generator Y_1 is:

$$Y_1 = \begin{bmatrix} A^{-2}\alpha^{-1} & -1 & A^{-2} \\ 0 & 0 & A^{-4}\alpha^{-1} \\ 0 & -A^2\alpha & A^{-2}\alpha^{-1}+\alpha \end{bmatrix}.$$

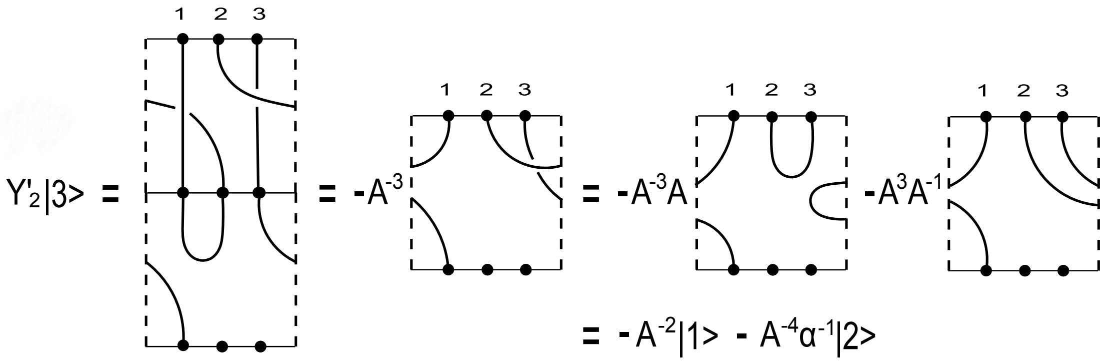
We now find the matrix representation of Y_2 in a similar fashion.



$$= (A^2\alpha^{-1}+\alpha)|1\rangle + A^{-2}|2\rangle + |3\rangle$$



$$= (1-A^4)|1\rangle + A^{-2}\alpha^{-1}|2\rangle$$



This gives:

$$Y_2' = \begin{bmatrix} A^2\alpha^{-1}+\alpha & 1-A^4 & -A^{-2} \\ A^{-2} & A^{-2}\alpha^{-1} & -A^{-4}\alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix}.$$

Using (2.9), we know that $Y_2 = T_1^{-1}Y_1T_1^{-1}$ meaning

$$\begin{aligned} Y_2 &= \begin{bmatrix} A^2 & 0 & 0 \\ 0 & A^2 & 0 \\ \alpha & 1 & -A^{-2} \end{bmatrix} \begin{bmatrix} A^{-2}\alpha^{-1} & -1 & A^{-2} \\ 0 & 0 & A^{-4}\alpha^{-1} \\ 0 & -A^2\alpha & A^{-2}\alpha^{-1}+\alpha \end{bmatrix} \begin{bmatrix} A^2 & 0 & 0 \\ 0 & A^2 & 0 \\ \alpha & 1 & -A^{-2} \end{bmatrix} \\ &= \begin{bmatrix} A^2\alpha^{-1}+\alpha & 1-A^4 & -A^{-2} \\ A^{-2} & A^{-2}\alpha^{-1} & -A^{-4}\alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix} \\ &= Y_2'. \end{aligned}$$

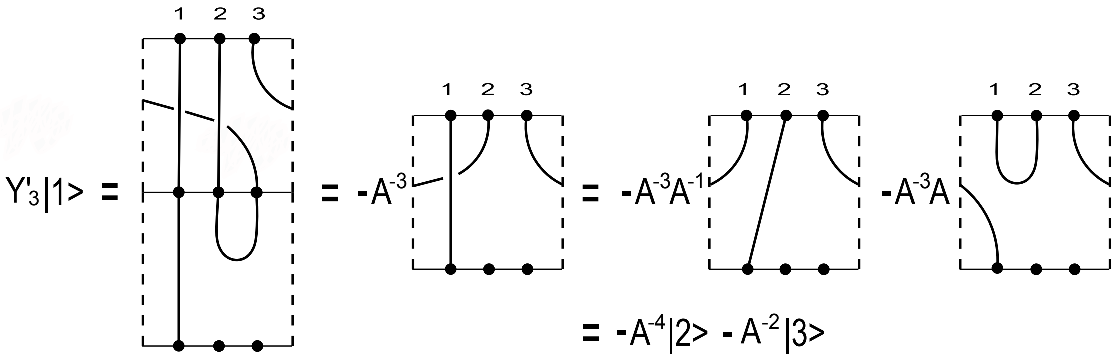
Again this is perfectly consistent with the tangle construction as

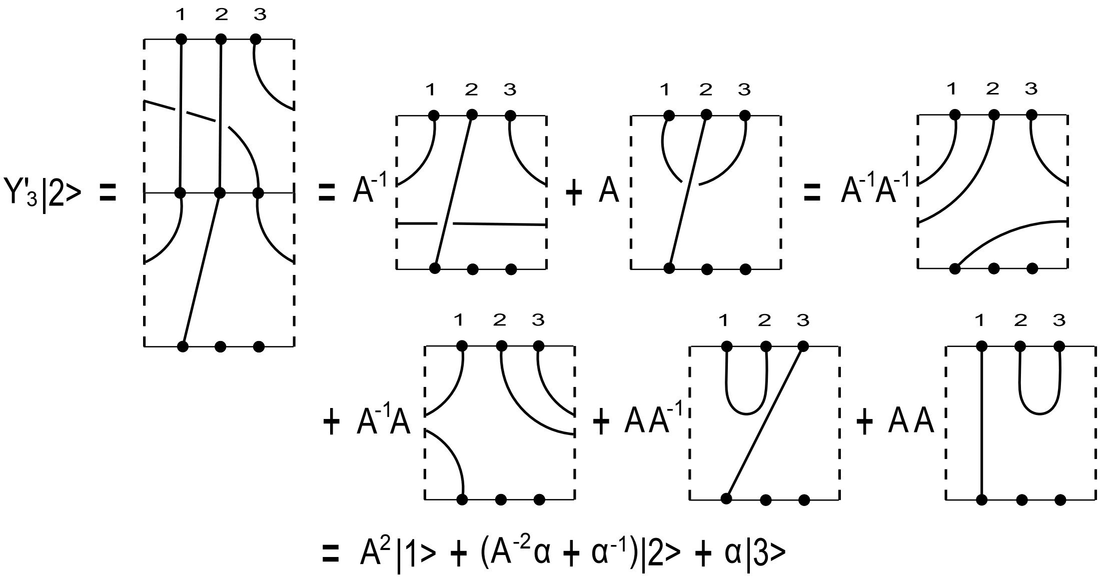
$$Y_2 = T_1^{-1}Y_1T_1^{-1} = AT_1'^{-1}A^{-2}Y_1'AT_1'^{-1} = T_1'^{-1}Y_1'T_1'^{-1} = Y_2'.$$

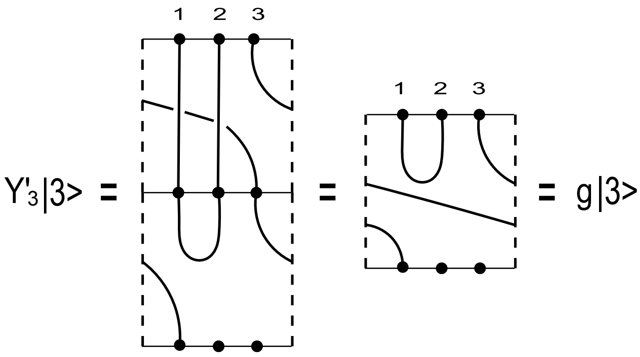
Therefore the affine Hecke algebra generator Y_2 is:

$$Y_2 = \begin{bmatrix} A^2\alpha^{-1}+\alpha & 1-A^4 & -A^{-2} \\ A^{-2} & A^{-2}\alpha^{-1} & -A^{-4}\alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix}.$$

Lastly, to complete the representation of $\mathcal{A}_3(t)$ we find the action of the affine Hecke algebra generator Y_3 on the pattern basis.







So the matrix for Y_3' is:

$$Y_3' = \begin{bmatrix} 0 & A^2 & 0 \\ -A^{-4} & A^{-2}\alpha + \alpha^{-1} & 0 \\ -A^{-2} & \alpha & g \end{bmatrix}.$$

By 2.9 we have that $Y_3 = T_2^{-1} Y_2 T_2^{-1}$ which gives the normalisation factor to be

$$
\begin{aligned}
Y_3 &= \begin{bmatrix} -A^{-2} & \alpha & \alpha^{-1} \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{bmatrix} \begin{bmatrix} A^2\alpha^{-1} + \alpha & 1 - A^4 & -A^{-2} \\ A^{-2} & A^{-2}\alpha^{-1} & -A^{-4}\alpha^{-1} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -A^{-2} & \alpha & \alpha^{-1} \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{bmatrix} \\
&= A^2 \begin{bmatrix} 0 & A^2 & 0 \\ -A^{-4} & A^{-2}\alpha + \alpha^{-1} & 0 \\ -A^{-2} & \alpha & \alpha^{-1} \end{bmatrix} \\
&= A^2 Y_3' \;\text{ when }\; g = \alpha^{-1}.
\end{aligned}
$$

This normalisation factor is in perfect agreement with our tangle construction since:

$$Y_3 = T_2^{-1} Y_2 T_2^{-1} = A T_2'^{-1} Y_2' A T_2'^{-1} = A^2 T_2'^{-1} Y_2' T_2'^{-1} = A^2 Y_3'.$$

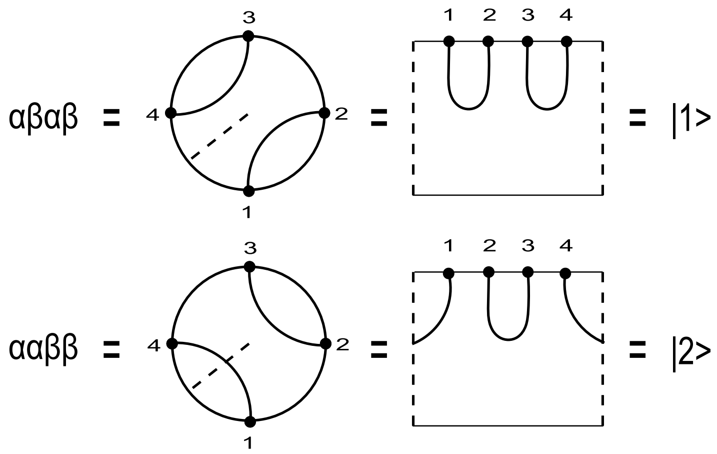
It is also consistent with the value we previously obtained for g, notably $g = \alpha^{-1}$. Finally, the last affine Hecke algebra generator in the $N = 3$ case is given by:

$$Y_3 = \begin{bmatrix} 0 & A^4 & 0 \\ -A^{-2} & A^2\alpha^{-1} + \alpha & 0 \\ -1 & A^2\alpha & A^2\alpha^{-1} \end{bmatrix}.$$

This completes our construction of the tangle representation of $\mathcal{A}_3(t)$.

5A.2 $N = 4$ Tangle Representation

We have a choice: either puncture every basis element or leave them unpunctured. Here we look only at the natural unpunctured case. By (5.1) there are two elementary patterns that make the basis of this representation. Using the rules of Subsection 5.2.1, these patterns are $\alpha\beta\alpha\beta$ and $\alpha\alpha\beta\beta$ and are illustrated below.



As in the previous two cases we obtain the Hecke algebra generators $\{T_i | i = 1, 2, 3\}$. These are given by:

$$T_1 = \begin{bmatrix} -A^2 & 1 \\ 0 & A^{-2} \end{bmatrix} \quad , \quad T_1^{-1} = \begin{bmatrix} -A^{-2} & 1 \\ 0 & A^2 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} A^{-2} & 0 \\ 1 & -A^2 \end{bmatrix} \quad , \quad T_2^{-1} = \begin{bmatrix} A^2 & 0 \\ 1 & -A^{-2} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} -A^2 & 1 \\ 0 & A^{-2} \end{bmatrix} \qquad T_3^{-1} = \begin{bmatrix} -A^{-2} & 1 \\ 0 & A^2 \end{bmatrix}.$$

In this particular two dimensional basis σ is represented as:

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \sigma^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Finally the affine Hecke algebra generators $\{Y_i | i = 1, 2, 3\}$ are:

$$Y_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad Y_1^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -A^{-4} & A^{-2} - A^2 \\ 0 & -A^4 \end{bmatrix} \quad , \quad Y_2^{-1} = \begin{bmatrix} -A^4 & A^2 - A^{-2} \\ 0 & -A^{-4} \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} -A^4 & A^2 - A^{-2} \\ 0 & -A^{-4} \end{bmatrix} \quad , \quad Y_3^{-1} = \begin{bmatrix} -A^{-4} & A^{-2} - A^2 \\ 0 & -A^4 \end{bmatrix},$$

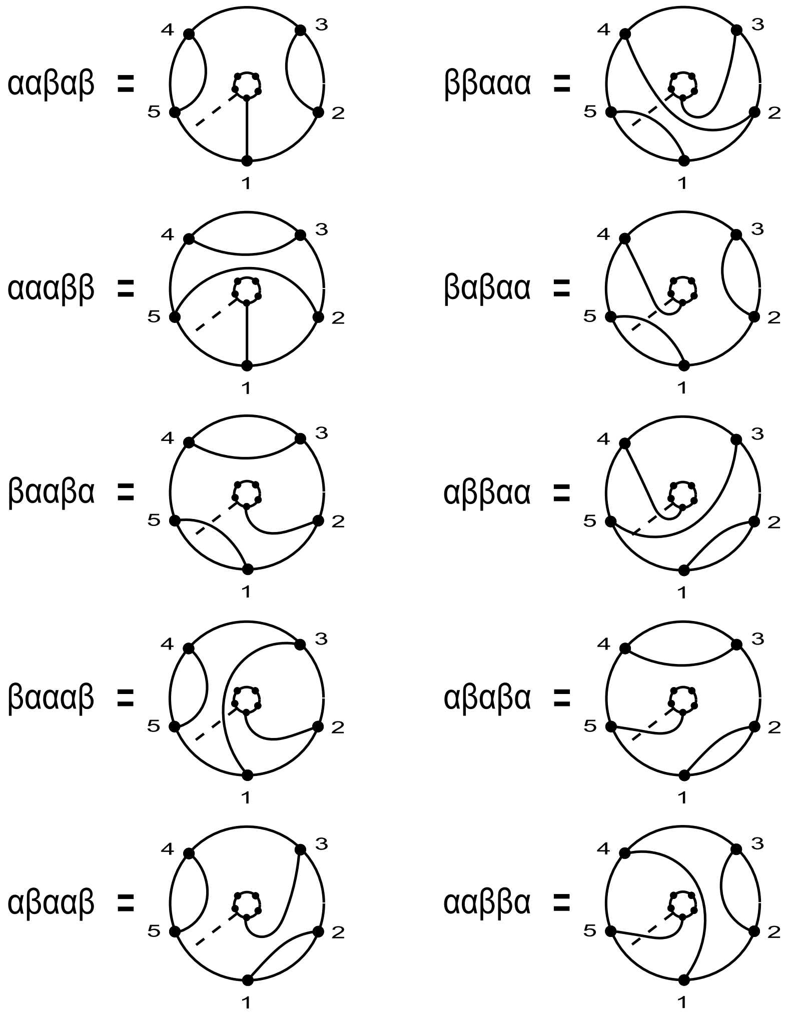
$$Y_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad Y_4^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We see that this representation is not particularly interesting since $\sigma^N = \sigma^4 = \mathbb{1}$ and $T_1 = T_3$. Also $Y_1 = Y_4$ and $Y_3 = Y_2^{-1}$ which means the product of the Y_i is simply the identity.

$$\prod_{i=1}^{4} Y_i = Y_1 Y_2 Y_2^{-1} Y_1 = Y_1^2 = \mathbb{1}.$$

5A.3 $N = 5$ Tangle Representation

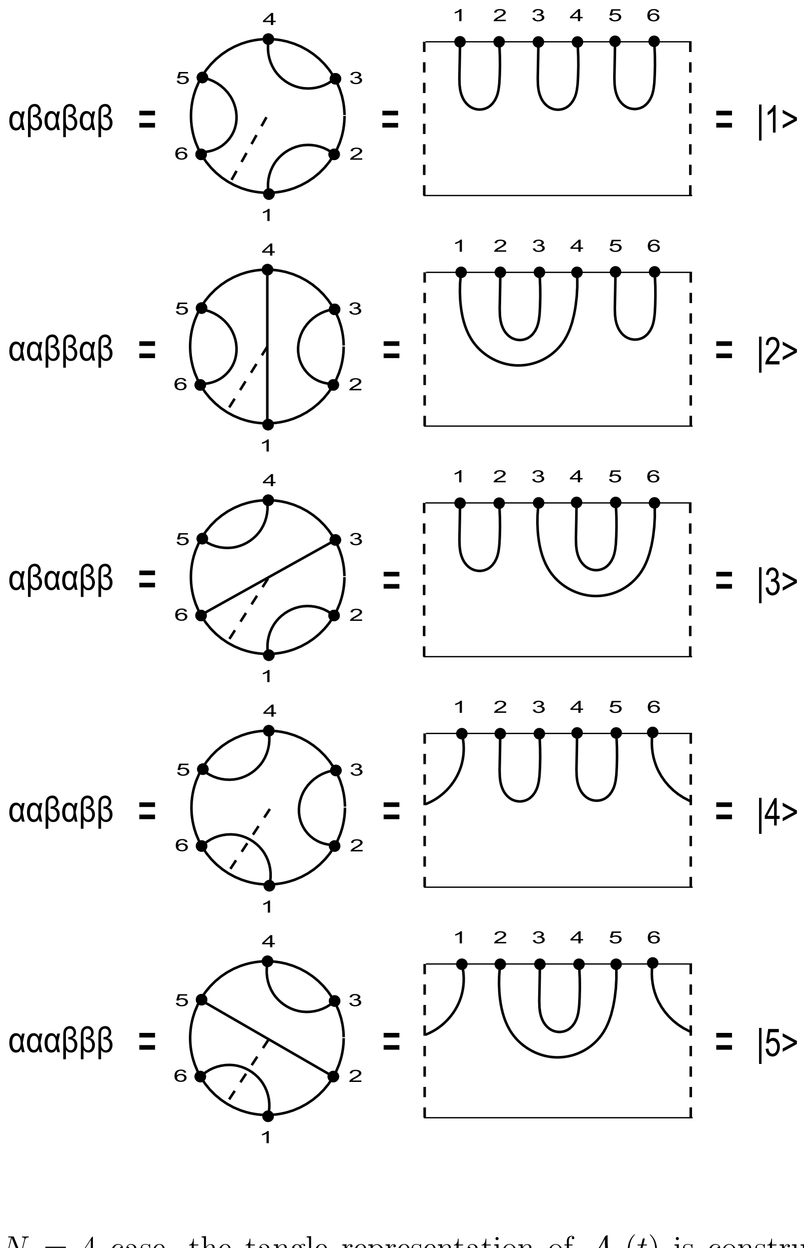
In the $N = 5$ case, by (5.2) the basis is composed of 10 elementary patterns. Following the rules for encoding patterns gives the basis elements pictured below.



The construction of the tangle representation for $\mathcal{A}_5(t)$ is completed by describing the action of the Hecke generators $\{T_i | i = 1, \ldots, 4\}$ and the affine Hecke generators $\{Y_i | i = 1, \ldots, 5\}$ as well as the operator σ on the elementary tangles. The result is a 10 dimensional matrix representation.

5A.4 $N = 6$ Tangle Representation

Using (5.1) we know that 5 elementary tangles complete the basis of this representation. All points are paired giving the basis $\{\alpha\beta\alpha\beta\alpha\beta, \alpha\alpha\beta\beta\alpha\beta, \alpha\beta\alpha\alpha\beta\beta, \alpha\alpha\beta\alpha\beta\beta, \alpha\alpha\alpha\beta\beta\beta\}$. These elementary tangles are illustrated below.



Following the $N = 4$ case, the tangle representation of $\mathcal{A}_6(t)$ is constructed in a very similar fashion.

Chapter 6

Knot Theory Connections

In this chapter we give our own understanding of the central role of the Hecke algebra in the development of knot theory and in particular in the discovery of the two variable knot polynomial known as the HOMFLY polynomial, discovered by Lickorish and Millett, Freyd and Yetter, Oceanu, and Hoste in [9]. We will give a detailed presentation of the construction of this polynomial following the approach of Jones in [25], that is via a specific trace function defined on the Hecke algebra. Though this construction is well known, we have included it due to the fact that it is very closely related to the tangle representation of the previous chapter. Furthermore, readers may also be familiar with the work contained in the remainder of this chapter. We nonetheless present it to give our own interpretation and a clear and detailed account highlighting the importance of the Hecke algebra to knot theory.

For example we explicitly derive the well known Jones polynomial [26] which is a one variable specialisation of the HOMFLY polynomial. We present a detailed account of a simple intuitive proof of its existence found by Kauffman in [23]. Furthermore we clearly illustrate with concrete examples how to calculate, in several different ways, the HOMFLY and Jones polynomial for some specific knots. Also, of particular interest to us is the Jones polynomial of the operator σ, which as we previously derived, can be used to determine the defining relations of the affine Hecke algebra. We will show that using the trace of its graphical representation, which we constructed in Chapter 3, we obtain all torus knots of arbitrary number of strands.

The connections between knot theory and the Hecke algebra, due to Jones, results from studying the representation theory of the braid group $\mathcal{B}_N$, which is rooted in the representation theory of the symmetric group $\mathcal{S}_N$. This is a consequence of the surjection $\mathcal{B}_N \longrightarrow \mathcal{S}_N$, given by mapping the elementary braid to the transposition $s_i = (i, i+1)$.

Therefore we can obtain new information about representations of $\mathcal{B}_N$ by studying the collection of irreducible representations of $\mathcal{S}_N$ and lifting them to representations of $\mathcal{B}_N$ by deforming them.

In particular, deforming the symmetric group gives the Hecke algebra $\mathcal{H}_N(\mathfrak{q})$ on which we define a Markov trace. The Markov trace on the Hecke algebra, which is due to Oceanu in [9], is unique and describes a relation between an N and an $N+1$ dimensional algebraic structure. Ensuring that this trace remains invariant under Markov moves, which one can think of as Reidemeister moves on closed braids, means that the closure of a braid in $\mathcal{B}_N$ represented within $\mathcal{H}_N(\mathfrak{q})$ must also be invariant. The result discovered by Jones in [25] is a two variable invariant of oriented links obtained via the closure of braids in $\mathcal{B}_N$. This invariant is essentially the HOMFLY polynomial which is obtained via reparameterisation.

6.1 Links via Braids

In Chapter 2 we defined the braid group $\mathcal{B}_N$ and the Hecke algebra $\mathcal{H}_N(\mathfrak{q})$ in a somewhat general fashion. Here to avoid confusion and as distinction between both sets of generators is needed, we redefine the braid group $\mathcal{B}_N$ as the group generated by elements $\{\sigma_i | i = 1, .., N-1\}$ with relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ for } |i-j| \geq 2, && (6.1)\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ otherwise .} && (6.2) \end{aligned}$$

Note that one may easily recover the presentation of the braid group of Chapter 1 by simply sending σ_i to T_i. Furthermore its pictorial representation is still described by defining σ_i and its inverse σ_i^{-1} to correspond to the exchange of the i^{th} and $(i+1)^{\text{th}}$ strands as illustrated in Section 3.1.

Given a braid $\alpha \in \mathcal{B}_N$ we can always form the oriented link $L = \hat{\alpha}$, called the trace closure of α in the following way: connect the strand at the rightmost top peg around to the right to the strand at the rightmost bottom peg, then connect in the same way the next to rightmost top and bottom strands and so on, until all strands top and bottom are connected. In Figure 6.1 the trace closure of the braid $\alpha = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ is the oriented link $\hat{\alpha}$ known as the figure-8 knot.

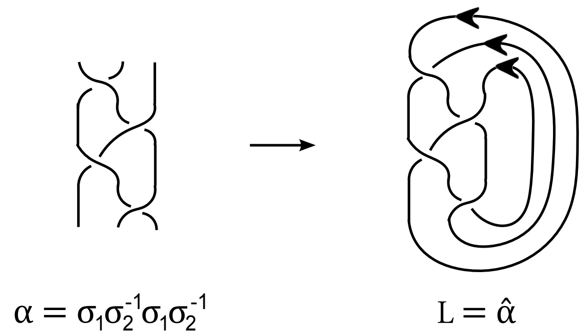


Figure 6.1: The trace closure of $\alpha = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in \mathcal{B}_3$ is the link L known as the figure-8 knot.

Figure 6.1 also highlights another point. In [27], J. Alexander asserts that any tame oriented link is isotopic to the closure of a braid in $\mathcal{B}_N$. We highlight though, that the representation of a link as a closed braid is non-unique; that is the closure of many different braids give rise to the same link.

However Markov's Theorem [28] states that if braids $\alpha \in \mathcal{B}_N$ and $\delta \in \mathcal{B}_M$ have isotopic closures, then there is a finite sequence of Markov moves which takes one to the other. For its complete proof readers are referred to Birman in [29] which published the first detailed proof. There are two types of permissible moves. A Markov move of type I, called *conjugation*, changes $\alpha \in \mathcal{B}_N$ to $\beta\alpha\beta^{-1} \in \mathcal{B}_N$ for any braid $\beta \in \mathcal{B}_N$, as described in Figure 6.2.

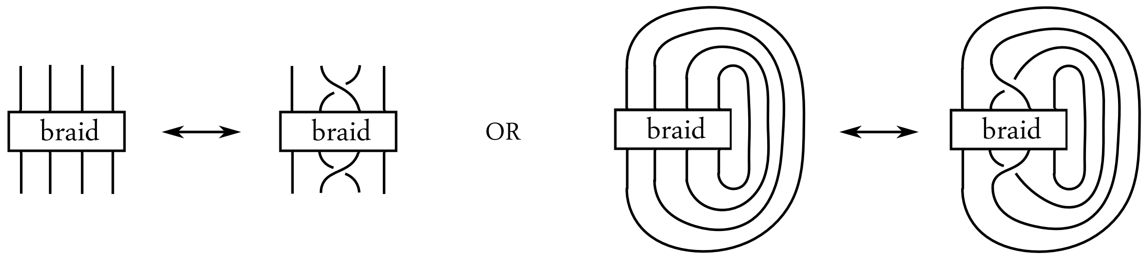


Figure 6.2: Markov move of type I. Conjugation by σ_j leaves the braid and the oriented link corresponding to the braid closure unchanged.

The Markov moves of type II, called *stabilisation* (and *destabilisation* respectively), change $\alpha \in \mathcal{B}_N$ to $\alpha\sigma_N^{\pm 1} \in \mathcal{B}_{N+1}$. We illustrate this in Figure 6.3.

Therefore under Markov moves, the oriented link corresponding to the closed braid remains unchanged. This property is crucial in describing the construction of a two variable polynomial invariant arising from the representations of the braid group $\mathcal{B}_N$.

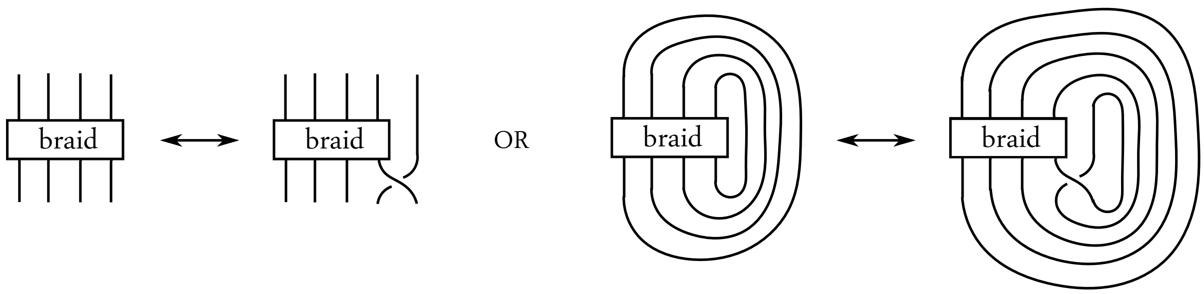


Figure 6.3: A Markov move of type II. Stabilisation adds a loop to the closed braid. Similarly destabilisation deletes a loop from the closed braid.

6.2 $\mathcal{H}_N(\mathfrak{q})$ and $\mathcal{S}_N$ revisited

In this section we describe all representations of the braid group in which its generators σ_i have at most two eigenvalues. Writing g_i for the image of σ_i, under such a representation we must have a quadratic equation of the form $g_i^2 + ag_i + b = 0$, where a, b are scalars. It is common convention to assume one of the eigenvalues is 1 and eliminate one of the variables to obtain the quadratic relation

$$g_i^2 = (\mathfrak{q} - 1)g_i + \mathfrak{q}, \qquad \mathfrak{q} \text{ a scalar.}$$

This familiar expression is of course the Hecke relation. Therefore it is now evident that knowledge of representations of $\mathcal{B}_N$ in which the σ_is have at most two eigenvalues is the same as knowledge of the Hecke algebra $\mathcal{H}_N(\mathfrak{q})$.
Recall that $\mathcal{H}_N(\mathfrak{q})$ is the algebra with generators $g_1, \ldots, g_{N-1}$, satisfying the relations

$$g_i g_j = g_j g_i \text{ for } |i - j| \geq 2, \tag{6.3}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ otherwise }, \tag{6.4}$$

$$g_i^2 = (\mathfrak{q} - 1)g_i + \mathfrak{q}. \tag{6.5}$$

From the above definition of $\mathcal{H}_N(\mathfrak{q})$ it is clear that for each $\mathfrak{q} \neq 0$, $\mathcal{B}_N$ has a representation inside $\mathcal{H}_N(\mathfrak{q})$ obtained by sending σ_i to g_i.

It is at this point useful to highlight the reason for our change in notation. In Chapter 2 we defined the Hecke algebra in terms of generators T_i and parameter t. Here we represent the generators by g_i and the parameter by $\mathfrak{q}$. The reason being, that in this section we describe *all* representations of $\mathcal{B}_N$ where its generators have at most two eigenvalues. This is the general case, whereas in Chapter 2 we defined $\mathcal{H}_N(t)$ where the braid group generators have eigenvalues given specifically by $t^{1/2}$ and $-t^{-1/2}$. Furthermore the generators g_i are a scaled version of the T_i and hence a change of notation is necessary. If one desires, simply substituting t for $\mathfrak{q}$ and sending $g_i \longrightarrow t^{1/2}T_i$ in the above relations, one recovers the initial definition of $\mathcal{H}_N(t)$ in Chapter 2.

Now recall that in Subsection 2.3, we saw pointed out that the Hecke algebra can be thought of as a "deformation" of the symmetric group $\mathcal{S}_N$, which is generated by $s_1, \ldots, s_{N-1}$ with relations

$$\begin{aligned} s_i s_j &= s_j s_i \text{ for } |i-j| \geq 2, && (6.6) \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ otherwise }, && (6.7) \\ s_i^2 &= 1. && (6.8) \end{aligned}$$

Irreducible representations of $\mathcal{S}_N$ are parameterised by Young Tableaux. Furthermore as a vector space, it is shown in [30] that $\mathcal{S}_N$ is spanned by $N!$ reduced words in the transpositions s_i:

$$\left\{ (s_{i_1} s_{i_1-1} \ldots s_{i_1-k_1})(s_{i_2} s_{i_2-1} \ldots s_{i_2-k_2}) \ldots (s_{i_p} s_{i_p-1} \ldots s_{i_p-k_p}) \right\}, \tag{6.9}$$

where $1 \leq i_1 < i_2 < \ldots < i_p \leq N-1$ and $i_j - k_j \geq 1$.

Now imagine trying to reduce words on the g_is and σ_is to words of minimal length. Then relation (6.8) is as good as the Hecke relation (6.5) for this purpose. Thus a system of reduced words of the $s_i \in \mathcal{S}_N$ will furnish a $N!$ dimensional basis for $\mathcal{H}_N(\mathfrak{q})$. By simply writing g_i for s_i in (6.9), a convenient such basis is:

$$\left\{ (g_{i_1} g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p} g_{i_p-1} \ldots g_{i_p-k_p}) \right\}, \tag{6.10}$$

where $1 \leq i_1 < i_2 < \ldots < i_p \leq N-1$ and $i_j - k_j \geq 1$ [25].

Before we introduce a trace function on $\mathcal{H}_N(\mathfrak{q})$, it is essential to give the following important statement due to Jones in [25]. By [31], given $\mathfrak{q}$ is not a root of unity or zero, the quadratic irreducible representations of $\mathcal{B}_N$ are in one-to-one correspondence with Young Tableaux. Therefore their decomposition rules and their dimensions are the same as for $\mathcal{S}_N$.

We are now in a position to define Oceanu's trace on the Hecke algebra.

6.3 Oceanu's trace on $\mathcal{H}_N(\mathfrak{q})$

In the following we will always consider $\mathcal{H}_N(\mathfrak{q})$ as embedded in $\mathcal{H}_{N+1}(\mathfrak{q})$ via (6.10), and the representation of $\mathcal{B}_N$ inside $\mathcal{H}_N(\mathfrak{q})$ will be denoted by f, so that $f(\sigma_i) = g_i$.

The trace on $\mathcal{H}_N(\mathfrak{q})$ is due to Oceanu [9]. It states that for every $z \in \mathbb{C}$, there is a linear trace function tr: $\bigcup_{N=1}^{\infty} \mathcal{H}_N(\mathfrak{q}) \longrightarrow \mathbb{C}$ uniquely defined by

1) $\mathrm{tr}(1) = 1$,

2) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,

3) $\mathrm{tr}(xg_n) = z\mathrm{tr}(x)$ for $x \in \mathcal{H}_N(\mathfrak{q})$.

The uniqueness of this trace function is proved inductively in [25] using the $N!$ element basis given by (6.10). Due to its significance, we outline this proof in Appendix 6A.1.

Note the similarity between the third property of the trace function and the Markov move of type II. Both expressions describe a relation between an N and a $N+1$ dimensional structure. As such this property is known as a Markov property and the trace tr is a Markov trace.

It is now clear that given the properties of the trace function in addition to the Hecke relation (6.5), it is possible to calculate the trace of any element of $\mathcal{H}_N(\mathfrak{q})$.
As an example consider the braid word $\alpha = \sigma_1^3 \in \mathcal{B}_2$, where we note that its trace closure $\hat{\alpha}$ is the right handed trefoil and that $f(\sigma_1^3) = g_1^3$.

$$\begin{aligned}
\mathrm{tr}(g_1^3) &= \mathrm{tr}(g_1^2 g_1) \\
&= \mathrm{tr}\left((\mathfrak{q}-1)g_1^2 + \mathfrak{q}g_1\right) && \text{by (6.5)} \\
&= (\mathfrak{q}-1)\mathrm{tr}(g_1^2) + \mathfrak{q}\mathrm{tr}(g_1) \\
&= (\mathfrak{q}-1)\mathrm{tr}\left((\mathfrak{q}-1)g_1 + \mathfrak{q}\right) + \mathfrak{q}z && \text{by (6.5) and property 3} \\
&= (\mathfrak{q}-1)^2 z + \mathfrak{q}(\mathfrak{q}-1) + \mathfrak{q}z && \text{by property 3} \\
&= (\mathfrak{q}^2 - \mathfrak{q} + 1)z + \mathfrak{q}(\mathfrak{q}-1).
\end{aligned}$$

The calculation of the trace of all other elements of $\mathcal{H}_N(\mathfrak{q})$ follows in a similar fashion.

6.4 The two variable Knot Polynomial

The key to the construction of this knot invariant lies in the fact that the trace function on the Hecke algebra is a Markov trace. Therefore we slightly modify the function f: $f(\sigma_i) = g_i$, to obtain from a given braid, a two variable knot polynomial which is also invariant under the stabilisation move. Such a polynomial will be Markov invariant and hence an invariant of the knot type of the closed braid.

Algebraically, stabilisation and destabilisation take the form $\alpha \in \mathcal{B}_N \to \alpha\sigma_N^{\pm 1} \in \mathcal{B}_{N+1}$. It is then necessary to rescale our representation f in such a way that both Markov moves of type II have the same effect on the trace function. We do this as follows:
Suppose there exists $\theta \in \mathbb{C}$ such that $\mathrm{tr}(\theta g_i) = \mathrm{tr}((\theta g_i)^{-1})$, then solving for θ gives

$$\begin{aligned} \mathrm{tr}(\theta g_i) &= \mathrm{tr}(g_i^{-1}\theta^{-1}) \\ \Rightarrow \theta^2 \mathrm{tr}(g_i) &= \mathrm{tr}(g_i^{-1}). \end{aligned}$$

But using the Hecke relation (6.5) and the third property of the trace tr, we obtain

$$\theta^2 = \frac{\mathrm{tr}(g_i^{-1})}{\mathrm{tr}(g_i)} = \frac{\mathrm{tr}(\mathfrak{q}^{-1}g_i - 1 + \mathfrak{q}^{-1})}{z} = \frac{z - \mathfrak{q} + 1}{\mathfrak{q}z}.$$

Letting $\lambda = \theta^2$ and solving for z gives

$$z = \frac{-(1-\mathfrak{q})}{1-\lambda\mathfrak{q}}. \tag{6.11}$$

Defining $f_\lambda : \mathcal{B}_N \to \mathcal{H}_N(\mathfrak{q})$ by $f_\lambda(\sigma_i) = \sqrt{\lambda}\sigma_i$, the action of the trace tr on the representation f of $\mathcal{B}_N$ now acts as follows:

$$\mathrm{tr}\left(f_\lambda(\sigma_i)\right) = z\sqrt{\lambda} = -\sqrt{\lambda}\left(\frac{1-\mathfrak{q}}{1-\lambda\mathfrak{q}}\right).$$

We would like to define a map $\mathcal{B}_N \to \mathbb{Z}[\mathfrak{q}^{\pm 1}, \lambda^{\pm 1}]$ which is Markov invariant. Presently we have that

$$\mathrm{tr}\left(f(\alpha \cdot \sqrt{\lambda}\sigma_N)\right) = -\sqrt{\lambda}\left(\frac{1-\mathfrak{q}}{1-\lambda\mathfrak{q}}\right) = \mathrm{tr}\left(f(\alpha \cdot \sqrt{\lambda}\sigma_N^{-1})\right),$$

for any $\alpha \in \mathcal{B}_N$. However we see that the quantity

$$\left(\frac{-(1-\lambda\mathfrak{q})}{\sqrt{\lambda}(1-\mathfrak{q})}\right)^{N-1} \mathrm{tr}\left(f_\lambda(\alpha)\right) \text{ for } \alpha \in \mathcal{B}_N,$$

depends only on the link formed by the closure of α, that is, $\hat{\alpha}$. Hence the two variable invariant $X_L(\mathfrak{q},\lambda)$ of the oriented link L is given by

$$X_L(\mathfrak{q},\lambda) \;=\; \left(\frac{-(1-\lambda\mathfrak{q})}{\sqrt{\lambda}(1-\mathfrak{q})}\right)^{N-1}\left(\sqrt{\lambda}\right)^{e}\operatorname{tr}\left(f(\alpha)\right), \tag{6.12}$$

where $\alpha \in \mathcal{B}_N$ is any braid with closure $L = \hat{\alpha}$, e being the the sum of the powers of α as a word on the σ_i and f the representation of $\mathcal{B}_N$ in $\mathcal{H}_N(\mathfrak{q})$, $f(\sigma_i) \to g_i$.

The two variable knot polynomial, a Laurent polynomial $\mathcal{P}_L(t,x)$, is obtained via substitution. Letting $t = \sqrt{\lambda}\sqrt{\mathfrak{q}}$ and $x = \sqrt{\mathfrak{q}} - 1/\sqrt{\mathfrak{q}}$ in (6.12), then $\mathcal{P}_L(t,x) = X_L(\mathfrak{q},\lambda)$.
Moreover, $\mathcal{P}_L(t,x)$ is uniquely defined by the skein relation

$$t^{-1}\mathcal{P}_{L_+} - t\mathcal{P}_{L_-} \;=\; x\mathcal{P}_{L_0}, \tag{6.13}$$

where L_+, L_- and L_0 are any three oriented links that are identical except near a point where they are as in Figure 6.4.

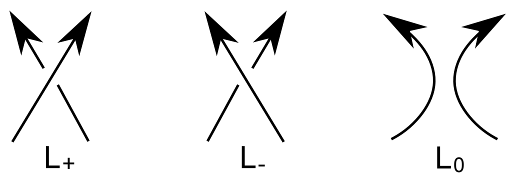


Figure 6.4: Crossing types L_+, L_- and L_0.

To illustrate in detail how to evaluate the knot polynomial $\mathcal{P}_L(t,x)$ by firstly calculating the invariant $X_L(\mathfrak{q},\lambda)$, we present the following example.

Example

We calculate $X_L(\mathfrak{q},\lambda)$ for the right handed trefoil. Recall that the right handed trefoil is given by the closure of the braid $\alpha = \sigma_1^3$.

We have that $\alpha = \sigma_1^3 = \sigma_1\sigma_1\sigma_1 \in \mathcal{B}_2 \Rightarrow N = 2$ and $e = 3$. Hence with $f(\sigma_1^3) = g_1^3$ (6.12) becomes

$$X_L(\mathfrak{q},\lambda) \;=\; \frac{-(1-\lambda\mathfrak{q})}{\sqrt{\lambda}(1-\mathfrak{q})}\left(\sqrt{\lambda}\right)^{3}\operatorname{tr}(g_1^3).$$

We already calculated that $\mathrm{tr}(g_1^3) = (\mathfrak{q}^2 - \mathfrak{q} + 1)z + \mathfrak{q}(\mathfrak{q} - 1)$, hence using the value of z given by (6.11), we find that

$$\begin{aligned}
X_L(\mathfrak{q}, \lambda) &= \left(\frac{-(1-\lambda\mathfrak{q})}{\sqrt{\lambda}(1-\mathfrak{q})}\left(\sqrt{\lambda}\right)^3\right)\left(-(\mathfrak{q}^2 - \mathfrak{q} + 1)\frac{(1-\mathfrak{q})}{1-\lambda\mathfrak{q}} + \mathfrak{q}(\mathfrak{q}-1)\right) \\
&= \left(\frac{-\lambda(1-\lambda\mathfrak{q})}{(1-\mathfrak{q})}\right)\left(\frac{-(1-\mathfrak{q})(\mathfrak{q}^2 + 1 - \lambda\mathfrak{q}^2)}{1-\lambda\mathfrak{q}}\right) \\
&= \lambda(\mathfrak{q}^2 + 1 - \lambda\mathfrak{q}^2).
\end{aligned}$$

Its corresponding two variable knot polynomial $\mathcal{P}_L(t, x)$ is given by the substitution $t = \sqrt{\lambda}\sqrt{\mathfrak{q}}$ and $x = \sqrt{\mathfrak{q}} - 1/\sqrt{\mathfrak{q}}$. For the right handed trefoil it is the polynomial

$$\begin{aligned}
\mathcal{P}_L(t, x) &= \lambda\mathfrak{q}\left((\sqrt{\mathfrak{q}} - 1/\sqrt{\mathfrak{q}})^2 + 2 - \lambda\mathfrak{q}\right) \\
&= t^2(x^2 + 2 - t^2) \\
&= x^2t^2 + 2t^2 - t^4.
\end{aligned}$$

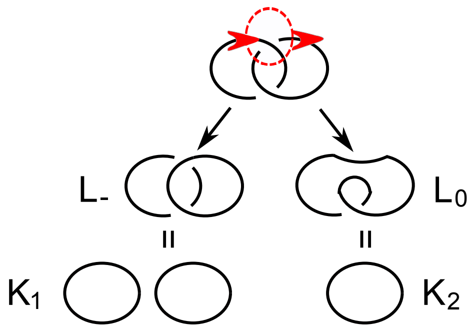
Let us now verify this result by explicitly evaluating $\mathcal{P}_L(t, x)$ using the skein relation (6.13). However before using the skein relation on the right handed trefoil, it is necessary to know the polynomial $\mathcal{P}_L(t, x)$ associated with the N-component unlink. This is particularly simple since the N-component unlink is given by the closure of the identity braid $\in \mathcal{B}_N$. Therefore (6.12) simply becomes

$$X_L(\mathfrak{q}, \lambda) = \left(\frac{-(1-\lambda\mathfrak{q})}{\sqrt{\lambda}(1-\mathfrak{q})}\right)^{N-1},$$

and hence after substitution its associated polynomial is

$$\mathcal{P}_L(t, x) = \left(\frac{t^{-1} - t}{x}\right)^{N-1}. \tag{6.14}$$

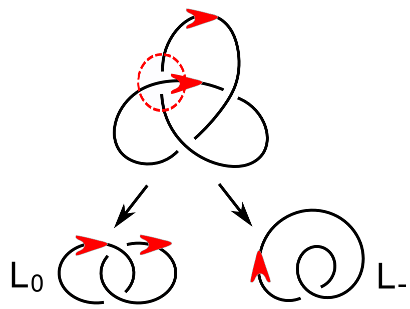
It is also convenient to firstly find $\mathcal{P}_L(t, x)$ for the Hopf link with orientation as pictured in the resolving tree below.



Note that this link is the closure of the braid σ_1^2, and since both links have the same orientation, the crossing encircled in red is an L_+ crossing, hence we replaced it with L_- and L_0 crossings respectively. So this Hopf link decomposes into the unknot in addition to the disjoint union of two unknots. Using the skein relation (6.13), as well as (6.14) we get

$$\begin{aligned}\mathcal{P}_{L_+}(t,x) &= xt\mathcal{P}_{L_0} + t^2\mathcal{P}_{L_-} \\ &= xt\mathcal{P}_{L_0}(K_2) + t^2\mathcal{P}_{L_-}(K_1) \\ &= xt(1) + t^2\left(\frac{t^{-1}-t}{x}\right) \\ &= xt + x^{-1}t - x^{-1}t^3.\end{aligned}$$

Similarly the right handed trefoil is the closure of the braid σ_1^3 and hence its resolving tree is given by the following diagram:



We see that the trefoil breaks into a Hopf link of L_+ crossing, where both links have the same orientation, in addition to the unknot. Hence its two variable polynomial $\mathcal{P}_L(t,x)$ is

$$\begin{aligned}\mathcal{P}_{L_+}(t,x) &= xt\mathcal{P}_{L_0} + t^2\mathcal{P}_{L_-} \\ &= xt(xt + x^{-1}t - x^{-1}t^3) + t^2(1) \\ &= x^2t^2 + 2t^2 - t^4,\end{aligned}$$

as expected. This is in perfect agreement with the polynomial obtained via the invariant trace function on $\mathcal{H}_N(\mathfrak{q})$.

6.5 When $\mathcal{B}_N$ has eigenvalues $t^{1/2}$, $-t^{-1/2}$

For completeness we derive $\mathcal{P}_L(t,x)$ in the case when $\mathcal{B}_N$ has two eigenvalues, namely $t^{1/2}$ and $-t^{-1/2}$. These eigenvalues correspond to the braid group $\mathcal{B}_N$ introduced in Chapter 2 with relations (2.1)-(2.2), and whose associated Hecke algebra $\mathcal{H}_N(t)$ satisfies the Hecke relation (2.3) given by $T_i^2 - \left(t^{1/2} - t^{-1/2}\right) T_i = \mathbb{1}$.

For convenience we let $k = \left(t^{1/2} - t^{-1/2}\right)\mathbb{1}$. This implies that the Hecke relation is given by $T_i^2 = kT_i + \mathbb{1}$ and furthermore we have $T_i^{-1} = T_i - k$. We now follow the derivation of $\mathcal{P}_L(t,x)$ in Section 6.4 to firstly find the two variable invariant $X_L(k,\lambda)$ of the oriented link L.

Simple calculations show that $z = k/(1-\lambda)$ and hence

$$\operatorname{tr}\left(f_\lambda(T_i)\right) = z\sqrt{\lambda} = \sqrt{\lambda}\left(\frac{k}{1-\lambda}\right).$$

Therefore in this specific case, equation (6.12) for the invariant $X_L(k,\lambda)$ becomes

$$X_L(k,\lambda) = \left(\frac{1-\lambda}{\sqrt{\lambda}k}\right)^{N-1}\left(\sqrt{\lambda}\right)^e \operatorname{tr}\left(f(\alpha)\right), \tag{6.15}$$

where as before $\alpha \in \mathcal{B}_N$ is any braid with closure $L = \hat{\alpha}$, e is the exponent sum of α as a word on the σ_i and f the representation of $\mathcal{B}_N$ in $\mathcal{H}_N(t)$.

Its two variable knot polynomial Laurent polynomial $\mathcal{P}_L(t,x)$ is obtained via the substitution $t = \sqrt{\lambda}$ and $x = k$.

As an example we consider the right handed trefoil again. We already know that $f(\alpha) = T_1^3$, $N = 2$ and $e = 3$. Using the properties of the trace function on $\mathcal{H}_N(t)$ from Section 6.3 we find that

$$\begin{aligned}
\operatorname{tr}(T_1^3) &= \operatorname{tr}(T_1^2T_1) \\
&= k\operatorname{tr}(T_1^2) + \operatorname{tr}(T_1) \\
&= (k^2+1)z + k \\
&= \frac{k(k^2+2-\lambda)}{1-\lambda}.
\end{aligned}$$

The invariant $X_L(k,\lambda)$ resulting from the trace on $\mathcal{H}_N(t)$ when the representation of $\mathcal{B}_N$ within it has eigenvalues $t^{1/2}$ and $-t^{-1/2}$ is from (6.15)

$$\begin{aligned} X_L(k,\lambda) &= \left(\frac{1-\lambda}{\sqrt{\lambda}k}\right)\left(\sqrt{\lambda}\right)^3\left(\frac{k(k^2+2-\lambda)}{1-\lambda}\right) \\ &= \lambda(k^2+2-\lambda). \end{aligned}$$

Substituting $t = \sqrt{\lambda}$ and $x = k$, we obtain

$$\mathcal{P}_L(t,x) \;=\; x^2t^2 + 2t^2 - t^4,$$

as anticipated.

6.6 HOMFLY Polynomial

It is important to note that the two variable knot polynomial $\mathcal{P}_L(t,x)$ derived in Section 6.4 is *essentially* the HOMFLY polynomial, but *not* exactly the original HOMFLY that appeared in [9] and *not* the one that is quoted in literature.

Given the invariant $X_L(\mathfrak{q},\lambda)$ defined by (6.12), the HOMFLY polynomial $P(L)$, of the oriented link L, is obtained by the reparameterisation of the variables t and x to $\ell = it^{-1}$ and $m = ix$, where $i^2 = -1$.
Therefore in terms of λ and $\mathfrak{q}$ we have

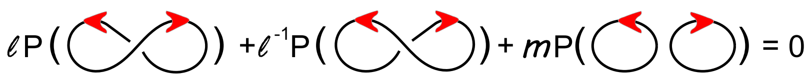
$$\begin{aligned} \ell &= it^{-1} &&\Rightarrow t = i\ell^{-1} = \sqrt{\lambda}\sqrt{\mathfrak{q}} \\ m &= ix &&\Rightarrow x = -im = \sqrt{\mathfrak{q}} - 1/\sqrt{\mathfrak{q}}. \end{aligned}$$

Furthermore the HOMFLY polynomial is defined by the following skein relation

$$\ell P(L_+) + \ell^{-1}P(L_-) + mP(L_0) \;=\; 0, \tag{6.16}$$

where L_+, L_- and L_0 are the crossings as in Figure 6.4. In addition to this $P(\bigcirc) = 1$, that is, the unknot has HOMFLY polynomial 1.

Using the skein relation (6.16), we can immediately evaluate the HOMFLY polynomial of two disjoint unknots.

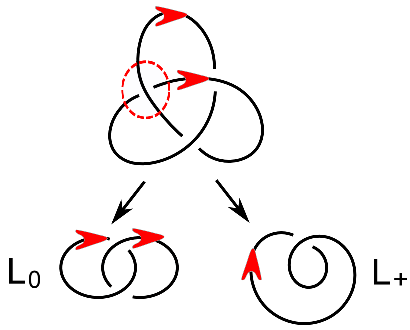


We see that $P(L_+)$ and $P(L_-)$ are simply slightly twisted unknots, therefore:

$$\begin{aligned} P(L_+) = P(L_-) &= 1 \\ \Rightarrow P(L_0) &= -m^{-1}(\ell + \ell^{-1}). \end{aligned}$$

So the HOMFLY polynomial of two disjoint unknots is given by $-m^{-1}(\ell+\ell^{-1})$. We will now use this result to evaluate the HOMFLY polynomial for the left handed trefoil.

In the resolving tree below, replacing the highlighted L_- crossing with L_0 and L_+ crossings, the left handed trefoil breaks down into a Hopf link in addition to the unknot.



Therefore we find that its HOMFLY polynomial is given by

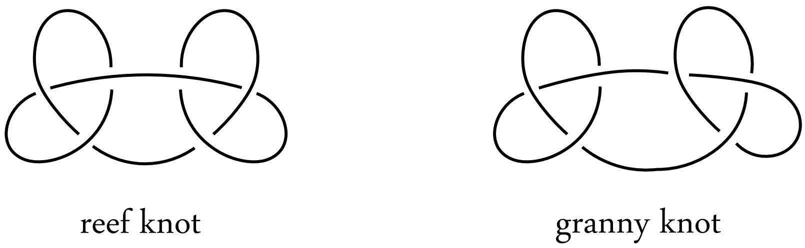
$$\begin{aligned} P(L_-) &= -m\ell P(L_0) - \ell^2 P(L_+) \\ &= -m\ell(-m\ell + m^{-1}\ell^3 + m^{-1}\ell) - \ell^2(1) \\ &= m^2\ell^2 - \ell^4 - 2\ell^2. \end{aligned}$$

Note that in the above calculation we used the fact that the HOMFLY polynomial of a Hopf link with L_- crossing is $P(L_-) = -m\ell + m^{-1}\ell^3 + m^{-1}\ell$. (We have included a complete description of Hopf links in Appendix 6A.2.)

In a similar fashion we can easily evaluate the HOMFLY polynomial for the right handed trefoil, by observing its decomposition into a Hopf link with L_+ crossing in addition to the unknot. In this case we obtain $P(L_+) = m^2\ell^{-2} - 2\ell^{-2} - \ell^{-4}$. Clearly this resulting polynomial is not the same as the one we calculated for the left handed trefoil. The point being that the HOMFLY polynomial differentiates between right handed and left handed trefoils. This leads us to introduce the following nice property of $P(L)$: the HOMFLY polynomial of a composite knot is simply the product of the individual link polynomials, that is

$$P(L_1 \# L_2) = P(L_1)P(L_2). \tag{6.17}$$

A particularly transparent example is that of the reef knot and the granny knot, which we illustrate below.



The reef knot combines a left handed trefoil and a right handed trefoil. Hence its HOMFLY polynomial by (6.17) in addition to the results we previously obtained, is

$$P(\text{reef}) \;=\; (-2\ell^2 - \ell^4 + m^2\ell^2)(-2\ell^{-2} - \ell^{-4} + m^2\ell^{-2}),$$

which can be easily verified using the skein relation (6.16).
Similarly the granny knot is the product of two left handed trefoils, so unsurprisingly its HOMFLY polynomial is

$$P(\text{granny}) \;=\; (-2\ell^2 - \ell^4 + m^2\ell^2)^2.$$

In the following section we will derive the Jones polynomial [26], which is a one variable specialisation of the HOMFLY polynomial. Interestingly the Jones polynomial is unable to distinguish between a granny knot and a reef knot.

6.7 The Jones Polynomial

The Jones polynomial [26], $V(L)$, is a one variable polynomial invariant for oriented links. It is a Laurent polynomial in the variable $t^{1/2}$ which satisfies

$$\begin{aligned} V(\text{unknot}) \;&=\; 1, \\ t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) \;&=\; 0, \end{aligned} \tag{6.18}$$

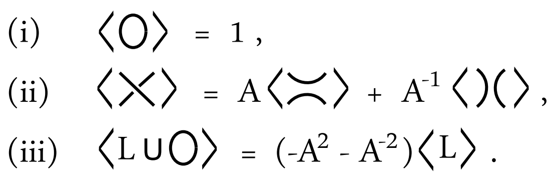
where L_+, L_- and L_0 are as in Figure 6.4.
It is important to note that in a projection of an oriented link L, the crossings are of two types; that of L_+ in Figure 6.4 is called positive, that of L_- is negative.

As we previously mentioned, the Jones polynomial is a specialisation of the HOMFLY polynomial. As such $V(L)$ is obtained from $P(L)$ by the substitution

$$(\ell, m) \;=\; \left(it^{-1}, i(t^{-1/2} - t^{1/2})\right), \qquad \text{where } i^2 \;=\; -1.$$

However, there exists a much more intuitive way to find $V(L)$. Following [32] we outline an almost trivial proof of the existence of the Jones polynomial found by L. H. Kauffman [23]. This remarkable proof provides the reader with an excellent insight in calculating the Jones polynomial of any link.

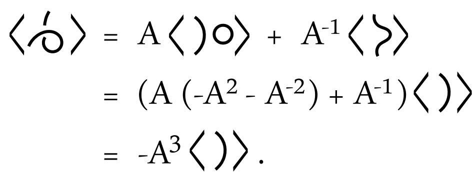
We begin by considering projections of unoriented links. For each such projection L define a Laurent polynomial $\langle L \rangle$ in one variable A by the following three rules:



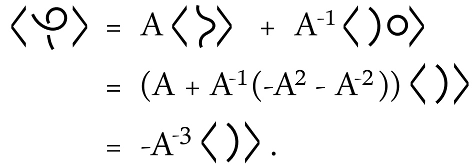
This $\langle L \rangle$ is called the *bracket polynomial* of L. (It is also known as the Kauffman bracket). Rule (i) states that 1 is the polynomial of the particular projection of the unknot that has no crossing at all. Rule (ii) describes how to write a crossing in terms of a combination of two other projections where that crossing has been destroyed. The last rule states that the polynomial of the union of L with a disjoint loop can be rewritten as the factor $(-A^2 - A^{-2})$ times the polynomial of L.

It is evident from the above rules that the choice of order in which the crossings are chosen is irrelevant. Furthermore, each application of rule (ii) reduces the number of crossings in the projections until there are no crossings at all. To verify that $\langle L \rangle$ is in fact an invariant of real links, we must see if it remains unchanged by the Reidemeister moves introduced in 5.2.

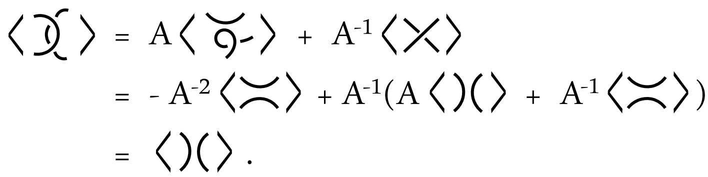
Recall that Reidemeister Move I allowed us to include or exclude twists:



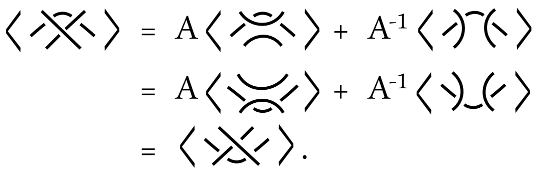
In a similar fashion we show that:



Thus the bracket polynomial fails to be invariant under Move I. Let us see if this is also true for Move II.



The calculation above shows that the bracket polynomial is in fact invariant under Move II. Finally Move III allowed us to move a string from one side of a crossing to the other side of the crossing:



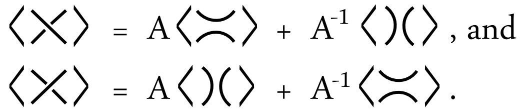
Hence there is also invariance under Move III.

Now give L an orientation. Let $w(L)$, the writhe of L, be the algebraic sum of the crossings of L, counting $+1$ for a positive crossing and -1 for a negative crossing. It is clear that Move I adds or subtracts 1 to $w(L)$, so the writhe of L is certainly not invariant under this move. However $w(L)$ is invariant under Moves II and III. Thus any combination of $w(L)$ and L will be invariant under Moves II and III, and their non-invariant behaviours under Move I cancel in the expression of the Kauffman polynomial $X(L)$ defined as:

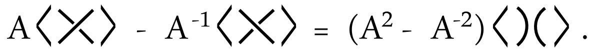
$$X(L) \quad = \quad (-A)^{-3w(L)}\langle L\rangle.$$

The above completes Kauffman's remarkable proof that the polynomial $X(L)$ is a well defined invariant of oriented links. Using the above expression for $X(L)$ we can now obtain the skein relation (6.18) of the original Jones polynomial $V(L)$ in several simple steps.

Rule (ii) implies that:



Therefore:



Suppose that orientations can be chosen for these last three projections so that the arrows point approximately upwards as in Figure 6.4. Call them L_+, L_- and L_0. Then $w(L_\pm) = w(L_0) \pm 1$ and direct substitution gives:

$$A(-A)^3 X(L_+) - A^{-1}(-A)^{-3} X(L_-) \;\; = \;\; (A^2 - A^{-2}) X_{L_0}.$$

Writing $A = t^{-1/4}$ this becomes

$$t^{-1} X(L_+) - t X(L_-) + (t^{-1/2} - t^{1/2}) X(L_0) \;\; = \;\; 0,$$

completing the fact that under the substitution $A = t^{-1/4}$, $X(L)$ is the original Jones polynomial $V(L)$, for they satisfy the same defining formula.

In the next section we describe explicitly how to evaluate the Jones polynomial of a given link by calculating its Kauffman bracket using the method outlined above.

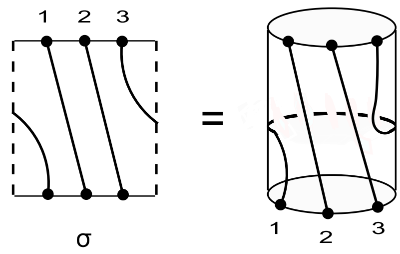
6.8 Tracing over $\mathcal{A}_N(t)$ elements

In Section 6.4 we presented Jones' discovery in [25] of a two variable knot polynomial $\mathcal{P}_L(t, x)$, arising from the study of representations of the Hecke algebra. The existence of this knot polynomial, which is related to the HOMFLY polynomial, is dependent on the Markov trace, tr, defined on the Hecke algebra. In this section we investigate the effect of tracing over elements of the affine Hecke algebra. In particular we look at the trace closure of the operator σ, (2.11), which yields interesting results.

Recall that in Subsection 2.2.1 we defined the affine Hecke algebra $\mathcal{A}_N(t)$, purely in terms of the Hecke algebra generators, T_i, and an operator σ defined as

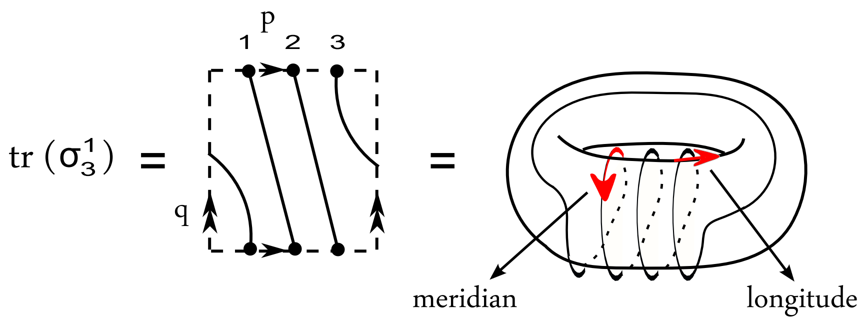
$$\sigma \;\; := \;\; T_{N-1}^{-1} T_{N-2}^{-1} \ldots T_1^{-1} Y_1. \tag{6.19}$$

We also derived that σ^N is central in $\mathcal{A}_N(t)$. Subsequently in Chapter 3 we constructed its pictorial representation and saw that it acts as a kind of raising operator on the indices. In the 3 strand case, $N = 3$, σ is described by



We now investigate the effect of the trace closure, as described in Section 6.1, on σ.

Graphically, tracing over the element σ is equivalent to identifying the top and bottom edges of the cylinder with each other. The trace closure of σ, denoted $\mathrm{tr}(\sigma)$, therefore represents a particular type of knot. The resulting knot is a torus knot T(p,q), where the strand wraps p times in the meridional direction and q times in the longitudinal direction of the torus.



Note that in the diagram above we depicted σ in the $N = 3$ strand case, where the subscript 3 denotes the number of strands and also p, the number of times, when opposite edges are identified, that the resulting strand wraps in the meridional direction of the torus.

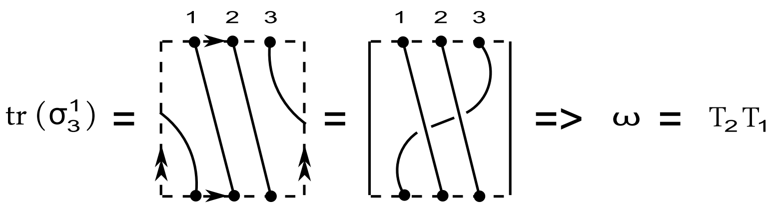
Thus we have just shown that tracing over σ_3^1, that is the element σ in the $N = 3$ case, generates the torus knot T(3,1). It is then clear that every torus knot can be written in terms of the trace closure of σ; increasing the number of strands gives the parameter p, while raising σ to different powers yields q:

$$\mathrm{tr}\left(\sigma_p^q\right) = \mathrm{T(p,q)}$$

As we have previously mentioned, every oriented link is isotopic to the closure of a braid in $\mathcal{B}_N$. Hence, associated to every torus knot is a *braid word* ω; for the torus knot T(p,q) the braid word given in terms of the Hecke algebra generators T_i is

$$\omega \;=\; (T_{p-1}\ldots T_1)^q.$$

So in the case of the trace closure of σ_3^1, we obtain the braid word $\omega = (T_{p-1}\ldots T_1)^q = (T_2T_1)^1$, as we have illustrated in the diagram below.



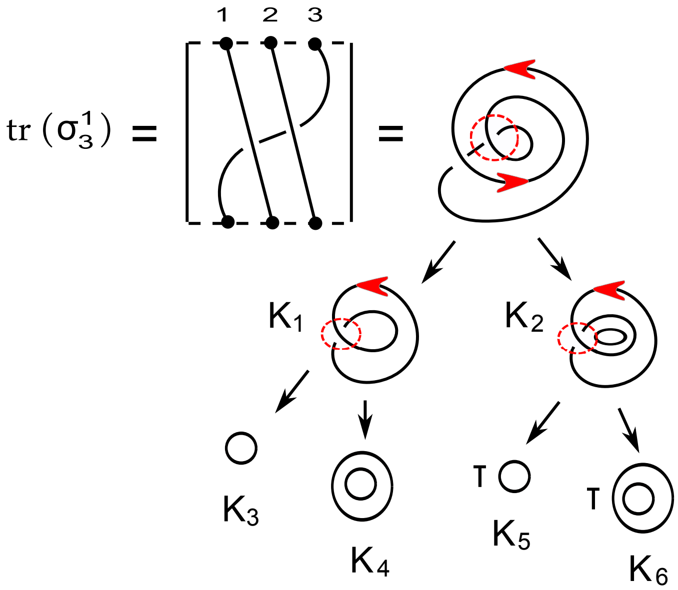
Having found that the tr(σ_3^1) is given by the closure of the braid T_2T_1, we can easily evaluate its HOMFLY polynomial as we described in Section 6.5. However before doing so we will firstly evaluate the Jones polynomial $V(L)$ of the trace closure of σ_3^1. We choose to do this as an explicit example to demonstrate Kauffman's intuitive approach to calculate the Jones polynomial of any link.

Following our description of Kauffman's proof in Section 6.7, we begin by finding the bracket polynomial of the tr(σ_3^1), before calculating its Kauffman polynomial $X(L)$ and then using the substitution $A = t^{-1/4}$ to obtain its associated Jones polynomial.

The Kauffman polynomial of the trace of σ_3^1 is given by

$$X(L) \;\;=\;\; (-A)^{-3w(L)}\langle L\rangle,$$

with $L = tr(\sigma_3^1)$. Firstly we find the bracket polynomial of the trace closure of σ_3^1. In the following diagram we provide the resolving tree for its evaluation. At each step we have encircled in red, the crossing chosen to be decomposed. We decompose every such crossing using the rules derived in Section 6.1. It is also worth mentioning that we have assigned an orientation to the knot to enable the calculation of its writhe.



From the resolving tree we find that the bracket polynomial of the $\mathrm{tr}(\sigma_3^1)$ is

$$\begin{aligned}
\langle \mathrm{tr}(\sigma_3^1) \rangle &= A\langle K_1 \rangle + A^{-1}\langle K_2 \rangle \\
&= A\left(A\langle K_3\rangle + A^{-1}\langle K_4\rangle\right) + A^{-1}\tau\left(A\langle K_5\rangle + A^{-1}\langle K_6\rangle\right) \\
&= A^2\langle o\rangle + \tau\langle o\rangle + \tau\langle o\rangle + A^{-2}\tau^2\langle o\rangle \\
&= A^{-6},
\end{aligned}$$

where we used the substitution $\tau = (-A^2 - A^{-2})$.

Now using the writhe of the $\mathrm{tr}(\sigma_3^1)$ we evaluate its Kauffman polynomial $X(L)$. With the given orientation there are two negative crossings implying that $w(L) = -2$ and hence:

$$X(\mathrm{tr}(\sigma_3^1)) \;=\; (-A)^{-3(-2)}(A^{-6}) \;=\; 1.$$

Since there are no As in the final expression, the Jones polynomial of the trace of σ_3^1 is

$$V(\mathrm{tr}(\sigma_3^1)) \;=\; 1.$$

But by (6.18), the Jones polynomial of the unknot has the value 1. Therefore σ_3^1 is a particular projection of the unknot. This somewhat surprising result is verified given the following expression for the Jones polynomial of a torus knot T(p,q):

$$V(t) \;=\; t^{(p-1)(q-1)/2}\left(\frac{1 - t^{p+1} - t^{q+1} + t^{p+q}}{1 - t^2}\right). \tag{6.20}$$

Clearly this equation is interchangeable for p and q as desired and given p=1, q arbitrary or q=1, p arbitrary we obtain the Jones polynomial of the unknot $V(t) = 1$. In the case of $\mathrm{tr}(\sigma_3^1)$, q=1 and hence it is completely anticipated that its Jones polynomial is that of the unknot.

Since the Jones polynomial of the trace closure of σ_3^1 is simply 1, then the HOMFLY polynomial of the $\mathrm{tr}(\sigma_3^1)$ must also necessarily be equal to 1. We will explicitly verify this.

To find its HOMFLY polynomial we evaluate its associated two variable invariant $X_L(k,\lambda)$, which by (6.15) is given by

$$X_L(k,\lambda) \;=\; \left(\frac{1-\lambda}{\sqrt{\lambda}k}\right)^{N-1}\left(\sqrt{\lambda}\right)^{e}\mathrm{tr}\left(f(\alpha)\right),$$

and then use the substitutions $t = i\ell^{-1} = \sqrt{\lambda}$ and $x = -im = k$. We already know that $f(\alpha) = T_2T_1 \in \mathcal{B}_3$, hence $N = 3$ and $e = 2$. Furthermore using the properties of the trace function on $\mathcal{H}_N(t)$ from Section 6.3, we find that

$$\mathrm{tr}(T_2T_1) \;=\; z\mathrm{tr}(T_1) \;=\; z^2.$$

Therefore, using the value for z, that is $z = k/(1-\lambda)$, the two variable invariant of the trace closure of σ_3^1 is

$$X_L(k,\lambda) \;=\; \left(\frac{1-\lambda}{\sqrt{\lambda}k}\right)^{2}\left(\sqrt{\lambda}\right)^{2}z^2 \;=\; \left(\frac{1-\lambda}{\sqrt{\lambda}k}\right)^{2}\left(\sqrt{\lambda}\right)^{2}\left(\frac{k}{1-\lambda}\right)^{2} \;=\; 1.$$

As a result, the HOMFLY polynomial of the $\mathrm{tr}(\sigma_3^1)$ is also equal to 1, as expected.

This concludes the final chapter of this thesis which described the importance of the Hecke algebra in the development of knot polynomials. In summary, we presented the discovery by Jones of a two variable invariant for oriented links obtained by studying representations of the braid group. The construction of this invariant was largely due to a specific trace, called the Markov trace, on the Hecke algebra. We showed that the resulting invariant is a two variable knot polynomial. Under reparameterisation this polynomial gave the HOMFLY polynomial, which we also described in terms of its skein relation. In addition to this we also gave a detailed account of Kauffman's construction of the Jones polynomial, which is a one variable specialisation of the HOMFLY polynomial.

Finally using the graphical representation of the operator σ, which we constructed in Chapter 3, we explicitly evaluated the Jones and HOMFLY polynomials of the trace closure of σ. We also highlighted that the fact that every torus knot can be written purely in terms of the trace closure of σ.

Appendix 6

6A.1 Oceanu's tr is unique

We provide a detailed description of Jones inductive proof which can be found in [25]. We prove that for every $z \in \mathbb{C}$, there is a linear trace function tr: $\bigcup_{N=1}^{\infty} \mathcal{H}_N(\mathfrak{q}) \longrightarrow \mathbb{C}$ uniquely defined by

1) $\mathrm{tr}(1) = 1,$

2) $\mathrm{tr}(ab) = \mathrm{tr}(ba),$

3) $\mathrm{tr}(xg_n) = z\mathrm{tr}(x)$ for $x \in \mathcal{H}_N(\mathfrak{q})$.

The uniqueness of this trace function is proved inductively using the $N!$ element basis given by

$$\left\{(g_{i_1}g_{i_1-1}\ldots g_{i_1-k_1})(g_{i_2}g_{i_2-1}\ldots g_{i_2-k_2})\ldots(g_{i_p}g_{i_p-1}\ldots g_{i_p-k_p})\right\}, \tag{6A.1}$$

where $1 \leq i_1 < i_2 < \ldots < i_p \leq N-1$ and $i_j - k_j \geq 1$.

It suffices to show that a trace on $\mathcal{H}_N(\mathfrak{q})$ can be uniquely extended to a trace on $\mathcal{H}_{N+1}(\mathfrak{q})$. The basic elements of $\mathcal{H}_{N+1}(\mathfrak{q})$ which do not belong to $\mathcal{H}_N(\mathfrak{q})$ are of the form xg_Ny with $x\,, y \in \mathcal{H}_N(\mathfrak{q})$. This follows from the fact that by (6A.1), we note that any word for $\mathcal{H}_{N+1}(\mathfrak{q})$ contains g_N at most once.

We must define the extension of the trace by:

$$\mathrm{tr}(xg_Ny) \;=\; z\mathrm{tr}(x) \text{ for } xg_Ny \in \frac{W_{N+1}(g_i)}{W_N(g_i)}\,,$$

where $W_{N+1}(g_i)$ is all the words of the g_i in $\mathcal{H}_{N+1}(\mathfrak{q})$, and $W_N(g_i)$ is all the words of the g_i in $\mathcal{H}_N(\mathfrak{q})$.

We have to show that the linear extension of this definition to $\mathcal{H}_{N+1}(\mathfrak{q})$ is in fact a trace. We are free to define a linear functional inductively from $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(xg_n) = z\mathrm{tr}(x)$ for $x\,, y \in \mathcal{H}_N(\mathfrak{q})$. Therefore we need to show the second property, namely that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$.

By induction we may suppose it for $a, b \in \mathcal{H}_N(\mathfrak{q})$. The only non trivial case, where all elements do not commute is

$$\operatorname{tr}(g_N x g_N y) \quad = \quad \operatorname{tr}(x g_N y g_N),$$

but since by (6A.1) any word for $\mathcal{H}_{N+1}(\mathfrak{q})$ contains g_N at most once, we need only consider the following four cases:

(A) When $x, y \in \mathcal{H}_{N-1}(\mathfrak{q})$,

(B) When $x = a g_{N-1} b$ with $a, b, y \in \mathcal{H}_{N-1}(\mathfrak{q})$,

(C) When $y = a g_{N-1} b$ with $a, b, x \in \mathcal{H}_{N-1}(\mathfrak{q})$,

(D) When $x = a g_{N-1} b$ and $y = c g_{N-1} d$ with $a, b, c, d \in \mathcal{H}_{N-1}(\mathfrak{q})$.

To verify all of the four cases we will use the following relations: the three properties of the trace function tr denoted (1), (2) and (3), the braid relation (6.4) and the Hecke relation (6.5).

(A) We begin by examining case (A). We have that $x, y \in \mathcal{H}_{N-1}(\mathfrak{q})$ and need to show that $\operatorname{tr}(g_N x g_N y) = \operatorname{tr}(x g_N y g_N)$.

This follows simply because since $x, y \in \mathcal{H}_{N-1}(\mathfrak{q})$ and $g_N \in \mathcal{H}_{N+1}(\mathfrak{q})$, then g_N commutes with both x and y. Therefore:

$$\begin{array}{ccc} g_N x = x g_N & \text{and} & g_N y = y g_N \\ \Rightarrow \operatorname{tr}(g_N x g_N y) & = & \operatorname{tr}(x g_N y g_N) \end{array}$$

(B) Now we look at case (B). We have that $x = a g_{N-1} b$ with $a, b, y \in \mathcal{H}_{N-1}(\mathfrak{q})$ and need to show that

$$\operatorname{tr}(g_N a g_{N-1} b g_N y) \quad = \quad \operatorname{tr}(a g_{N-1} b g_N y g_N). \tag{6A.2}$$

We look at the LHS of (6A.2) firstly and note that a and b commute with g_N; hence

$$\begin{aligned} \operatorname{tr}(g_N a g_{N-1} b g_N y) &= \operatorname{tr}(a g_N g_{N-1} g_N b y) \\ &= \operatorname{tr}(a g_{N-1} g_N g_{N-1} b y) && \text{by (6.4)} \\ &= z \operatorname{tr}(a g_{N-1} g_{N-1} b y) && \text{by (3)} \\ &= z \operatorname{tr}(a g_{N-1}^2 b y). \end{aligned}$$

Now we use the Hecke relation (6.5) to rewrite g_{N-1}^2 as $g_{N-1}^2 = (\mathfrak{q}-1)g_{N-1} + \mathfrak{q}$. Substitution means we now have that

$$\begin{aligned}\operatorname{tr}(g_N a g_{N-1} b g_N y) &= z\operatorname{tr}(a(\mathfrak{q}-1)g_{N-1}by) + z\operatorname{tr}(a\mathfrak{q}by)\\ &= (\mathfrak{q}-1)z\operatorname{tr}(ag_{N-1}by) + \mathfrak{q}z\operatorname{tr}(aby).\end{aligned}$$

We look at the RHS of (6A.2) and see that b and y commute with g_N; therefore

$$\begin{aligned}\operatorname{tr}(ag_{N-1}bg_Nyg_N) &= \operatorname{tr}(ag_{N-1}bg_N^2y)\\ &= \operatorname{tr}(ag_{N-1}b(\mathfrak{q}-1)g_Ny + ag_{N-1}b\mathfrak{q}y) && \text{by (6.5)}\\ &= (\mathfrak{q}-1)\operatorname{tr}(ag_{N-1}bg_Ny) + \mathfrak{q}\operatorname{tr}(ag_{N-1}by)\\ &= (\mathfrak{q}-1)z\operatorname{tr}(ag_{N-1}by) + \mathfrak{q}z\operatorname{tr}(aby) && \text{by (3).}\end{aligned}$$

Since the expressions we derived for the LHS and RHS are equal, then

$$\operatorname{tr}(g_Nag_{N-1}bg_Ny) = \operatorname{tr}(ag_{N-1}bg_Nyg_N). \tag{6A.3}$$

(C) Case (C) follows analogously from case (B) above since the roles of x and y are simply reversed.

(D) Lastly we show case (D). We have that $x = ag_{N-1}b$ and $y = cg_{N-1}d$ with $a, b, c, d \in \mathcal{H}_{N-1}(\mathfrak{q})$ and need to show that

$$\operatorname{tr}(g_Nag_{N-1}bg_Ncg_{N-1}d) = \operatorname{tr}(ag_{N-1}bg_Ncg_{N-1}dg_N). \tag{6A.4}$$

We look at the LHS of (6A.4) firstly and note that a and b commute with g_N; hence

$$\begin{aligned}\operatorname{tr}(g_Nag_{N-1}bg_Ncg_{N-1}d) &= \operatorname{tr}(ag_Ng_{N-1}g_Nbcg_{N-1}d)\\ &= z\operatorname{tr}(ag_{N-1}^2bcg_{N-1}d) && \text{by (6.4) and (3)}\\ &= (\mathfrak{q}-1)z\operatorname{tr}(ag_{N-1}bcg_{N-1}d) + \mathfrak{q}z\operatorname{tr}(abcg_{N-1}d) && \text{by (6.5)}\\ &= (\mathfrak{q}-1)z\operatorname{tr}(ag_{N-1}bcg_{N-1}d) + \mathfrak{q}z^2\operatorname{tr}(abcd) && \text{by (3).}\end{aligned}$$

We look at the RHS of (6A.4) and see that b and d commute with g_N; therefore

$$\begin{aligned}
\mathrm{tr}(ag_{N-1}bg_Ncg_{N-1}dg_N) &= \mathrm{tr}(ag_{N-1}bcg_Ng_{N-1}g_Nd) && \\
&= z\mathrm{tr}(ag_{N-1}bcg_{N-1}^2d) && \text{by (6.4) and (3)} \\
&= (\mathfrak{q}-1)z\mathrm{tr}(ag_{N-1}bcg_{N-1}d) + \mathfrak{q}z\mathrm{tr}(ag_{N-1}bcd) && \text{by (6.5)} \\
&= (\mathfrak{q}-1)z\mathrm{tr}(ag_{N-1}bcg_{N-1}d) + \mathfrak{q}z^2\mathrm{tr}(abcd) && \text{by (3).}
\end{aligned}$$

Clearly we now have that the LHS is equal to the RHS and hence

$$\mathrm{tr}(g_Nag_{N-1}bg_Ncg_{N-1}d) = \mathrm{tr}(ag_{N-1}bg_Ncg_{N-1}dg_N). \tag{6A.5}$$

As we have proved all four cases this completes the proof that the linear trace function tr is uniquely defined.

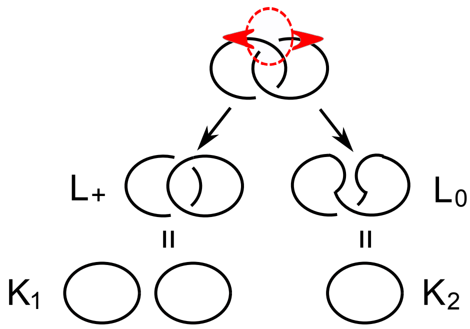
6A.2 Hopf Links

This appendix is included to highlight the subtleties involved in calculating the HOMFLY and Jones polynomials of Hopf links with various orientations. It is an important example because the Hopf link is a two component link, hence we must account for the orientation of each individual link. An important feature of the HOMFLY polynomial is that it gives the same polynomial for a knot with given orientation as for the same knot with the orientation reversed. This is not true for the Jones polynomial.

A knot is a one component link, therefore with Hopf links reversing the orientation of both links leaves its HOMFLY polynomial unchanged, whereas we obtain a different polynomial is we reverse the orientation of one of the links. The same holds true for the Jones polynomial. We demonstrate these subtleties in the following examples.

Case 1:

Consider the HOMFLY polynomial for the Hopf link where both links have opposite orientations. In the diagram below the first link has anticlockwise orientation and the second link has clockwise orientation. The crossing encircled in red is a negative L_- crossing, hence we find the HOMFLY polynomial for this Hopf link for L_-. Replace this crossing with an L_+ and with an L_0 crossing and evaluate the resulting knots.



From skein relation (6.16) and resolving tree above:

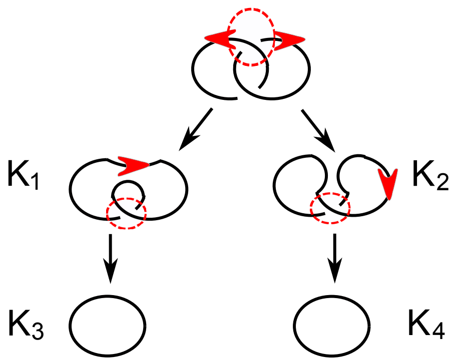
$$\begin{aligned} P(L_-) &= -m\ell P(L_0) - \ell^2 P(L_+) \\ &= -m\ell P(K_2) - \ell^2 P(K_1) \\ &= -m\ell(1) - \ell^2(-m^{-1}(\ell + \ell^{-1})) \\ &= -m\ell + m^{-1}\ell^3 + m^{-1}\ell, \end{aligned}$$

where we used that the HOMFLY polynomial of two disjoint unknots is given by $-m^{-1}(\ell + \ell^{-1})$.

Since the Jones polynomial $V(L)$ is obtained from the HOMFLY polynomial $P(L)$ by the substitution $(\ell, m) = (it^{-1}, i(t^{-1/2} - t^{1/2}))$, the Jones polynomial for the Hopf link where both links have opposite orientations is given by

$$\begin{aligned} P(L_-) &= -m\ell + m^{-1}\ell^3 + m^{-1}\ell \\ \Rightarrow V(t) &= \ell\left(-m + m^{-1}(1 + \ell^2)\right) \\ &= it^{-1}\left(-i(t^{-1/2} - t^{1/2}) + \frac{(1 - t^{-2})}{i(t^{-1/2} - t^{1/2})}\right) \\ &= t^{-1}\left(t^{-1/2} - t^{1/2} - \frac{(1 - t^{-1})(1 + t^{-1})}{t^{1/2}(1 - t^{-1})}\right) \\ &= -t^{-1/2} - t^{-5/2}. \end{aligned}$$

We verify this result in the following calculation.



The bracket polynomial is given by

$$\begin{aligned}\langle \text{Hopf} \rangle &= A\langle K_1\rangle + A^{-1}\langle K_2\rangle \\ &= -AA^3\langle K_3\rangle - A^{-1}A^{-3}\langle K_4\rangle \\ &= -A^4 - A^{-4}.\end{aligned}$$

This Hopf link contains two negative crossings, hence its writhe $w(L) = -2$ and its Kauffman polynomial $X(L)$ is

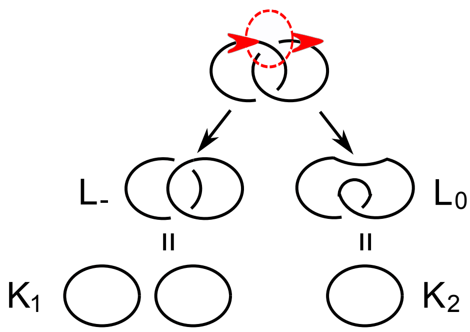
$$\begin{aligned}X(L) &= (-A)^{-3w(L)}\langle L\rangle \\ &= (-A)^{-3(-2)}\left(-A^4 - A^{-4}\right) \\ &= -A^{10} - A^2.\end{aligned}$$

Finally with the substitution $A = t^{-1/4}$ we obtain the Jones polynomial to be, as expected

$$V(t) = -t^{-1/2} - t^{-5/2}.$$

Case 2:

Now consider the HOMFLY polynomial of the same Hopf link with the orientation of the first link reversed. So both links are in the clockwise direction. This time the crossing circled in red is a positive L_+ crossing so the resulting polynomial will be for L_+.



From skein relation and resolving tree above:

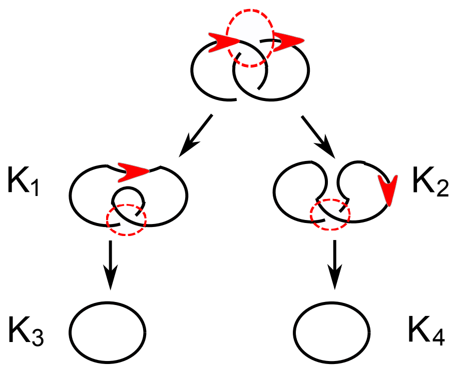
$$\begin{aligned} P(L_+) &= -m\ell^{-1}P(L_0) - \ell^{-2}P(L_-) \\ &= -m\ell^{-1}P(K_2) - \ell^{-2}P(K_1) \\ &= -m\ell^{-1}(1) - \ell^{-2}(-m^{-1}(\ell + \ell^{-1})) \\ &= -m\ell^{-1} + m^{-1}\ell^{-1} + m^{-1}\ell^{-3}. \end{aligned}$$

Comparing this polynomial with the one obtained in Case 1, it is clear that changing the orientation of one of the links results in a different polynomial, $\ell \longrightarrow \ell^{-1}$.

Via the substitution $(\ell, m) = (it^{-1}, i(t^{-1/2} - t^{1/2}))$, the Jones polynomial for this particular Hopf link is therefore

$$\begin{aligned} P(L_+) &= -m\ell^{-1} + m^{-1}\ell^{-1} + m^{-1}\ell^{-3} \\ \Rightarrow V(t) &= \ell^{-1}\left(-m + m^{-1}(1 + \ell^{-2})\right) \\ &= -it\left(-i(t^{-1/2} - t^{1/2}) + \frac{(1-t^2)}{i(t^{-1/2} - t^{1/2})}\right) \\ &= -t\left(t^{-1/2} - t^{1/2} + \frac{(1-t)(1+t)}{t^{-1/2}(1-t)}\right) \\ &= -t^{1/2} - t^{5/2}. \end{aligned}$$

Reversing the orientation of one of the links changes the Jones polynomial as $t \longrightarrow t^{-1}$, which we verify in the following calculation.



The bracket polynomial is given by

$$\begin{aligned} \langle \text{Hopf} \rangle &= A\langle K_1 \rangle + A^{-1}\langle K_2 \rangle \\ &= -AA^3\langle K_3 \rangle - A^{-1}A^{-3}\langle K_4 \rangle \\ &= -A^4 - A^{-4}. \end{aligned}$$

This Hopf link contains two positive crossings, hence its writhe $w(L) = 2$ and its Kauffman polynomial $X(L)$ is

$$\begin{aligned} X(L) &= (-A)^{-3w(L)}\langle L\rangle \\ &= (-A)^{-3(2)}\left(-A^4 - A^{-4}\right) \\ &= -A^{-2} - A^{-10}. \end{aligned}$$

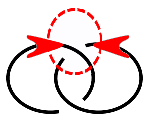
Finally with the substitution $A = t^{-1/4}$ we obtain the Jones polynomial to be, as expected

$$V(t) = -t^{1/2} - t^{5/2}.$$

The last two cases are merely different projections of the first two cases. However we have included them to complete all of the possible orientations of the Hopf link.

<u>Case 3:</u>

The third case involves calculating the HOMFLY and Jones polynomial of the Hopf link where the orientation of both links has been reversed. One can see that it is a different projection of the Hopf link in Case 1, obtained by turning it upside-down. However changing the orientation of both of the links does not affect these polynomials as we retain the same crossings. Consider the Hopf link pictured below:

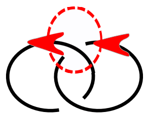


The first link is in the clockwise direction and the second link is in the anticlockwise direction. This is equivalent to reversing the orientation of both links from Case 1. Therefore we expect to obtain the same polynomial.
Clearly the encircled crossing is a negative L_- crossing, and one can calculate, as expected that:

$$\begin{aligned} P(L_-) &= -m\ell + m^{-1}\ell^3 + m^{-1}\ell, \\ V(t) &= -t^{-1/2} - t^{-5/2}. \end{aligned}$$

Case 4:

Finally for completeness we consider the Hopf link pictured below. We expect to get the same results as in Case 2, since, with respect to it, we have reversed the orientation of both links here. In fact it is just a different projection of the Hopf link in Case 2 obtained by turning it upside-down. In this case both links are oriented in the anticlockwise direction.



Examining the highlighted crossing tells us it is a positive L_+ crossing and hence as expected we find

$$\begin{aligned} P(L_+) &= -m\ell^{-1} + m^{-1}\ell^{-1} + m^{-1}\ell^{-3}, \\ V(t) &= -t^{1/2} - t^{5/2}. \end{aligned}$$

6A.3 Glossary

Bijection: A bijection is a one-to-one and onto correspondence giving an exact pairing between all elements of two sets.

Homomorphism: A homomorphism is a structure preserving map between two algebraic structures.

Homeomorphism: A homeomorphism is a continuous bijection from one topological space to another, with continuous inverse.

Isomorphism: An isomorphism is a bijective homomorphism. Therefore it is a structure preserving map with a one-to-one and onto correspondence between all elements of two sets.

Automorphism: An automorphism of an algebra is a mapping of the algebra onto itself which preserves all of its structure.

Antiisomorphism: An antiisomorphism θ between A and B is an isomorphism from A to the opposite of B and vice versa which must satisfy:

$$\theta(ab) \quad = \quad \theta(b)\theta(a).$$

Involution: An involution θ from $G \longrightarrow G$, is a mapping which satisfies the following relations:

$$\theta(g_1 g_2) \quad = \quad \theta(g_2)\theta(g_1), \qquad \text{with } \theta^2 \; = \; 1.$$

Normal Subgroup: A subgroup H of G is called a normal subgroup if for every $g \in G$ and $h_1 \in H$, there exists a $h_2 \in H$ such that

$$h_1 g \quad = \quad g h_2 \quad \text{ or } \quad g h_2 g^{-1} \; = \; h_1.$$

$SL(2,\mathbb{Z})$: Is the special linear group of 2×2 matrices with integer entries and determinant 1.

Modular Group $PSL(2,\mathbb{Z})$: Is the projective special linear group of 2×2 matrices with integer entries and determinant 1. It is a quotient of $SL(2,\mathbb{Z})$ by $\mathbb{Z}$ and can be shown to be generated by the two transformations $S : z \to z+1$ and $T : z \to -1/z$. S is a translation and T is inversion in the unit circle followed by reflection about $\mathrm{Re}(z) = 0$. The modular group Γ can be written as

$$\Gamma \cong \langle S, T \,|\, S^2 = \mathbb{1}, (ST)^3 = \mathbb{1} \rangle.$$

We point out that the above definition of the modular group is the most widely used one, however in this thesis we take the modular group to be generated by the two transformations S and U instead of S and T. This is to avoid confusion with the braid group generators denoted T_i. Therefore we say that

$$\Gamma \cong \langle S, U \,|\, S^2 = \mathbb{1}, (SU)^3 = \mathbb{1} \rangle.$$

Ambient Isotopy: An ambient isotopy between two subspaces X and Y of $\mathbb{R}^N$ is a continuous function $H : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ satisfying:

$$\begin{aligned} H(\cdot, 0) &= \textit{identity}\,, \\ H(X, 1) &= Y\,, \\ H(\cdot, t) &: \mathbb{R}^N \to \mathbb{R}^N \text{ is a homeomorphism for all } t \in [0,1]. \end{aligned}$$

If H exists, then X and Y are called ambient isotopic.

Knot: A knot is an embedding of a circle S^1 into Euclidean 3-space, $\mathbb{R}^3$, or the 3-sphere, S^3.

Unknot: The unknot is the trivial knot, that is, the unknotted circle.

Link: A link, also known as a knot of multiplicity $\mu > 1$, is an embedding of a disjoint union of 1-spheres S^1_i, $i \leq \mu \leq 1$, into S^3 or $\mathbb{R}^3$.

Chapter 7

Conclusions

In conclusion we reiterate the most notable points of this thesis and refer to possible further work.

This thesis centres around the investigation of Hecke-type algebraic structures. In Chapter 3 we construct such a structure called a double affine Q-dependent braid group $\mathcal{D}_N(t,q)$. The underlying structure of this particular group is the N-strand braid group, which is rendered "Q-dependent" by appending to it a set of N commuting operators $\{Q_i\}$. Using all of the knowledge acquired of affine and double affine Hecke algebras from Chapter 1, the Q-dependent braid group is extended firstly to an affine Q-dependent braid group and then to a double affine Q-dependent braid group. This is achieved by the addition of two extra sets of generators; the affine generators Y_i, and the double affine generators Z_i.

To complement the algebraic definition of the double affine Q-dependent braid group we put forward a graphical representation that fully describes its structure. We extend the pictorial representation of the braid group, whose generators braid strands in an infinitely long strip, to one of an affine braid group by defining braiding on the surface of a cylinder. Further extension to describe the action of the double affine braid group generators is obtained in the cube representation. In the cube representation, strands run from the top to the bottom face of the cube. As opposite faces are identified, an equivalent form of these cubes is in terms of toroids, which we illustrate in detail in Chapter 3.

We show that when strands are used in the cube representation, we cannot fully capture all off the structure of the double affine Q-dependent braid group. Instead we have a pictorial description of the elliptic braid group. However using ribbons, and not strands, enables the complete description of the structure of $\mathcal{D}_N(t,q)$. Of particular importance is the interpretation of the action of the Q_i operators; we can consistently show that the operator Q_i generates a full anticlockwise twist in the i^{th} ribbon.

Also of note is that when the operators Q_i are simply parameters, that is $Q_i = q\mathbb{1}$, and the double affine braid group generators satisfy the Hecke relation, then $\mathcal{D}_N(t,q)$ reduces to a double affine Hecke algebra. In terms of the cube-ribbon representation, we now have a graphical representation for any DAHA, for all values of the parameter q.

In Chapter 4, we presented the polynomial representation U of the double affine Hecke algebra and described its close connection to Macdonald polynomials. We also explicitly obtained Macdonald polynomials by simultaneously diagonalising matrices representing the action of affine Hecke algebra generators within U. As we know that any DAHA can be obtained from $\mathcal{D}_N(t,q)$, possible future work is to construct a polynomial representation for the double affine Q-dependent braid group. By restricting the action of the operators Q_i one should recover the polynomial representation of DAHAs. This is particularly interesting as it may lead to the discovery of a wide family of polynomials, of which Macdonald polynomials are given by setting the $Q_i = q\mathbb{1}$.

The tangle representation which we developed in Chapter 5 provides finite dimensional representations of the affine Hecke algebra $\mathcal{A}_N(t)$. As the $\mathcal{A}_N(t)$ generators can be simply mapped to the Temperley-Lieb generators, we also have finite dimensional matrix representations of the Temperley-Lieb algebra in terms of the elementary pattern basis. The pattern basis has many similarities to the path model representation of the braid group described in [8]. As such, using the tangle representation as an algebra of pictures to calculate the Kauffman bracket and hence the Jones polynomial of certain knots is certainly a direction worthy of research.

An even bigger challenge is the extension and generalisation of the tangle representation of $\mathcal{A}_N(t)$, to one valid for all double affine Hecke algebras. This exciting prospect would therefore result in finite dimensional representations of DAHAs. The foundations for such an extension are already in place as in the cube-ribbon representation we can successfully illustrate any DAHA generator. However, as stated at the end of Chapter 5, we must also find a means of decomposing non planar crossings into a linear combination of elementary patterns. A possible way of achieving this may be by assigning various weights to these crossings and then evaluating them using the defining relations of a DAHA.

Lastly, in Chapter 6 we gave our own interpretation of previous works by presenting a clear and detailed account of the central role of the Hecke algebra in the development of knot theory. Though readers may have been familiar with the majority of this work we included it due to its close connections with our tangle representation and also as a application of the algebraic description of Hecke algebras that we presented in Chapter 2. Many of the ideas in this chapter may be generalised given all of the new insights provided in this thesis. For example, of particular interest for future work is to investigate the Markov trace on the affine Hecke algebra. In the second chapter we detailed the construction of $\mathcal{A}_N(t)$ in terms of the operator σ, and the Hecke algebra generators T_i. Therefore we can write all words in $\mathcal{A}_N(t)$ purely in terms of σ and any one of the T_i. In addition to this, since we showed that the Jones and HOMFLY polynomial of the trace of σ is 1, then by the Markov property of the Markov trace there is an interesting prospect of defining a trace invariant on the affine Hecke algebras.

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Acknowledgments

I would firstly like to offer my gratitude to my advisor Jiří, who not only introduced me to the beautiful area of Hecke algebras, but also expertly guided and motivated me throughout my studies. I also thank Dr. Paul Watts who helped me immensely in every possible way and who always willingly made as much time as was needed to answer all of my queries. Without his super factoring abilities and seemingly endless knowledge of mathematics, I would not have been able to achieve all of the results presented in this thesis. I thank my collaborator Dr. Vincent Pasquier who shared with me much of his knowledge and experience and also provided me with a constant source of interesting ideas.

I appreciate all of the support of my colleagues in the Mathematical Physics department for their approachability and guidance throughout my PhD. In particular I thank professors Daniel Heffernan and Charles Nash. In addition I thank Ms. Monica Harte for all her help and all of my fellow office mates who were always on hand to share a problem.

I also offer my sincerest thanks to my family and friends who were a constant source of encouragement and who were always there when I needed them the most.

Finally, as an Irish Research Council (IRC) scholar I acknowledge the financial support of IRC who provided me with the opportunity for postgraduate research. The work in this thesis was also supported by Science Foundation Ireland through the grant 10/IN.1/I3013.