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Synthesis of Resonance by Nonlinear Distortion Methods

Abstract: This article explores techniques for synthesizing resonant sounds using the principle of nonlinear distortion. These methods can be grouped under the heading of "subtractive synthesis without filters," the case for which has been made in the literature. Starting with a simple resonator model, this article looks at how the source-modifier arrangement can be reconstructed as a heterodyne structure made of a sinusoidal carrier and a complex modulator. From this, we examine how the modulator signal can be created with nonlinear distortion methods, looking at the classic case of phase-aligned formant synthesis and then our own modified frequency-modulation technique. The article concludes with some application examples of this sound-synthesis principle.

The synthesis of resonant sounds is an area that has generally been studied from the perspective of subtractive synthesis techniques. Some of these studies have been carried out under the general heading of virtual analog modeling, in the analyses of typical filter and oscillator configurations (Stilson and Smith 1996; Lane et al. 1997; Lowenfels 2003; Välimäki and Huovilainen 2006; Lazzarini and Timoney 2010). Beyond these, it is well established that the resonance characteristics of instruments (including the voice) and environments are a defining aspect of their acoustic properties (Backus 1977; Howard and Angus 2009). Methods of synthesizing resonant sounds thus clearly have a wide application in music.

In most of the existing studies, a typical method for the generation of resonance has been to use spectrum modifiers of a complex waveform. Alternatively, resonance regions can be created directly by a variety of techniques. The case for methods of resonant synthesis without the use of source-modifier arrangements has been made in the literature (Lazzaro and Wawrzynek 2004) and in the introduction of new techniques, such as formant wave synthesis (*fonction d'onde formantique*, FOF) (Rodet 1984) and VOSIM (for "voice simulation") (Kaegi and Tempelaars 1978).

In addition to these, there are a variety of nonlinear distortion techniques (Lazzarini 2009) that can be used to recreate these sonic results. By carefully controlling the amount of distortion applied to a sinusoidal signal, resonant regions can be created and shaped to provide a variety of sonorities.

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Some of these techniques, such as simple frequency-modulation (FM) synthesis (Chowning 1973) and polynomial transfer-function waveshaping (Arfib 1978; LeBrun 1979) are, however, not very well suited to the creation of realistic resonant sounds. They will generally produce unnatural changes in the spectrum in response to changes in the amount of distortion applied. As a consequence, these will be quite poor in modeling the effects of a spectrum modifier.

Other techniques, such as various forms of summation formulae (Windham and Steiglitz 1970; Moorer 1976), modified FM (Timoney, Lazzarini, and Lysaght 2008; Lazzarini and Timoney 2010), and other means of waveshaping (Lazzarini and Timoney 2010), will allow a more natural handling of spectral evolutions. Recent work has indeed formally confirmed this hypothesis (Macret, Pasquier, and Smyth 2012). In this article, we further explore these possibilities and discuss some useful application scenarios.

In the following sections, we will examine a series of techniques of resonant synthesis by means of distortion methods. They offer some interesting alternatives to the typical oscillator-filter setups, dispensing with the need for infinite impulse response filters, which those setups generally use. In particular, some of these techniques can provide accurate control of signal phases, which is problematic with infinite impulse response designs, as discussed by Puckette (1995). They also allow for efficient implementations that are less costly than comparable methods. In the proposed synthesis algorithms, a periodic wideband source is assumed—namely, an impulse train. The source-filter arrangement is then decomposed into a sequence of impulse responses, which are generated at the fundamental frequency rate.

Simple Resonator Model

A very simple model of resonance is given by an all-pole second-order resonator defined as:

$$y(n) = x(n) + 2R\cos(\omega)y(n-1) - R^2y(n-2) \quad (1)$$

where R is the pole radius $(0 \leq R < 1)$, $\omega = 2\pi f/f_s$ is the pole angle, f *is* the frequency in Hz, and f_s is the sampling rate. For sharp resonances, we can make the assumption that the resonance center frequency f_c is very close to the pole frequency f (Steiglitz 1996). Given that we are interested in this particular case, we will assume the approximation:

$$\omega_c \approx \omega \quad (2)$$

where $\omega_c = 2\pi f_c/f_s$. The impulse response of this filter can be written as

$$s(n) = R^n \frac{\sin(\omega_c(n+1))}{\sin(\omega_c)} \quad (3)$$

which models a damped sinusoid. Recasting Equation 3, leaving out the linear phase shift by ω_c yields a simple heterodyne synthesis model (Lazzarini and Timoney 2012):

$$s(n) = M(n)\sin(\omega_c n) \quad (4)$$

where we have a sinusoidal carrier at the center frequency being amplitude-modulated by $M(n)$, which can be used to describe a complex periodic modulator, phase-synchronous to the sinusoid. In the following sections, we will show how this model can be used in distortion synthesis configurations for the synthesis of resonant sounds.

Nonlinear Distortion Modulators

A flexible method for dynamically controlling the modulator $M(n)$ waveshape (and consequently its spectrum) is provided by nonlinear distortion techniques. In this section, we outline two options for this purpose, one represented by phase-aligned formant (PAF) synthesis (Puckette 1995) and another based on modified FM (ModFM) synthesis (Timoney, Lazzarini, and Lysaght 2008; Lazzarini and Timoney 2010).

Phase-Aligned Formant Synthesis

In his original formulation of PAF, Puckette describes it as a distortion method and compares it to various other ways of producing a complex spectrum synthetically. He highlights the fact that, unlike other techniques, his has the advantages of providing an easily described spectrum, including predictable phases, that can be synthesized efficiently. Starting with a desired additive-synthesis spectrum, Puckette decomposes it into a carrier-modulator arrangement, similar to the one in Equation 4 ($\omega_c = 2\pi f_c$ and f_c as the frequency at the center of the resonance):

$$s(n) = M(n)\cos(\omega_c n) \quad (5)$$

where $M(n)$ is defined as (with $\omega_o = 2\pi f_0/f_s$):

$$M(n) = \sum_{k=-\infty}^{\infty} (ge^{i\omega_0 n})^{|k|} = \frac{1-g^2}{1+g^2-2g\cos(\omega_0 n)} \quad (6)$$

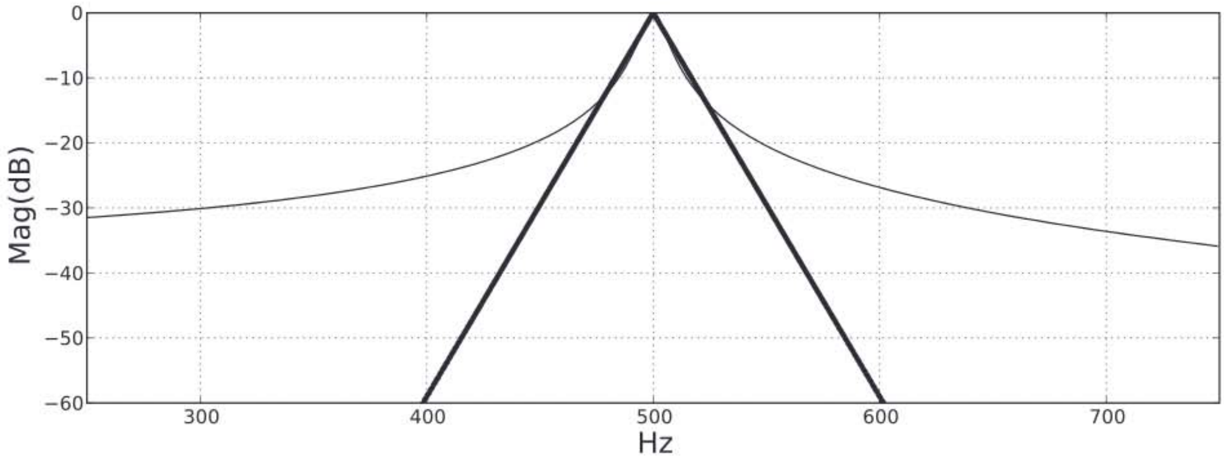
where $i = (-1)^{0.5}$, k is a sinusoid component index, and $g^{|k|}$ is an amplitude of each component. In his algorithm, the right-hand side of Equation 5 is implemented using waveshaping, with a cosine input, in a manner similar to LeBrun's version (LeBrun 1979) of Moorer's summation formula (Moorer 1976). In fact, it is easy to demonstrate that PAF is basically one of Moorer's double-sideband synthesis formulae. In its implementation, however, the synthesis of PAF is defined as:

$$M(n)\cos(\omega_c n) = \frac{1+g}{1-g} \times f\left(\frac{2\sqrt{g}}{1-g}\sin\left(\frac{\omega_0 n}{2}\right)\right)\cos(\omega_c n) \quad (7)$$

with

$$f(x) = \frac{1}{1+x^2} \quad (8)$$

Figure 1. Comparison between PAF (thick line) and band-pass resonator (thin line) magnitude spectra, with $f_c = 500$ Hz and $B = 10$ Hz.



The efficiency of the method is related to the fact that it only requires two oscillators and a function-table mapping for $f(x)$. Dynamic spectra are possible by varying the g factor in Equation 7. This parameter is effectively a distortion index, which is related to the bandwidth of the signal. Puckette defines it as:

$$g = e^{-\frac{f_0}{B}} \tag{9}$$

with B as the resonance bandwidth in Hz. If we want to measure the bandwidth at the usual –3 dB (half-power) points, however, we need to adjust that formula, as it actually calculates a –4.35 dB bandwidth ($20 \log_{10} e^{-0.5}$). The correction factor that should be applied to f_0/B is $\ln(2)$:

$$g = e^{-\frac{\ln(2)}{B}} = 2^{-\frac{f_0}{B}} \tag{10}$$

The resulting spectral envelope produced by this method is triangular in shape (if dB magnitudes are used). Its slope is therefore a constant $3/B$ dB per Hz–e.g., if the bandwidth B is 10 Hz, then the spectral decay on each side of the center frequency will be 0.3 dB per Hz. We can contrast this behavior with that of the second-order resonator (Equation 1), whose slope beyond the passband is closer to a constant-Q –12 dB/octave, rather than constant bandwidth (see Figure 1).

Modified FM Synthesis

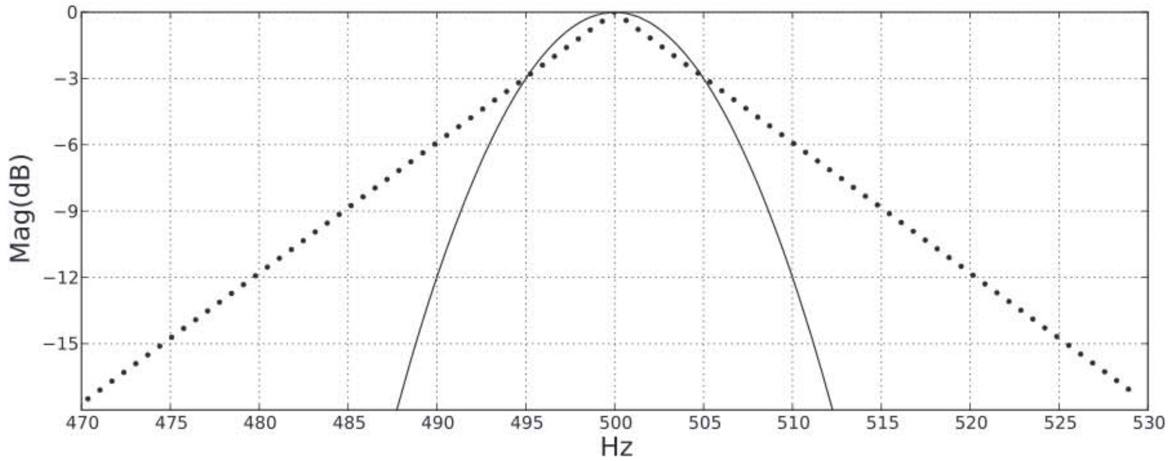
As an alternative to PAF, we propose here using the ModFM synthesis algorithm to model a source-filter arrangement. This technique has already been demonstrated to be useful in the design of virtual analog oscillators for classic waveshapes (Timoney, Lazzarini, and Lysaght 2008). One of its properties is that it allows for easy control of signal bandwidth, which can be used to reproduce filter effects.

The ModFM method is a variation on classic FM synthesis that implements a summation formula based on modified Bessel functions, rather than the ordinary Bessel functions found in "classic" FM.

$$\Re\{e^{i\theta + k\cos(\omega)}\} = e^{k\cos(\omega)}\cos(\theta) = I_o(k)\cos(\theta) + \sum_{n=1}^{\infty} I_n(k)\left(\cos(\theta - n\omega) + \cos(\theta + n\omega)\right) \tag{11}$$

with $I_n(k) = i^{-n} J_n(ik)$, the modified Bessel function of order n (a special case of that function for purely imaginary arguments) (Watson 1944). ModFM can be obtained from a classic FM equation by effecting the change of variable $z = -ik$. To complete the synthesis formula, we scale the output by e^{-k}.

Figure 2. Comparison between PAF (dots) and ModFM (line) magnitude spectra, with $f_c = 500$ Hz and $B = 10$ Hz.



The equation for the basic ModFM resonant synthesis can be written as

$$s(n) = e^{(k\cos(\omega_m n)-k)}\cos(\omega_c n) \tag{12}$$

with the carrier and modulation frequencies f_c and f_m, respectively, and $\omega_m = 2\pi f_m/f_s$ and $\omega_c = 2\pi f_c/f_s$.

Scaled modified Bessel functions are unipolar and decaying. In addition, and most importantly for filter modeling, we have $I_{n+1}(k) < I_n(k)$. In other words, they always produce a decaying spectral envelope. Notice that Equation 12 is effectively another version of Equation 4, which forms the basis of our model for the synthesis of resonance. Therefore, if we tune our modulator to the fundamental frequency f_0 and the carrier to the resonance frequency f_c, we will have yet another way of recreating a source-modifier arrangement, with different spectral characteristics. A key parameter in ModFM is represented by k in Equation 12, which controls the amount of modulation in the output signal. This is the distortion index in the present algorithm that will control the bandwidth of the signal, according to Equation 10.

It is particularly useful to draw up some connections between ModFM and PAF. It is possible to see that the two modulators share similarities in terms of their waveshaping functions. The PAF waveshaper approximates ModFM for low values of the distortion index. If we consider that

$$\frac{1}{1+x^2} \approx e^{-x^2} \quad \text{for } x \ll 1 \tag{13}$$

then we can compare the two modulator signals:

$$\frac{1}{1+\frac{4g}{(1-g)^2}\sin^2\left(\frac{\omega}{2}\right)} \approx \exp\left(\frac{2g}{(1-g)^2}\left[\cos(\omega)-1\right]\right) \tag{14}$$

and k, the distortion index, can be approximated by:

$$k \approx \frac{2g}{(1-g)^2} \tag{15}$$

The advantage of this is that we can then get closer to finding a relationship between the bandwidth and distortion index in ModFM. By observing Equation 16 and heuristically adapting it, we can obtain an expression that will allow us to do so. The approximation error in the previous expressions amounts to the –3 dB bandwidth's being off by about a factor of 0.295:

$$k = \frac{2g}{(1-g)^2}, \quad \text{with } g = 2^{-\frac{f_0}{0.295B}} \tag{16}$$

Although the time-domain shapes of the ModFM and PAF modulation functions are closely related (they are both Gaussian-like), they will lead to very different spectral envelopes. ModFM exhibits a much steeper rolloff in the passband (below –3 dB), but a less steep slope in the stopband (0 to –3 dB) (see Figure 2). The effect on the synthetic signals is quite significant, both techniques producing very distinct sonorities.

Sweeping the Frequency

Although it is possible to sweep the center frequency of resonance in ModFM, leading to non-integral f_c: f_0 ratios, a certain amount of aliasing-induced distortion will inevitably occur. In order to allow for a better result, we apply a method similar to the one proposed for PAF. Instead of a single carrier, we use two, tuned to adjacent harmonics of f_0, and we interpolate their output, depending on the position of the desired spectral peak. The revised synthesis algorithm is shown here (for positively defined f_c and f_0 and $\omega_0 = 2\pi f_0$):

$$s(t) = e^{(k\cos(\omega_0 t)-k)}\,[(1-a)\cos(n\omega_0 t) + a\cos([n+1]\omega_0 t)] \quad (17)$$

where

$$n = \mathrm{int}\left(\frac{f_c}{f_0}\right) \quad (18)$$

and

$$a = \frac{f_c}{f_0} - n \quad (19)$$

This allows us to dynamically vary the resonance frequency f_c with no additional cost to the synthesis signal quality.

Variations on the ModFM Method

The PAF and ModFM resonant synthesis methods are both designed to realize band-pass spectral shapes. It is possible, however, to reproduce other amplitude response curves that are commonly found in different filter designs. Particularly interesting is the modeling of resonant low-pass spectral envelopes. The resulting spectrum of ModFM is purely cosine-phase, unlike the resonator model described earlier in the section "Simple Resonator Model." Therefore, it is possible to add an extra modulator signal to cover the lower region of the stopband with no difficult phase-interference issues. This second signal will be centered at 0 Hz (i.e., it will not need to be heterodyned), and its distortion index can be controlled so that the signal bandwidth extends from 0 Hz to f_c. Therefore, this algorithm can be expressed as follows:

$$s(t) = Ae^{(k\cos(\omega_0 t)-k)} \times [(1-a)\cos(n\omega_0 t) + a\cos([n+1]\omega_0 t)] + (1-A)e^{(l\cos(\omega_0 t)-l)} \quad (20)$$

where $0 \leq A \leq 1$ and

$$l = \frac{2h}{(1-h)^2}, \quad \text{with } h = 2^{-\frac{f_0}{0.59 f_c}} \quad (21)$$

The amount of resonance can be controlled in two ways: (1) by modifying the parameter A, increasing it for a more prominent peak; and (2) by decreasing the bandwidth of the resonance region. In fact, it is also possible to control both bandwidths with a single Q value:

$$B_{lp} = mB_{res} = m\frac{f_c}{Q_{res}} \quad (22)$$

In this case, m is the ratio between the two values of B, for the low-pass and resonant signal components. By varying m, it is possible to modify not only the peak of resonance but also the slope of the filter beyond the resonant frequency. The case of Equation 21 is a special one, where $m = 2 f_c B_{res}^{-1}$; for a given A, values of m above that will increase the resonance effect. This is the case plotted in Figure 3.

Conversely, high-pass responses may be obtained by centering the second modulator signal at the Nyquist frequency. This can be done by multiplying it by a cosine wave at half the sampling rate (or, effectively, alternating the sign of each sample in the signal). This design does not prove to be very practical, however, as it will have to use very large bandwidths in the typical applications where the resonance frequency is in the lower side of the spectrum. This will have the effect of reducing the signal to a pulse with a very short duration, which can lead to significant aliasing distortion.

In addition to low- and high-pass designs, it is possible to recreate a band-reject response by carefully subtracting two or more resonant ModFM signals. For example, the expression of Equation 20

Figure 3. ModFM low-pass resonant spectrum, with $B_{res} = 200$ Hz, $f_c = 1$ kHz, and $A = 0.1$.

Figure 4. Band-reject ModFM magnitude plot, with $B = 200$ Hz, $f_c = 500$ Hz, and $A = 0.2$.

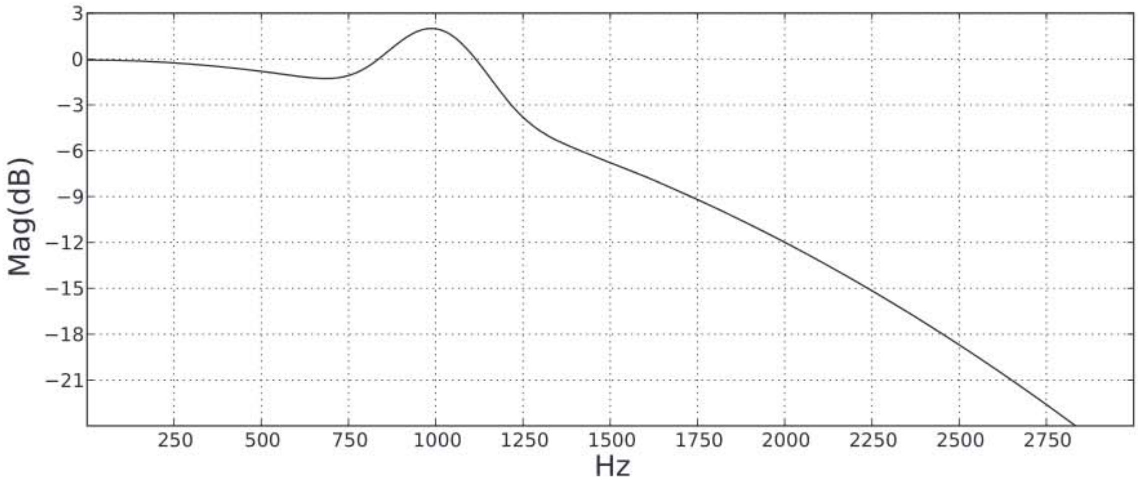


Figure 3

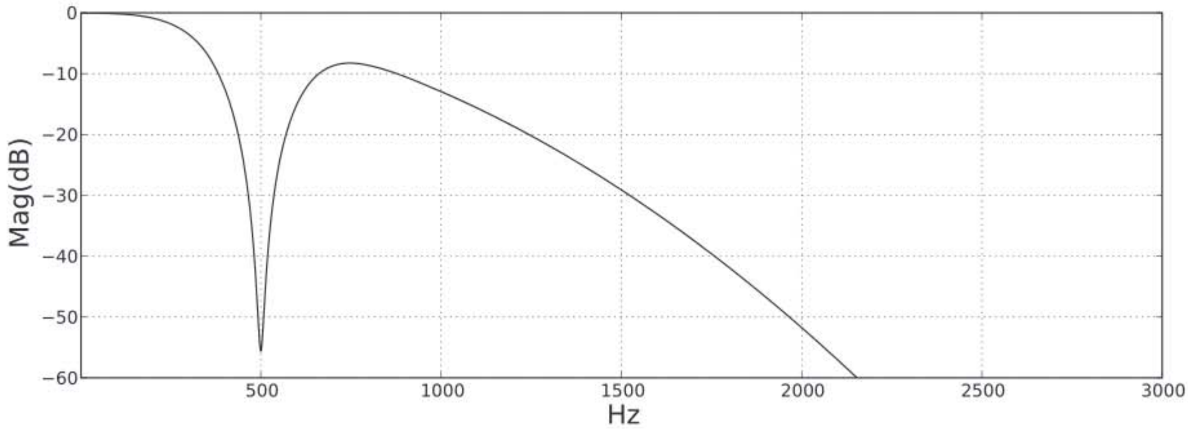


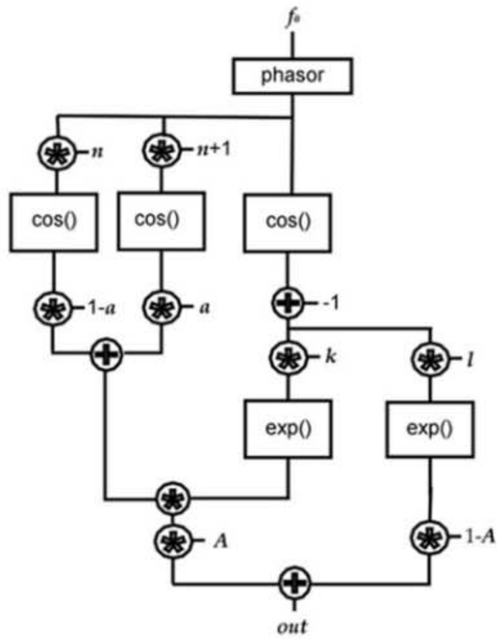
Figure 4

can be made into a subtraction of signals, generating the magnitude response seen in Figure 4. It is necessary to control the amplitudes of each modulator element judiciously, however, to be able to create good notches in the spectrum, which can be swept to generate a "phaser" effect. In addition, the notch frequency will most likely need a small correction factor, dependent on the bandwidth and the amplitudes of each modulator. (In the case of Figure 4, a factor of 0.96 was applied.) A method for this might be inferred from examining the modified-Bessel function expansion of this synthesis method, but the possibilities raised here have yet to be explored.

ModFM Implementation

The ModFM design presented here includes one section to realize the band-pass resonance and another to provide the low-pass signal, which are summed together to produce the output signal. It

Figure 5. Block diagram of resonant ModFM synthesis.



uses a single phase generator or modulo counter (the commonly used "phasor" unit generator of MUSIC *N* programming). This will be used to synchronize the phases of the double carrier (cosine) and the modulator. The latter, on its turn, is then waveshaped by two individual exponential transfer functions. The low-pass/band-pass mix is controlled by a single variable *A*, in the 0–1 range. The sinusoids and waveshaping functions are implemented using table lookup. A block diagram for resonant ModFM synthesis is shown in Figure 5.

The flowcharts indicate that the described methods can be efficiently computed. To demonstrate this, a test has been carried out to compare the processor times of a standard FOF generator with those of the ModFM synthesis. For this test, a 60-sec tone of similar characteristics (same fundamental, resonance center frequency, and bandwidth) was synthesized using Csound 5 on a 2.8-GHz Intel Core Duo processor. The average timing of five performances of each method were: FOF, 0.375 secs; and ModFM, 0.107 secs. These results indicated that ModFM is almost four times less expensive than FOF. We can conclude that the methods proposed here are sufficiently efficient and low-cost.

Some Applications

The techniques described in this article have a number of obvious uses. In particular, the emulation of resonant filters for synthesizer sounds can be described as a typical usage. In this section we would like to briefly examine two other application scenarios: the synthesis of vocal sounds and the specific case of ModFM as a component in an analysis-synthesis system.

Voice Synthesis

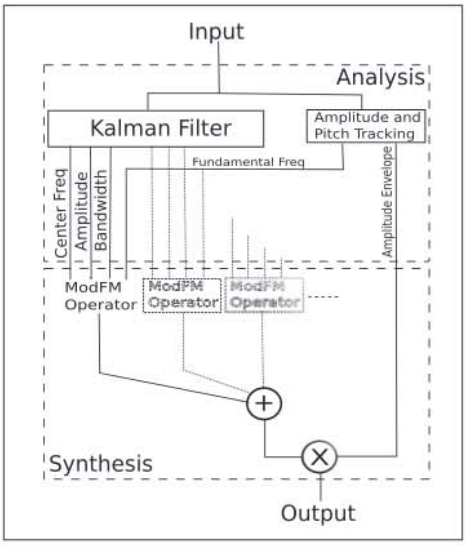
The emulation of vocal formants has been explored in many synthesis methods in the literature. Among these, we find subtractive synthesis (Gold and Rabiner 1968; Klatt 1980); already-discussed direct-synthesis algorithms such as VOSIM, FOF, and PAF; and frequency-domain methods, such as spectral modeling synthesis (Serra 1997) and the phase vocoder (Flanagan and Golden 1966). Given the similarities of the resonator model and ModFM to FOF and PAF, respectively, it is possible to propose that these techniques can be used in vocal (vowel) synthesis applications.

By considering the flowchart of Figure 5 as a description of a "formant operator," borrowing the description used in some FM synthesis implementations, we can design instruments for vocal synthesis. These would have parametric controls for formant center frequencies and bandwidths. Each operator would model a formant peak, and its output could then be given the correct amplitude scaling for a given vowel model. The ModFM approach has the advantage that it guarantees all signal components to be in cosine phase, so the combination of several operators would be absolutely additive.

Analysis-Synthesis with ModFM

The predictable phases of ModFM components also allow it to be used as an element of a general-purpose formant analysis-synthesis system. This can be designed with the use of some means of extracting

Figure 6. *An analysis-synthesis system based on Kalman filters and ModFM formant synthesis (adapted from Lazzarini and Timoney 2009).*



parametric information from arbitrary input signals. As an example of this, we have used an approach (Lazzarini and Timoney 2009) based on an unscented Kalman filter (Wan and Van de Merwe 2000) to provide the analysis section, which is fed into a multiple-operator ModFM synthesizer (see Figure 6). Formant parameters (frequency, bandwidth, and relative amplitude) are obtained for N spectral peaks, and these are suitably resynthesized by the ModFM resonant method. Implementation details for this application are given in the cited article.

This approach, although optimized for sounds with strong formant characteristics, can actually be used as a general-purpose analysis-synthesis design. As the number of formants searched for (N) gets larger, the method starts to resemble the tracking of partials followed by additive synthesis. In addition, parametric manipulation can be applied to provide various transformations on the original analysis data, such as time and frequency scaling, formant metamorphosis, cross-synthesis, and spectral morphing.

In fact, other analysis methods can potentially be used to generate parameters for ModFM. For instance, we have put forward a new design (Lazzarini and Timoney 2011) of the traditional channel vocoder, using a ModFM-operator synthesis method. Similarly, spectral matching algorithms, such as the one proposed by Horner (1998) for FM synthesis, would produce suitable parameterizations for this synthesis method.

Conclusions

The combined methods of heterodyning and distortion synthesis can offer very interesting alternatives for the implementation of source-modifier synthesis without the use of filters. We have shown here a number of principles that offer an alternative for sound designers wanting to reproduce and manipulate the effect of resonance. In particular, these methods have been shown to be low-cost in computation. They can be realized efficiently both in software and in hardware. Another advantage, particularly in the case of the distortion-based techniques, is the precise control of output signal phases, which can be problematic in filtering applications. This makes possible the addition of signals without problems of frequency-dependent phase interference in general-purpose synthesis applications. In summary, the methods discussed in this article form a useful set of techniques for the design of digital synthesis instruments.

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