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Phase transitions in topological lattice models via topological symmetry breaking

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Abstract. We study transitions between phases of matter with topological order. By studying these transitions in exactly solvable lattice models we show how universality classes may be identified and critical properties described. As a familiar example to elucidate our results concretely, we describe in detail a transition between a fully gapped achiral 2D p-wave superconductor ($p + ip$ for pseudo-spin up/$p - ip$ for pseudo-spin down) to an s-wave superconductor. We show in particular that this transition is of the 2D transverse field Ising universality class.

Motivated in part by growing interest in topological quantum computation, a considerable effort has been invested in understanding systems which realize topologically ordered phases of matter. Though much is understood about the topological properties of these phases and their possible applications to quantum computing [1–3], little is presently known about transitions between proximate phases of different topological order. While a powerful formalism [4], known as 'topological symmetry breaking' (TSB), has been worked out to identify when pairs of phases can in principle be connected by a continuous transition, the critical properties of these transitions have not been systematically studied. Recently increasing interest has been focused on such phase transitions, highlighting both their importance to understanding physical systems

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such as quantum Hall bilayers [5, 6], and their potential as theoretical models displaying critical behavior unlike that of more conventional statistical mechanical systems [7].

One context in which such phase transitions are relatively well-understood is in Ising gauge theories, or Kitaev's toric code (TC) model [2]. Here there are two possible second-order transitions: a confining transition in the gauge theory, which is of the 3D Ising type [8, 9] (or equivalently 2D transverse field Ising), and a 2D Ising transitions obtained by deforming the model's 'loop gas' ground state [10–12]. In the TC model, first-order transitions are also known to occur in the presence of a magnetic field [13].

In this paper, we take a step toward building a systematic understanding of the relationship between TSB transitions in systems with non-Abelian anyons and second-order phase transitions involving a broken symmetry. Focusing on confining-type transitions, we build on the work of Fradkin and Shenker [8], who showed that the confining transition in an Ising gauge theory is dual to the transition in a 2D transverse field Ising model (TFIM). Here we show that this duality mapping can be generalized to phase transitions between pairs of phases with anyonic excitations, and much more complicated topological orders. Specifically, we study such phase transitions in the lattice models of topological matter introduced by Levin and Wen [14]. We add a perturbation to the Hamiltonian which condenses a bosonic field, and show that at low energies the resultant phase transition is dual to that of a TFIM. The lattice models we construct also give interesting proof-of-principle examples of the TSB scenario [4], in which we can track explicitly the fate of the topological order after condensation. For simplicity, we will focus on the example of the transition between a phase with non-Abelian topological order ($SU(2)_2 \times \overline{SU(2)}_2$), and the Ising gauge theory, though our analysis applies to a family of phases with more exotic topological orders described by $SU(2)_k \times \overline{SU(2)}_k$. In [15], we extend this analysis to families of transitions in the Potts class.

We emphasize that though the critical theory we find in our models is the same as that of the TC, the relationship between the excitations in the two gapped phases is considerably more complex. In addition to confining some excitations, condensation also engenders 'splitting' (the appearance of new particles after condensation). In this sense our results are very different from those obtained from previous studies on lattice models.

While the lattice models we study are best viewed as toy models, their long-wavelength behavior should be that of analogous real physical systems. Consequently we will center our discussion around the example probably most familiar from the literature, where the topological order is that of a chiral p-wave superconductor [16]. Specifically, the Levin–Wen model with $SU(2)_2 \times \overline{SU(2)}_2$ topological order can be viewed as a toy model for the long-wavelength properties of a bi-layer net achiral p-wave superconductor, in which the order parameter is $p + \mathrm{i}p$ for pseudo-spin up (top layer), and $p - \mathrm{i}p$ for pseudo-spin down (bottom layer). The Ising phase transition we describe in this case is to a phase which is topologically an s-wave superconductor.

Let us first describe briefly the exactly solvable Levin–Wen Hamiltonians for the honeycomb lattice [14]. For the $SU(2)_2 \times \overline{SU(2)}_2$ topological order, the states in the lattice model are given by assigning a label $i \in \{1, \sigma, \psi\}$ to each edge. Two types of operators act on these states: string operators $\hat{S}_r(C)$, which raise and lower all edge labels along the trajectory of the path C, and vertex projectors B_V, which project onto specific 'allowed' combinations of labels entering a vertex. In both cases, their action is determined by the fusion rules of $SU(2)_2$ Chern–Simons theory ($\psi \times \psi = 1$, $\sigma \times \psi = \sigma$, and $\sigma \times \sigma = 1 + \psi$). B_V projects onto states in which the three labels entering the vertex V appear in one of the following 'allowed' combinations:

$$B_V|{}^{i}\!\curlywedge^{k}_{j}\rangle = \begin{cases} 1 & (1,1,1) \quad (1,\psi,\psi) \quad (\sigma,\sigma,\psi) \quad (\sigma,\sigma,1) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

which are chosen such that the three labels incident at each vertex can fuse to give the identity in $SU(2)_2$. The string operators act on all edges along some path C in the lattice. Their action is described by

$$\hat{S}_s(C)|\Psi\rangle = \prod_{k\in C} \alpha_{i_k,i_{k+1},e_k} \hat{s}_s(k)|\Psi\rangle. \tag{2}$$

Here the coefficients of proportionality α_{i_k,i_{k+1},e_k} depend on all three edge labels at each vertex along c. Their precise form is dictated by properties of the $SU(2)_2$ theory, as described in [14], but will not be crucial for our discussion. The operator $\hat{s}_s(k)$ acts on the edge k according to:

$$\begin{aligned} &\hat{s}_\sigma|1\rangle \sim |\sigma\rangle, \quad \hat{s}_\sigma|\psi\rangle \sim |\sigma\rangle, \quad \hat{s}_\sigma|\sigma\rangle \sim |1\rangle + |\psi\rangle, \\ &\hat{s}_\psi|1\rangle \sim |\psi\rangle, \quad \hat{s}_\psi|\sigma\rangle \sim |\sigma\rangle, \quad \hat{s}_\psi|\psi\rangle \sim |1\rangle. \end{aligned} \tag{3}$$

The Hamiltonian we study is

$$H = -\epsilon_V \sum_V B_V - \frac{J_0}{2}\sum_P \left(B_P^{(0)} + B_P^{(\phi)}\right) - \frac{J_z}{2}\sum_P \left(B_P^{(0)} - B_P^{(\phi)}\right) - J_x \sum_e (-1)^{n_\sigma(e)}, \tag{4}$$

where $n_\sigma(e)$ is 1 if edge e carries label σ and is 0 otherwise, and

$$\begin{aligned} B_P^{(0)} &= \frac{1}{2}\left(1 + \sqrt{2}\hat{S}_\sigma(\hexagon) + \hat{S}_\psi(\hexagon)\right) \\ B_P^{(\phi)} &= \frac{1}{2}\left(1 - \sqrt{2}\hat{S}_\sigma(\hexagon) + \hat{S}_\psi(\hexagon)\right) \end{aligned} \tag{5}$$

are projectors composed of string operators acting simultaneously on all of the edges bordering the plaquette P. Physically, these project onto states with no flux through the plaquette P, and ϕ flux through the plaquette P, respectively.

The solvable model of [14] is obtained by taking $J_z = J_0$, $J_x = 0$. Since B_V and $B_P^{(0)}$ commute [14], the spectrum at this point—which determines the topological order of the uncondensed phase—can be obtained exactly. Excitations consist of eight types of quasi-particles $\sigma_{L,R}$, $\psi_{L,R}$, $\sigma_L\sigma_R$, $\sigma_L\psi_R$, $\psi_L\sigma_R$ and $\psi_L\psi_R$. The subscripts R and L denote the chiralities of the particles. In the lattice model, R particles violate only the vertex projectors, L particles violate both the vertex and plaquette projectors, and their achiral combinations are pure plaquette violations.

We may understand this spectrum by considering the corresponding excitations in the superconducting system. The $p+\mathrm{i}p/p-\mathrm{i}p$ superconductor should be thought of as two independent layers with opposite (L, R) chiralities. $\sigma_{R,L}$ corresponds to the superconducting vortex (together with its associated 0-energy Majorana fermion bound state) in the top (R) or bottom (L) layer; $\psi_{R,L}$ are fermionic quasi-particles in the two layers, and the other six excitations are composites of fermions and/or vortices in both layers. The mutual statistics of particles in the same layer are those of the chiral p-wave superconductor whereas excitations in different layers have trivial mutual statistics. The ψ particles have $\mathbb{Z}_2$ symmetry, meaning they fuse with themselves to give 1. In the superconducting picture, this reflects the fact that fermions are conserved only mod 2.

The particle $\phi \equiv \psi_L \psi_R$ is a boson and can therefore in principle condense in a second-order TSB-type phase transition. In terms of the superconducting bilayer, ϕ represents a Cooper pair in the pseudo-spin (inter-layer) singlet channel. Here we study this transition, showing that as ϕ is a $\mathbb{Z}_2$ field, the transition is in the TFIM class.

To condense ϕ in the model (4), we simultaneously decrease J_z and increase J_x. In the lattice description, ϕ excitations live entirely on plaquettes, as they do not violate the vertex condition imposed by B_V [17]. A plaquette with (without) a ϕ excitation has eigenvalues 0 and 1 (1 and 0) under $B_P^{(0)}$ and $B_P^{(\phi)}$, respectively. Thus J_z sets the mass of the ϕ excitation, which varies from J_0 at $J_z = J_0$ to 0 at $J_z = 0$. The operator $J_x(-1)^{n_\sigma(e)}$—which assigns an energy penalty $2J_x$ if the edge e carries the label σ—creates ϕ particles on the pair of plaquettes bordering e (or hops an existing ϕ from one plaquette to another).

To describe the resulting phase transition, we exploit the fact that all operators in (4) commute with B_V at every vertex, and with $B_P^{(0)} + B_P^{(\phi)}$ on each plaquette. We can therefore study the phase transition in the reduced Hilbert space consisting only of the ground state plus some number of ϕ particles created. Since ϕ is a $\mathbb{Z}_2$ field, we describe the states in the low-energy sector of the lattice model by an Ising variable $n_\phi \equiv \frac{1}{2}(S_z + 1) = 0, 1$ on each plaquette (together with a label α identifying the ground state sector if the system is on the torus). The vertex projectors, as well as the plaquette term $\frac{J_0}{2}(B_P^{(0)} + B_P^{(\phi)})$, act as a multiple of the identity on this space. Within this low-energy sector, the plaquette term $\frac{J_z}{2}(B_P^{(0)} - B_P^{(\phi)})$ acts like the spin operator $J_z S_z$, diagonal in the n_ϕ basis, while the ϕ pair-creation operator $J_x(-1)^{n_\sigma(e)}$ acts like $-J_x \sum_{\langle ij \rangle} S_x^{(i)} S_x^{(j)}$ as it flips pairs of spins on neighboring plaquettes. One can check that the operators in the second line of equations (4) satisfy the appropriate commutation relations.

In the reduced Hilbert space, the effective theory for the phase transition is exactly the TFIM, with spins on sites of the dual lattice. The paramagnetic phase, in which J_z dominates, corresponds to the initial $\mathrm{SU}(2)_2 \times \overline{\mathrm{SU}(2)}_2$ topological phase. The ferromagnetic phase, in which J_x dominates, contains an indefinite number of ϕ particles in each plaquette, and corresponds to the condensed phase. (Recall that ϕ is dual to an Ising spin oriented along the transverse field). The critical theory on the boundary between these two phases is described by the critical point of the 2D TFIM.

At this juncture, the reader may wonder how it is that we have mapped a transition which changes the topological order onto one which breaks a global symmetry. Since topological order is associated with long-ranged statistical interactions (or generalized Berry phases) which can only occur in the presence of gauge fields, this is indeed a surprising result. One way to understand it is to note that ϕ is essentially the vortex of a $\mathbb{Z}_2$ gauge theory; the special form of the Hamiltonian we use here ensures that no objects behaving like electric sources of this gauge field are present in the ground state anywhere in the phase diagram. It is because of this that we may exploit the duality between the TFIM and Ising gauge theory [18] to study the phase transition. For a more generic choice of Hamiltonian this mapping will no longer hold, as the appropriate dual theory would again be an Ising gauge theory. However, it is known that weak matter sources do not affect criticality in the Ising gauge theory [13]—hence we expect that the transition will still be described by an Ising critical point away from the solvable limit considered here.

Having identified a mechanism for condensing the particle ϕ, we now describe the topological order of the condensed phase. We will see that the condensed phase of the lattice model can be mapped exactly onto Kitaev's TC [2], which is known to be the same topological order as that of an s-wave superconductor [19].

To understand the topological order of the condensed phase, it is helpful to consider the limit $J_z = 0$, in which the lattice Hamiltonian is again exactly solvable. Here the plaquette term takes on a particularly simple form:

$$\frac{J_0}{2}\left(B_P^{(0)} + B_P^{(\varphi)}\right) = J_0\left(1 + \hat{S}_\psi(\hexagon)\right) \tag{6}$$

which leaves n_σ on each edge unchanged. When $J_x > 0$, this implies that the exact ground state contains no edges labeled σ, as each such edge has an energy cost $2J_x$. In this limit, the label σ is confined and we drop it from the low-energy theory. This leaves a solvable lattice model containing only the two edge labels 1 and ψ, with the Hamiltonian of the TC [2] on the honeycomb lattice.

Our next task is to understand how the eight excitations of the initial model are related to the TC's 3. To identify the spectrum of the condensed phase, we must account for three effects [4]. First, ψ_L and ψ_R are mixed by fusion with the condensate, and become indistinguishable. Such identification of particles is a generic feature of condensation transitions. In the language of the superconductor, once the inter-layer order parameter $\langle c_\uparrow c_\downarrow \rangle$ becomes non-zero, fermion number is not conserved mod 2 in each layer independently, but only in the overall system as a whole (i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is broken to a single $\mathbb{Z}_2$). Second, the label σ becomes confined, effectively eliminating the four quasi-particles $\sigma_L, \sigma_R, \psi_L\sigma_R$ and $\sigma_L\psi_R$ from the spectrum. In the superconducting picture, this is because a vortex in a single layer engenders an energetically costly 'branch cut' in the inter-layer s-wave condensate. Finally, TSB predicts that the bound state $\sigma_L\sigma_R$ splits into two distinct types of excitation in the condensed phase [4]. We will show that in the lattice model this splitting can be understood as a result of a new conservation law.

The energetics of confinement are easily understood in terms of the Ising model representation. The operator S^z, which measures the flux of ϕ through a plaquette P, is simply

$$\frac{1}{2}\left(B_P^{(0)} - B_P^{(\varphi)}\right) = \sqrt{2}\hat{S}_\sigma(\hexagon) \tag{7}$$

i.e. the σ-labeled raising operator applied to all edges surrounding this plaquette. In the absence of vertex violations, the σ label always forms closed loops on the lattice, which constitute domain walls in the S^x basis, since we identify $(-1)^{n_\sigma(ij)} \equiv S^x(i)S^x(j)$. In the ferromagnetic phase of the Ising model, these domain walls are confined. The energy cost of a string of edges labeled σ is thus linear in the string length, and the label σ disappears from the long-wavelength theory.

Alternatively, we may exploit the duality between the TFIM and Ising gauge theory, given by

$$S^z_P = \pm 1 \to \prod_{i \text{ borders } P} \sigma^z_i = \pm 1, \quad S^x_i S^x_j = \pm 1 \to \sigma^z_{\tilde{i}}, \tag{8}$$

where $\tilde{i}$ is the edge between the two plaquettes i and j. This maps the domain walls of the Ising model—given in our lattice model by edges where $n_\sigma = 1$—onto the electric field $\sigma_z = -1$ of the Ising gauge theory. The symmetry-breaking transition in the TFIM is thus a confining transition in the gauge theory [8, 9]. This allows us to conclude that open σ strings (which join pairs of $\sigma_{L,R}$ vortex excitations) are also confined. Readers might object that this identification of σ strings with the Ising electric flux is not exact, as $\sigma_{L,R}$ are not bosons. This disagreement

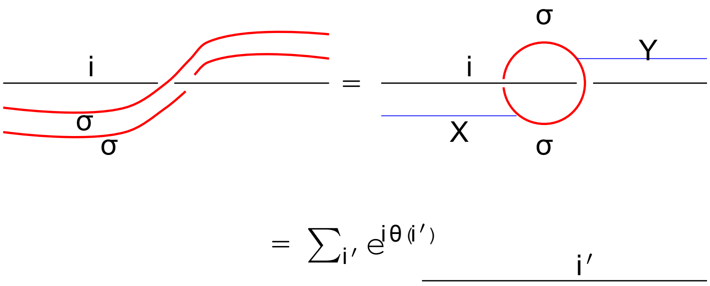

Figure 1. The action of the operator $\hat{S}_{\sigma\sigma}$ which creates the quasi-particle $\sigma_L\sigma_R$ which splits in the condensed phase. The product $\hat{S}_{\sigma_L}\hat{S}_{\sigma_R}$ can be resolved in terms of strings $X, Y = 1, \psi$ which act only on the edge labels, and a phase (depicted diagrammatically by the ring labeled σ) each time the string crosses an edge in the lattice. (On edges without crossings, the phase must be 1). The action for the different possible values of X, Y, i, i' is given in table 1. In the uncondensed phase $X, Y = 1$ or ψ independently on each edge; after condensation $X = Y$ and there are two distinct excitations $X = Y = \psi$ and $X = Y = 1$.

Table 1. The action of $\hat{S}_{\sigma\sigma}$ on an edge label i has two components: the action of a string $\hat{S}_X$ which maps i to i', and a phase $e^{i\theta}$. (Combinations not shown give 0.) X and Y may be distinct only if the edge i crossed by the string operator carries the label σ. After condensation σ is confined, and the value of X is conserved along the string.

X	Y	i	i′	θ
1	1	1	1	0
ψ	ψ	1	ψ	0

X	Y	i	i′	θ
1	1	ψ	ψ	π
ψ	ψ	ψ	1	0

X	Y	i	i′	θ
ψ	1	σ	σ	−π/4
1	ψ	σ	σ	π/4

in statistics is, however, immaterial to the energetics of confinement, for which we require only static sources.

To trace the origins of the splitting, we first must understand the form of the operators which create $\sigma_L\sigma_R$. In the $p + ip/p - ip$ superconductor, this is a pair of vortices, one in each layer, which are bound together. This excitation may appear in one of two states: the total fermion number of the pair of vortices may be 1 or ψ. However, this fermion number will change if a single $\sigma_{L,R}$ vortex in one of the layers braids with this bound pair, and is not a topologically protected quantity. Hence it is conserved in the phase with s-wave pairing, where $\sigma_{L,R}$ vortices are confined, but not in the pure p-wave phase.

In the lattice model, the same effect is seen in the quasi-particle creation operators. The operator $\hat{S}_{\sigma_R}\hat{S}_{\sigma_L}$ which creates $\sigma_L\sigma_R$ particles acts with a combination of a phase factor at each edge crossed by the string operator, and by acting with $\hat{S}_{1,\psi} \in \{\hat{S}_1, \hat{S}_\psi\}$ on each edge that s runs along. (This is depicted diagrammatically in figure 1). The label $1, \psi$ carried by this string corresponds to the net fermion number of the vortex pair discussed in the previous paragraph. Table 1 gives the action of the components of $\hat{S}_{\sigma\sigma}$. Here again we find that $\hat{S}_{\sigma\sigma}$ occurs in two distinct flavors, $\hat{S}^1_{\sigma\sigma}$ and $\hat{S}^\psi_{\sigma\sigma}$, distinguished by whether or not they interchange the edge labels 1 and ψ. Before condensation only the symmetric combination of these comprises a topological excitation (other combinations have an energy cost which grows with string length). After confining the σ strings, however, we find two distinct quasi-particles, $\sigma\sigma_1$ and $\sigma\sigma_\psi$. This can occur because in the absence of edges labeled σ, an operator cannot raise the edge label on one edge by ψ unless it either also raises the edge label by ψ on an adjacent edge, or creates a vertex violation. Hence the two operators $\hat{S}^1_{\sigma\sigma}$ and $\hat{S}^\psi_{\sigma\sigma}$ now create physically distinct quasi-particle excitations, one of which violates vertices and plaquettes, and the other of which violates only plaquettes.

To summarize, in the condensed phase we find three quasi-particles $\psi, \sigma\sigma_1$, and $\sigma\sigma_\psi$. These correspond exactly to the three excitations in the TC: the two bosons $\sigma\sigma_1$ and $\sigma\sigma_\psi$ are the vortex m and electric source e respectively; the fermion ψ is the combination em. The phases inherited from the original $\sigma_L\sigma_R$ creation operators ensure that braiding any two distinguishable particles around each other incurs a phase of -1.

To clarify the fate of the topological order across the phase boundary, we consider the ground state degeneracy on the torus. In the $\mathrm{SU}(2)_2 \times \overline{\mathrm{SU}(2)}_2$ phase there are 9 ground states $|\Omega_\alpha\rangle$, which can be identified by the flux α through one of the non-contractible curves on the torus. These fall into two classes, distinguished by the operators $\hat{L}_i = \prod_{e \text{ on } c_i} (-1)^{n_\sigma(e)}$, where c_i, $i = 1, 2$ run along edges in the dual lattice around the two non-contractible curves on the torus. Four of the ground states have at least one $L_i = -1$. Because σ can only appear in closed loops, this means that these states must have a string of edges labeled σ which run across the width of the system in one of the two directions; in the confined phase, where such a string incurs an energetic cost linear in the system size, they are no longer ground states. Of the remaining five, the two anti-symmetric combinations $(|\Omega_{\psi_L}\rangle - |\Omega_{\psi_R}\rangle$ and $(|\Omega_{\psi_L\psi_R}\rangle - |\Omega_1\rangle)$ vanish in the condensed ground state. Hence these also become gapped and split off from the ground-state sector, leaving only the two symmetric combinations $(|\Omega_{\psi_L}\rangle + |\Omega_{\psi_R}\rangle$ and $(|\Omega_{\psi_L\psi_R}\rangle + |\Omega_1\rangle)$. Finally, as ψ appears only in closed loops after condensation, there are two copies of the state $|\Omega_{\sigma_L\sigma_R}\rangle$ distinguished by the eigenvalue of $\prod_{e \text{ on } c_1} (-1)^{n_\psi(e)}$. This gives four ground states on the torus, as expected for the TC.

The transition studied so far, between $\mathrm{SU}(2)_2 \times \overline{\mathrm{SU}(2)}_2$ and the TC, is prototypical of a class of transitions between pairs of topological phases, all of which exhibit transverse-field Ising criticality. We give a detailed description of these other transitions (together with related transitions where an achiral $\mathbb{Z}_q$ boson condenses to give a q-state transverse field Potts transition) in a related work [15]; here we merely outline their general features. Consider a Levin–Wen Hamiltonian describing the topological theory $\mathrm{SU}(2)_k \times \overline{\mathrm{SU}(2)}_k$, which has quasi-particles of spin $0, \frac{1}{2}, \ldots, \frac{k}{2}$ in two mutually non-interacting sectors R and L. We condense the boson $\phi_k \equiv \frac{k}{2}_L \frac{k}{2}_R$ by decreasing the gap J_z to creating these plaquette violations, and increasing the coefficient J_x of the pair- creation term $\sum_e (-1)^{2s(e)}$, where $s(e)$ is the spin of the label on edge e. As ϕ_k is always a $\mathbb{Z}_2$ field which commutes with the vertex projectors, the dynamics of the phase transition is described by the TFIM. All half-odd integer spin edge labels map to the domain wall; hence particles of net half-odd integer spin (of the form $\frac{2j}{2}_L \frac{2i+1}{2}_R$, or vice versa) are confined in the condensed phase. As before, pairs of particles $(i_L j_R, \frac{k-2i}{2}_L \frac{k-2j}{2}_R)$ which mix by fusion with $\frac{k}{2}_L \frac{k}{2}_R$ are no longer distinct in the condensed phase. This leaves $\frac{k^2}{4} + \frac{k}{2} + 1$ potential quasi-particles.

As in the $k = 2$ case, it remains to ask whether any of these quasi-particles will split. If k is odd, there is no splitting. This leads to a theory whose topological structure is

$SO(3)_k \times \overline{SO(3)}_k$ – meaning that the only effect of condensation is to eliminate half-odd integer spin excitations from the theory. The final theory is simple in this case because the initial particle spectrum is of the form $\{a_1, a_2, \ldots, a_{r/2}; \phi \times a_1, \phi \times a_2, \ldots, \phi \times a_{r/2}\}$, and condensation merely confines particles of net half-odd integer spin, and identifies pairwise excitations $\frac{2j+1}{2}_L \frac{2i+1}{2}_R \equiv \frac{k-2j-1}{2}_L \frac{k-2j-1}{2}_R$. If k is even, the criteria of [4] dictate that the $\frac{k}{4}_L \frac{k}{4}_R$ particle splits into two components. In the lattice model, these are distinguished by whether the associated edge string operator carries even or odd integer spin. In this case, the final theory is not a doubled theory with decoupled right- and left- handed sectors, but a more involved type of achiral theory known as a Drinfeld double [20].

Here we have described a phase transition between phases exhibiting the topological orders of a $p+\mathrm{i}p/p-\mathrm{i}p$ superconductor, and that of an s-wave superconductor (or equivalently, the TC [2, 19]). By realizing the initial phase in a solvable Levin–Wen [14] type lattice model, where the phase transition can be induced by a simple deformation of the Hamiltonian, we show that the phase transition is that of the 2D TFIM[6]. This critical theory is well-understood, and is known to be only part of a wider range of possible phase transitions separating the TC from the vacuum [21, 22]. We have identified the 2D TFIM critical theory as the description of a whole class of condensation transitions, in phases including those of $SU(2)_k \times \overline{SU(2)}_k$ topological order, identifying these as the TSB analogue of transitions which break global Ising symmetry. The TC to vacuum and $SU(2)_2 \times \overline{SU(2)}_2$ to TC transitions are the simplest Abelian and simplest non-Abelian examples in this class.

As mentioned in the introduction, in the TC model it is now understood that the scenario studied by Fradkin and Shenker, which we generalized here to certain anyon theories, is one of two possible types of second-order phase transitions. This leaves the interesting open question of what the analogues of the 2D Ising critical point and the first-order transition in an external field would be in more general anyonic systems.

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