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Simultaneous Diophantine approximation in two metrics and the distance between conjugate algebraic numbers in $\mathbb{C} \times \mathbb{Q}_p$

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Abstract

A lower bound for the number of integer polynomials which simultaneously have "close" complex roots and "close" p-adic roots is obtained.

Keywords: Diophantine approximation; Root separation

In this paper, information is obtained regarding the number of integer polynomials of degree at least three which have close conjugate roots in the complex and p-adic fields simultaneously. Before we proceed, some notation is needed. Throughout the paper, P is an integer polynomial, so $P \in \mathbb{Z}[x]$ where

$$P(f) = a_n f^n + \cdots + a_1 f + a_0,$$

and has degree $\deg P \leq n$ and height $H(P) = \max_{0 \leq j \leq n} |a_j|$. We assume from now on that $n \geq 3$. In general, a complex root of P will be denoted by $\alpha = \alpha(P)$ and a p-adic root of P will be denoted by $\gamma = \gamma(P)$.

Let $\mu_1(A_1)$ be the Lebesgue measure of a measurable set $A_1 \subset \mathbb{R}$, and $\mu_2(A_2)$ the Haar measure of a measurable set $A_2 \subset \mathbb{Q}_p$. Using these definitions, define the product measure μ

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on $\mathbb{R} \times \mathbb{Q}_p$ by setting $\mu(A) = \mu_1(A_1)\mu_2(A_2)$ for a set $A = A_1 \times A_2$. Throughout the paper, $\#S$ stands for the cardinality of a set S. By $\ll$ ($\gg$), we will mean the Vinogradov symbols with implicit constants depending only on n. If the phrase "Q sufficiently large" is used then this size will also depend only on n.

Let α_1 and α_2 be complex roots of an irreducible polynomial P. Define λ_n to be the infimum of real numbers w for which the inequality

$$|\alpha_1 - \alpha_2| \gg H(P)^{-w}$$

holds for $H(P)$ sufficiently large and $\deg P \le n$. For any distinct roots α_1, α_2 of $P \in \mathbb{Z}[x]$, of degree $\deg P = n$, $n \ge 2$, it is well known that the inequality

$$|\alpha_1 - \alpha_2| \gg H(P)^{-n+1}$$

holds (see [8] for details), i.e. $\lambda_n \le n - 1$.

The question of how sharp this inequality is remains open. It was proved by Evertse [7] that $\lambda_3 = 2$ and it is not difficult to show that $\lambda_2 = 1$. In [6], a special type of polynomial P is constructed which demonstrates that $\lambda_n \ge \frac{n+2}{4}$ for odd $n \ge 5$ and $\lambda_n \ge \frac{n}{2}$ for even $n \ge 4$. Recently, Bugeaud and Dujella [5] have improved all known lower bounds for λ_n when they showed that $\lambda_n \ge \frac{n}{2} + \frac{n-2}{4(n-1)}$ for $n \ge 4$. The polynomials constructed in these papers are exotic and nothing is known of their quantity. An alternative approach was taken in [1] where it was proved that $\lambda_n \ge (n+1)/3$ for all n. Let $\mathbf{P}_n(Q) = \{P \in \mathbb{Z}[f] : \deg P \le n, H(P) \le Q\}$. In that paper, the authors obtained an estimate for the number of polynomials with at least two "close" real roots. More precisely in their Corollary 2 they show that there exist at least $Q^{\frac{n+1}{3}}$ polynomials $P \in \mathbf{P}_n(Q)$ for which at least two of the roots of P satisfy the inequality

$$|\alpha_1 - \alpha_2| \ll Q^{-\frac{n+1}{3}}.$$

When we use the Vinogradov symbol we suppress the dependence of the constant on n which we consider to be fixed. In the present paper, it will similarly be demonstrated that there exist a large number of polynomials which have both close real and close p-adic roots. Usually the subset $\mathcal{P}_n(Q) \subset \mathbf{P}_n(Q)$ will be considered where $\mathcal{P}_n(Q)$ is the set of irreducible $P \in \mathbf{P}_n(Q)$ such that

$$|a_n|_p \gg 1, \qquad |a_n| \ge H(P)/2, \quad \gcd(a_0, \dots, a_n) = 1. \tag{1}$$

(The 2 is not essential but used for convenience, any positive constant greater than 1 will do.)

Theorem 1. *Fix v_1 with $0 \le v_1 < 1/3$ and let $Q_0(n) \in \mathbb{R}$ be a large constant. Let $N_n(Q)$ be the number of polynomials $P \in \mathcal{P}_n(Q)$ which have at least two roots $\alpha_1, \alpha_2 \in \mathbb{C}$ and at least two roots $\gamma_1, \gamma_2 \in \mathbb{Q}_p^*$ satisfying*

$$|\alpha_1 - \alpha_2| \ll Q^{-v_1}, \qquad |\gamma_1 - \gamma_2|_p \ll Q^{-v_1}.$$

Then, for all $Q > Q_0$,

$$N_n(Q) \gg Q^{n+1-4v_1}.$$

(Here $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers.)

This question is closely related to the question of how many polynomials have "small" discriminant. Such problems were considered by Bernik et al. in the real case [2] and the p-adic case [3]. In [4], these results were combined using similar methods to [2,3] to obtain a lower bound on the number of polynomials with small discriminant in the real and p-adic metrics simultaneously. Some of the methods used in [1–3] will be used to prove Theorem 3 below which lies at the heart of Theorem 1 and is of interest in its own right. It is a substantial improvement of Theorem 1.2 of [4], formulated here as Theorem 2, which did not contain the restrictions on the second derivative. The latter part of the proof of Theorem 3 (for the *inessential domains*) is very similar to that in [4] and therefore will not be done in full. The proof of Theorem 1 will be done after the proof of Theorem 3.

Fix a set $I \times K$ where I is an interval contained in $[0, 1) \subset \mathbb{R}$ and K is a cylinder contained in $\mathbb{Z}_p$. From now on, v_0 and v_1 will be fixed real numbers such that

$$0 \le v_1 < 1/3 \quad \text{and} \quad v_0 + v_1 = n/2. \tag{2}$$

For real numbers c_0, δ_0, Q two sets are defined. First, $\mathcal{L}_n(v_0, v_1, c_0, \delta_0, Q)$ is the set of points $(x, w) \in I \times K$ such that the system of inequalities

$$|P(x)| < c_0 Q^{-v_0}, \qquad |P(w)|_p < c_0 Q^{-v_0}, \tag{3}$$

$$\delta_0 Q^{1-v_1} < |P'(x)| < c_0 Q^{1-v_1}, \qquad \delta_0 Q^{-v_1} < |P'(w)|_p < c_0 Q^{-v_1}, \tag{4}$$

holds for some $P \in \mathcal{P}_n(Q)$. Similarly, define the set $\mathcal{K}_n(v_0, v_1, c_0, \delta_0, Q)$ to be the set of points $(x, w) \in I \times K$ which satisfy (3) and (4) together with

$$|P''(x)| > \delta_0 Q, \qquad |P''(w)|_p > \delta_0 \tag{5}$$

for some $P \in \mathcal{P}_n(Q)$. Also, define the set $\mathcal{P}_n^{\mathcal{K}}(Q)$ of polynomials $P \in \mathcal{P}_n(Q)$ for which (3)–(5) are satisfied for some $(x, w) \in I \times K$. It will be shown that if $P \in \mathcal{P}_n^{\mathcal{K}}(Q)$ then two of the roots of P are "close"; thus only a lower bound on the cardinality of $\mathcal{P}_n^{\mathcal{K}}(Q)$ is needed to prove Theorem 1.

The following theorem was proved in [4].

Theorem 2 (*Theorem 1.2 from [4]*)**.** *For all real numbers κ' where $0 < \kappa' < 1$ there exist constants δ_0 and c_0 depending only on n such that*

$$\mu(\mathcal{L}_n(v_0, v_1, c_0, \delta_0, Q)) > \kappa' \mu(I \times K)$$

for Q sufficiently large.

This will be used to prove the next theorem.

Theorem 3. *For all real numbers κ where $0 < \kappa < 1$ there exist constants δ_0 and c_0 depending only on n such that*

$$\mu(\mathcal{K}_n(v_0, v_1, c_0, \delta_0, Q)) > \kappa \mu(I \times K)$$

for Q sufficiently large.

We should point out that the final phrase *for Q sufficiently large* was omitted from the statement of Theorem 1.2 of [4]. This is an error which we correct here.

Before the main results are proved some more notation is introduced together with some preliminary calculations. Let $P \in \mathcal{P}_n(Q)$. For α, a complex root of P, and γ a p-adic root of P, define the sets

$$S_P(\alpha) = \left\{ x \in \mathbb{R} : |x - \alpha| = \min_{\alpha' \in \mathbb{C}: P(\alpha')=0} |x - \alpha'| \right\},$$

$$T_P(\gamma) = \left\{ w \in \mathbb{Q}_p : |w - \gamma|_p = \min_{\gamma' \in \mathbb{Q}_p^*: P(\gamma')=0} |w - \gamma'|_p \right\}.$$

Thus, if $x \in S_P(\alpha)$ then the closest root of P to x is α. Clearly, each point (x, w) lies in at least one set $S_P(\alpha) \times T_P(\gamma)$ and there are at most n^2 distinct sets $S_P(\alpha) \times T_P(\gamma)$ for each P.

From (1), it is not difficult to show that the roots of P for each $P \in \mathcal{P}_n(Q)$ are bounded, more precisely,

$$|\alpha_i| \le 2n, \qquad |\gamma_i|_p \le p^{-n}, \quad i = 1, \ldots, n. \tag{6}$$

This is proved in [9, pages 13 & 85].

The next lemma contains some inequalities which will be used throughout the rest of the paper.

Lemma 1. *Let $P \in \mathcal{P}_n(Q)$ such that (3) and (4) hold for some $(x, w) \in S_P(\alpha) \times T_P(\gamma)$. The following inequalities hold:*

$$|x - \alpha| < 2nc_0 Q^{-v_0} |P'(\alpha)|^{-1} \tag{7}$$

$$< 4nc_0 \delta_0^{-1} Q^{v_1 - v_0 - 1}, \tag{8}$$

$$|w - \gamma|_p < c_0 Q^{-v_0} |P'(\gamma)|_p^{-1} \tag{9}$$

$$< c_0 \delta_0^{-1} Q^{v_1 - v_0}. \tag{10}$$

Proof. It was shown in [9, pages 13 & 75], and in fact is easy to prove, that for $P \in \mathcal{P}_n(Q)$, $x \in S_P(\alpha)$ and $w \in T_P(\gamma)$

$$|x - \alpha| < n|P(x)||P'(x)|^{-1}, \tag{11}$$

$$|w - \gamma|_p < |P(w)|_p |P'(w)|_p^{-1}. \tag{12}$$

By considering the Taylor series of P', the values of P' at α and γ are now compared to the values of P' for points $(x, w) \in S_P(\alpha) \times T_P(\gamma)$. The details will only be provided for the real coordinate. The arguments for the p-adic coordinate are similar. Note that, since $v_1 < \frac{1}{3}$ the equation $v_0 + v_1 = \frac{n}{2}$ implies that

$$v_0 > 2v_1 + \beta, \tag{13}$$

for any β with $0 < \beta < \frac{1}{2}$. Each term of the Taylor series

$$P'(x) = \sum_{i=1}^{n} ((i-1)!)^{-1} P^{(i)}(\alpha)(x - \alpha)^{i-1}$$

is estimated for $x \in S_P(\alpha)$ satisfying (3) and (4). From (11) and (13)

$$\frac{1}{(j-1)!} |P^{(j)}(\alpha)||x - \alpha|^{j-1} \ll Q^{1+(j-1)(v_1 - v_0 - 1)} < Q^{1 - v_1 - \beta}, \quad 2 \le j \le n, \tag{14}$$

for Q sufficiently large. Here, using (6), the trivial bound $|P^{(i)}(\alpha)| \ll Q$ has been used. Therefore, for $x \in S_P(\alpha)$ satisfying (3) and (4) it follows that

$$\frac{1}{2}\delta_0 Q^{1-v_1} < \frac{1}{2}|P'(x)| < |P'(\alpha)| < 2|P'(x)| < 2c_0 Q^{1-v_1}. \tag{15}$$

Similarly, using (12), (13) and the properties of the ultrametric we obtain

$$\delta_0 Q^{-v_1} < |P'(w)|_p = |P'(\gamma)|_p < c_0 Q^{-v_1} \tag{16}$$

for $w \in T_P(\gamma)$ satisfying (3) and (4). Using this it is easily shown that (7)–(10) hold. □

1. Proof of Theorem 3

By Theorem 2, there exist c_0 and δ_0' such that

$$\mu(\mathcal{L}_n(v_0, v_1, c_0, \delta_0', Q)) > \kappa'\mu(I \times K).$$

It should be clear that for any $\delta_0 < \delta_0'$

$$\mu(\mathcal{L}_n(v_0, v_1, c_0, \delta_0, Q)) \geq \mu(\mathcal{L}_n(v_0, v_1, c_0, \delta_0', Q)) > \kappa'\mu(I \times K).$$

To prove Theorem 3, it is sufficient to demonstrate that the set points (x, w) which satisfy (3), (4) and either $|P''(x)| \leq \delta_0 Q$ or $|P''(w)|_p < \delta_0$ is sufficiently small. There are two sets to consider. Let $\mathcal{F}_1$ be the set of points $(x, w) \in I \times K$ such that if (3) and (4) hold for some $P \in \mathcal{P}_n(Q)$ then $|P''(x)| < \delta_0 Q$ also holds. Similarly, let $\mathcal{F}_2$ be the set of points $(x, w) \in I \times K$ such that if (3) and (4) hold for some $P \in \mathcal{P}_n(Q)$ then $|P''(w)|_p < \delta_0$ also holds. It will be shown that if $0 < \kappa < \kappa'$, then δ_0 can be chosen so that $\mu(\mathcal{F}_i) < \frac{\kappa'-\kappa}{2}\mu(I \times K)$, $i = 1, 2$. The proofs are almost exactly the same except that at one step different lemmas are used which will be detailed later.

Define the set of polynomials $P \in \mathcal{P}_n(Q)$ which satisfy (3), (4) and $|P''(x)| < \delta_0 Q$ for some point $(x, w) \in I \times K$ as $\mathcal{P}_n^{\mathcal{F}}(Q)$. Let $A(P)$ be the set of complex roots of P and define $A_{\mathcal{F}}(P) \subseteq A(P)$ to be the set of roots α for which there exists $x \in S_P(\alpha)$ satisfying (3), (4) and $|P''(x)| < \delta_0 Q$. Similarly, let $G(P)$ be the set of p-adic roots of P and define $G_{\mathcal{F}}(P) \subseteq G(P)$ to be the set of roots γ for which there exists $w \in T_P(\gamma)$ satisfying (3) and (4). For a polynomial P with complex root α and p-adic root γ define $\sigma(\alpha, \gamma, P)$ to be the set of solutions of (7) and (9) and define $\sigma(P) = \cup_{\alpha \in A_{\mathsf{F}}(P)} \cup_{\gamma \in G_{\mathsf{F}}(P)} \sigma(\alpha, \gamma, P)$. It should be clear that $\mathcal{F}_1 \subseteq \cup_{P \in \mathcal{P}_n^{\mathsf{F}}(Q)} \sigma(P)$. It will be shown that the measure of this union is small.

Now, we follow the proof of Theorem 1.2 in [4]. The initial details will be done in full as the constants are different. Choose two real numbers u_1 and u_2 with the following properties:

$$\begin{aligned} &u_1 + u_2 = 1 - 2v_1, \\ &v_0 > u_1 > 2v_1 - 1 \geq v_1 - 1, \\ &v_0 > u_2 > 2v_1. \end{aligned} \tag{17}$$

That this is possible can be readily verified using the conditions (2) on v_1 and v_0. The first equation of (17) is necessary as the measures of two different sets which need to be "small", will be shown to have bounds depending on $\delta_0 Q^{u_1+u_2-1+2v_1}$ and $\delta_0 Q^{-u_1-u_2+1-2v_1}$. Clearly, if equality does not hold then one of these sets could be large. For a polynomial P with complex root α and p-adic root γ define the set $\sigma_1(\alpha, \gamma, P)$ of points (x, w) for which the inequalities

$$|x - \alpha| < Q^{-u_1}|P'(\alpha)|^{-1}, \qquad |w - \gamma|_p < Q^{-u_2}|P'(\gamma)|_p^{-1}, \tag{18}$$

hold. Clearly, from (17), $\sigma(\alpha, \gamma, P) \subset \sigma_1(\alpha, \gamma, P)$. The Taylor series of P is considered for each point in $\sigma_1(\alpha, \gamma, P)$ in the neighbourhood of the roots and each term is estimated from above. Once more this will be demonstrated for the real coordinate with similar estimates for the p-adic coordinate. Using (15), (17), (18) and the trivial bound $|P^{(j)}(\alpha)| \ll Q$, it can be readily verified that there exists $\varepsilon > 0$ such that

$$\begin{aligned} &|P'(\alpha)||x - \alpha| < Q^{-u_1}, \\ &\frac{1}{j!}|P^{(j)}(\alpha)||x - \alpha|^j \ll Q^{1-j(u_1+1-v_1)} < Q^{-u_1-\varepsilon}, \quad 2 \leq j \leq n. \end{aligned} \tag{19}$$

Thus,

$$|P(x)| < 2Q^{-u_1} \quad \text{on } \sigma_1(\alpha, \gamma, P). \tag{20}$$

Using a similar argument to that of Lemma 1 by considering the Taylor series of P'' in the neighbourhoods of α and γ it is easy to obtain

$$|P''(\alpha)| < 2\delta_0 Q, \qquad |P''(\gamma)|_p < c_0$$

on $\sigma(\alpha, \gamma, P)$. For the real example use the estimate

$$\frac{1}{(j-2)!} P^{(j)}(\alpha)|x-\alpha|^{j-2} \ll Q^{1+(j-2)(v_1-v_0-1)} < Q^{1-\varepsilon}, \quad j = 3, \ldots, n$$

which is obtained in the same way as (14).

In exactly the same way as above, using this, (15) and (17), estimates for the Taylor series of $P'(x)$ and $P''(x)$ are extended to $\sigma_1(\alpha, \gamma, P)$. For this, instead of (19), the bounds

$$\frac{1}{(j-1)!}|P^{(j)}(\alpha)||x-\alpha|^{j-1} \ll Q^{1-(j-1)(u_1+1-v_1)} < Q^{1-v_1-\varepsilon}, \quad j = 2, \ldots, n$$

$$\frac{1}{(j-2)!}|P^{(j)}(\alpha)||x-\alpha|^{j-2} \ll Q^{1-(j-2)(u_1+1-v_1)} < Q^{1-\varepsilon}, \quad j = 3, \ldots, n$$

are used to show that

$$|P'(x)| < 4c_0 Q^{1-v_1} \quad \text{and} \quad |P''(x)| < 4\delta_0 Q \tag{21}$$

on $\sigma_1(\alpha, \gamma, P)$. It can similarly be readily verified by $u_2 > 2v_1$ that the inequalities

$$|P(w)|_p < Q^{-u_2}, \qquad |P'(w)|_p < c_0 Q^{-v_1}, \qquad |P''(w)|_p < c_0 \tag{22}$$

also hold on $\sigma_1(\alpha, \gamma, P)$.

The polynomials in $\mathcal{P}_n^{\mathcal{F}}(Q)$ are now partitioned into sets which have the same coefficients for x^2 to x^n. For integers a_i, $i = 2, \ldots, n$ let $\mathbf{b}$ be the $(n-1)$-tuple $(a_n, \ldots, a_2)$ and let $\mathcal{P}_n^{\mathbf{b}}(Q)$ be the set of polynomials in $\mathcal{P}_n^{\mathcal{F}}(Q)$ for which the coefficient of x^i is a_i for $i = 2, \ldots, n$. An adaptation of Sprindzuk's method of essential and inessential domains is now used (see [9] for details). An interval $\sigma_1(\alpha, \gamma, P)$ is called *essential* if $\mu(\sigma_1(\alpha, \gamma, P) \cap \sigma_1(\tilde{\alpha}, \tilde{\gamma}, \tilde{P})) \le \frac{1}{2}\mu(\sigma_1(\alpha, \gamma, P))$ for all polynomials $\tilde{P} \in \mathcal{P}_n^{\mathbf{b}}(Q)$, $\tilde{P} \neq P$ and all roots $\tilde{\alpha}, \tilde{\gamma}$ of $\tilde{P}$. It is called *inessential* otherwise. These definitions imply that a point (x, w) can lie in an essential interval for at most two distinct polynomials. Clearly,

$$\mathcal{F}_1 \subseteq \bigcup_{\mathbf{b}\in\mathbb{Z}^{n-1}:|\mathbf{b}|\le Q} \; \bigcup_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \; \bigcup_{\substack{\alpha\in A_{\mathsf{F}}(P)\\ \gamma\in G_{\mathsf{F}}(P)}} \sigma(\alpha, \gamma, P)$$

and

$$\bigcup_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \; \bigcup_{\substack{\alpha\in A_{\mathsf{F}}(P)\\ \gamma\in G_{\mathsf{F}}(P)}} \sigma(\alpha, \gamma, P) = \left(\bigcup_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \; \bigcup_{\substack{\alpha\in A_{\mathsf{F}}(P)\\ \gamma\in G_{\mathsf{F}}(P)\\ \sigma_1(\alpha,\gamma,P)\ \text{essential}}} \sigma(\alpha, \gamma, P) \right) \bigcup \left(\bigcup_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \; \bigcup_{\substack{\alpha\in A_{\mathsf{F}}(P)\\ \gamma\in G_{\mathsf{F}}(P)\\ \sigma_1(\alpha,\gamma,P)\ \text{inessential}}} \sigma(\alpha, \gamma, P) \right).$$

Note that by $|\mathbf{b}|$ we mean the sup norm so that $|\mathbf{b}| = \max_{i=2,\ldots,n} |b_i|$.

First we consider the essential intervals. The next lemma will be used to count the number of $\mathbf{b}$ for which $P \in \mathcal{P}_n^{\mathbf{b}}(Q)$ satisfies $|P''(x)| < 4\delta_0 Q$.

Lemma 2. *Let f be a continuously differentiable real valued function on $J = [a, b] \subset \mathbb{R}$ which satisfies* $\max_{x\in J} |f'(x)| < M$. *Let $K(J, B)$ be the set of integers d such that the inequality $|f(x) + d| < B$, $B > 1$, has a solution for $x \in J$. Then $\#K(J, B) \leq 2B + M|J|$.*

Proof. It is necessary to obtain an upper bound for the number of integers d for which the curves $y = f(x)+d$ intersect the box $[a, b]\times[-B, B]$. As the curves are continuous they must intersect the boundary of this box at least twice. Since the vertical distance between the curves is 1 the maximum number which can intersect one of the vertical boundary lines (i.e. $x = a$ or $x = b$) is $2B$. Using the mean value theorem the horizontal distance between the curves is at least $1/M$. Thus the maximum number which can intersect one of the horizontal boundary lines (i.e. $y = B$ or $y = -B$) is $(b - a)M$. Therefore, $\#K(J, B) \leq 2B + M|J|$ as required. □

It can be readily verified that $P^{(3)}(x) \leq n^4 Q$ for every $P \in \mathcal{P}_n(Q)$, $x \in I$. Let J be the interval defined by (8) so that $|J| \leq 8nc_0\delta_0^{-1} Q^{v_1-v_0-1}$. To use the lemma take $M = n^4 Q$ and $B = 4\delta_0 Q$. Fix integers $a_3, \ldots, a_n$ and let $P(x) = a_n x^n + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$. Then, the number of coefficients a_2 such that $|P''(x)| < 4\delta_0 Q$ is given by

$$\#a_2 \leq 8\delta_0 Q + 16n^5 c_0\delta_0^{-1} Q^{v_1-v_0} < 9\delta_0 Q$$

from (17) for Q sufficiently large since $v_1 - v_0 < 0$. The number of vectors $\mathbf{b} = (a_n, \ldots, a_2)$ with a_2 fixed is $(2Q + 1)^{n-2}$. Thus, the total number of vectors $\mathbf{b}$ such that $|P''(x)| < 4\delta_0 Q$ for $P \in \mathcal{P}_n^{\mathbf{b}}(Q)$ is at most

$$9\delta_0 Q(2Q + 1)^{n-2} < 3^n \delta_0 Q^{n-1}.$$

Denote the set of these vectors by D.

Note that $\mu(\sigma(\alpha, \gamma, P)) \leq 2nc_0^2 Q^{-2v_0+u_1+u_2} \mu(\sigma_1(\alpha, \gamma, P))$. As a point (x, w) can lie in an essential interval for at most two polynomials, we have

$$\sum_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \sum_{\substack{\alpha\in A_F(P)\\ \gamma\in G_F(P)\\ \sigma_1(\alpha,\gamma,P)\ \text{essential}}} \mu(\sigma_1(\alpha, \gamma, P)) \leq 2n^2 \mu(I \times K).$$

The n^2 comes from the fact that there are at most n^2 pairs of roots α, γ for each P. Summing over all $\mathbf{b} \in D$ gives

$$\begin{aligned}\sum_{\mathbf{b}\in D} \sum_{P\in\mathcal{P}_n^{\mathbf{b}}(Q)} \sum_{\substack{\alpha\in A_F(P)\\ \gamma\in G_F(P)\\ \sigma_1(\alpha,\gamma,P)\ \text{essential}}} \mu(\sigma(\alpha, \gamma, P)) &< 3^n \delta_0 Q^{n-1} 2nc_0^2 Q^{-2v_0+u_1+u_2} 2n^2 \mu(I \times K)\\ &\leq \frac{\kappa' - \kappa}{4} \mu(I \times K)\end{aligned}$$

for δ_0 chosen appropriately and because $n - 1 - 2v_0 + u_1 + u_2 = -1 + u_1 + u_2 + 2v_1 = 0$ from (17). When calculating the measure for the inessential sets the same power $(-1 + u_1 + u_2 + 2v_1)$ will also appear but multiplied by -1.

Now, the inessential sets are considered so assume that $\sigma_1(\alpha, \gamma, P)$ is inessential. Thus, there exists $\tilde{P} \in \mathcal{P}_n^{\mathbf{b}}(Q)$ such that $\mu(\sigma_1(\alpha, \gamma, P) \cap \sigma_1(\tilde{\alpha}, \tilde{\gamma}, \tilde{P})) \geq \frac{1}{2}\mu(\sigma_1(\alpha, \gamma, P))$. Let $R = P - \tilde{P}$

so that $R(f) = b_1 f + b_0$ for some $b_0, b_1 \in \mathbb{Z}$ with $|b_i| \leq 2Q$. Then, from (20)–(22), R satisfies

$$|b_1 x + b_0| < 4Q^{-u_1}$$
$$|R'(x)| = |b_1| < 8c_0 Q^{1-v_1}$$
$$|b_1 w + b_0|_p < Q^{-u_2}$$
$$|R'(w)|_p = |b_1|_p < c_0 Q^{-v_1}$$

on $\sigma_1(\alpha, \gamma, P) \cap \sigma_1(\tilde{\alpha}, \tilde{\gamma}, \tilde{P})$, (cf [4, (4.4)] and further). The proof from this point is now exactly the same as that in [4]. Using that proof it is shown that the set of points which lie in at least one inessential interval has measure at most

$$\delta_0 C Q^{1-u_1-u_2-2v_1} \mu(I \times K)$$

where C is a constant depending on n. By (17), this is equal to $\delta_0 C \mu(I \times K)$. Thus, again δ_0 can be chosen so that the set of points which lie in at least one inessential interval has measure at most $\frac{\kappa' - \kappa}{4} \mu(I \times K)$.

To obtain a bound for the measure of $\mathcal{F}_2$ the only difference to above is in counting the number of possible $\mathbf{b}$ for which $P \in \mathcal{P}_n^{\mathbf{b}}(Q)$ satisfies $|P''(w)|_p < \delta_0$. For this instead of Lemma 2 the next lemma is used. The aim is to count the number of polynomials P which satisfy $|P''(w)|_p < \delta_0$ on a ball defined by (10).

Lemma 3. *Let p be a prime and $M \subseteq \mathbb{Z}_p$ be a cylinder such that $\mu_2(M) = p^{-l_1}$, $l_1 \geq 1$. Let $T \in P_n(Q)$. Define l_2 by the inequalities $p^{-l_2} < \delta_0 \leq p^{-l_2+1}$ and assume that $l_1 \geq l_2 + 1$. Then, for $w \in M$, the inequality*

$$|T(w) + d|_p < \delta_0$$

has at most $2Q\delta_0 + 1$ solutions in integers d with $|d| \leq Q$.

Proof. First, fix a point $w_0 \in M$. If $d_0 \in \mathbb{Z}$ satisfies the inequality

$$|T(w_0) + d_0|_p < \delta_0,$$

then all other solutions of $|T(w_0) + d|_p < \delta_0$ are of the form $d = d' + \sum_{i=1}^{\infty} m_i p^{l_2+i}$, $m_i \in \{0, \ldots, p-1\}$. For any point $w \in M$ which satisfies $|w - w_0|_p < p^{-l_1}$ we have

$$|T(w) + d_0|_p \leq \max\{|T(w) - T(w_0)|_p, |T(w_0) + d_0|_p\} < \delta_0$$

since, by the p-adic mean value theorem for polynomials,

$$|T(w) - T(w_0)|_p \leq |w - w_0|_p < \delta_0.$$

Hence, $|T(w) + d_0|_p < \delta_0$. Therefore, if d_0 satisfies $|T(w_0) + d_0|_p < \delta_0$ then it also satisfies $|T(w) + d_0|_p < \delta_0$ and the only solutions are of the form $d = d_0 + \sum_{i=1}^{\infty} m_i p^{l_2+i}$ for $m_i \in \{0, \ldots, p-1\}$. Thus, the number of such d satisfying $|d| \leq Q$ is at most $\frac{2Q}{p^{l_2}} + 1 \leq 2\delta_0 Q + 1$ which completes the proof of the lemma. □

Therefore, for any fixed δ_0 and Q sufficiently large (so that $c_0 \delta_0^{-1} Q^{v_1 - v_0} < \delta_0$) the number of vectors $\mathbf{b}$ which satisfy $|P''(w)| < \delta_0$ on the ball defined by (10) is at most $2\delta_0 Q + 1$. The proof from hereon is exactly the same as for the real case.

Thus, finally, it has been shown that $\mu(\mathcal{F}_i) < \frac{\kappa'-\kappa}{2}\mu(I \times K)$ so that

$$\mu(\mathcal{K}_n(v_0, v_1, c_0, \delta_0, Q)) > \kappa\mu(I \times K).$$

2. Proof of Theorem 1

Let $P \in \mathcal{P}_n^{\mathcal{K}}(Q)$. Fix roots $\alpha_1 \in A(P)$ and $\gamma_1 \in G(P)$; the other roots of P are then ordered so that

$$|\alpha_1 - \alpha_2| \le |\alpha_1 - \alpha_3| \le \cdots \le |\alpha_1 - \alpha_n|,$$
$$|\gamma_1 - \gamma_2|_p \le |\gamma_1 - \gamma_3|_p \le \cdots \le |\gamma_1 - \gamma_n|_p.$$

The arguments below are for the p-adic coordinate; the arguments for the real coordinate are similar. Using the inequalities above and the fact that $|P'(\gamma_1)|_p = |a_n|_p|\gamma_1 - \gamma_2|_p \ldots |\gamma_1 - \gamma_n|_p$, it should be clear that $|P'(\gamma_1)|_p \ge |a_n|_p|\gamma_1 - \gamma_2|_p^{n-1}$. Therefore, by (2), (10), (16) and because $n \ge 3$, the inequalities $|\gamma_1-\gamma_2|_p \ll Q^{\frac{-v_1}{n-1}}$ and $|w-\gamma_2|_p \le \max\{|w-\gamma_1|_p, |\gamma_1-\gamma_2|_p\} \ll Q^{\frac{-v_1}{n-1}}$ hold for $w \in T_P(\gamma_1)$. Consider the second derivative of P where

$$P''(w) = a_n \sum (w - \gamma_{i_1}) \cdots (w - \gamma_{i_{n-2}})$$

and the sum is taken over all distinct $(n-2)$-tuples $(i_1, \ldots, i_{n-2})$ with $i_j \in \{1, \ldots, n\}$ for $j = 1, \ldots, n-2$. Every summand in this second derivative contains at least one of the factors $w - \gamma_k$ for $k = 1, 2, 3$. Since $|P''(w)|_p \ge \delta_0$ it can be readily verified that $|w-\gamma_n|_p, \ldots, |w-\gamma_3|_p \gg 1$. Then, since $|w - \gamma_1|_p \ll Q^{-v_0+v_1}$ it follows that

$$|\gamma_1 - \gamma_n|_p \ge \ldots \ge |\gamma_1 - \gamma_3|_p = |(w - \gamma_3) - (w - \gamma_1)|_p = |w - \gamma_3|_p \gg 1.$$

Therefore, from (6), (16), and the fact that $P'(\gamma_1) = a_n(\gamma_1 - \gamma_2)\prod_{3\le i\le n}(\gamma_1 - \gamma_i)$ we have

$$|\gamma_1 - \gamma_2|_p \ll Q^{-v_1}|a_n|_p^{-1} \ll Q^{-v_1}, \tag{23}$$

by (1). Similarly, for the real case, it can be shown that

$$|\alpha_1 - \alpha_2| \ll Q^{1-v_1}|a_n|^{-1} \ll Q^{-v_1}. \tag{24}$$

Hence, every polynomial $P \in \mathcal{P}_n^{\mathcal{K}}(Q)$ has at least two complex roots α_1, α_2 and at least two p-adic roots γ_1, γ_2 which satisfy (23) and (24), respectively. Also, (8) and (10) hold so that

$$\mu\left(\bigcup_{\substack{\alpha\in A(P)\\ \gamma\in G(P)}} \sigma(\alpha,\gamma,P)\right) \le \sum_{\substack{\alpha\in A(P)\\ \gamma\in G(P)}} \mu(\sigma(\alpha,\gamma,P)) < n^2(4n)c_0^2\delta_0^{-2}Q^{2v_1-2v_0-1}.$$

Thus, since

$$\mathcal{K}_n(v_0, v_1, c_0, \delta_0, Q) \subset \bigcup_{P\in\mathcal{P}_n^{\mathrm{K}}(Q)} \bigcup_{\substack{\alpha\in A(P)\\ \gamma\in G(P)}} \sigma(\alpha,\gamma,P)$$

we have, by Theorem 3,

$$\kappa\mu(I \times K) < \mu(\mathcal{K}_n(v_0, v_1, c_0, \delta_0, Q)) < \#\mathcal{P}_n^{\mathcal{K}}(Q)4n^3c_0^2\delta_0^{-2}Q^{2v_1-2v_0-1}.$$

Finally, this implies that $\#\mathcal{P}_n^{\mathcal{K}}(Q) \gg Q^{2v_0-2v_1+1} = Q^{n+1-4v_1}$ which proves the theorem.

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References

[1] V. Beresnevich, V.I. Bernik, F. Götze, The distribution of close conjugate algebraic numbers, Compositio Math. 146 (2010) 1165–1179.

[2] V. Bernik, F. Götze, O. Kukso, Lower bounds for the number of integral polynomials with given order of discriminants, Acta Arith. 133 (2008) 375–390.

[3] V. Bernik, F. Götze, O. Kukso, On the divisibility of the discriminant of an integral polynomial by prime powers, Lith. Math. J. 48 (2008) 380–396.

[4] N. Budarina, D. Dickinson, J. Yuan, On the number of polynomials with small discriminants in the Euclidean and p-adic metrics, Acta Math. Sinica (2012) (in press).

[5] Y. Bugeaud, A. Dujella, Root separation for irreducible integer polynomials, Bull. Lond. Math. Soc. (2011) Advance Access, published October 2011 (doi:10.1112/blms/bdr085) in press.

[6] Y. Bugeaud, M. Mignotte, On the distance between roots of integer polynomials, Proc. Edinb. Math. Soc. (2) 47 (2004) 553–556.

[7] J.H. Evertse, Distances between conjugates of an algebraic number, Publ. Math. Debrecen 65 (2004) 323–340.

[8] K. Mahler, An inequality for the discriminant of a polynomial, Michigan Math. J. 11 (1964) 257–262.

[9] V. Sprindžuk, Mahler's Problem in the Metric Theory of Numbers, vol. 25, Amer. Math. Soc., Providence, RI, 1969.