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SIMULTANEOUS DIOPHANTINE APPROXIMATION OF INTEGRAL POLYNOMIALS IN THE DIFFERENT METRICS

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Throughout, let

$$P(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0$$

be an integer polynomial of degree $\deg P \leqslant n$ and height $H = H(P) = \max_{1 \leqslant j \leqslant n} |a_j|$. In this paper we will consider a problem of Diophantine approximation on such polynomials in the real, complex and p–adic fields simultaneously. That is, we will study the approximation of zero by the values of $|P(x)|$, $|P(z)|$ and $|P(w)|_p$, where $x \in \mathbb{R}$, $z \in \mathbb{C}$, $w \in \mathbb{Q}_p$.

Let Ψ be a monotonically decreasing function. In [9] it is shown that if the volume sum $\sum_{r=1}^{\infty} \Psi(r)$ converges then the set of points $(x, z, w) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ which satisfy the inequalities $|P(x)| \leqslant H^{-v_1} \Psi^{\lambda_1}(H)$, $|P(z)| \leqslant H^{-v_2} \Psi^{\lambda_2}(H)$ and $|P(w)|_p \leqslant H^{-v_3} \Psi^{\lambda_3}(H)$, where $v_1 + 2v_2 + v_3 = n - 3$ and $\lambda_1 + 2\lambda_2 + \lambda_3 = 1$, for infinitely many integer polynomials P has measure zero.

A more specialised result is that of V.N. Borbat in [8] who showed that the system of inequalities

$$\begin{cases} |P(x)| < H^{-n+v'}, \\ |P'(x)| < H^{1-v'-\epsilon'}, \ 0 \leqslant v' < 1, \end{cases}$$

for any $\epsilon' > 0$ has infinitely many solutions $P \in \mathbb{Z}[x]$ only for a set of measure zero. Borbat's result allows us to find a lower bound for the Hausdorff dimension of the set of real numbers x which are approximated by special algebraic numbers at which the derivative of the minimal polynomial is relatively small.

In the present paper, we generalize this result to simultaneous approximation on $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ and consider some applications.

Before we proceed, some notation is needed. Let $\mu_1(A)$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, $\mu_2(A)$ the Lebesgue measure of a measurable set $A \subset \mathbb{C}$ and $\mu_3(A)$ the Haar measure of a measurable set $A \subset \mathbb{Q}_p$. Using these definitions, define the measure μ on a set $A \subseteq \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ by $\mu(A) = \mu_1(A)\mu_2(A)\mu_3(A)$.

Let $L_n(v)$ denote the set of points lying in a parallelepiped $T = I \times K \times D$, where I is an interval in $\mathbb{R}$, K is a disc in $\mathbb{C}$ and D is a cylinder in $\mathbb{Q}_p$, for which the system of inequalities

$$\begin{aligned} \max(|P(x)|, |P(z)|, |P(w)|_p|) &< H^{-\frac{n-3}{4}+v} \Psi^{\frac{1}{4}}(H), \\ \max(|P'(x)|, |P'(z)|) &< H^{1-v}, \\ |P'(w)|_p &< H^{-v}, \end{aligned} \tag{1}$$

has infinitely many solutions $P \in \mathbb{Z}[x]$.

Theorem 1. *If* $n \geqslant 3$ *and* $\sum_{H=1}^{\infty} \Psi(H) < \infty$ *then* $\mu(L_n(\nu)) = 0$ *with* $0 \leqslant \nu \leqslant 0.027$.

For $n = 3$ this theorem is easily proved. Hereafter, only the case $n \geqslant 4$ will be considered.

As Ψ is monotonic and the series $\sum_{H=1}^{\infty} \Psi(H)$ converges it is easy to show that on average $\Psi(H) < c_1 H^{-1}$, where c_1 is independent of H. Therefore, instead of the first inequality of (1) the weaker inequality

$$\max(|P(x)|, |P(z)|, |P(w)|_p) \ll H^{-\frac{n-2}{4}+\nu}, \tag{2}$$

may be considered at some stages for simplicity. Here and throughout $A \ll B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$.

In the main, positive constants which depend only on n will be denoted by $c(n)$. Where necessary these constants will be numbered $c_k(n)$, $k = 1, 2, \ldots$.

It is shown in [1, 7, 10] and [11] that, without loss of generality, it is enough to prove the theorem for the set of polynomials P satisfying (1) which are irreducible and also satisfy

$$H(P) = |a_n|, \quad |a_n|_p > p^{-n}. \tag{3}$$

Let $\mathcal{P}_n(H)$ denote this set and define $\mathcal{P}_n = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H)$.

Let $P \in \mathcal{P}_n(H)$ have roots $\alpha_1', \alpha_2', \ldots, \alpha_n'$ in $\mathbb{C}$ and roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. From (3) it is shown in [3] and [5] that

$$|\alpha_i'| \leqslant 2, \quad |\gamma_i|_p < p^n, \quad i = 1, \ldots, n.$$

From among the roots α_i' choose a real root α_1 and a non–real root α_j' which will hereafter be denoted by β_1. Order the roots α_i' according to their distance from α_1 or β_1 as follows:

$$\begin{aligned} |\alpha_1 - \alpha_2| &\leqslant \ldots \leqslant |\alpha_1 - \alpha_{n_1}|, \\ |\beta_1 - \beta_2| &\leqslant \ldots \leqslant |\beta_1 - \beta_{n_2}|, \\ |\gamma_1 - \gamma_2|_p &\leqslant \ldots \leqslant |\gamma_1 - \gamma_n|_p, \end{aligned}$$

with $n_1 + n_2 = n$, and define the sets

$$\begin{aligned} S_1(\alpha_1) &= \{x \in \mathbb{R} : |x - \alpha_1| = \min_{1 \leqslant j \leqslant n_1} |x - \alpha_j|\}, \\ S_2(\beta_1) &= \{z \in \mathbb{C} : |z - \beta_1| = \min_{1 \leqslant j \leqslant n_2} |z - \beta_j|\}, \\ S_p(\gamma_1) &= \{w \in \mathbb{Q}_p : |w - \gamma_1|_p = \min_{1 \leqslant j \leqslant n} |w - \gamma_j|_p\}. \end{aligned}$$

For example, $S_p(\gamma_1)$ is the set of those points $w \in \mathbb{Q}_p$, for which γ_1 is the nearest root.

Fix $\varepsilon > 0$ where ε is sufficiently small and suppose that $\varepsilon_1 = \varepsilon N^{-1}$ where $N = N(n) > 0$ is sufficiently large and let $T = [\varepsilon_1^{-1}]$. For a polynomial P define the real numbers ρ_{ij}, $i = 1, 2, 3$, and the integers k_j, l_j, m_j by

$$\begin{aligned} |\alpha_1 - \alpha_j| &= H^{-\rho_{1j}}, \quad 2 \leqslant j \leqslant n_1 \\ |\beta_1 - \beta_j| &= H^{-\rho_{2j}}, \quad 2 \leqslant j \leqslant n_2 \\ |\gamma_1 - \gamma_j| &= H^{-\rho_{3j}}, \quad 2 \leqslant j \leqslant n, \end{aligned}$$

and

$$\frac{k_j - 1}{T} \leqslant \rho_{1j} < \frac{k_j}{T}, \quad \frac{l_j - 1}{T} \leqslant \rho_{2j} < \frac{l_j}{T}, \quad \frac{m_j - 1}{T} \leqslant \rho_{3j} < \frac{m_j}{T}.$$

Further define the numbers q_i, r_i, s_i by

$$\begin{aligned} q_i &= \frac{k_{i+1} + \ldots + k_n}{T}, \quad (1 \leqslant i \leqslant n_1 - 1) \\ r_i &= \frac{l_{i+1} + \ldots + l_n}{T}, \quad (1 \leqslant i \leqslant n_2 - 1) \\ s_i &= \frac{m_{i+1} + \ldots + m_n}{T}, \quad (1 \leqslant i \leqslant n - 1). \end{aligned}$$

Each polynomial $P \in \mathcal{P}_n(H)$ is now associated with three integer vectors $\mathbf{q} = (k_2, \ldots, k_{n_1})$, $\mathbf{r} = (l_2, \ldots, l_{n_2})$ and $\mathbf{s} = (m_2, \ldots, m_n)$ and the number of these vectors is finite (and depends only on n, p and T). Let $\mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$ denote the set of polynomials $P \in \mathcal{P}_n(H)$ with the same triple of vectors $(\mathbf{q}, \mathbf{r}, \mathbf{s})$.

Fix $\delta_1 > 0$. Any complex number z lying in the parallelepiped $\mathbf{T}$ with $|\mathrm{Im}\, z| < \delta_1$ will be excluded. As δ_1 is arbitrary this can be done without loss of generality. Hence, from now on we assume that $|\mathrm{Im}\, z| \geqslant \delta_1$. Later, there will be inequalities of the kind $|z - \beta| < H(P)^{-\nu}$, $\nu > 0$; as the RHS tends to zero it will follow that there exists a root β such that $|\mathrm{Im}\, \beta| > \frac{1}{2}\delta_1$. In this case there is also a conjugate root $\bar{\beta}$ of P such that $|\beta - \bar{\beta}| > \delta_1$, and for any real root α of P the inequalities $|\beta - \alpha| = |\bar{\beta} - \alpha| > \frac{1}{2}\delta_1$ hold. Collecting this information, we have

$$|\mathrm{Im}\, \beta| > \frac{1}{2}\delta_1, \quad |\mathrm{Im}\, z| \geqslant \delta_1, \quad |\beta - \bar{\beta}| > \delta_1, \quad |\beta - \alpha| > \frac{1}{2}\delta_1. \tag{4}$$

1 Preliminary Results

From now on it will be assumed without loss of generality that $x \in S_1(\alpha_1)$, $z \in S_2(\beta_1)$, $w \in S_p(\gamma_1)$. In many places in the proof of the theorem values of polynomials will be estimated by means of a Taylor series. To obtain an upper bound on the terms in the Taylor series (and for other purposes) the following two lemmas (proved in [4] and [10]) will be used.

Lemma 1. *If* $P \in \mathcal{P}_n(H)$ *then*

$$\begin{aligned} |u-\alpha| &\leqslant 2^n |P(u)||P'(\alpha)|^{-1}, \\ |w-\gamma_1| &\leqslant |P(w)|_p |P'(\gamma_1)|_p^{-1}, \\ |u-\alpha| &\leqslant \min_{2\leqslant j\leqslant n} \left(2^{n-j}|P(u)||P'(\alpha)|^{-1}\prod_{k=2}^{j}|\alpha-\alpha_k|\right)^{\frac{1}{j}}, \\ |w-\gamma_1|_p &\leqslant \min_{2\leqslant j\leqslant n} \left(|P(w)|_p|P'(\gamma_1)|_p^{-1}\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p\right)^{\frac{1}{j}} \end{aligned}$$

where u *represents* x *or* z *and* α *is* α_1 *or* β_1 *as required.*

Lemma 2. *Let* $P \in \mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$. *Then*

$$\begin{aligned} |P^{(l)}(\alpha_1)| &< c(n)H^{1-q_l+(n-l)\varepsilon_1}, \\ |P^{(l)}(\beta_1)| &< c(n)H^{1-r_l+(n-l)\varepsilon_1}, \\ |P^{(l)}(\gamma_1)|_p &< c(n)H^{-s_l+(n-l)\varepsilon_1}, \end{aligned}$$

for $1 \leqslant l \leqslant n-1$.

The next lemma is proved in [12].

Lemma 3. *Let* $G(v)$ *be the set of points* (x, z, w) *for which the inequality*

$$|P(x)||P(z)|^2|P(w)|_p < H^{-v}, \quad n = \deg P \geqslant 3, \quad H = H(P),$$

has infinitely many solutions $P \in \mathbb{Z}[x]$. *Then, for* $v > n-2$

$$\mu(G(v)) = 0.$$

The following lemma is proved in [6]. At several points in the proof of the theorem there are various cases (of different types of polynomial) to consider; usually the existence of one case is disproved by finding a contradiction to the final inequality in the lemma below.

Lemma 4. *Let* P_1 *and* P_2 *be two integer polynomials of degree at most* n *with no common roots and* $\max(H(P_1), H(P_2)) \leqslant H$. *Let* $\delta > 0$ *and* $\eta_i > 0$ *for* $i = 1, 2, 3$. *Let* $I \subset \mathbb{R}$ *be an interval,* $K \subset \mathbb{C}$ *be a disk and* $D \subset \mathbb{Q}_p$ *be a cylinder with* $\mu_1(I) = H^{-\eta_1}$, $\operatorname{diam} K = H^{-\eta_2}$ *and* $\mu_p(D) = H^{-\eta_3}$. *If there exist* $\tau_1 > -1$, $\tau_2 > -1$ *and* $\tau_3 > 0$ *such that for all* $(x, z, w) \in I \times K \times D$

$$\begin{aligned} \max_{x\in I}(|P_1(x)|, |P_2(x)|) &< H^{-\tau_1}, \\ \max_{z\in K}(|P_1(z)|, |P_2(z)|) &< H^{-\tau_2}, \\ \max_{w\in D}(|P_1(w)|_p, |P_2(w)|_p) &< H^{-\tau_3}, \end{aligned}$$

then

$$\tau_1+2\tau_2+\tau_3+3+2\max(\tau_1+1-\eta_1, 0)+4\max(\tau_2+1-\eta_2, 0)+2\max(\tau_3-\eta_3, 0) < 2n+\delta.$$

Finally, we state two classical results. The first is proved in [2] and is an adaptation of Cauchy's Condensation Test. The second is the convergence half of the Borel–Cantelli Lemma which will be used throughout the proof of the theorem.

Lemma 5. *Let* $\Psi(H)$, $H = 1, 2, \ldots$, *be a monotonically decreasing sequence of positive numbers. If the series* $\sum_{H=1}^{\infty} \Psi(H)$ *converges, then for any number* $c > 0$ *the series* $\sum_{k=0}^{\infty} 2^k \Psi(c2^k)$ *converges respectively.*

Lemma 6 (Borel–Cantelli). *Let* (Ω, μ) *be a measure space with* $\mu(\Omega)$ *finite and let* A_i, $i \in \mathbb{N}$ *be a family of measurable sets. Let*

$$A = \{\omega \in \Omega : \omega \in A_i \textit{ for infinitely many } i \in \mathbb{N}\}$$

and suppose the sum $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. *Then* $\mu(A) = 0$.

2 Proof of the Theorem

Since $|\alpha_i| \leqslant 2$, $|\gamma_i|_p < p^n$ for $1 \leqslant i \leqslant n$ and $|w|_p \ll 1$ it follows from Lemma 1 (using $j = n$ and $H \leqslant H_0$) that the set of points (x, z, w), for which (1) is satisfied, is a subset of the set $\mathbf{T} = I \times K \times D$, where $I = [-3, 3]$, $K = \{z : |z| \leqslant 3\}$, $D = \{w : |w|_p \ll 1\}$.

The proof of the theorem will consist of a series of propositions. As a reminder, it is only necessary to consider irreducible polynomials P over the rational numbers.

Let

$$\mathcal{P}^t = \mathcal{P}^t(n, \mathbf{q}, \mathbf{r}, \mathbf{s}) = \bigcup_{2^t \leqslant H < 2^{t+1}} \mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$$

and suppose that the polynomials $P \in \mathcal{P}^t$ are irreducible and satisfy (3). In much of what follows system (2) will be used rather than (1). A polynomial is called (i_1, i_2, i_3)–linear if for $i_j = 0$, $j = 1, 2, 3$, the system of inequalities

$$\begin{aligned} q_1 + k_2 T^{-1} &< \frac{n+2}{4} - v, \\ r_1 + l_2 T^{-1} &< \frac{n+2}{4} - v, \\ s_1 + m_2 T^{-1} &< \frac{n-2}{4} - v, \end{aligned} \tag{5}$$

holds, and for $i_j = 1$, $j = 1, 2, 3$, the inequality signs in (5) are reversed. For example, $(0, 1, 1)$–linearity means that in (5) the first inequality has $<$ and the second and third have $\geqslant$. Denote by $\mathcal{P}^t(i_1, i_2, i_3) \subset \mathcal{P}^t$, $i_j = 0, 1$, $j = 1, 2, 3$, the class of (i_1, i_2, i_3)–linear polynomials. As there are only 8 kinds of linearity we shall consider them in turn.

We will use the constants

$$d_1 = q_1 + 2r_1 + s_1, \quad d_2 = (k_2 + 2l_2 + m_2)T^{-1}$$

heavily for the rest of the proof with different ranges of $d_1 + d_2$ considered separately.

Proposition 1. *If* $\sum_{H=1}^{\infty}\Psi(H) < \infty$ *then* $\mu(L_n(\nu)) = 0$ *when the polynomials are restricted to the subclass* $\mathcal{P}^t(0,0,0)$ *for which* $d_1 + d_2 > n + \varepsilon$.

Proof. By Lemma 1, all $\mathbf{u} = (x, z, w) \in S(\alpha_1) \times S(\beta_1) \times S(\gamma_1)$ satisfying (2) belong to the parallelepiped $\sigma(P)$ defined as the set of points $\mathbf{u}$ satisfying

$$\begin{aligned} |x-\alpha_1| &\ll 2^{-t(\frac{n+2}{4}-q_1-\nu)},\\ |z-\beta_1| &\ll 2^{-t(\frac{n+2}{4}-r_1-\nu)},\\ |w-\gamma_1|_p &\ll 2^{-t(\frac{n-2}{4}-s_1-\nu)}. \end{aligned} \tag{6}$$

The initial parallelepiped $\mathbf{T}$ is divided into smaller parallelepipeds $M = I_M \times K_M \times D_M$ such that

$$\mu_1(I_M) = 2^{-tk_2T^{-1}}, \quad \text{diam}\,(K_M) = 2^{-tl_2T^{-1}}, \quad \mu_p(D_M) = 2^{-tm_2T^{-1}}. \tag{7}$$

It will be said that the polynomial P belongs to the parallelepiped M if there exists $\mathbf{u} \in M$ such that (2) holds; we will denote this by $P(\mathbf{u}) \in M$. Let $P(\mathbf{u}) \in M$ and develop P as a Taylor series on M remembering that $P(\alpha_1) = P(\beta_1) = P(\gamma_1) = 0$ to obtain

$$P(t) = \sum_{j=1}^{n} (j!)^{-1}P^{(t)}(\zeta_1)(x-\zeta_1)^j \tag{8}$$

for $t = x, z, w$ and $\zeta_1 = \alpha_1, \beta_1, \gamma_1$ repectively. An upper bound for $|P(\mathbf{u})|$ is found using (7) and Lemma 2. As an example we will show how to estimate $|P(z)|$. The following inequalities obtained from the definitions of r_j and $l_jT_0^{-1}$ are used:

$$r_j + jl_2T_0^{-1} = r_j + l_2T_0^{-1} + (j-1)l_2T_0^{-1} \geqslant r_j + l_2T_0^{-1} + (l_2 + \ldots + l_{j-1})T_0^{-1} = r_1 + l_2T_0^{-1}.$$

These imply

$$\begin{aligned} |P'(\beta_1)||z-\beta_1| &\ll 2^{t(1-r_1+(n-1)\varepsilon_1-l_2T^{-1})} \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)},\\ |P^{(j)}(\beta_1)||z-\beta_1|^j &\ll 2^{t(1-r_j+(n-j)\varepsilon_1-jl_2T^{-1})} \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}, \quad 2 < j \leqslant n. \end{aligned}$$

Clearly these further imply that $|P(z)| \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}$. It is not difficult to acquire similar estimates for $|P(x)|$ and $|P(w)|_p$ so that

$$\begin{aligned} |P(x)| &\ll 2^{-t(q_1+k_2T^{-1}-1-(n-1)\varepsilon_1)},\\ |P(z)| &\ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)},\\ |P(w)|_p &\ll 2^{-t(s_1+m_2T^{-1}-(n-1)\varepsilon_1)}. \end{aligned} \tag{9}$$

We now consider the case where at most one polynomial belongs to each parallelepiped M. The number of such polynomials is at most $c(n)2^{t(k_2+2l_2+m_2)T^{-1}} = c(n)2^{td_2}$. Hence, from (6) the total measure of the set of $\mathbf{u} \in M$ satisfying (2) is

$$\leqslant c(n)2^{-t(n+1-d_1-d_2-4\nu)}.$$

From (5) it follows that $d_1 + d_2 < n + 1 - 4v$ so the series $\sum_{t=1}^{\infty} 2^{-t(n+1-d_1-d_2-4v)}$ converges and the proposition follows from the Borel–Cantelli lemma.

Now assume that the parallelepipeds M contain two or more polynomials P. All of these polynomials are irreducible, with degree at most n and height at most 2^{t+1}. For two such polynomials $P_1, P_2 \in M$ the system of inequalities (9) holds. Using Lemma 4, with $\tau_1 = q_1 + k_2T^{-1} - 1 - (n-1)\varepsilon_1$, $\tau_2 = r_1 + l_2T^{-1} - 1 - (n-1)\varepsilon_1$, $\tau_3 = s_1 + m_2T^{-1} - (n-1)\varepsilon_1$, $\eta_1 = k_2T^{-1}$, $\eta_2 = l_2T^{-1}$, $\eta_3 = m_2T^{-1}$, we obtain

$$3q_1 + k_2T^{-1} + 6r_1 + 2l_2T^{-1} + 3s_1 + m_2T^{-1} - 12(n-1)\varepsilon_1 < 2n + \delta.$$

Replacing q_1 by k_2T^{-1}, $2r_1$ by $2l_2T^{-1}$ and s_1 by m_2T^{-1} gives

$$2(d_1 + d_2) - 12(n-1)\varepsilon_1 < 2n + \delta,$$

which for $\delta = \varepsilon_1$ and $\varepsilon > 6n\varepsilon_1$ contradicts the condition in Proposition 1. This completes the proof.

Proposition 2. *If* $\sum_{H=1}^{\infty} \Psi(H) < \infty$ *then* $\mu(L_n(v)) = 0$ *when the polynomials are restricted to the subclass* $\mathcal{P}^t(0,0,0)$ *for which* $d_1 + d_2 < 4 - \varepsilon$.

Proof. We denote by $L_n'(v)$ the set of solutions (x, z, w) of the system of inequalities

$$\begin{array}{l} \max(|P(x)|, |P(z)|, |P(w)|_p|) < H^{-\frac{n-3}{4}+v}\Psi^{\frac{1}{4}}(H), \\ H^{0.9-v} < \max(|P'(x)|, |P'(z)|) < H^{1-v}, \\ H^{-0.1-v} < |P'(w)|_p < H^{-v}, \end{array} \tag{10}$$

Denote by $L_n''(v)$ the set $L_n(v) \setminus L_n'(v)$. Then for all $(x, z, w) \in L_n''(v)$ we have

$$\begin{array}{l} \max(|P(x)|, |P(z)|, |P(w)|_p|) < H^{-\frac{n-3}{4}+v}\Psi^{\frac{1}{4}}(H), \\ \max(|P'(x)|, |P'(z)|) < H^{0.9-v}, \\ |P'(w)|_p < H^{-0.1-v}. \end{array} \tag{11}$$

We replace $\Psi(H)$ by H^{-1} in (11). Further, we use the method which was introduced by Borbat [8] to get that the new system of inequalities has infinitely many solutions only for a set (x, z, w) of measure zero.

Now we investigate the set $L_n'(v)$. By Lemma 1, all solutions (x, z, w) for a fixed $P \in \mathcal{P}^t$ satisfying (1) are contained in the parallelepiped $\sigma_2(P)$ defined by the inequalities

$$\begin{array}{rcl} |x - \alpha_1| & \ll & 2^{-t(\frac{n-3}{4}-v)}\Psi(2^t)^{1/4}|P'(\alpha_1)|^{-1}, \\ |z - \beta_1| & \ll & 2^{-t(\frac{n-3}{4}-v)}\Psi(2^t)^{1/4}|P'(\beta_1)|^{-1}, \\ |w - \gamma_1|_p & \ll & 2^{-t(\frac{n-3}{4}-v)}\Psi(2^t)^{1/4}|P'(\gamma_1)|_p^{-1}. \end{array} \tag{12}$$

Define a second parallelepiped $\sigma_4(P)$ to be the set of points satisfying the inequalities

$$\begin{aligned} |x-\alpha_1| &\ll 2^{-t(\frac{1}{2}-\nu)}|P'(\alpha_1)|^{-1}, \\ |z-\beta_1| &\ll 2^{-t(\frac{1}{2}-\nu)}|P'(\beta_1)|^{-1}, \\ |w-\gamma_1|_p &\ll 2^{-t(\frac{1}{2}-\nu)}|P'(\gamma_1)|_p^{-1}. \end{aligned} \tag{13}$$

Clearly, $\sigma_2(P) \subset \sigma_4(P)$.

Using the Mean Value Theorem for the polynomial P in $\sigma_4(P)$ we obtain

$$P(x) = P'(\alpha_1)(x-\alpha_1) + 1/2P''(\xi_1)(x-\alpha_1)^2, \;\; \xi_1 \in (\alpha_1, x).$$

Estimating each term in the last equality individually gives

$$\begin{aligned} &|P'(\alpha_1)||x-\alpha_1| \ll 2^{-t(\frac{1}{2}-\nu)}, \\ &|P''(\alpha_1)||x-\alpha_1|^2 \ll 2^{-t(\frac{9}{5}-4\nu)}. \end{aligned}$$

For $3\nu < 1.3$ we obtain that $|P(x)| \ll 2^{-t(0.5-\nu)}$ for $x \in \sigma_4(P)$. It is easy to do the same for $|P(z)|$ and $|P(w)|_p$ so that for $\nu < 0.1$

$$\begin{aligned} |P(x)| &\ll 2^{-t(\frac{1}{2}-\nu)}, \\ |P(z)| &\ll 2^{-t(\frac{1}{2}-\nu)}, \\ |P(w)|_p &\ll 2^{-(\frac{1}{2}-\nu)}. \end{aligned} \tag{14}$$

We similarly estimate $P'(x) = P'(\alpha_1) + P''(\xi_2)(x-\alpha_1), \;\; \xi_2 \in (\alpha_1, x)$ on $\sigma_4(P)$. As before, each term is estimated individually so that

$$\begin{aligned} |P'(\alpha_1)| &\ll 2^{-t(\nu-1)}, \\ |P^{('')}(\xi_2)||x-\alpha_1| &\ll 2^{-t(-1+0.5-\nu+1-\nu-0.1)} \ll 2^{-t(-2\nu+0.4)}. \end{aligned}$$

Hence, $|P'(x)| \leqslant 2|P'(\alpha_1)| \ll 2^{-t(\nu-1)}$ for $\nu < 0.1$. From this and similar inequalities for $P'(x)$ the following inequalities hold on $\sigma_4(P)$ for $\nu < 0.1$

$$\begin{aligned} |P'(z)| &\ll 2^{-t(\nu-1)}, \\ |P'(w)|_p &\ll 2^{-t\nu}. \end{aligned} \tag{15}$$

Fix the vector $d = (a_6, a_7, \ldots, a_n)$, $|a_j| \leqslant 2^{t+1}$ and let $\mathcal{P}_d^t$ denote the set of polynomials $P \in \mathcal{P}^t$ with the same vector d. The parallelepiped $\sigma_4(P_1)$ is called *essential* if for all polynomials $P_2 \in \mathcal{P}_d^t$

$$\mu(\sigma_4(P_1) \cap \sigma_4(P_2)) < \frac{1}{2}\mu(\sigma_4(P_1)).$$

If, on the other hand, there exists $P_2 \in \mathcal{P}_d^t$ such that

$$\mu(\sigma_4(P_1) \cap \sigma_4(P_2)) \geqslant \frac{1}{2}\mu(\sigma_4(P_1)),$$

then the parallelepiped $\sigma_4(P_1)$ is called *inessential.*

First, assume that $\sigma_4(P_1)$ is essential. Then, it follows that

$$\sum_{P_1 \in \mathcal{P}_d^t} \mu(\sigma_4(P_1)) \ll \mu(\mathbf{T}).$$

Also, from (12) and (13),

$$\mu(\sigma_2(P_1)) \ll \mu(\sigma_4(P_1))2^{t(-n+5)}\Psi(2^t).$$

Since the number of classes $\mathcal{P}_d^t$ is at most $c(n)2^{t(n-5)}$ from the above two displayed inequalities we have

$$\sum_d \sum_{P_1 \in \mathcal{P}_d^t} \mu(\sigma_2(P_1)) \ll 2^t\Psi(2^t)\mu(T).$$

By Lemma 5, the series $\sum_{t=1}^{\infty} 2^t\Psi(2^t)$ converges and the proof for the case of essential intervals can be completed using the Borel–Cantelli Lemma.

Now, assume that $\sigma_4(P_1)$ is inessential so that there exists $P_2 \in \mathcal{P}_d^t$ such that

$$\sigma(P_1, P_2) = \sigma_4(P_1) \cap \sigma_4(P_2), \qquad \mu(\sigma(P_1, P_2)) \geqslant \frac{1}{2}\mu(\sigma_4(P_1)).$$

The systems of inequalities (14) and (15) hold simultaneously on $\sigma(P_1, P_2)$ for both P_1 and P_2. Hence, if $R(f) = P_2(f) - P_1(f) = b_5f^5 + \ldots + b_1f + b_0$ then R satisfies

$$\begin{aligned} |R(x)| &\ll 2^{-t(\frac{1}{2}-\nu)}, \\ |R(z)| &\ll 2^{-t(\frac{1}{2}-\nu)}, \\ |R(w)|_p &\ll 2^{-t(\frac{1}{2}-\nu)}, \\ |R'(x)| &\ll 2^{-t(\nu-1)}, \\ |R'(z)| &\ll 2^{-t(\nu-1)}, \\ |R'(w)|_p &\ll 2^{-t\nu}. \end{aligned} \tag{16}$$

If $\theta_1, \ldots, \theta_5$ are the roots of R then

$$\begin{aligned} R(f) &= b_5(f-\theta_1)(f-\theta_2)\ldots(f-\theta_5), \\ R'(\theta_1) &= b_5(\theta_1-\theta_2)\ldots(\theta_1-\theta_5). \end{aligned}$$

From (4) and (16) it follows that there must be another real root close to the real root α. By the same argument, the complex root β has another complex root which is close to it, and similarly, for its conjugate $\bar{\beta}$. Hence, there is a contradiction as R cannot have 6 roots.

Proposition 3. *If* $\sum_{H=1}^{\infty}\Psi(H)<\infty$ *then* $\mu(L_n(v))=0$ *when the polynomials are restricted to the subclass* $\mathcal{P}^t(0,0,0)$ *for which*

$$4-\varepsilon\leqslant d_1+d_2\leqslant n+\varepsilon. \tag{17}$$

Proof. Instead of system (1) we use system (2). Exactly as in (7) the parallelepiped T is divided into parallelepipeds M. Let $P\in M$ and develop P as a Taylor series to obtain (9). For some $\theta>0$ consider only parallelepipeds which contain at most $c(n)2^{t\theta}$ polynomials. Then, by Lemma 1, the measure of the set of points $\mathbf{u}\in T$ which satisfy (2) is at most the measure of the parallelepiped $\sigma(P)$ (defined in (6)) multiplied by the number of parallelepipeds M and $c(n)2^{t\theta}$, that is

$$c(n)2^{-t(n+1-d_1-d_2-\theta-4v)}.$$

If $\theta<n+1-d_1-d_2-4v$ then the series $\sum_{t=1}^{\infty}2^{-t(n+1-d_1-d_2-\theta-4v)}$ converges and the Borel–Cantelli Lemma can be used to complete the proof. Thus, from now on, we assume that $\theta\geqslant u=n+1-d_1-d_2-4v$. From (17), $1-4v-\varepsilon\leqslant u\leqslant n-3-4v+\varepsilon$. Let $u_1=u-d$ where $d=0.14$. Writing u_1 as a sum of integer and fractional parts $[u_1]+\{u_1\}$ calculate

$$p=n-[u_1]=d_1+d_2-1+\{u_1\}+d+4v. \tag{18}$$

According to the Dirichlet box principle, there are at least $k=c(n)2^{t(d+fu_1g)}$ polynomials $P_1,\dots,P_k$ among these $c(n)2^{tu}$ polynomials whose first $[u_1]$ highest coefficients are the same. Consider the $k-1$ polynomials $R_j(f)=P_j(f)-P_1(f)$ for $2\leqslant j\leqslant k$. It can be readily verified that

$$\begin{aligned}|R_j(x)| &\ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)},\\ |R_j(z)| &\ll 2^{t(1-r_1-l_2T^{-1}+(n-1)\varepsilon_1)},\\ |R_j(w)|_p &\ll 2^{t(-s_1-m_2T^{-1}+(n-1)\varepsilon_1)},\end{aligned} \tag{19}$$

with $2\leqslant j\leqslant k$, $\deg R_j\leqslant n-[u_1]$ and $H(R)\leqslant 2^{t+2}$. The polynomials $R_j(f)=b_{n-[u_1]}f^{n-[u_1]}+\cdots+b_1f+b_0$ are now divided into sets. In each set the values of the coefficients $b_{n-[u_1]},\dots,b_1$ lie in an interval of length $2^{t(1-h_1)}$ where $h_1=\{u_1\}(n-[u_1])^{-1}$. Again apply Dirichlet's box principle to obtain that there are at least $L=c(n)2^{td}$ polynomials R_j in one such set. These will be renumbered $R_1,\dots,R_L$. Develop the $R_j'(f)$ as a Taylor series on M and consider the polynomials $S_i(f)=R_i(f)-R_1(f)$, which satisfy

$$\begin{aligned}&|S_i(x)|\ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)}, && |S_i'(x)|\ll 2^{t(1-q_1+(n-1)\varepsilon_1)}\\ &|S_i(z)|\ll 2^{t(1-r_1-l_2T^{-1}+(n-1)\varepsilon_1)}, && |S_i'(z)|\ll 2^{t(1-r_1+(n-1)\varepsilon_1)}\\ &|S_i(w)|_p\ll 2^{t(-s_1-m_2T^{-1}+(n-1)\varepsilon_1)}, && |S_i(w)|_p\ll 2^{t(-s_1+(n-1)\varepsilon_1)},\end{aligned} \tag{20}$$

with $2\leqslant i\leqslant L$, $\deg S_i\leqslant n-[u_1]$, and $H(S_i)\ll 2^{t(1-h_1)}$. Note that $\min(q_1,r_1,s_1)\geqslant v$ in this case.

There are three possibilities to consider. First assume that all the polynomials S_i have the form $i_1S, i_2S, \dots, i_LS$ for some fixed polynomial S. Then $i' = \max_{1\leqslant j\leqslant L}|i_j| \geqslant c(n)2^{td}$ and (20) holds for $i'S_0$ with $H(S_0) \ll 2^{t(1-h_1-d)}$. By (20),

$$|S_0(x)||S_0(z)|^2|S_0(w)|_p \ll 2^{t(3-d_1-d_2-3d+4(n-1)\varepsilon_1)}. \tag{21}$$

Then we apply for the system (20) the strengthening of the Lemma 3 which we can get by using the induction method in the Sprindzuk's theory of essential and inessental domains [11]. The proof will be complete if it can be shown that

$$|S_0(x)||S_0(z)|^2|S_0(w)|_p < H(S_0)^{2-\deg S_0+4\nu-\epsilon_2}. \tag{22}$$

By passing to the height of the polynomial S in (21),

$$\begin{aligned} &d_1+d_2-3+3d-4(n-1)\varepsilon_1 > (n-[u_1]-2-4\nu+\epsilon_2)(1-h_1-d),\\ &pd-4\nu d-2\{u_1\}/p-4\nu\{u_1\}/p-4(n-1)\varepsilon_1-\epsilon_2(1-\{u_1\}/p-d)>0, \end{aligned}$$

This is true for $d = 0.14$, $\nu \leqslant 0.027$, $p \geqslant 4$ and ϵ_1, ϵ_2 sufficiently small.

For the second case, assume that one of the polynomials S_i, $1 \leqslant i \leqslant L$ (say, S_0), is reducible, i.e. $S_0 = S_0^{(1)}S_0^{(2)}$. Then, for one of these, for example $S_0^{(1)}(f)$ the system (20) holds and $\deg S_0^{(1)}(f) \leqslant n-[u_1]-1$. In this case Lemma 3 can be applied if it can be proved that the inequalities

$$\begin{aligned} &d_1+d_2-3-4(n-1)\varepsilon_1 > (d_1+d_2-4+\{u_1\}+d)(1-h_1),\\ &1-4\nu-d-4(n-1)\varepsilon_1-3\{u_1\}/p>0 \end{aligned} \tag{23}$$

hold. It is not difficult to show that this is true for $d = 0.14$, $\nu \leqslant 0.027$, $p \geqslant 4$ and ε_1 sufficiently small.

Finally assume that among the S_i there are at least two polynomials (say S_1 and S_2) which have no common roots. Pass to the height of the polynomials S_i in (20) and apply Lemma 4 with $h = 1-h_1$. Then,

$$\begin{aligned} \tau_1 &= (q_1+k_2T^{-1}-1-(n-1)\varepsilon_1)h^{-1}, \quad \eta_1 = k_2T^{-1}h^{-1},\\ \tau_2 &= (r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)h^{-1}, \quad \eta_2 = l_2T^{-1}h^{-1},\\ \tau_3 &= (s_1+m_2T^{-1}-(n-1)\varepsilon_1)h^{-1}, \quad \eta_3 = m_2T^{-1}h^{-1},\\ &\deg S \leqslant n-[u_1], \end{aligned}$$

and the inequality

$$3q_1+k_2T^{-1}+6r_1+2l_2T^{-1}+3s_1+m_2T^{-1}-12(n-1)\varepsilon_1-9h_1 < 2(n-[u_1])h+\delta$$

must hold. Reduce the LHS by replacing q_1 with k_2T^{-1}, $2r_1$ with $2l_2T^{-1}$ and s_1 with m_2T^{-1}. This gives that

$$\delta > 2-2d-8\nu-\frac{9\{u_1\}}{p}-12(n-1)\varepsilon_1.$$

If $n-[u_1]\geqslant 6$ then the above inequality is a contradiction for $d=0.14$, $\nu\leqslant 0.027$ and sufficiently small δ and ε_1. Hence, the set of (x,z,w) for which the inequalities hold for infinitely many polynomials like S_i of which two have no common roots is empty.

The proof when $n-[u_1]=4$ and $n-[u_1]=5$ can be done exactly as in [9]. The proof of the proposition is complete.

Now we consider the case when each coordinate is equal to 1 in the vector (i_1,i_2,i_3) of the definition of linearity. So the system

$$\begin{aligned} q_1+k_2T^{-1} &\geqslant \frac{n+2}{4}-\nu,\\ r_1+l_2T^{-1} &\geqslant \frac{n+2}{4}-\nu,\\ s_1+m_2T^{-1} &\geqslant \frac{n-2}{4}-\nu, \end{aligned} \tag{24}$$

holds together with system (2).

Proposition 4. *If* $\sum_{H=1}^{\infty}\Psi(H)<\infty$ *then* $\mu(L_n(\nu))=0$ *when the polynomials are restricted to the subclass* $\mathcal{P}^t(1,1,1)$.

Proof. Using (2) and Lemma 1 we obtain

$$\begin{aligned} |x-\alpha_1| &\ll 2^{-t\left(\frac{\frac{n+2}{4}-q_2-\nu}{2}\right)}=2^{-t\mu_1},\\ |z-\beta_1| &\ll 2^{-t\left(\frac{\frac{n+2}{4}-r_2-\nu}{2}\right)}=2^{-t\mu_2},\\ |w-\gamma_1|_p &\ll 2^{-t\left(\frac{\frac{n-2}{4}-s_2-\nu}{2}\right)}=2^{-t\mu_3}. \end{aligned} \tag{25}$$

Let $\sigma_5(P)$ be the parallelepiped defined by these inequalities. Divide the parallelepiped T into smaller parallelepipeds M with sidelengths $2^{-t(\mu_1-\gamma)}$, $2^{-t(\mu_2-\gamma)}$ and $2^{-t(\mu_3-\gamma)}$ where $\gamma=\frac{1}{10n}$. Let $P\in M$ and develop it as a Taylor series on M. As before, obtain an upper bound for all the terms in the series. The estimates for the real coordinate are presented below.

$$\begin{aligned} |P'(\alpha_1)||x-\alpha_1| &\ll 2^{t\gamma}|P'(\alpha_1)2^{-t\mu_1}|\ll 2^{t(1-q_1+\gamma+(n-1)\varepsilon_1+\nu/2+q_2/2-(n+2)/8)}\\ &\ll 2^{t(\nu+2\gamma+(2-n)/4+(n-1)\varepsilon_1)},\\ |P''(\xi_1)||x-\alpha_1|^{(2)} &\ll 2^{2t\gamma}|P''(\alpha_1)2^{-2t\mu_1}|\ll 2^{t(\nu+2\gamma+(2-n)/4+(n-1)\varepsilon_1)}. \end{aligned}$$

Obtain similar estimates for $|P(z)|$ and $|P(w)|_p$ so that the inequalities

$$\begin{aligned} |P(x)| &\ll 2^{t(\nu+2\gamma+(2-n)/4+(n-1)\varepsilon_1)},\\ |P(z)| &\ll 2^{t(\nu+2\gamma+(2-n)/4+(n-1)\varepsilon_1)},\\ |P(w)|_p &\ll 2^{t(\nu+2\gamma+(2-n)/4+(n-1)\varepsilon_1)} \end{aligned} \tag{26}$$

hold. Two cases are now considered. First, assume that at most one polynomial P belongs to each parallelepiped M. The number of these parallelepipeds is $c(n)2^{t(\mu_1+2\mu_2+\mu_3-4\gamma)}$ so the measure of the set of $\mathbf{u} \in M$ satisifying (2) and (24) (using (25)) is

$$c(n)2^{-t(\mu_1+2\mu_2+\mu_3-\mu_1-2\mu_2-\mu_3+4\gamma)} = c(n)2^{-4t\gamma}.$$

Clearly the series $\sum_{t=0}^{\infty} c(n)2^{-4t\gamma}$ is convergent which is enough to complete the proof in this case.

Now assume that the parallelepipeds M contain two or more polynomials P_1 and P_2 (remember that we may assume P_1 and P_2 are irreducible). For such polynomials the system of inequalities (26) holds and they do not have common roots. Use Lemma 4, with

$$\begin{aligned} \tau_1 = \tau_2 = \tau_3 = -\nu - 2\gamma - \frac{2-n}{4} - (n-1)\varepsilon_1, \\ \eta_1 = -\frac{1}{2}(\nu + q_2 + \frac{-n-2}{4}) - \gamma, \\ \eta_2 = -\frac{1}{2}(\nu + r_2 + \frac{-n-2}{4}) - \gamma, \\ \eta_1 = -\frac{1}{2}(\nu + s_2 + \frac{2-n}{4}) - \gamma, \end{aligned}$$

to obtain

$$2 + 2n - 8\nu - 16\gamma - 12(n-1)\varepsilon_1 + (q_2 + 2r_2 + s_2) < 2n + \delta,$$

so that

$$\delta > 2 - 8\nu - 16\gamma - 12(n-1)\varepsilon_1 + (q_2 + 2r_2 + s_2).$$

If $16\gamma + 12(n-1)\varepsilon_1 < 0.5$ then $\delta > 1.5 - 8\nu$. Hence, for $\delta = 0.1$ and $\nu < 0.175$ this is a contradiction. Thus, there do no exist parallelepipeds M containing two or more irreducible polynomials and Proposition 4 is proved.

In the cases when one or two coordinates are equal to 1 in the linearity vector (i_1, i_2, i_3) we must combine the calculation for the subclass $\mathcal{P}^t(0,0,0)$ and $\mathcal{P}^t(1,1,1)$. Putting all the propositions together completes the proof of the theorem (more details are in [9]).

We indicate the following important applications of Theorem 1.

First, we adapt Theorem 1 to the problem for polynomials with small discriminant. The discriminant of the polynomial can be written as the determinant

$$D(P)=(-1)^{\frac{n(n-1)}{2}}\begin{vmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\ n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & 0 & \cdots & 0 \\ 0 & na_n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & na_n & (n-1)a_{n-1} & \cdots & a_1 \end{vmatrix} \tag{27}$$

or as the product of squares of root differences

$$D(P) = a_n^{2n-2} \prod_{1\leqslant j<i\leqslant n} (\alpha_i - \alpha_j)^2. \tag{28}$$

From (28)it follows that $D(P) = 0$ if and only if the polynomial P has multiple roots. By (27), we obtain that if $D(P) \neq 0$ then $D(P) \geqslant 1$.

If the first coefficient a_n of the polynomial $P(f)$ is a sufficiently large integer and the inequality $\min_{1\leqslant j<i\leqslant n} |\alpha_i - \alpha_j| > \delta > 0$ holds for the roots of $P(f)$, then

$$|D(P)| > c(\delta)a_n^{2n-2}.$$

Further, let Q be a sufficiently large number with

$$H(P) \leqslant Q. \tag{29}$$

Denote by $\mathbf{P}_n$ the set of polynomials satisfying (29). From (27) – (29) it can be seen that all the values of $D(P)$ belong to the interval

$$[-c(n)Q^{2n-2}, c(n)Q^{2n-2}]. \tag{30}$$

By (29), we also note that the set $\mathbf{P}_n$ contains exactly $(2Q + 1)^{n+1}$ polynomials (including the zero polynomial).

For some prime number q, positive integer l and $\rho > 0$ denote by $\mathbf{P}_n(Q, q, l, \rho)$ the subset of polynomials $P \in \mathbf{P}_n$, for which

$$D(P) \leqslant Q^{2n-2-2\rho}, \tag{31}$$

$$q^l \,\|\, D(P). \tag{32}$$

Here $q^l \| D(P)$ means that $q^l | D(P)$ and $q^{l+1} \nmid D(P)$. The question of how many polynomials satisfy (31) or (32) or both together is a natural problem in the theories of Diophantine approximation and the theory of Diophantine equations.

From (28) we obtain that (31) holds if the distance between two roots of the polynomial $P_n(f)$ decreases as Q increases. In particular, it will hold if for some j, $1 \leqslant j \leqslant n$, the derivative

$$|P'(\alpha_j)| = |a_n(\alpha_j - \alpha_1)\cdots(\alpha_j - \alpha_{j-1})(\alpha_j - \alpha_{j+1})\cdots(\alpha_j - \alpha_n)|$$

tends to 0 as $Q \to \infty$. If (32) holds then the p-adic norm $|P'(\alpha_i)|_p$ is small for some i, $1 \leqslant i \leqslant n$.

From Theorem 1 we can obtain the lower bounds for the derivatives of the polynomial P in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ for the set of the point $\mathbf{u}_1 = (x, z, w) \in \mathbf{T}_1$, $\mathbf{T}_1 \subset \mathbf{T}$, for which $\mu(\mathbf{T}_1) > \frac{\mu(T)}{2}$. By Lemma 1, for every point $\mathbf{u}_1 \in \mathbf{T}_1$ there exists a point with three algebraic coordinates. The value of the derivative of the polynomial P_1 in the algebraic coordinate for every metric will satisfy the system of inequalities (1) if $<$ is replaced by $\ll$. This gives that the inequalities (31) and (32) hold for the parameters ρ and l. These parameters depend on ν because the discriminant contains the derivative of the polynomial at the roots. Then, we can choose a point $\mathbf{u}_2 \in \mathbf{T}_1$ for which there exists a polynomial $P_2 \not\equiv P_1$. Such a point $\mathbf{u}_2$ exists because $\mu(\mathbf{T}_1) > \frac{\mu(T)}{2}$. This procedure allows us to construct a large number of polynomials satisfying conditions (31) and (32).

As a second application of a Theorem 1, we would like to investigate the more general question when the first inequality in (1) is

$$\max(|P(x)|, |P(z)|, |P(w)|_p|) < H^{-\frac{n}{4}+t+\nu}\Psi^{1-t}(H), \ 0 < t < 1.$$

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