JOURNAL OF ECONOMIC THEORY 34, 13–31 (1984)

A Unified Beta Pricing Theory*

GREGORY CONNOR

J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201

Received December 31, 1982; revised January 3, 1984

This paper derives Ross's mutual fund separation theory and a new, equilibrium version of Ross's arbitrage pricing theory as special cases of a general theory. The paper also reveals that the two theories are identical in their predictions of asset prices and portfolio returns. The capital asset pricing model (a restricted case of the mutual fund separation theory) receives special treatment. *Journal of Economic Literature* Classification Numbers: 021, 313.

1. INTRODUCTION

This paper proves fundamental similarities between two asset pricing theories. It compares the mutual fund separation theory (the general case of the popular capital asset pricing model) with a new, competitive equilibrium version of the well-known arbitrage pricing theory. The paper presents both a theoretical and empirical unification of the two theories. First, it derives both of the theories simultaneously as two cases of a general theory. Second, the paper shows that the testable implications of the two are empirically indistinguishable if the analyst only observes asset prices and investors' portfolio returns.

The capital asset pricing model (CAPM) receives special treatment as an important restricted case of the mutual fund separation theory (MFST). Previous authors have remarked that the CAPM provides a convenient "black box" which mimics the intuitive understanding of portfolio choice and asset pricing in a large, diverse economy. This paper goes a step further: it formally constructs a model which follows this intuitive understanding, and shows an exact empirical equivalence between this model and the CAPM.

Some of the results of this paper have appeared elsewhere in different form. The mutual fund separation theory is due to Ross [10]. The version

* This is a revised version of the second chapter of my Ph.D. dissertation, completed under the direction of Stephen Ross at the Department of Economics, Yale University. I would like to thank Carol Mershon Connor, Philip Dybvig, Jonathan Ingersoll, Alvin Klevorick, Stephen Ross, Daniel Siegel, and Ludo van der Heyden for their many helpful comments.

0022-0531/84 $3.00

642/34/1-2

described herein simplifies some of the arguments in his proof but sacrifices the generality of his result. The arbitrage pricing theory (APT) is also due to Ross [8]. This paper derives a competitive equilibrium version of the APT. Ross's arbitrage theory describes the prices of a large, but unspecified, subset of the assets. This competitive equilibrium theory describes the prices of all the assets in the economy. The competitive equilibrium approach also makes it possible to see close relationships between the pricing of assets in a large economy and classic principles of market efficiency and portfolio diversification.

Section 2 defines a *factor economy* as an economy in which asset payoffs obey a factor model. The theory applies both to a factor economy with a finite number of assets and to one with an infinite number of assets. Although these two types of economies seem quite different, the general theory rarely needs to distinguish between them. The paper uses general-dimensional linear algebra to treat both cases simultaneously.

Section 3 defines *market insurance* as the ability of market trading to completely eliminate idiosyncratic risk from investor portfolios. Section 4 proves the general pricing theory for any factor economy obeying the market insurance conditions.

Section 5 proves that it is impossible to empirically distinguish the finite assets from the infinite assets version of the theory. Section 6 focuses on the CAPM as a special case of the finite assets model, and argues that the CAPM serves well as an "as if" model, mimicking the predictions of the more intuitive limit theory. Section 7 provides a summary of the paper.

2. The Definition of a Factor Economy

This section gives some basic definitions for an economy in which asset payoffs follow a factor model.

The vector of per-share, gross payoffs for the ν risky assets can be written as[1]

$$x = c + Bf + i. \tag{1}$$

The κ-vector of random variables, $f = (f_1, f_2, \ldots, f_\kappa)$, consists of the *market factors*. The market factors describe the economy-wide random influences which (linearly) affect the payoffs of assets. They are normalized so that $E[f_\lambda] = 0$.

[1] Throughout the paper, upper case Roman letters represent matrices, lower case letters represent either vectors or functions, and Greek letters represent scalars. Subscripted terms are an exception to this rule: x_3 would be a scalar, the third component of the vector x. An apostrophe denotes the transpose of a vector or matrix.

The *beta matrix* B is a $(\nu \times \kappa)$-matrix of constants; the γth row of B is the vector of *factor betas* for the γth asset. Without loss of generality, it can be assumed that B has full column rank. Otherwise, there would exist an equivalent factor model with fewer factors. (See Appendix 1 of [3] for a discussion of equivalent factor models.)

The nonrandom ν-vector c measures the per-share expected payoffs of the assets. The paper assumes throughout that there exists a riskless asset with per-share payoff x_0.

The vector of *idiosyncratic variates*, $i = (i_1, i_2, \dots, i_\nu)$, represents the extra random variation specific to individual assets. These random variates are normalized so that $E[i_\gamma] = 0$. The covariance matrix of idiosyncratic terms is assumed to exist and is denoted by V:

$$V = E[ii'].$$

If V is singular, let V^{-1} denote the Moore–Penrose inverse[2] of V.

The paper assumes that none of the ν risky assets is redundant, and therefore $E[(x-c)(x-c)']$ is nonsingular. A *portfolio* (a_0, a) is a linear functional on $R^{\nu+1}$. The product of a portfolio with the asset payoffs $a_0 x_0 + a'x$ is the *portfolio payoff*. A norm is defined on the space of portfolios by using the second moment of the portfolio payoffs:

$$\|(a_0, a)\| = E[(a_0 x_0 + a'x)^2]^{1/2}.$$

Ross's arbitrage pricing theory uses a sequential-economy approach. Ross considers a sequence of economies with an increasing number of assets and proves an approximate pricing result. The approximation increases in accuracy as the number of assets grows large.

Chamberlain and Rothschild [2] have recently shown that many of Ross's results can be succinctly restated by examining a *fixed* economy with an *infinite* number of assets. This allows the approximating properties of the sequential model to become exact properties. This paper adapts the Chamberlain–Rothschild technique to a competitive equilibrium version of the APT.[3]

The paper models an economy with a countably infinite collection of risky assets whose payoffs obey a factor model:

$$x = c + Bf + i, \tag{2}$$

where the terms have the same definitions as in the finite case, replacing

[2] For any finite matrix X there exists a unique matrix X^{-1} (the Moore–Penrose inverse of X) having the properties: $XX^{-1}X = X$, $X^{-1}XX^{-1} = X^{-1}$, $(XX^{-1})' = XX^{-1}$, and $(X^{-1}X)' = X^{-1}X$.

[3] See [4] for a sequential-economy model of the equilibrium arbitrage pricing theory.

vectors in R^{ν} with vectors in R^{∞}. Again, I assume that there exists a riskless asset with payoff x_0.

For simplicity, the limit model is specified with a finite number of investors. Without loss of rigor, one can view each of these investors as representing an infinite number of investors of μ different types. Within each type, all investors are identical.

As in the finite number case, a *portfolio*, (a_0, a), is a linear functional on the space of asset payoffs, R^{∞}. The product of a portfolio with the asset payoffs $a_0 x_0 + a'x$ is the *portfolio payoff.* Portfolios are restricted to those linear functionals whose payoffs have a finite second moment:

$$E[(a_0 x_0 + a'x)^2] < \infty.$$

This second moment defines the norm on the space:

$$\|(a_0, a)\| = E[(a_0 x_0 + a'x)^2]^{1/2}. \tag{3}$$

This definition of a portfolio means that two collections of asset holdings which produce identically the same portfolio payoff constitute the same portfolio.[4] The definition eliminates spuriously different portfolios. The space of portfolios is a Hilbert space.

An example will illustrate the definition. An investor might choose to hold one share of the riskless asset and one-half share of each of the first two risky assets. This portfolio could be represented by the array of real numbers: $(1, \frac{1}{2}, \frac{1}{2}, 0, 0, 0,\ldots)$; or the investor might hold $\frac{1}{3}$ share of the first three risky assets: $(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0,\ldots)$. This can be extended to any number ν of assets: $(1, 1/\nu, 1/\nu,\ldots, 1/\nu, 0, 0,\ldots)$. This model also lets an investor hold the limit of such a sequence:

$$\operatorname*{limit}_{\nu\to\infty}(1, 1/\nu, 1/\nu, 1/\nu,\ldots).$$

Although such a portfolio cannot be represented by a fixed array of real numbers, it is well defined as an element in the vector space of linear functionals on R^{∞} under the norm (3).

Define B^{ν} as the matrix consisting of the first ν rows of B and i^{ν} as the vector consisting of the first ν elements of i. Let $V^{\nu} = E[i^{\nu} i^{\nu\prime}]$. In the infinite assets case it will be assumed that B^{ν} has full column rank and V^{ν} is nonsingular for every ν. Note that the sequence must begin at some value of ν greater than or equal to κ.

[4] Using this norm creates equivalence classes containing all linear functionals whose difference in payoff has a zero second moment. That is, if (a_0, a) and (b_0, b) are such that $E[(a_0 x_0 + a'x - b_0 x_0 - b'x)^2] = 0$ then $(a_0, a) = (b_0, b)$.

The following definitions apply to both the finite and limit factor economies. Let (q_0, q) be the market portfolio, that is, the per-capita supply of assets.

A vector of *asset prices* (p_0, p) is a vector of real numbers with dimension equal to the number of assets. The *cost* of a portfolio (a_0, a) is the product $a_0 p_0 + a'p$.

An *allocation* $\{(a_0^1, a^1), (a_0^2, a^2), ..., (a_0^\mu, a^\mu)\}$ is a collection of portfolios, one for each investor, which satisfies the resource constraint:

$$\sum_{\gamma=1}^{\mu} a^\gamma = \mu q, \qquad \sum_{\gamma=1}^{\mu} a_0^\gamma = \mu q_0,$$

where μ is the number of investors. This is just the economy-wide constraint and does not include any budget constraint.

An *initial allocation* $\{(e_0^1, e^1), ..., (e_0^\mu, e^\mu)\}$ is an allocation representing the endowments of investors. A *competitive equilibrium* consists of an *equilibrium allocation* $\{(a_0^1, a^1), ..., (a_0^\mu, a^\mu)\}$ and *equilibrium prices* (p_0, p) such that

$$\text{(budget feasibility)} \qquad a_0^\gamma p_0 + a^{\gamma\prime} p \leqslant e_0^\gamma p_0 + e^{\gamma\prime} p$$

and

$$\text{(optimality)} \qquad E[u^\gamma(a_0^\gamma x_0 + a^{\gamma\prime} x)] \geqslant E[u^\gamma(g_0 x_0 + g'x)]$$

for any portfolio (g_0, g) which is budget feasible.

The main assumptions of the theory will be summarized in two definitions. A *finite factor economy* is a set of investors, assets, and endowments such that:

(F1) There are μ investors, all of whom have risk-averse, von Neumann–Morgenstern utility functions.

(F2) There are ν risky assets with per-share payoffs obeying (1), and $V = E[ii']$ and $E[ff']$ exist.

(F3) There exists a riskless asset with per-share payoff x_0.

(F4) $(B'B)$ and $E[(x-c)(x-c)']$ are nonsingular.

(F5) $E[i \mid f] = 0$.

(F6) The economy has a competitive equilibrium.

A *limit economy* is a set of investors, assets, and endowments such that:

(L1) There are μ investors all of whom have risk-averse von Neumann–Morgenstern utility functions.

(L2) There is a countably infinite collection of risky assets obeying (2), and $V^\nu = E[i^\nu i^\nu]$ and $E[ff']$ exist for every ν.

(L3) There exists a riskless asset with per-share payoff x_0.

(L4) $(B^{v\prime}B^{v})$ and V^{v} are nonsingular for every v.

(L5) $E[i \mid f] = 0$.

Note that in the finite economy the covariance matrix of idiosyncratic terms may be singular. If the matrix is nonsingular, it is not possible to completely eliminate idiosyncratic risk from portfolios with a finite number of assets. In the limit economy, one does not need singularity—investors can eliminate idiosyncratic risk by diversifying (i.e., holding many assets, each in very small quantity).

Assumption (F6) can be made exogenous by using Hart's [5] results.[5] The existence of equilibrium in the limit case will follow from its existence in the finite case (Theorem 4).

3. Insurable Factor Economies

This section gives conditions under which all investors in a factor economy (finite or limit case) are able to diversify away idiosyncratic risk. It also shows, by a Pareto-efficiency argument, that if investors can diversify away idiosyncratic risk, they will do so in competitive equilibrium.

First, the section defines a well-diversified portfolio as one with zero idiosyncratic risk. It defines an insured allocation as one in which all portfolios are well diversified. An economy is insurable if there exists an insured allocation for every distribution across investors of expected payoff and market risk. It is shown that if an economy is insurable then the competitive equilibrium allocation is insured.

It is unnecessary to distinguish between the finite and limit cases of the theory in this section. The linear algebra is general dimensional, covering both cases simultaneously.

Definition. A *well-diversified portfolio* (w_0, w) is one with no idiosyncratic risk: $E[(w'i)^2] = 0$.

Although the definition is the same in the finite and limit cases, the "diversification mechanism" behind it differs in the two cases. In the finite assets model, investors eliminate risk by exploiting the singularity in the covariance matrix of idiosyncratic terms. Investors hold particular combinations of assets whose idiosyncratic risks exactly offset one another. In the infinite assets model, investors diversify by holding many assets, each in small quantity. The infinite number formalism of the model expresses the limit of

[5] If one adds the distributional assumptions that i, f are bounded and that $x \geqslant 0$, then Hart's Theorem 3.3 applies and competitive equilibrium exists.

this process: investors hold an infinite number of assets, each in infinitesimally small quantity.

An investor who holds a well-diversified portfolio is effectively insured against idiosyncratic risk. If assets are allocated so that all investors hold well-diversified portfolios, then all investors are insured.

DEFINITION. An allocation is *insured* if it consists entirely of well-diversified portfolios.

The next definition of this section draws an equivalence between portfolios which have identical expected payoffs and factor risk but different idiosyncratic risk.

DEFINITION. Two portfolios (a_0, a) and (w_0, w) are *factor equivalent* if $a'B = w'B$ and $a_0 x_0 + a'c = w_0 x_0 + w'c$. Two allocations,

$$\{(a_0^1, a^1), (a_0^2, a^2), ..., (a_0^\mu, a^\mu)\} \quad \text{and} \quad \{(w_0^1, w^1), (w_0^2, w^2), ..., (w_0^\mu, w^\mu)\},$$

are *factor equivalent* if all of their corresponding portfolios are factor equivalent: (a_0^γ, a^γ) is factor equivalent to (w_0^γ, w^γ) for $\gamma = 1, 2, ..., \mu$.

Economies in which all investors can be assigned well-diversified portfolios, for any distribution of expected payoff and factor risk, are called *insurable.*

DEFINITION. A factor economy is *insurable* if for any allocation there exists a factor-equivalent, insured allocation.

The next theorem gives the insurability conditions for any factor economy.

THEOREM 1. *A factor economy is insurable if and only if*

$$E[(q'i)^2] = 0 \tag{4}$$

and for any κ*-vector* b *there exists a well-diversified portfolio* (a_0, a) *such that*

$$a'B = b'. \tag{5}$$

The Appendix states condition (5) in terms of the primitive elements of the factor model.

Proof (Necessity). Given that an economy is insurable, let $\{(w_0^1, w_1), (w_0^2, w^2), ..., (w_0^\mu, w^\mu)\}$ be any insured allocation. Note that

$$0 = \frac{1}{\mu} \sum_{\gamma=1}^{\mu} E[(w^{\gamma\prime} i)^2]^{1/2} \geqslant E[(q'i)^2]^{1/2},$$

where the inequality follows from the Cauchy–Schwartz inequality. Therefore $E[(q'i)^2]=0$.

Let b be any κ-vector. Since B has full column rank, for any b there exists a portfolio (z_0, z) such that $z'B=b'$. Under the hypothesis of insurability there exists a well-diversified portfolio (w_0, w) which is factor equivalent to (z_0, z); hence,

$$w'B=z'B=b',$$

which proves that condition (5) is necessary for insurability.

(Sufficiency). Let $\{(a_0^1, a^1), (a_0^2, a^2), \ldots, (a_0^\mu, a^\mu)\}$ be any allocation. Replace $a^1, a^2, \ldots, a^{\mu-1}$ with well-diversified, factor-equivalent portfolios $w^1, w^2, \ldots, w^{\mu-1}$. Let

$$w^\mu=\mu q-\sum_{\gamma=1}^{\mu-1} w^\gamma$$

$$w_0^\gamma=\frac{1}{x_0}(a_0^\gamma x_0+a^{\gamma\prime}c-w^{\gamma\prime}c).$$

The allocation $\{(w_0^1, w^1), (w_0^2, w^2), \ldots, (w_0^\mu, w^\mu)\}$ is an insured factor-equivalent alternative to $\{(a_0^1, a^1), (a_0^2, a^2), \ldots, (a_0^\mu, a^\mu)\}$. It is insured because $w^1, w^2, \ldots, w^{\mu-1}$ are well diversified by construction, and w^μ is well diversified by the Cauchy–Schwartz inequality:

$$E[(w^{\mu\prime}i)^2]^{1/2}\leqslant\mu E[(q'i)^2]^{1/2}+\sum_{\gamma=1}^{\mu-1}E[(w^{\gamma\prime}i)^2]^{1/2}.$$

It is factor equivalent because w^1, $w^2, \ldots, w^{\mu-1}$ are factor equivalent by construction and for w^μ:

$$w^{\mu\prime}B=\mu q'B-\sum_{\gamma=1}^{\mu-1} w^{\gamma\prime}B=\mu q'B-\sum_{\gamma=1}^{\mu-1} a^{\gamma\prime}B=a^{\mu\prime}B.$$

The collection of portfolios is an allocation because

$$\sum_{\gamma=1}^{\mu} w^\gamma=\mu q-\sum_{\gamma=1}^{\mu-1} w^\gamma+\sum_{\gamma=1}^{\mu-1} w^\gamma=\mu q$$

and

$$\sum_{\gamma=1}^{\mu} w_0^\gamma=\sum_{\gamma=1}^{\mu} a_0^\gamma+\frac{1}{x_0}(\mu q'c-\mu q'c)=\mu q_0.$$

Q.E.D

Given the assumptions of a factor economy, an investor prefers a well-diversified portfolio to a factor-equivalent, undiversified one.

REMARK 1. In a factor economy, any investor strictly prefers a well-diversified portfolio to a factor-equivalent portfolio with nonzero idiosyncratic variance.

Proof. Let (w_0, w) be well diversified and factor equivalent to (a_0, a) which has nonzero idiosyncratic variance. Using $w_0 x_0 + w'c = a_0 x_0 + a'c$ and $w'B = a'B$:

$$\begin{aligned} &E[u(a_0 x_0 + a'x)] \\ &\quad = E[u(w_0 x_0 + w'c + w'Bf + a'i)] < E[u(w_0 x_0 + w'c + w'Bf)] \\ &\quad = E[u(w_0 x_0 + w'x)], \end{aligned}$$

where the inequality follows from $E[i \mid f] = 0$ and Jensen's inequality.

Q.E.D.

The next result is fundamental. It is based on the classic principle that competitive markets efficiently allocate risk. Since insured allocations are preferable (by Remark 1) and possible (in an insurable economy), the Pareto-efficiency of competitive equilibrium guarantees that the competitive allocation is insured.

THEOREM 2. *In an insurable factor economy, the competitive allocation is insured.*

Proof. Let $\{(a_0^1, a^1),..., (a_0^\mu, a^\mu)\}$ be a competitive allocation. Construct the factor-equivalent, insured allocation $\{(w_0^1, w^1), (w_0^2, w^2),..., (w_0^\mu, w^\mu)\}$ described in Theorem 1. By Remark 1, $E[u(w_0^\gamma x_0 + w^{\gamma\prime}x)] \geqslant E[(a_0^\gamma x_0 + a^{\gamma\prime}x)]$, where the inequality is an equality if (a_0^γ, a^γ) is well diversified. Comparing the two allocations gives

$$\begin{aligned} E[u(w_0^1 x_0 + w^{1\prime}x)] &\geqslant E[u(a_0^1 x_0 + a^{1\prime}x)] \\ E[u(w_0^2 x_0 + w^{2\prime}x)] &\geqslant E[u(a_0^2 x_0 + a^{2\prime}x)] \\ \vdots \qquad & \qquad \vdots \\ E[u(w_0^\mu x_0 + w^{\mu\prime}x)] &\geqslant E[u(a_0^\mu x_0 + a^{\mu\prime}x)]. \end{aligned}$$

Unless all of the inequalities are equalities, the allocation $\{(w_0^1, w^1), (w_0^2, w^2),..., (w_0^\mu, w^\mu)\}$ Pareto-dominates $\{(a_0^1, a^1), (a_0^2, a^2),..., (a_0^\mu, a^\mu)\}$. Therefore, by efficiency, the competitive allocation $\{(a_0^1, a_1), (a_2^2, a^2),..., (a_0^\mu, a^\mu)\}$ is insured. Q.E.D.

Theorem 2 has as a corollary a mutual fund separation theorem.

COROLLARY 2.1. *In an insurable factor economy, each investor's equilibrium portfolio consists of a linear combination of $\kappa + 1$ mutual funds.*

Proof. By condition (5) there exists a well-diversified portfolio for every factor beta position. Construct the κ well-diversified portfolios $\{(a_0^1, a^1), (a_0^2, a^2), \ldots, (a_0^\kappa, a^\kappa)\}$ with factor beta positions:

$$a^{1\prime}B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad a^{2\prime}B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \ldots, \qquad a^{\kappa\prime}B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Theorem 2 implies that each investor holds a portfolio which can be described as a linear combination of these "mutual fund" portfolios and the riskless asset. Q.E.D.

The standard statement of mutual fund separation is that investors are *indifferent* between a mutual fund portfolio and any other portfolio. Corollary 2.1 may seem surprisingly strong—they always *hold* a mutual fund portfolio. This is due to the different definition of a portfolio. The usual definition distinguishes between portfolios which have the same return (i.e., redundant portfolios) whereas the definition used in this paper does not.

The separation result of the CAPM (Sharpe [14]) is a special case of Corollary 2.1. Corollary 2.1 is less general than Ross's [10] portfolio separation theorem for the MFST since Ross describes sufficient *and* necessary conditions. However, this paper's version has the strength that it extends Ross's small-economy theory to the large-economy case.

Ross also treats the pricing consequences of his portfolio separation theorem. The pricing results in the next section parallel his, but apply to both the finite and limit versions of the theory.

4. PRICE LINEARITY IN AN INSURABLE FACTOR ECONOMY

This section proves the paper's pricing thorem. Equilibrium prices in an insurable factor economy are linear in the expected payoffs and factor betas of the assets. All of the theorems in this section apply to both the finite and limit cases. Most of the linear algebra is general dimensional: separate proofs for the finite and limit cases are rarely necessary.

The proof of the pricing theorem relies on the result that investors hold well-diversified portfolios. An investor with such a portfolio is risk neutral at the margin with respect to idiosyncratic risk. This "marginal risk neutrality" means that the investor cannot be in competitive equilibrium unless prices are linear in the factor coefficients. Otherwise, he can improve his expected utility by undiversifying his portfolio.

DEFINITIONS. Let (p_0, p) be the vector of per-share prices of the assets. Normalize the prices so that the per-dollar return on the riskless asset equals one $(x_0/p_0 = 1)$. This normalization on prices is retained throughout the paper. Prices are *linear* if a vector m exists such that

$$p_0 = x_0, \qquad p = c + Bm.$$

The vector m is called the vector of *factor prices.*

A portfolio with zero cost, zero risk, and a positive expected payoff is an arbitrage portfolio. The next lemma treats a portfolio which is "almost" an arbitrage portfolio, except that it contains idiosyncratic risk.

LEMMA 1. *In any factor economy there exists a portfolio* (h_0, h) *with zero cost, zero factor risk, and positive expected payoff:*

$$h_0 p_0 + h'p = 0, \qquad h'B = 0, \qquad h_0 x_0 + h'c > 0,$$

if and only if prices are not linear.

Proof (Sufficiency). Consider first a finite economy. Suppose that prices are not linear. Let m be the vector which minimizes the sum of squares $d'd$ in the equation

$$p = c + Bm + d.$$

By a well-known property of least-squares residuals, the vector d so chosen is orthogonal to B. Construct the portfolio

$$h_0 = \frac{1}{x_0}(c'd + d'd), \qquad h = -d.$$

This portfolio has zero cost, zero factor risk, and an expected payoff of

$$h_0 x_0 + h'x = c'd + d'd - c'd = d'd$$

which is greater than zero whenever $d \neq 0$.

For the limit economy case, construct the ν-vector p^ν which consists of the first ν elements of p. Similarly, construct the vector c^ν and the $(\nu \times \kappa)$-matrix B^ν. Let m^ν be the vector which minimizes the sum of squares $d^{\nu\prime} d^\nu$ in the equation

$$p^\nu = c^\nu + B^\nu m^\nu + d^\nu.$$

If $d^{\nu}=0$ for all ν, then $m^{\nu}=m^{*}=(B^{\nu\prime}B^{\nu})^{-1}(p^{\nu}-c^{\nu})$ for all ν. This implies that $p=c+Bm^{*}$.

Given that $d^{\nu}\neq 0$ for some ν, in the limit economy let $d^{*}=(d^{\nu},0,0,0,...)$ and construct the zero cost, zero factor risk portfolio:

$$h_0=\frac{1}{x_0}(d^{*\prime}c+d^{*\prime}d^{*}), \qquad h=-d^{*}$$

which has an expected payoff of $d^{\nu\prime}d^{\nu}>0$.

(Necessity, both finite and limit case). Suppose that prices are linear. Let (h_0,h) be any hedge portfolio. It has cost:

$$h_0p_0+h'p=h_0x_0+h'c+h'd.$$

Since $d=0$, $h'd=0$. Hence

$$h_0p_0+h'p=h_0x_0+h'c>0$$

and the portfolio has positive cost. Q.E.D.

Lemma 1 above shows that if prices are not linear, then there exists some "risk premium" for idiosyncratic risk: an investor can costlessly earn positive expected profit by incurring idiosyncratic risk. The next theorem shows that there cannot exist any such risk premium in an insurable factor economy. Therefore, prices must be linear.

THEOREM 3. *In an insurable factor economy, competitive equilibrium prices are linear.*

Proof. Let (a_0,a) be the competitive equilibrium portfolio of some investor. By Theorem 2 it is well diversified. Let (h_0,h) be any zero cost, zero factor risk portfolio. By the optimality of (a_0,a) and the zero cost of (h_0,h):

$$\frac{d}{d\varepsilon}E[u(a_0x_0+a'x+\varepsilon(h_0x_0+h'x))]_{\varepsilon=0}=0, \tag{6}$$

otherwise the investor could costlessly increase expected utility by adding an increment of (h_0,h) to his portfolio. Solving for the derivative (6):

$$(h_0x_0+h'c)\,E[u'(a_0x_0+a'x)]+E[(h'i)\,u'(a_0x_0+a'x)].$$

Using $E[i\,|f]=0$ and $a'i\equiv 0$ gives

$$(h_0x_0+h'c)\,E[u'(a_0x_0+a'x)]. \tag{7}$$

Since $(7) = 0$ from (6), and $E[u'(a_0 x_0 + a'x)] > 0$ by the risk-aversion assumption, this implies

$$(h_0 x_0 + h'c) = 0,$$

and this is equivalent to price linearity. Q.E.D.

The finite economy case in Theorem 3 is the same as Ross's MFST pricing theorem. The limit economy case is a strengthened version of Ross's APT.[6] The next section unifies these two theories in another way by showing an empirical equivalence.

5. The Factor Pricing Theory Isomorphism

This section shows that an insurable finite economy and an insurable limit economy are not empirically distinguishable. Given identity of investor preferences and endowments in the two economies, the economies produce "isomorphic" competitive equilibria. The equilibrium portfolio return of each investor is the same in the two economies, and the linear formula for competitive equilibrium prices is the same as well.

Definition. A finite economy and a limit economy *correspond* if they have the same investor preferences and the same market factors, and their endowments are factor equivalent.

A price vector in a finite economy p and one in a corresponding limit economy p^* are superficially different, since one belongs to R^v and the other to R^∞. However, if each of the price vectors is linear in the expected payoffs and factor betas of its assets:

$$p = c + Bm, \qquad p^* = c^* + B^*m,$$

[6] Ross's APT guarantees that the sum of squared deviations from price linearity are bounded. That is, there exists m, d, ω such that $p_0 = x_0$, $p = c + Bm + d$, and $d'd < \omega < \infty$. This finite bound on the total sum of squares implies that the mean-squared pricing error goes to zero:

$$\lim_{v \to \infty} (1/v) \sum_{\gamma=1}^{v} (d^\gamma)^2 = 0. \tag{8}$$

The competitive equilibrium version guarantees that

$$d = 0. \tag{9}$$

See Shanken [13] for a discussion of the difference in testable implications between (8) and (9).

and the factor price vector m is the same in each, then the price vectors represent identical prices. The "underlying prices" (factor prices) are identical. A portfolio in a finite economy and a portfolio in a corresponding limit economy are superficially different, since one belongs to the dual space of $R^{\nu+1}$ and the other to the dual space of R^{∞}. However, if the portfolio payoffs are identical, then in an important sense the two portfolios are the same.

DEFINITION. Competitive equilibria of a finite economy and of a corresponding limit economy are *isomorphic* if prices are linear in each economy, the factor prices are the same in each, and every investor has the same portfolio payoff in the two equilibrium allocations.

The use of the term "isomorphic" deserves explanation. Note that in either economy there is a linear function from an asset's expected payoff and factor betas to its equilibrium price. If the two competitive equilibria are isomorphic, then this function is the same in each economy. Similarly, for any investor (specified by preferences and an endowment of factor risk and expected payoff) there is a function[7] to an equilibrium portfolio payoff. If two economies are isomorphic then this function is the same. The term "isomorphic" denotes the equivalence across these two functions.

THEOREM 4. *Consider an insurable finite economy and a corresponding insurable limit economy. For any competitive equilibrium in either economy there is an isomorphic one in the other economy.*

Proof. Consider a competitive allocation $\{(a_0^1, a^1),..., (a_0^u, a^u)\}$ and price vector (p_0, p) in an insurable finite economy. By Theorem 2, the allocation is insured. Construct (see Theorem 1) a factor-equivalent, insured allocation in the corresponding limit economy, $\{(a_0^{*1}, a^{*1}),..., (a_0^{*u}, a^{*u})\}$. Construct the price vector:

$$p_0 = x_0, \qquad p^* = c^* + B^*m,$$

where m is the factor price vector from the linear prices in the finite economy. This allocation and price vector form a competitive equilibrium in the limit economy. Competitive equilibrium requires two properties, budget feasibility and budget optimality.

(Budget feasibility). Budget feasibility of the competitive equilibrium in the finite economy requires

$$e_0^{\gamma} p_0 + e^{\gamma\prime} p \geqslant a_0^{\gamma} p_0 + a^{\gamma\prime} p.$$

[7] If there are multiple competitive equilibria then this will be a relation rather than a function, but that causes no difficulty.

If two economies have linear prices with the same factor prices, then factor-equivalent portfolios in them have the same cost (as the reader can quickly prove). By the correspondence of the two economies, each investor's endowments are factor equivalent and $e_0^\gamma p_0 + e^{\gamma\prime} p = e_0^{*\gamma} p_0 + e^{*\gamma\prime} p^*$. Hence $(a_0^{*\gamma}, a^{*\gamma})$ is budget feasible:

$$e_0^{*\gamma} p_0 + e^{*\gamma\prime} p^* \geqslant a_0^{*\gamma} p_0 + a^{*\gamma\prime} p^*.$$

(Budget optimality). In an insurable economy with linear prices, an investor only needs to consider well-diversified portfolios to find his optimal portfolio. For any poorly diversified portfolio, there exists a factor-equivalent (and therefore equal cost) portfolio which he will strictly prefer.

Let (g_0, g) be any budget-feasible portfolio in the finite economy. By the optimality of competitive portfolios, (a_0, a) is weakly preferred to (g_0, g). Let (g_0^*, g^*) be any budget-feasible portfolio in the limit economy. If $E[u(g_0^* x_0 + g^{*\prime} x)] > E[u(a_0^* x_0 + a^{*\prime} x)]$ then there is a budget-feasible (g_0, g) in the finite economy such that $E[u(g_0 x_0 + g' x)] > E[u(a_0 x_0 + a' x)]$, contradicting the competitive equilibrium in the finite economy.

The proof reverses exactly for the mapping from an equilibrium in the limit economy to an isomorphic one in the finite economy. Let $\{(a_0^{1*}, a^{1*}), \ldots, (a_0^{\mu *}, a^{\mu *})\}$, (p_0^*, p^*) be an equilibrium in the limit economy. By Theorem 3 the prices are linear: $p^* = c^* + B^* m$. Construct the factor-equivalent, insured allocation, $\{(a_0^1, a^1), \ldots, (a_0^\mu, a^\mu)\}$ in the finite economy and define p by $p = c + Bm$, using the factor price vector from the limit economy equilibrium. Repeat the proof of feasibility and optimality given above. Q.E.D.

Theorem 4 is my second step in building upon Ross's evidence that the APT and MFST are closely related theories (Ross [10, p. 278]). My first step is the unified derivation of the MFST and (equilibrium) APT presented in Section 4. Theorem 4 provides a more formal connection. It is an "impossibility theorem": one can never distinguish between the finite and limit models of this paper, i.e., between the MFST and equilibrium APT, based on prices or portfolio returns. This is true even if one fully knows the prices which will occur in competitive equilibrium under the two models (see Scarf [12] for how one might compute the prices). It would be treacherous to claim that there is absolutely *no* testable distinction between the two. The theorem only guarantees that portfolio return and asset price observations alone will not suffice to distinguish them.

6. Choosing between the Cases

Most of this paper has been dedicated to minimizing the distinctions between the finite and limit cases of the theory. This section attempts to choose which provides a better model of asset pricing based on the differences that remain. It argues that the limit case is superior, since it requires a weaker assumption on asset supplies. However, the CAPM, a restricted version of the finite-assets case, avoids this untenable assumption.

The most telling comparison of the finite and limit cases centers on the assumption that the market portfolio is well diversified:

$$E[(q'i)^2] = 0. \tag{4}$$

As discussed earlier, the "diversification mechanism" behind (4) differs between the two cases. In the finite case, asset supplies must be in a particular proportion such that the idiosyncratic risks exactly cancel out of the market portfolio. This "singularity" assumption is very restrictive on asset supplies. It is also nongeneric: a perturbation of asset supplies would almost surely destroy this property.

In the limit case, (4) arises naturally. Asset supplies need not be in any special ratio. Rather, the supply of each asset must be "very small" (infinitesimal) on a per-capita basis. Together with reasonable bounds on the correlations of the idiosyncratic variates this guarantees (4).

The CAPM does not use the framework of the more general MFST and does not require the strong assumption on asset supplies. The CAPM does not assume that a factor model *generates* asset payoffs. Instead, it uses asset payoffs to *construct* a factor model. Given a market portfolio q and asset payoffs x consider the *market model* defined by:

$$\begin{aligned}
f &= q'x - E[q'x] \\
c &= E[x] \\
B &= \operatorname{cov}(x, q'x)/\operatorname{var}(q'x) \\
x &= c + Bf + i.
\end{aligned}$$

This is an "ex-post" factor model: asset payoffs determine the factor value, rather than the factor value affecting asset payoffs. The CAPM uses this "ex-post" factor model to generate the finite-case theory while avoiding its untenable restriction on asset supplies. The well-diversified condition on the market portfolio follows immediately from the definition of the factor model, since $q'i = q'x - (\operatorname{var}(q'x)/\operatorname{var}(q'x))\, q'x \equiv 0$. If x is joint normal (the usual CAPM assumption), then the market model satisfies the other distributional requirement $(E[i \mid f]=0)$ and price linearity follows by the theorems of Section 4.

The CAPM obviously has been a useful model of asset pricing. Some theorists have objected to the model because its derivation is unintuitive. This paper provides a formal justification for the simple model. The CAPM produces the same empirical predictions as a more sophisticated theory which follows the analyst's intuition. The CAPM provides a convenient "black box" which mimics the structure of asset prices and portfolio returns in a large, diverse economy.

7. Summary

This paper unifies the mutual fund separation theory with a new, equilibrium version of the arbitrage pricing theory. It considers two economies, one with a fixed, finite number of assets and another with an infinite number of assets. The mutual fund separation theory is proven on the finite assets economy and the equilibrium-version arbitrage pricing theory on the infinite assets economy. The paper describes a common framework of assumptions for the two economies and proves the two theories simultaneously under this framework.

The paper next proves an empirical equivalence. Given that investor preferences and endowments are the same in the two economies, it demonstrates that equilibrium prices and portfolio returns are identical in them. Hence it is impossible to distinguish between the two theories by observing competitive equilibrium prices or portfolio returns.

Although the capital asset pricing model is mathematically a special case of the finite assets model, it requires separate treatment. The CAPM takes a different viewpoint from the more general MFST. By doing so, it avoids the finite model's most objectionable assumption.

Various authors (e.g., Roll and Ross [7]) have argued that the CAPM is flawed because it does not capture the intuition which motivates its use. Others have countered that the model seems to serve as a convenient "black box"—generating predictions similar to those one would expect from a more intuitively plausible model. This paper supports the "black box" justification of the CAPM with a more rigorous argument. The infinite assets theory herein directly follows the large-economy intuition that motivates the CAPM. The finite assets theory is empirically identical to this infinite assets theory. The CAPM, a restricted version of the finite assets theory, is empirically identical to a (restricted version of) the infinite assets theory. Hence the CAPM provides a remarkably accurate black box representation of the large-economy intuition.

APPENDIX

LEMMA 2. *In a finite factor economy, condition* (5) *holds if and only if* $(I - V^{-1}V)B$ *has full column rank.*

Proof (Sufficiency). Suppose $(I - V^{-1}V)B$ has full column rank. Then for any b there exists a solution z to the equation:

$$z'(I - V^{-1}V)B = b'.$$

For any b the portfolio of risky assets $w = (I - V^{-1}V)z$ is well diversified and solves the equation:

$$w'B = b';$$

hence condition (5) holds.

(Necessity). Suppose condition (5) holds. Then for any b there exists a w such that

$$w'B = b' \qquad \text{and} \qquad w'V = 0.$$

The second condition implies that $w'(I - V^{-1}V) = w'$; therefore, for any b there exists a w such that

$$w'(I - V^{-1}V)B = b'$$

which implies that $(I - V^{-1}V)B$ has full column rank. Q.E.D.

Let $\|\cdot\|_\kappa$ denote a matrix norm on $R^{\kappa\times\kappa}$. In general, any norm will do since all norms on $R^{\kappa\times\kappa}$ generate the same neighborhood of zero. For convenience, the proof below uses the Euclidean norm

$$\|X\|_\kappa = \max_{\substack{g'g=1\\ g\in R^\kappa}} |g'Xg|.$$

LEMMA 3. *In a limit factor economy, condition* (5) *holds if*

$$\lim_{v\to\infty} \|(B^{v\prime}(V^v)^{-1}B^v)^{-1}\|_\kappa = 0.$$

Proof. Suppose that $\lim_{v\to\infty} \|(B^{v\prime}(V^v)^{-1}B^v)^{-1}\|_\kappa = 0$. Consider the sequence of risky asset portfolios defined by

$$w^v = (V^v)^{-1}B^v(B^{v\prime}(V^v)^{-1}B^v)^{-1}b'$$
$$w^{*v} = (w^v, 0, 0, 0, \ldots).$$

Each portfolio in this sequence has a factor beta vector of b because

$$w^{*v\prime}B = w^{v\prime}B^{v} = b'.$$

The variances of the portfolios are:

$$w^{*v\prime}Vw^{*v} = w^{v\prime}V^{v}w^{v} = b'(B^{v\prime}(V^{v})^{-1}B^{v})^{-1}b \leqslant (b'b)\,\|(B^{v\prime}(V^{v})^{-1}B^{v})^{-1}\|_{\kappa}.$$

Given $\lim_{v\to\infty}\|(B^{v\prime}(V^{v})^{-1}B^{v})^{-1}\|_{\kappa} = 0$, this implies $\lim_{v\to\infty} w^{v\prime}V^{v}w^{v} = 0$. The sequence of portfolios (w^{*v}) has constant factor risk and idiosyncratic variance approaching zero. Therefore, the sequence converges. Let $w = \lim_{v\to\infty} w^{*v}$. Note that

$$w'B = b' \qquad \text{and} \qquad E[(w'i)^2] = 0. \qquad \text{Q.E.D.}$$

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