The provision of mathematics support and the role of the history of mathematics

by Ciarán Mac an Bhaird

Abstract

There is a well-documented crisis in mathematics education both nationally and internationally. Increasing numbers of students are struggling with many of the basics of mathematics when they enter third level. One response has been the widespread establishment of mathematics support services for students who are deemed at risk. These supports are constantly analysed and quantified to establish best practice and to measure the impact they have on students. In this paper we give an overview of these issues. We also discuss how introducing struggling students to topics in the historical development of mathematics can help them deal successfully with many of the issues that they have.

1. Introduction

In this paper we start by giving an overview of the crisis in mathematics education. We describe the response that many third level institutes nationally and internationally have taken by establishing extra mathematics support services to target students who are struggling. Most of these supports do not include the use of the history of mathematics to any large extent. We consider research which investigates the effectiveness of these various initiatives and the type of student who will engage with these supports. We also gain some insight into the type of problems that students have and the reasons why they are often reluctant or afraid to engage with mathematics at any level.

We briefly considered the idea of introducing students with weak mathematical skills to the history of mathematics during

conversations with Professor John Cleary at the National University of Ireland Maynooth (NUIM) in 2004. However, it was some particular incidents in the Mathematics Support Centre (MSC) in 2007 that caused us to seriously consider this method. On one occasion, while a student was struggling with certain aspects of Calculus he asked 'Is Calculus just made up to torture students?' On another occasion, while addressing issues with zero and infinity, a student asked 'Why are there all these crazy rules?' The MSC provides a very friendly and non-judgmental atmosphere and students who attend are much more open about the issues they have with mathematics than they would be in a normal classroom environment. This is quite rewarding because it allows us to get to the heart of the problems they are having. It is a common occurrence during weaker students' initial visits that they comment when struggling with a concept that they think they are 'really stupid' or 'slow'. They get frustrated because they are struggling with basic concepts, whilst in the middle of a higher level mathematics course.

When we started to explain the historical background to these mathematical topics, we discovered that these students had rarely thought of this before. These students have little or no concept of mathematics as a continuous and growing subject. They generally have no idea why they are studying their course material. They try to apply the rules and hope to get close enough to the answer. Hourigan and O'Donoghue mention this lack of understanding.[1]

In general students with weak mathematical backgrounds have little or no exposure to the background and development of mathematics. They are also unlikely to investigate the history of mathematics independently. It is well documented that students who struggle often think that they are the first or only people to have difficulties with these problems and thus rarely ask for help due to fear of embarrassment. They also often perceive mathematics knowledge as an instant knowledge

[1] Hourigan and O'Donoghue 2007.

instead of something that can be worked out over a period of time. We find that explaining that some mathematical concepts have taken thousands of years to develop into their current form actually reassures the students. They see the bigger picture and realize that mathematical understanding is not automatic, it involves success and failure. These methods are an essential part of the process of understanding. Students who realize this can overcome the fear of tackling new mathematical material, a major issue as outlined by Grehan, Mac an Bhaird and O'Shea.[2]

We want to introduce the history of mathematics to weaker students, students who are unlikely to attend history of mathematics courses, read the textbooks or be involved in similar initiatives. Simply pointing the students towards the available resources would not have any significant impact. If they were already having difficulties with mathematics, why would they research the background of the material that is confusing them? In this paper, we use the example of an Italian solution to cubic equations and the subsequent development of imaginary numbers to demonstrate how many of the concerns that struggling students deal with on a regular basis have similarities to issues dealt with in the development of mathematics.

An initial description of attempts to address these issues is presented in Mac an Bhaird.[3] These include using the history of mathematics when designing a Mathematics Refresher Course for incoming mature students, using the history of mathematics when explaining concepts in the MSC and a project with **sigma** (The Centre for Excellence in Mathematics and Statistics Support) to develop history of mathematics resources to accompany existing resources on various mathematical topics.

[2] Grehan, Mac an Bhaird and O'Shea 2010.

[3] Mac an Bhaird 2009.

2. The crisis in mathematics education and the provision of mathematics support services

Considerable research has been carried out on the reasons why a significant number of students entering third level institutions in Ireland have basic mathematical problems and the repercussions that this has for education. Hourigan and O'Donoghue[4] and Lynch et al.,[5] contain detailed analyses of these issues in the teaching and learning of mathematics at second level in Ireland. Some of the main factors listed include: bad publicity for mathematics, negative attitudes towards the subject, little understanding of the context or background of mathematics and rote learning. The National Council for Curriculum and Assessment review,[6] and Cosgrove et al.,[7] have highlighted possible impacts of these problems. Low attainment in mathematics is cited as a contributing factor in low enrolment and retention rates in science and technology courses.[8] There is also widespread international concern about the numbers of students who have basic mathematical problems, for example Picker and Berry.[9]

One response from third level institutions in Ireland and the UK to try and address these issues has been the widespread provision of mathematics support services. The supports include a wide range of services such as: drop-in centres; workshops; peer-tutoring; online courses; podcasts; videos; leaflets and books. An excellent example of the type of resources being developed is available at http://www.mathcentre.ac.uk/. The main aim of these services is to assist students in overcoming their mathematical difficulties. Mathematics support is also active in helping students with different challenges and backgrounds such as Access, Disability and Mature students. It aims to provide a better mathematics learning experience for

[4] Hourigan and O'Donoghue 2007.
[5] Lynch et al. 2003.
[6] National Council for Curriculum and Assessment 2005.
[7] Cosgrave et al. 2004.
[8] Irish Department of Education and Science 2002.
[9] Picker and Berry 2001.

students from all these backgrounds. Gill, Johnson and O'Donoghue[10] and Perkin and Croft[11] contain a comprehensive audit of mathematics learning support services in Ireland and the UK.

Organisations such as NCE-MSTL (The National Centre for Excellence in Mathematics and Science Teaching and Learning) based in the University of Limerick and **sigma** based in the UK have been established to promote and expand, amongst other things, the area of mathematics support. **sigma**, for example, has helped establish mathematics support networks (hubs) in various parts of the UK. The Irish Mathematics Learning Support Network was established in 2009 to promote mathematics support on a national basis.

3. The Impact of Mathematics Support

Continuous and thorough evaluations of mathematics support services are of critical importance to the establishment of best practice and the maintenance of these services for the students who need them.

A number of studies have been carried out at Irish Universities. Gill and O'Donoghue,[12] Dowling and Nolan[13] and Mac an Bhaird, Morgan and O'Shea[14] all look at the effectiveness of these supports. Further evaluations have been carried out in the UK, for example Pell and Croft,[15] Patel and Little[16] and Lee et al.[17] The majority of the evidence that suggests that mathematics support is having a positive impact on student grades and student retention. Many of these papers look at the benefits of these supports to students with weak mathematical backgrounds.

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[10] Gill, Johnson and O'Donoghue 2008.
[11] Perkin and Croft 2004.
[12] Gill and O'Donoghue 2007.
[13] Dowling and Nolan 2006.
[14] Mac an Bhaird, Morgan and O'Shea 2009.
[15] Pell and Croft 2008.
[16] Patel and Little 2006.
[17] Lee et al. 2008.

Croft and Grove[18] look at the type of students who avail of support. There is evidence that not all students are engaging; in particular it should be noted that a significant minority of at-risk students are neither availing of support nor engaging with mathematics. Some authors, for example Ryan, Pintrich and Midgley,[19] have found that the fear of showing a lack of knowledge or ability negatively impacts on students' willingness to ask questions. Grehan et al.[20] focus on the fears that students expressed and how these fears prevented them from engaging with mathematics during their first year at university. This fear manifested itself in four different ways: fear of failure; fear of showing a lack of knowledge or ability; fear of being singled out; and fear of the unknown. Students also displayed a lack of awareness of services or structures within mathematics. Many of these factors were also identified in a study of students at Loughborough University.[21]

4. Existing resources and similar ideas

Using the history of mathematics in mathematics education is not a novel idea. Extensive material is available which discusses this issue, for example Katz[22] contains several interesting papers. Grugetti[23] states that 'An historical analysis allows teachers to understand why a certain concept is difficult for the student' and 'In observing the historical evolution of a concept, pupils will find that mathematics is not fixed and definitive'. Student teachers who are being trained to use the History of Mathematics in their teaching are interviewed by Isaacs, Mohan Ram and Richards.[24] One student commented: 'I have never been taught maths in this way before, rather I have always been given a set of problems and been told to solve them, which has made maths a boring subject. Yet, when faced with a question

[18] Croft and Grove 2006.
[19] Ryan, Pintrich and Midgley 2001.
[20] Grehan, Mac an Bhaird and O'Shea 2010.
[21] Symonds, Lawson and Robinson 2008.
[22] Katz 2000
[23] Grugnetti 2000, 29-35.
[24] Isaacs, Mohan Ram and Richards 2000, 123-33.

to answer, and the history behind a particular thing, it makes further questions easier to handle and I found it stays ingrained in the memory better.' These are issues we would like to address. However, almost without exception, the articles and material are geared towards specialist mathematical students. This gives us valuable insight into the process of introducing the history of mathematics to students in general. The question is how do we address this issue with the weaker students?

There is a wealth of material available on the history of mathematics, for example there is the comprehensive MacTutor website http://www-history.mcs.st-and.ac.uk/. However, we believe that students with weak mathematical backgrounds are unlikely to use these resources independently; at the very least it is safe to assume that a student struggling with basic mathematics will find a text on the history of mathematics very daunting. Many universities offer a History of Mathematics module as a standard undergraduate course. In general these courses are available for specialist mathematics students only, whereas the weaker students get little or no exposure to the background and origins of mathematics.

5. A topic from the history of mathematics

In this section we consider the origins and development of imaginary or complex numbers. We chose this topic because it is a common area of concern for students. We believe we can use this topic to show students that development and progression in mathematical methods has come from struggling with issues, concepts and problems. It is interesting that mathematical outcomes can often contradict or at least challenge the prevalent views of the time and we can learn a lot from how the different mathematicians through the ages reacted to answers which were, to them, quite strange or unexpected.

We also see the power of generalization in mathematical development and a demonstration that the formulae that students regularly use do not just appear, they developed over time (quite often thousands of years) to their current form. The fact that many of the conceptual problems that students have

today are echoed in the past is a very revealing and informative lesson for students and should reassure them that it is in thinking about a problem that true understanding is achieved, not simply from learning off its current format.

In the interests of brevity and clarity we write the methods and solutions almost exclusively in modern notation. Note that before we tackle these problems in such an algebraic way we discuss with students the importance of algebra by allowing them to see some mathematical problems in rhetorical form first. They quickly develop an appreciation for the use of algebra in representing mathematical problems. For more detail on these areas of mathematics see texts such as Boyer,[25] Eves[26] and Fauvel and Gray.[27]

5.1. The square root of a negative number

Students are introduced to quadratic equations $ax^2 + bx + c = 0, a, b, c \in \mathbb{R},\ a \neq 0$ and their general solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

relatively early in second level, and it is a technique that they use on a regular basis. They rarely see the origins of quadratic equations or the development of the general formula.

Regardless, students are quickly made aware that if $b^2 - 4ac < 0$ then this leads to the square root of a negative number. They should already have received an explanation based on the fact that the product of both two negatives and two positives yields a positive, thus the square root of a negative number does not exist. So a quadratic equation with $b^2 - 4ac < 0$ has no (real) solution. Some time later in their second level education many of them are then introduced to the concept of $i = \sqrt{-1}$ and (the very unfortunately named) complex or imaginary numbers. Students are now exposed to techniques for dealing with a mathematical situation where

[25] Boyer 1991.
[26] Eves 1990.
[27] Fauvel and Gray 1987.

$b^2 - 4ac < 0$. This can cause much confusion amongst students, particularly the weaker ones. They struggle with concepts of real and complex numbers and do not understand why the square root of a negative number is essentially not allowed in one section and then accepted in a later section. It is of fundamental importance to show these students that this struggle on this issue is almost a mirror image of the initial struggle and development of the topic hundreds of years earlier.

We turn to perhaps one of the most controversial times in the history of mathematics, 16th century Italy and the algebraic solution to cubic equations. This area of the history of mathematics is well documented in standard books on the History of Mathematics, for example Eves.[28] There are tales of intrigue, plagiarism and deception involving a number of different mathematicians, the main characters being Cardano and Tartaglia. We believe students can benefit significantly from being exposed to these scandals and some excitement in mathematics. We focus here on a general method for solving cubics of the form $x^3 + px = q$ which was described in *Ars Magna* written by Cardano in 1545. The process for determining the solution is interesting in its own right and follows a series of steps until the general solution is presented as

$$x = \sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} - \left(\frac{q}{2}\right)}$$

Several worked examples are presented in the book but the key moment comes when the formula is used to try to solve the equation $x^3 = 15x + 4$ and we encourage students to do this calculation. Following the steps described, the equation gave rise to $\sqrt{-121}$. Until this period square roots of negative numbers were essentially ignored. However, this cubic equation has the real solution $x = 4$ and thus cannot be dismissed.

It is remarkable that Cardano included this problem at all

[28] Eves 1990.

given all the stories of intrigue that surround him. However, clearly there was a realization that the square root of a negative number could no longer be ignored. Cardano commented on what had happened and that he was not quite sure what to do about it. This is a very important process in the world of mathematics, a constant reassessment of what is known and unknown and a continuous changing of views. Students really benefit from seeing how mathematicians deal with unexpected answers and they can see how much can be gained from considering strange and unexpected results in detail.

As a consequence of this unexpected solution to a cubic equation the concept of complex numbers was discovered and subsequently developed over a period of time into its current format. It is also rather unexpected that complex numbers (negative square roots) had their origins in dealing with cubic equations and not quadratic equations. In fact, this was the key to the realization that negative square roots could no longer be ignored.

6. Conclusions

Using the history of mathematics is one approach that should help with basic understanding of the material. We believe that this can help alter the students' image of mathematics and show that there is much more to mathematics than memorization. We also believe that these students could see many parallels between their own struggles and the struggles in the development of mathematics. The focus should not all be on getting the answer right or wrong, rather trying the problem and learning from what works and what does not work is much more likely to help their overall understanding.

Our work in this area is ongoing and we at a very preliminary stage of developing a comprehensive collection of historical materials to help students on a range of mathematical topics. A similar paper[29] was given at the History of Mathematics and Teaching of Mathematics Conference in

[29] Mac an Bhaird 2010.

Szeged, Hungary in May 2010. In that paper, the author goes into more detail in four areas in the history of mathematics: the Egyptian method of false position; Babylonian solutions to simultaneous non-linear equations; an Arabic geometric solution to cubic equations; and the Italian solution to cubic equations.

This paper is a brief overview of a paper presented at the John Cleary Memorial Conference in Trinity College Dublin in May 2010. To a large extent the content of the talk was inspired by Professor John Cleary. The author tutored his History and Philosophy of Mathematics module at the NUIM for many years. This module was introduced in NUIM to help potential teachers of mathematics gain an understanding of the origins and background of many of the topics that they would one day teach. It was during one of our weekly discussions in 2005 that the author first considered the possibility of introducing the History of Mathematics to students outside the module group. The author was especially interested in students with weaker mathematical backgrounds and discussed this at length with Professor Cleary. Professor Cleary's concern about the apparent downward trends in education were well known and examples of his thoughts on the area are available in Cleary.[30]

There is a significant problem in mathematics education and we believe that the history of mathematics can play a crucial role in helping all students with their understanding in mathematics. This is particularly true of weaker students for whom lack of context and background has been highlighted as an area in their mathematics education that needs attention. In particular, focus should be centered on particular aspects of and stories from the history of mathematics that would help students with their understanding of mathematics and mathematical processes.

The outstanding issue we face is how to expose weaker students to this material in an organized and coherent way; this is not an easy task. These students are less likely to be interested

[30] Cleary 2006.

in the background of material that they are already struggling to understand.

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