# Bose condensation and branes

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ABSTRACT: When the cosmological constant is considered to be a thermodynamical variable in black hole thermodynamics, analogous to a pressure, its conjugate variable can be thought of as a thermodynamic volume for the black hole. In the AdS/CFT correspondence this interpretation cannot be applied to the CFT on the boundary but, from the point of view of the boundary $SU(N)$ gauge theory, varying the cosmological constant in the bulk is equivalent to varying the number of colors in the gauge theory. This interpretation is examined in the case of $AdS_5 \times S^5$, for $\mathcal{N} = 4$ SUSY Yang-Mills at large $N$, and the variable thermodynamically conjugate to $N$, a chemical potential for color, is determined. It is shown that the chemical potential in the high temperature phase of the Yang-Mills theory is negative and decreases as temperature increases, as expected. For spherical black holes in the bulk the chemical potential approaches zero as the temperature is lowered below the Hawking-Page temperature and changes sign at a temperature that is within one part in a thousand of the temperature at which the heat capacity diverges.

# Contents

## 1 Introduction

It was suggested in [1] that in the presence of a cosmological constant a black hole with mass $M$ should be viewed as a thermodynamic system for which the mass is interpreted as the enthalpy, $M = H(S, P)$, with entropy $S$ and pressure $P = -\frac{\Lambda}{8\pi G_N}$. The variable thermodynamically conjugate to $P$ would then have the natural interpretation of a volume, [1, 2],

$$V = \frac{\partial H}{\partial P}\bigg|_S, \tag{1.1}$$

although it does not in general have any a priori connection to a notion of volume in the geometric sense [3].

This picture has been extensively investigated recently, for a review see [4], with most work focusing on the case of negative $\Lambda$ (an exception being [5]). A natural question to ask in this context is what the role of the cosmological constant might be on the boundary CFT, at finite temperature, in the AdS/CFT correspondence. As $\Lambda$ is then related to the number, $N$, of branes in the bulk, and this translates to the number of colors in the boundary gauge theory, the variable thermodynamically conjugate to $\Lambda$ should behave as a chemical potential for color,[1] [1, 6, 7].

The chemical potential $\mu$ is calculated in $AdS_5 \times S^5$, with a black hole in $AdS_5$, corresponding to a finite temperature $\mathcal{N} = 4$ SUSY gauge theory on the boundary at large $N$, [8]. The Hawking-Page phase transition in the bulk is equivalent to a phase transition in the boundary gauge theory with a mass gap in the high temperature phase, [9, 10]. We show that, in the high temperature phase of the boundary theory, $\mu$ is negative and is a decreasing function of temperature, consistent with general expectations for a chemical potential, [11]. Conversely if the Hawking temperature is lowered below that of the Hawking-Page phase transition the chemical potential associated with a 5-dimensional asymptotically AdS spherical black hole can become zero or even positive. Though the black hole solution is not the one most relevant for the physics when that happens, the temperature at which $\mu$ vanishes is very close to, within one part in a thousand, the

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[1]This should not be confused with a chemical potential for gluons or quarks, the former should vanish just as for photons and the latter vanishes in the absence of baryon violating processes. The chemical potential for color does not break supersymmetry.

temperature at which the heat capacity diverges. The phase with $\mu \leq 0$ could conceivably be accessed by supercooling the quark gluon plasma.

In the AdS/CFT correspondence the cosmological constant is normally considered to be a fixed parameter that does not vary. In $AdS_5 \times S^5$ however it is not given a priori, it is just a parameter in a solution of 10-dimensional supergravity, and is no more fundamental to the theory than a black hole mass or any other parameter in the metric. Indeed in some scenarios it can be dynamically determined by a scalar potential, [12, 13]. In [14] the thermodynamic energy of the boundary conformal field theory was calculated as a function of volume, temperature and $N$, but the bulk metric only has one parameter if $\Lambda$ is fixed, and even allowing $\Lambda$ to vary gives only two parameters which is not enough to provide thermodynamic potentials as a function of three independent parameters, that requires an extra assumption. Our philosophy here is to trade the two geometric parameters in the $AdS_5$ black hole, $r_h$ and $L$ below, for two thermodynamic variables, which are taken to be entropy and $N^2$ in the micro-canonical ensemble.

## 2 The chemical potential

The line element for an asymptotically AdS Schwarzschild black hole in 5-dimensions is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d^2\Omega_{(3)}, \tag{2.1}$$

where $d^2\Omega_{(3)}$ is the (dimensionless) line element on a compact 3-dimensional space $\Sigma_3$ and

$$f(r) = k - \frac{8G_{(5)}M}{3\pi r^2} + \frac{r^2}{L^2}. \tag{2.2}$$

Here $L$ is the anti-de Sitter length scale, with $\Lambda = -\frac{6}{L^2}$, and we shall consider the two cases $k = 1$ ($\Sigma_3$ a unit 3-sphere) and $k = 0$ ($\Sigma_3$ a 3-torus).

The event horizon radius, $r_h$, is the largest root of $f(r) = 0$, allowing $M$ to be expressed as a function of $r_h$ and $L$,

$$M = \frac{3\pi r_h{}^2(kL^2 + r_h{}^2)}{8G_{(5)}L^2}. \tag{2.3}$$

One must bear in mind however that, in the AdS/CFT correspondence, $G_{(5)}$ is itself a function of $L$ since

$$\frac{1}{16\pi G_{(5)}} = \frac{V_{S^5}}{16\pi G_{(10)}} = \frac{\pi^2 L^5}{16G_{(10)}}, \tag{2.4}$$

where $V_{S^5} = \pi^3 L^5$ is the volume of the 5-dimensional sphere with radius $L$ and $G_{(10)}$ is the (fixed) 10-dimensional Newton constant. Hence

$$M = \frac{3\pi^4 r_h{}^2 L^3(kL^2 + r_h{}^2)}{8G_{(10)}}. \tag{2.5}$$

We now wish to express $M$ as a function of the entropy, $S$, and the number of colors of the gauge theory, $N$, with $N$ large. The Bekenstein-Hawking entropy of the black hole is

$$S = \frac{1}{4}\frac{A}{\hbar G_{(5)}} = \frac{\pi^5 L^5 r_h{}^3}{2\ell_P^8}, \tag{2.6}$$

where $A = 2\pi^2 r_h{}^3$ is the "area" of the black hole (for $k = 1$ the volume of a unit 3-sphere is $2\pi^2$ and for simplicity we have used the same value for the 3-torus, this is not necessary for $k = 0$ but any other choice does not materially affect the argument). $\hbar G_{(5)} = \frac{\hbar G_{(10)}}{\pi^3 L^5}$, is the cube of the 5-dimensional Planck length and $\hbar G_{(10)} = \ell_P^8$ where $\ell_P$ is the 10-dimensional Planck length, which is kept fixed throughout.

The other relation we need is [8]

$$L^4 = 4\pi g_s(\hbar\alpha')^2 N = \frac{\sqrt{2}N}{\pi^2}\ell_P^4. \tag{2.7}$$

Using (2.6) and (2.7) in (2.5) gives the mass as

$$M(S, N) = \frac{3\,\widetilde{m}_P}{4}\left\{ k\left(\frac{S}{\pi}\right)^{\frac{2}{3}}N^{\frac{5}{12}} + \left(\frac{S}{\pi}\right)^{\frac{4}{3}}N^{-\frac{11}{12}} \right\}, \tag{2.8}$$

with $\widetilde{m}_P = \frac{\sqrt{\pi}\,m_P}{2^{1/8}}$ and $m_P = \frac{\ell_P^7}{G_{(10)}}$ the 10-dimensional Planck mass.

The Hawking temperature follows from the standard thermodynamic relation

$$T = \left.\frac{\partial M}{\partial S}\right|_N = \frac{\widetilde{m}_P}{2\pi}\left\{ k\left(\frac{S}{\pi}\right)^{-\frac{1}{3}}N^{\frac{5}{12}} + 2\left(\frac{S}{\pi}\right)^{\frac{1}{3}}N^{-\frac{11}{12}} \right\}, \tag{2.9}$$

(with Boltzmann's constant set to one). There is no magic here, this is just a trivial re-writing of Hawking's formula relating temperature to the surface gravity of the black hole,

$$T = \frac{\hbar f'(r_h)}{4\pi} = \frac{\hbar(kL^2 + 2r_h{}^2)}{2\pi r_h L^2}. \tag{2.10}$$

Note that, at fixed $N$, the $k = 1$ temperature as a function of $S$ has a minimum of $\frac{\sqrt{2}\widetilde{m}_P}{\pi N^{\frac{1}{4}}}$. For any value of $T$ above this there are two values of $S$ with the same temperature, giving large black holes and small black holes. This minimum in $T$ corresponds to a divergence in the heat capacity, $C_N = T\left(\frac{\partial S}{\partial T}\right)_N$, small black holes have negative heat capacity and are always unstable.

The Gibbs free energy is

$$G(T, N) = M - TS = \frac{\widetilde{m}_P}{4}\left\{ k\left(\frac{S}{\pi}\right)^{\frac{2}{3}}N^{\frac{5}{12}} - \left(\frac{S}{\pi}\right)^{\frac{4}{3}}N^{-\frac{11}{12}} \right\}. \tag{2.11}$$

For spherical black holes, with $k = 1$, this is negative for $N^2 < \frac{S}{\pi}$, corresponding to $L < r_h$, and black holes in this regime are more stable than 5-dimensional AdS with thermal radiation at the same temperature, this is the Hawking-Page phase transition [15] which is distinct from the instability due to negative heat capacity mentioned above. When $G$ is positive spherical black holes are susceptible to decay to pure AdS plus radiation and this happens at the Hawking-Page transition temperature

$$T_* = \frac{3}{2\pi}\frac{\widetilde{m}_P}{N^{1/4}}. \tag{2.12}$$

$k = 1$ black holes are stable against Hawking-Page decay only or $T > T_*$.

The derivative of $M$ with respect to $N^2$ will be interpreted as a chemical potential for the number of colors, [1, 6, 7],

$$\mu := \left.\frac{\partial M}{\partial N^2}\right|_S = \frac{\widetilde{m}_P}{32}\left\{5k\left(\frac{S}{\pi}\right)^{\frac{2}{3}}N^{-\frac{19}{12}} - 11\left(\frac{S}{\pi}\right)^{\frac{4}{3}}N^{-\frac{35}{12}}\right\} \tag{2.13}$$

($N^2$ is used in the definition of $\mu$, rather than $N$, because all fields in the boundary $\mathcal{N} = 4$ SUSY Yang-Mills theory are in the adjoint representation of SU($N$)). $\mu$ is a measure of the energy cost to the system of increasing the number of colors.

# 3 Discussion

For flat $k = 0$ black holes $\mu$ is always negative, but for spherical $k = 1$ black holes it becomes positive when

$$N^2 > \left(\frac{11}{5}\right)^{\frac{3}{2}}\frac{S}{\pi}. \tag{3.1}$$

For a bosonic system the vanishing of the chemical potential would be a signal of Bose-Einstein condensation, for a fermionic system it is a signal that the exclusion principle is coming into play. In terms of temperature the bound (3.1) is saturated at a temperature some 6% below $T_*$,

$$T_0 = \frac{21}{2\pi\sqrt{55}}\frac{\widetilde{m}_P}{N^{\frac{1}{4}}} = 0.944\,T_*. \tag{3.2}$$

The condition that the black hole be stable against Hawking-Page decay to AdS plus thermal radiation is, in geometric variables, $r_h > L$ and the inequality (3.1) translates to

$$r_h{}^2 < \frac{5}{11}L^2, \tag{3.3}$$

putting positive $\mu$ into the region of phase space where black holes are unstable against the Hawking-Page transition, i.e. in the low temperature phase of the gauge theory. Such black holes are not necessarily irrelevant though, for temperatures just below $T_*$ one expects black holes to make their presences felt through thermal fluctuations and they would also be important if the quark-gluon plasma were supercooled.

As stated earlier the Hawking-Page instability at $r_h = L$ is distinct from the instability due to negative heat capacity, the latter is in the regime $r_h{}^2 < \frac{L^2}{2}$. The singularity in heat capacity at

$$T_\infty = \frac{\sqrt{2}\,\widetilde{m}_P}{\pi N^{\frac{1}{4}}}, \tag{3.4}$$

is the lowest value the $k = 1$ black hole temperature (2.9) can achieve and it is only marginally below $T_0$,

$$T_\infty = 0.999\,T_0. \tag{3.5}$$

It should be borne in mind however that the singularity in heat capacity at $T_\infty$ is not a cusp, as $T$ cannot go below $T_\infty$: as a function of $S$, with $N$ fixed, $T$ has two branches and $C_N$ is negative on the low $S$ branch and positive on the high $S$ branch.

For a classical gas $\mu$ is negative and becomes more negative as $T$ is increased, quantum effects become important when $\mu$ approaches zero and switches sign. Indeed these are general properties of a chemical potential [11] and we see that they are satisfied in the present case. In the high temperature phase, where the entropy per degree of freedom $\frac{S}{N^2}$ is large, equations (2.9) and (2.13) show that indeed

$$\mu \approx -\frac{11N^{\frac{3}{4}}}{2\widetilde{m}_P^3}\left(\frac{\pi T}{2}\right)^4 \tag{3.6}$$

is negative and a decreasing function of $T$. (3.6) is a strict equality for all $T$ when $k = 0$.

An important parameter in the discussion of chemical potentials is the fugacity

$$\xi = e^{\frac{\mu}{T}}. \tag{3.7}$$

$\xi < 1$ in the classical regime and tends to zero as $T \to \infty$, with quantum effects corresponding to $\xi \approx 1$. The figure below shows a plot of $\xi$ as a function of the dimensionless temperature

$$t = 2\pi\frac{N^{\frac{1}{4}}T}{\widetilde{m}_P} \tag{3.8}$$

for $k = 1$. On the lower branch $\xi$ is less than unity and is a decreasing function of $T$ above $T_\infty$. At fixed $T$, $\xi \to 0$ as $N \to \infty$ on the lower branch and $\xi \to 1$ on the upper branch, but finite $t$ is possible in the regime of validity of the solution with $\frac{T}{\widetilde{m}_P} \ll 1$ and $N \gg 1$ at the same time. Note that as the curve is traced from the high $T$, low $\xi$ branch, through $T_*$, $T_\infty$ is encountered before $\mu = 0$, even though $T_0 > T_\infty$.

Bose-Einstein condensation and/or Fermion repulsion can viewed in terms of flux density of the 5-form flux on $S^5$. The density of flux on $S^5$ decreases as $n = \frac{N}{\pi^3 L^5} \sim N^{-\frac{1}{4}}l_P^{-5}$ as $N$ is increased. For large $N$ the flux is of course classical but as $N$ is decreased quantum effects will become important at some stage, when $\mu$ approaches zero. If the flux is quantised and each unit of flux has a size $\lambda$ then the classical picture is only valid for $n \ll 1/\lambda^5$ or $N \gg \left(\frac{\lambda}{l_P}\right)^{20}$.

In the brane picture, when the event horizon is flat rather than spherical, the first term in (2.8) is not present and the chemical potential is always negative, which is a good thing as there is no Hawking-Page phase transition in this topology, all such black branes, no matter how small $r_h$ is, are stable against decay to AdS plus thermal radiation. The limit of $N$ co-incident branes can be obtained by first taking a stack of branes with all adjacent branes having the same separation, $s$, and then letting $s \to 0$. If the branes each have large mass $M_B$ then their Compton wavelength, $\lambda_B$, will be small. We want to take the limit $s \to 0$ and $M_B \to \infty$ and an important parameter characterising the physics is the ratio $\frac{s}{\lambda_B}$, with $s \to 0$ and $\lambda_B \to 0$. Provided this ratio is greater than unity the brane wave functions do not overlap in the large $N$ limit and quantum effects are not expected to be important, so we should be taken with $s \gg 1$ for consistency.

For simplicity, the discussion here has been restricted to maximally symmetric $\mathcal{N} = 4$ Yang-Mills, in which the number of flavours is not independent of the number of colors and all fields are in the adjoint of SU($N$). It would be interesting to extend the analysis
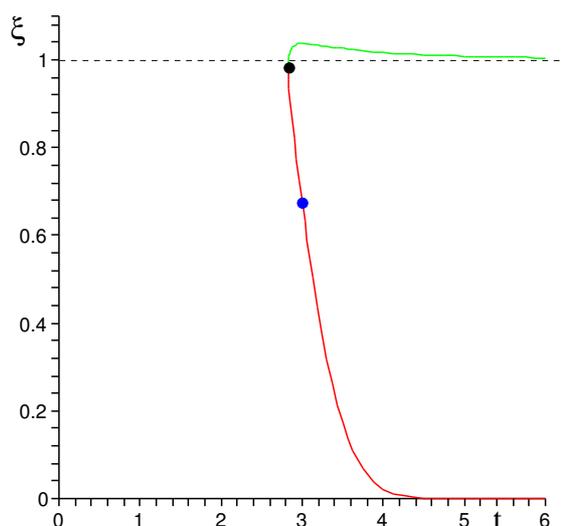
**Figure 1**. Fugacity as a function of $t$. The blue (lower) dot denotes the Hawking-Page phase transition, where $t = 3$ and $\xi = e^{-\frac{\pi}{8}}$: black holes are stable on the branch below this point. The black (upper) dot is the point $T_\infty$ where $C_P$ diverges, it lies marginally below $\xi = 1$ (horizontal dotted line). $C_P > 0$ on the red (lower) branch of the curve and $C_P < 0$ on the green (upper) branch.

to $\mathcal{N} = 1$ and varying $N_f$, by using classical solutions corresponding to less symmetric situations such as finite temperature black hole versions of the $\mathcal{N} = \infty$ solutions in [16] or [17] for example.

If the gauge symmetry is broken, e.g. $U(N) \rightarrow U(N_1) \times U(N_2)$, then one could envisage extending the analysis to look for non-supersymmetric bulk gravity solutions for which the black hole mass depends on $N_1$ and $N_2$ separately, generating two chemical potentials. Thermodynamic equilibrium would then dictate the values of $N_1$ and $N_2$ in a similar manner to the way chemical equilibrium is achieved in a system consisting of a molecular soup of different chemical species.

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