Duality in strongly interacting systems: $\mathcal{N} = 2$ SUSY Yang-Mills and the quantum Hall effect

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Classical solutions of the vacuum Maxwell's equations exhibit a $SO(2)$ duality symmetry, which is enhanced to $Sl(2, \mathbf{R})$ when dilaton and axion fields are included. Quantum effects break this symmetry but semi-classically $Sl(2, \mathbf{Z})$ symmetry, or a sub-group thereof, survives in Dirac-Schwinger-Zwanziger quantisation. Even this symmetry is expected to be broken in the full theory of quantum electrodynamics, but a modular sub-group survives as an infinite discrete symmetry of the vacua of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. An analogous situation occurs in the quantum Hall effect, where different quantum Hall states are related by a modular symmetry which is a sub-group of $Sl(2, \mathbf{Z})$. The similarities between the quantum Hall effect and supersymmetric Yang-Mills are reviewed and a possible link via the gauge/gravity correspondence is described. Scaling exponents in the quantum Hall effect are derived using the gauge-gravity correspondence.

1 Duality in electromagnetism

In classical electromagnetism the source free Maxwell equations enjoy a symmetry that is not present at the level of the action. Defining the complex vector field $\mathbf{Z} := \mathbf{B} + i\mathbf{E}$, the source free Maxwell's equations can be written

$$\nabla.\mathbf{Z} = 0, \qquad \nabla \times \mathbf{Z} = -i\dot{\mathbf{Z}}.$$

These equations are invariant under $\mathbf{Z} \to \mathrm{e}^{i\alpha}\mathbf{Z}$, or

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}. \tag{1}$$

This $SO(2)$ symmetry can be used to generate new solutions from old ones, but it is not associated with any conservation law as the Lagrangian $\mathcal{L} = -\frac{1}{2} Re(\mathbf{Z}.\mathbf{Z})^2 = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ is not invariant, unless α is an integral multiple of 0 or π,

$$\frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \quad \longrightarrow \quad \frac{1}{2}\cos(2\alpha)(\mathbf{E}^2 - \mathbf{B}^2) + \sin(2\alpha)\mathbf{E}.\mathbf{B}.$$

Including a scalar field ϕ (dilaton) and an axion χ facilitates an extension to $Sl(2, \mathbf{R})$ symmetry. The action

$$\mathcal{S} = \int \left\{ \frac{1}{2\kappa^2} \left(R - 2\Lambda - \frac{1}{2} \left(\partial\phi.\partial\phi + \mathrm{e}^{2\phi}\partial\chi.\partial\chi \right) \right) - \frac{1}{2}\mathrm{e}^{-\phi}F^2 - \frac{\chi}{2}F\widetilde{F} \right\} \sqrt{-g}d^4x, \tag{2}$$

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where $\widetilde{F}^{\mu\nu} = \frac{1}{2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\rho\sigma}$, gives the constitutive relations $D_i = G_{i0}$, $H^i = \frac{1}{2}\epsilon^{ijk} G_{jk}$ with

$$G^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \qquad \text{and} \qquad \begin{aligned} \mathbf{D} &= e^{-\phi}\mathbf{E} + \chi\mathbf{B} \\ \mathbf{H} &= e^{-\phi}\mathbf{B} - \chi\mathbf{E}. \end{aligned}$$

Defining the complex fields $\tau := \chi + ie^{-\phi}$; $\mathcal{F} = F - i\widetilde{F}$ and $\mathcal{G} = -\widetilde{G} - iG$, Gibbons and Rasheed [1] showed that the equations of motion are invariant under the $Sl(2, \mathbf{R})$ transformations
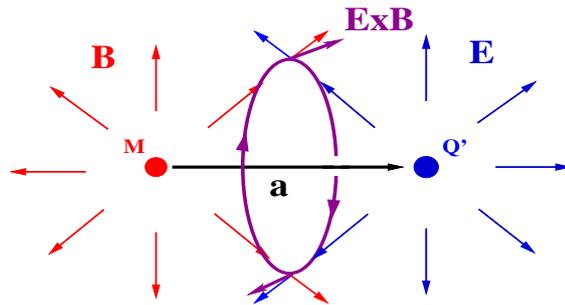
$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} \mathcal{G} \\ \mathcal{F} \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{G} \\ \mathcal{F} \end{pmatrix} \tag{3}$$

where $ad - bc - 1$.

Introducing sources spoil this symmetry in general unless one postulates the existence of, so far unobserved, magnetic charges. Dirac showed [2] that postulating megnetic monopoles necessarily leads to quantisation of electric charge. Consider a magnetic charge M at the origin and an electric charge Q' at a point $\mathbf{a}$, these generate the fields

$$\mathbf{B}(\mathbf{r}) = \frac{M}{4\pi}\frac{\mathbf{r}}{r^3}, \qquad \mathbf{E}(\mathbf{r}) = \frac{Q'}{4\pi}\frac{(\mathbf{r} - \mathbf{a})}{|\mathbf{r} - \mathbf{a}|^3} \qquad (\epsilon_0 = \mu_0 = 1).$$

This field configuration has circulating momentum $\mathbf{E} \times \mathbf{B}$

(Here and for the following figures online colour at: www.fp-journal.org.)

The total angular momentum is finite

$$\mathbf{J} = \int \mathbf{r} \times \left[\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\right] d^3\mathbf{r} = -\frac{Q'M}{4\pi}\hat{\mathbf{a}}$$

and quantisation of angular momentum, $J_z = N\frac{\hbar}{2}$ where N is an integer, leads to the famous Dirac quantisation condition

$$Q'M = 2\pi N\hbar. \tag{4}$$

The very existence of a magnetic charge necessarily implies that the electric charge $Q' = \frac{2\pi N\hbar}{M}$ is quantised.

Suppose the minimum electric charge is e so that $Q' = n'_e e$ with n'_e an integer. Then there is also a minimum magnetic charge m with $M = n_m m$ where n_m is an integer. Then, with $n'_e = n_m = 1$ and $N = 1$ in (4), $m = \frac{2\pi\hbar}{e}$. There is then a symmetry of Maxwell's equations with sources under the interchange $e \rightarrow \frac{2\pi\hbar}{m}$, $m \rightarrow -\frac{2\pi\hbar}{e}$, $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$. This $\mathbf{Z}_2$ symmetry is the quantum mechanical vestige the classical $SO(2)$ symmetry of the source-free Maxwell's equations, (1), quantum mechanics restricts α to be an integral multiple of $\frac{\pi}{2}$.

By extending this argument to two dyons, hypothetical particles carrying both electric and magnetic charges (Q', M') and (Q, M), the Dirac quantisation condition generalises to the Dirac-Schwinger-Zwanziger quantisation condition [3],

$$Q'M - M'Q = 2\pi N\hbar \qquad \Leftrightarrow \qquad n_m n'_e - n'_m n_e = N. \tag{5}$$

This breaks the classical $Sl(2, \mathbf{R})$ symmetry (3) down to the smaller group $Sl(2, \mathbf{Z})$, defined by restricting the four real numbers in (3) to be integers.

The derivation of the Dirac quantisation condition (4), and its generalisation to (5), uses semi-classical arguments, one cannot expect this simple picture to hold in the full theory of QED – there is no reason to expect $Sl(2, \mathbf{Z})$ symmetry in QED even when hypothetical monopoles are included.
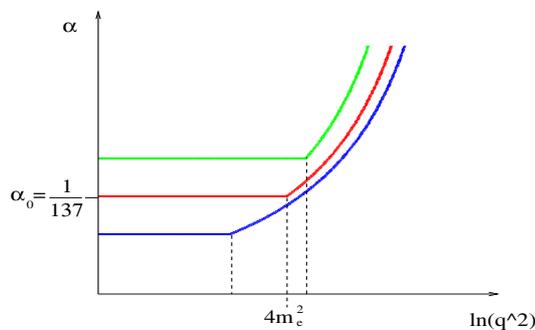
2　$\mathcal{N} = 2$ SUSY Yang-Mills

For $\mathcal{N} = 2$ supersymmetric Yang-Mills it was shown by Seiberg and Witten, [4] that a remnant of the semi-classical $Sl(2, \mathbf{Z})$ does survive full quantisation. The action for $SU(2)$, $\mathcal{N} = 2$ SUSY in 4-dimensions is:

$$S = \int dx^4 \left\{ \left(-\frac{1}{4g^2} tr(F_{\mu\nu} F^{\mu\nu}) + \frac{\theta}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} tr(F_{\mu\nu} F_{\rho\sigma}) \right) \right.$$
$$\left. + \frac{1}{g^2} tr \left((D_\mu \phi)^\dagger D^\mu \phi - \frac{1}{2} [\phi^\dagger, \phi]^2 \right) \right\}$$
$$+ \text{Fermionic terms}$$

The field content consists of three gauge bosons, a single complex scalar field and a pair of Weyl fermions (not explicitly shown above) all in the adjoint of $SU(2)$. Supersymmetry dictates that there are only two independent couplings, g and θ: the Higgs quadratic and quartic couplings are determined by the gauge coupling, they are not independent parameters. Degenerate vacua are parameterised by $<\phi>$ with $[\phi^\dagger, \phi] = 0$, or alternatively by $u = \frac{1}{2} tr <\phi^2>$ which has the advantage of being gauge invariant. If $<\phi> \neq 0$, $SU(2)$ is broken to $U(1)$ and $<\phi>$ gives two of the gauge fields, the W^+ and W^-, and the gluinos a mass. There is an effective low energy coupling for the residual $U(1)$ gauge symmetry, $\tau(u) := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, which depends on u or, equivalently, the mass and $\tau(u)$ runs with the mass in a manner completely analogous to the Callan-Symanzik β-function of QED. In QED the fine structure 'constant', $\alpha(m_e^2, q^2)$, depends on the electron mass and the momentum transfer, due to virtual creation of electron-positron pairs,

At energies less than twice the electron mass virtual pair-creation is suppressed and the running effectively stops so the infra-red value, $\alpha(m_e, 0)$, depends on the electron mass.

In QED this running can only be determined in perturbation theory, but for $\mathcal{N} = 2$ SUSY Seiberg and Witten determined $\tau(u)$, and hence the low energy α, analytically, including non-perturbative corrections due to instantons. They showed that the vacua of the low energy effective action have a symmetry under
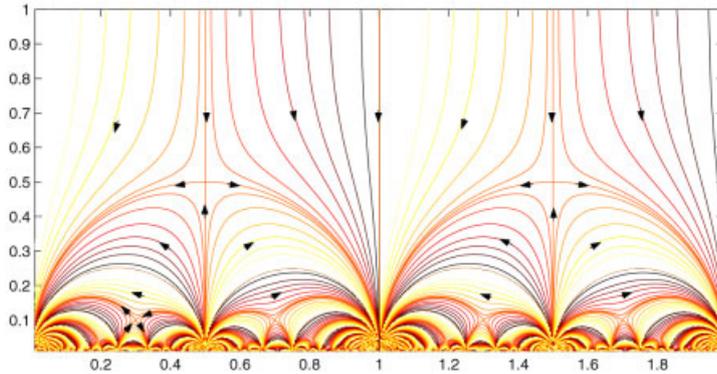
$$\tau \to \frac{a\tau + b}{c\tau + d}$$

where a, b, c, $d \in \mathbf{Z}$; $ad - bc = 1$ with b and c even. The set of all such transformation constitutes a sub-group of $Sl(2, \mathbf{R})$ formed by matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $\det \gamma = 1$ and b and c both even. This group is denoted by $\Gamma(2) \subset \Gamma(1) \cong Sl(2, \mathbf{Z})/\mathbf{Z}_2$ in the mathematical literature.

$\Gamma(2)$ can be generated by repeated action of the two elements

$$\tau \to \tau + 2, \qquad \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\tau \to \frac{1}{2\tau + 1}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Below is shown the flow in the upper-half τ-plane generated by Seiberg and Witten's function $\tau(u)$, [5]

(this flow is actually invariant under a larger sub-group of $\Gamma(1)$ where b can be even or odd and only c is constrained to be even – this sub-group is denoted $\Gamma_0(2)$). There are fixed points for strong coupling at all $\theta/2\pi = p/q$, with q even for ultra-violet fixed points (u-large) and q odd for infra-red fixed points (which correspond to $u = \pm 1$ in Seiberg and Witten's solution). These fixed points fall into three categories which are the images under $\Gamma(2)$ of the three points:

$\tau = i\infty$ (weak coupling) basic electric charge, gluino;

$\tau = 0$ ($p = 0, q = 1$), magnetic monopole;

$\tau = 1$ ($p = 1, q = 1$), dyon.

All other fixed points can be obtained by acting on one of $\tau = 0$, $\tau = 1$ or $\tau = i\infty$ with an element of $\Gamma(2)$. There are low energy effective degrees of freedom at each infra-red fixed point carrying monopole quantum number q, with q odd.

Important points to note are:

The infra-red, strong-coupling regime ($g \to \infty$) gives a topological field theory, $\tau \to \frac{\theta}{2\pi} = \frac{p}{q}$, with q odd. In general the dyons carry total electric charge Q, due to the Witten effect [6], in units of g

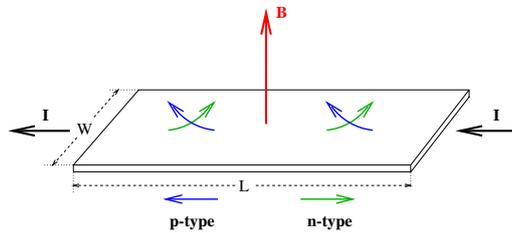$$Q = \left(n_e + n_m \frac{\theta}{2\pi}\right) g = \left(n_e + n_m \frac{p}{q}\right) g.$$

For any mutually prime pair p and q, there always exist two integers n_e and n_m such that $n_e q + n_m p = 1$, giving $Q = \left(\frac{1}{q}\right) g$. Thus there are always effective degrees of freedom with fractional electric charge $Q = \left(\frac{1}{q}\right) g$.

3   The quantum Hall effect

The Hall effect was discovered in 1879 by Edwin Hall and at the time it came as a surprise as it contradicts an earlier statement of James Clerk-Maxwell [7] (the statement originally appeared in the first edition in 1873, before Hall's experiment, and it was retained in the 3rd edition):

"It must be carefully remembered that the mechanical force which urges a conductor . . . acts, not on the electric current, but on the conductor which carries it."

In a typical experimental setup a current is passed through a two-dimensional slab of conducting or semi-conducting material, in the presence of a transverse magnetic field, B, and the Lorentz force on the charge carriers causes a transverse voltage, the Hall voltage, to build up,
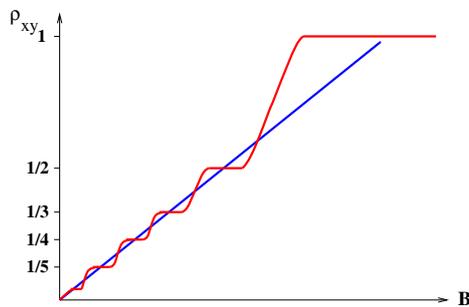

The current density is proportional to the applied electric field and the conductivity is a tensor, $J_\alpha = \sigma_{\alpha\beta} E^\beta$ (we shall restrict our consideration to isotropic samples for which $\sigma_{xx} = \sigma_{yy}$).

The resistivity is the inverse of the conductivity matrix. Classically the Hall resistivity $\rho_{xy}^{cl} = -\frac{B}{en}$ is proportional to the magnetic field, and the Ohmic conductivity can be expressed in terms of the collision time, τ_c, and the charge carrier mass, m, as $\rho_{xx}^{cl} = \frac{m}{e^2 n \tau_c}$. It is convenient to use complex co-ordinates in the two-dimensional plane of the sample,

$$z = x + iy \ \Rightarrow \ \rho := \rho_{xy} + i\rho_{xx}, \quad \sigma = \sigma_{xy} + i\sigma_{xx} = -1/\rho.$$

Note that positivity of the Ohmic resistivity imposes the constraint $\text{Im}(\rho) \geq 0 \ \Leftrightarrow \ \text{Im}(\sigma) \geq 0$, so σ is restricted to the upper-half complex plane.

For low temperature and large magnetic fields, in samples with high purity and high particle density, von Klitzing [8] discovered that the Hall resistance is quantised in units of $R_H = h/e^2 = 25.812807449(86) k\Omega$. $\rho = \frac{1}{p}\left(\frac{h}{e^2}\right), p \in \mathbf{Z}$, so $\sigma = p\left(\frac{e^2}{h}\right)$,

In the figure above, the straight (blue) line is the classical result and the stepped (red) line is von Klitzing's discovery – the Hall resistivity increases in a series of sharply defined steps between accurately quantised
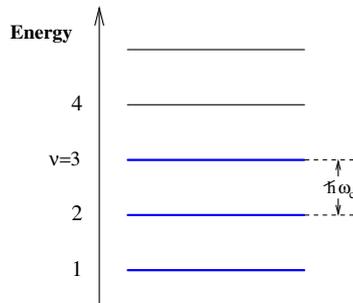
plateaux while the Ohmic resistivity vanishes on the plateaux. This is the integer quantum Hall effect (QHE) For even purer samples, as discovered by Tsui and Störmer, [9], the steps take fractional values, $\sigma = \frac{p}{q}\left(\frac{e^2}{h}\right)$, $p, q \in \mathbf{Z}$, q odd, giving rise to the fractional QHE.

A key ingredient in the understanding of the QHE effect is Landau's solution of the Schrödinger equation for free electrons moving in a transverse magnetic field. The energy eigenvalues are the same as the harmonic oscillator problem: equally spaced with separation $\hbar\omega_c$ where ω_c is the cyclotron frequency, $\omega_c = \frac{eB}{m}$. The energy levels (Landau levels) are degenerate with degeneracy per unit area: $g = \left|\frac{eB}{h}\right| = \left|\frac{B}{e}\right|\left(\frac{e^2}{h}\right)$. Since electrons are Fermions, at low temperatures ($k_B T << \hbar\omega_c$) the energy levels are filled up sequentially as the ratio of the particle density to the magnetic field is increased. The *filling factor*, ν, is defined as

$$\nu := n/g = \frac{ne}{B}\left(\frac{h}{e^2}\right)$$

in terms of which $|\sigma_{xy}^{cl}| = \nu\left(\frac{e^2}{h}\right)$, when $\sigma_{xx}^{cl} = 0$ (from now on we shall adopt 'natural' units for the QHE in which $\left(\frac{e^2}{h} = 1\right)$).

Below is a representation of the filled Landau levels for $\nu = 3$, for example,

There is an energy gap for integer ν, which would imply that integer ν should be very stable robust states were it not for the fact that the degeneracies are huge, typically g is of order 10^{11} cm^{-2}. Perturbations would be expected to broaden the Landau levels into a continuum of states, numerical investigations however indicate that, when perturbations broaden the levels, most states are localised and do not contribute to the conductivity – only states with energy $\left(n + \frac{1}{2}\right)\omega_c$ carry charge through the system (for a general introduction to these ideas in the QHE see [10]). The naïve picture of a stable energy gap at integer filling factor in fact survives the introduction of perturbations.

Then filled Landau levels are inert and do not affect the physics of the Landau levels above implying that the physics of pseudo-particle excitations should be the same for

$$\sigma \to \sigma + 1,$$

the dynamics of the pseudo-particle excitations in the top Landau level is oblivious to how many filled Landau levels there are below. This is analogous to the symmetry of the periodic table of the elements, where chemical properties of elements in the same column of the periodic table are similar.

If, in addition, the dynamics enjoys a symmetry under particle-hole interchange then the dynamics will also be the same for $\nu - [\nu] \to 1 - (\nu - [\nu])$, where $[\nu]$ is the integral part of ν, so $\nu - [\nu]$ is the fractional part of ν. In a sense, one-third full = one-third empty, for the top Landau level or, in terms of σ,

$$\sigma \to 1 - \bar{\sigma}.$$

An effective action description of the electromagnetic response in the QHE is given by Maxwell-Chern-Simons theory. The classical relation

$$B = -en\rho_{xy}^{cl} \Rightarrow \sigma_{xy}^{cl}B = J^0,$$

where $J^0 = en$ is the charge density, follows from and effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}}[A_0] = -\sigma_{xy} A_0 B + A_0 J^0.$$

Extending this to a Lorentz co-variant form gives a $U(1)$ Chern-Simons action
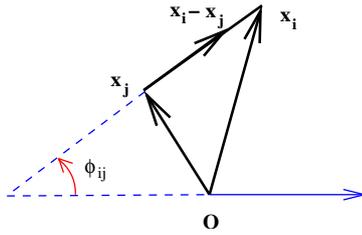
$$\mathcal{L}_{\text{eff}}[A] = -\frac{\sigma_{xy}}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu.$$

A non-zero Ohmic conductivity can be incorporated into the effective action description by thinking of the effective action as giving the response function, a frequency dependent electric permittivity. The characteristic of a conductor is that the electric permittivity has a pole at zero frequency and the residue gives the conductivity, $\sigma_{xx} = i \lim_{\omega \to 0}(\omega \epsilon(\omega))$, so the low energy, long wavelength effective action is

$$\mathcal{L}_{\text{eff}}[A] = -\frac{\epsilon}{4} F^2 - \frac{\sigma_{xy}}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu + \cdots$$

$$\approx \frac{i\sigma_{xx}}{4\omega} F^2 - \frac{\sigma_{xy}}{4} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + A_\mu J^\mu + \cdots,$$

where the dots denote non-zero frequency and finite wavelength corrections The effective action is complex because the system is dissipitave for $\sigma_{xx} > 0$.

A second symmetry can be obtained by introducing an auxiliary gauge field, the statistical gauge field [11]. Suppose we have a system of N charge carriers with multi-particle wave function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Associate an angle ϕ_{ij} with each pair of particles, as shown in the diagram below,

(this angle is equal to the complex phase of $z_i - z_j$). Choose a specific particle i and perform a gauge transformation which changes Ψ by a phase, to a new wave-function $\widetilde{\Psi}$ given by

$$\widetilde{\Psi}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = e^{\frac{i\vartheta}{\pi}(\Sigma_{i<j}\phi_{ij})} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$$
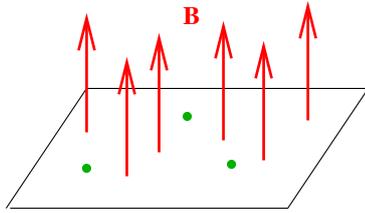
In this new gauge we see that, under interchange of particle i with any other particle j, $\phi_{ij} \to \phi_{ij} + \pi$ and the phase of $\widetilde{\Psi}$ changes by an amount ϑ more than that of Ψ. If $\vartheta = 2\pi k$ they behave the same way under particle interchange while if $\vartheta = \pi(2k + 1)$ they differ by a minus sign: so, in the latter case, if Ψ is a fermionic wave-function, which changes sign under the interchange of any two particles, then $\widetilde{\Psi}$ is a bosonic wave-function and vice-versa.

Under such a gauge transformation we must make the substitution $-i\hbar\nabla - e\mathbf{A} \to -i\hbar\nabla - e(\mathbf{A} + \mathbf{a})$ in the Hamiltonian, where

$$a_\alpha(\mathbf{x}_i) = \frac{\hbar\vartheta}{e\pi} \sum_{j\neq i} \nabla_\alpha^{(i)} \phi_{ij} \Rightarrow \epsilon^{\beta\alpha} \nabla_\beta^{(i)} a_\alpha(\mathbf{x}_i) = \frac{2\hbar\vartheta}{e} \sum_{j\neq i} \delta(\mathbf{x}_i - \mathbf{x}_j).$$
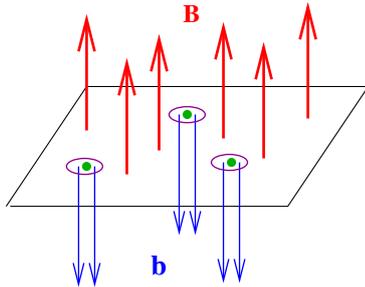
a is called the **statistical gauge field** and it generates a satistical magnetic field $b(\mathbf{x}) := \epsilon^{\beta\alpha}\nabla_\beta a_\alpha(\mathbf{x}) = \frac{2\hbar\vartheta}{e} n(\mathbf{x})$, where $n(\mathbf{x})$ is the charge carrier density. b is non-zero since this is a singular gauge transformation if any of the particles coincide.

The notion of the statistical magnetic field can be used to simplify the description of the dynamics of charged pseudo-particles moving in a strong external field. Consider a system of such particles
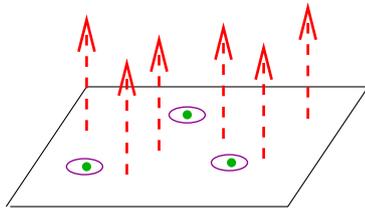
in the above picture, for example, the green dots are electrons and the red arrows represent magnetic flux units, so $\nu = \frac{1}{2}$.

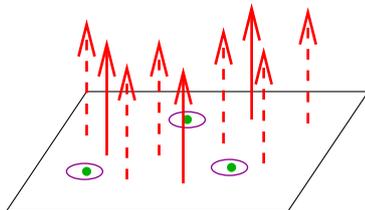Now introduce a statistical gauge field with $k = -2$,

This has the effect of attaching two units of statistical gauge field flux to every electron, exactly cancelling the external magnetic **B** in the Hamilton. Mathematically $A_\mu \to A'_\mu = A_\mu + a_\mu$ with $b := \epsilon^{\beta\alpha}\nabla_\beta a_\alpha$ such that $\frac{b}{n} = \frac{\vartheta}{\pi}\left(\frac{h}{e}\right) \overset{(\vartheta=-2\pi)}{=} -2\left(\frac{h}{e}\right)$.

Under the dynamics of this new modified Hamiltonian the charge particles (which are composite objects made up of electrons bound to two flux units of statistical gauge field – Jain's composite fermions [12]) behave as though there were no external field at all and we can bring to bear the full arsenal of perturbation theory for charged particles with no external field.

If we now increase the external field by another three units of magnetic flux, the filling factor for the composite fermions is unity, while the filling factor for the original electrons is now $\frac{1}{3}$.

A filling factor of $\frac{1}{3}$ for electrons is equivalent to a filling factor of 1 for composite fermions, $\nu = 1/3 \Leftrightarrow \nu_{CF} = 1$, or $\left(\frac{1}{\nu} \Leftrightarrow \frac{1}{\nu} + 2\right)$. In this picture the Fractional QHE can be viewed as the integer QHE for composite Fermions, [12].

These ideas were applied to the conductivity at the level of the complex response function in a Maxwell-Chern-Simons effective action, in the infra-red limit, by Lütken and Ross [13, 14] and by Kivelson, Lee and Zhang [15]. These authors proposed that the following transformations map between different phases of the two-dimensional quantum Hall electron gas:

$$
\begin{aligned}
\text{Landau Level Addition (\textbf{L}):} \quad & \sigma \rightarrow \sigma + 1 \\
\text{Flux Attachment (\textbf{F}}^2\text{):} \quad & \frac{1}{\sigma} \rightarrow \frac{1}{\sigma} + 2 \\
\text{Particle-Hole Interchange (\textbf{P}):} \quad & \sigma \rightarrow 1 - \overline{\sigma}
\end{aligned}
\tag{7}
$$

(outer automorphism)

The first two of these transformations generate a sub-group, $\Gamma_0(2)$, of the modular group whose general element has the following action on σ:
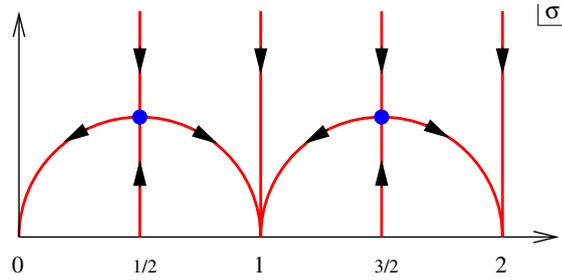
$$
\sigma \rightarrow \frac{a\sigma + b}{c\sigma + d}
$$

where a, b, c, $d \in \mathbf{Z}$, and $ad - bc = 1$ with c even. The idea here is that different quantum Hall states correspond to different phases of the two-dimesnional electron gas and the transition between two states, $\nu\colon p/q \rightarrow p'/q'$, as the external magnetic field is varied is a Quantum Phase Transition, as described bi Fisher [16].
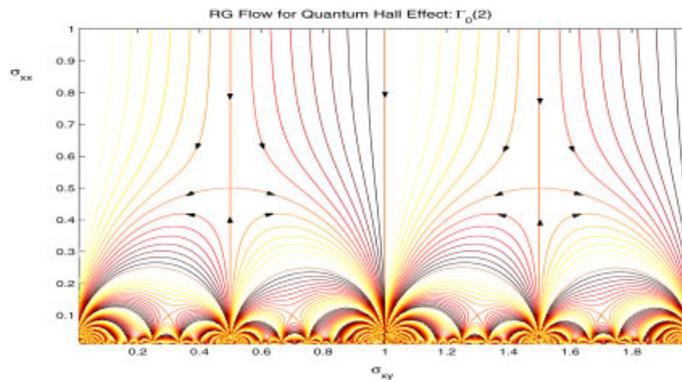
Kivelson, Lee and Zhang [15] called (7) the **law of corresponding states**: just as different gases satisfy an equation of state that is very similar when written in reduced form, the Van der Waals equation of state, so the equation of state for different quantum Hall phases of the electron gas should be the same. This is not a symmetry of all of the physics, the conductivities are different in different phases and critical parameters between phases are different, but nevertheless there is a correspondence between the different phases. Indeed it is an experimental fact that the physics of the second Landau level is very to different to that of the first: there is an even denominator state at $\nu = 5/2$ that so far has no counterpart in the first Landau level. It is a prediction of the law of corresponding states that there should be such a state at $\nu = 1/2$, if the law of corresponding states is correct then presumably mobilities are not yet high enough or temperatures not yet low enough for this state to be seen.

There is not only a mathematical analogy between (6) and (7), they are also closely analogous physically: the particles in $\mathcal{N} = 2$ SUSY are dyons and in the quantum Hall effect at $\nu = 1/q$ the quasi-particles have electric charge e/q, as argued by Laughlin [17]

There is a second order phase transition between different phases in the QHE, there is a critical value of the magnetic field B_c where the correlation length diverges $\xi \approx |\Delta B|^{-\nu_\xi}$, where $\Delta B = B - B_c$. The conductivity depends on a number of parameters, the most important of which we can expect to be the temperature T, the external field B and the charge carrier density n. Since σ is dimensionless (in natural units with $\frac{e^2}{h}$) simple scaling arguments imply that it can depend on only two arguments and not all three independently, we can therefore choose to write $\sigma(T, \Delta B, n) = \sigma(\Delta B/T^\kappa, n/T^{\kappa'})$ where κ and κ' are exponents (anomalous dimensions) that are in principle determined by the underlying Hamiltonian (experimentally $\kappa = 0.42 \pm 0.01$ [18]). As a consequence of this σ flows as T is varied. The flow is strongly restricted by demanding that this flow commutes with the action of the modular group. In particular it implies that fixed points of $\Gamma_0(2)$ are fixed points of flow (by a fixed point we mean only that that there exists a $\gamma \in \Gamma_0(2)$ such that $\gamma(\sigma_*) = \sigma_*$ for some σ_* – it is not necessary that all γ leave σ_* invariant). With some mild extra assumptions, including that the integers are attractive fixed points of the flow and that σ_{xx} decreases as T decrease, as is typical for a semi-conductor, it can be shown [14, 19], that modular symmetry then implies that even denominators are repulsive. Indeed modular symmetry strongly constrains the topology of the flow to have the form shown below
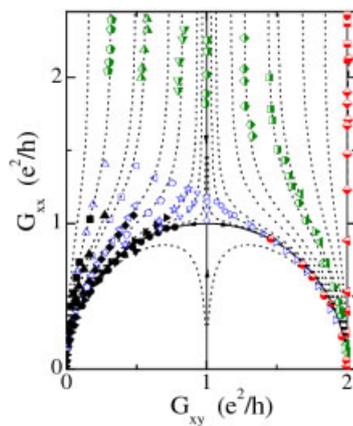
Interpolating the above figure, [14, 20], we derive the flow diagram below

RG Flow for Quantum Hall Effect: $\Gamma_0(2)$

This diagram has the following features:

- there are attractive fixed points at $\sigma_{xy} = p/q$, q odd; repulsive points for q even.

- In the composite Fermion picture $q = 2m + 1$, with $2m$ the number of vortices.

- There is a fractal structure near real axis, though of course there are no true fractals in Nature and we expect the picture to break down in certain limits: for example a Wigner crystal is expected to form for $\nu < \frac{1}{7}$, so the modular group is not valid in this regime, and $\hbar\omega_c > k_B T$ is only valid if B is not too small or T not too large.
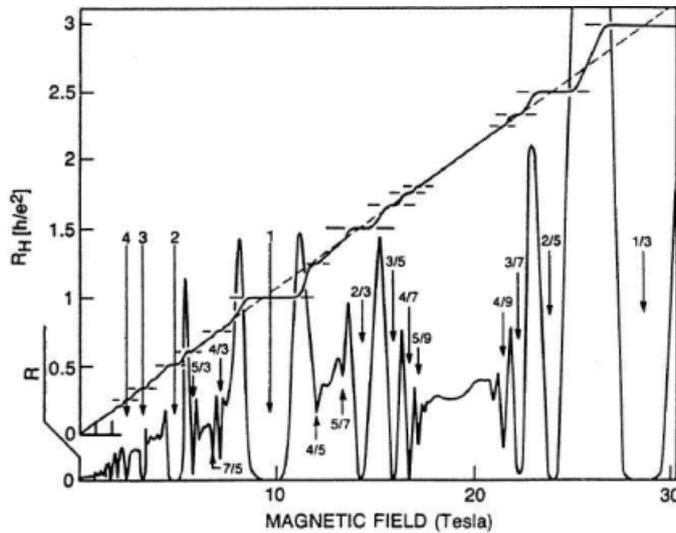
The figure below shows, taken from [21], shows some experimetental temperature flows in a GaAs sample. The dotted lines are the theoretical prediction from the previous figure and each symbol represents a fixed magnetic field (the values of the conductivities are doubled because the electron spins are degenerate).

Modular symmetry predicts a selection rule for quantum Hall phase transitions. It requires that any transition between two plateaux $\nu\colon \frac{p}{q} \to \frac{p'}{q'}$ can be obtained from $\nu\colon 0 \to 1$ by some $\gamma \in \Gamma_0(2)$, so $\gamma(0) = \frac{p}{q}$ and $\gamma(1) = \frac{p'}{q'}$. This requires $\gamma = \begin{pmatrix} p' - p & p \\ q' - q & q \end{pmatrix}$ and since $\det\gamma = 1$ we derive the selection rule [22]:

$$p'q - pq' = 1.$$

This is closely analogous to the Schwinger-Zwanziger quantisation rule for dyons in four-dimensions. It is well supported by experimental data [23] as shown below, where the plateaux are indicated by their fractions

4 AdS/CFT correspondence

Given the remarkable modular symmetry parallel between the two ostensibly very different physical systems, $\mathcal{N} = 2$ SUSY Yang-Mills in $(3+1)$-dimensions and the QHE effect in $(2+1)$-dimensions, it is tempting to look for a relation between them from perspective of the AdS/CFT correspondence. In this context we shall construct a model with the following properties:
• The $(2+1)$-dimensional sample is the boundary of a $(3+1)$-dimensional gravity coupled to matter;
• The boundary theory describes strongly interacting electrons in $2+1$ dimensions. Since the conductivity is dimensionless the boundary theory will be related to a CFT in $(2+1)$-dimensions.
• The bulk theory will be a classical gravitational theory with matter, chosen to exhibit $Sl(2, \mathbf{R})$ in its classical solutions.
    To this end we take a bulk metric of the form

$$ds^2 = L^2 \left\{ -f(v)\frac{dt^2}{v^{2z}} + \frac{dr^2}{f(v)v^2} + \frac{dx^2 + dy^2}{v^2} \right\} \tag{8}$$

where the cosmological constant is negative, $\Lambda = -\frac{3}{L^2}$, and the co-ordinate v is related to the usual radial Schwarzschild co-ordinate r by $v = \frac{L}{r}$. In the AdS/CFT framework z is a Lifshitz scaling exponent describing the relative scaling of time and space, $x \to \ell x$, $y \to \ell y$, $t \to \ell^z t$. The special case $z = 1$ corresponds to a Lorentz invariant theory in $(2+1)$-dimensions. If $f(v_h) = 0$ for some finite value v_h then

there is an event horizon with an associated Hawking temperature, $T = \frac{|f'(v_h)|}{4\pi v_h^{z-1}L}$ (v_h is the smallest value of v for which $f(v)$ vanishes).

We want the bulk matter to exhibit classical $Sl(2, \mathbf{R})$ symmetry in its solutions and a suitable system with this property is Einstein-Maxwell-dilaton-axion system [1] (one can also replace the Maxwell action with a Dirac-Born-Infeld action and retain the desired property).

A classical solution of (2), with the metric (8), is given by

$$f(v) = 1 - \left(\frac{v}{v_h}\right)^{z+2}, \quad e^{-\phi} = K^2 v^4, \quad \chi = 0$$

$$G^{vt} = \frac{Qv^6}{L^2}, \quad F^{vt} = \frac{Qv^2}{K^2 L^2}, \quad \text{with} \quad z = 5$$

[24] , see also [25].

The value $z = 5$ is determined by the relative normalisations of the Einstein and the dilaton terms in the action (2), it can be varied by changing the relative weights of these two terms in the action and the classical solutions still have $Sl(2, \mathbf{R})$ symmetry. (I thank Elias Kiritsis for pointing this out). $z = 5$ is the value that is required for the action to be the bosonic part of a supersymmetric theory in the bulk.

Using the techniques in [26] and [27] we can calculate the conductivity in the boundary theory using a probe brane. Scaling arguments imply the conductivity has the form

$$\sigma\left(\frac{B}{T^{\frac{2}{z}}}, \frac{n}{T^{\frac{2}{z}}}\right),$$

and with $z = 5$ this leads to QHE scaling dimensions κ and κ' given by

$$\kappa = \kappa' = \frac{2}{z} = 0.4$$

[28]. This is in remarkable agreement with the experimental value of 0.42 ± 0.01, [18].

Summary

There are remarkable parallels between $\mathcal{N} = 2$ SUSY Yang-Mills in $(3+1)$-dimensions and the quantum Hall effect in $(2+1)$-dimensions. Modular transformations on the complex coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ map between vacua of SUSY Yang-Mills, $\tau \to \frac{a\tau+b}{c\tau+d}$. This is a symmetry of the SUSY vacua with excitations that are composite objects consisting of gauginos bound to monopoles.

In the quantum Hall effect modular transformations on the complex conductivity , $\sigma = \sigma_{xy} + i\sigma_{xx}$, map between different QHE phases, $\sigma \to \frac{a\sigma+b}{c\sigma+d}$. This map is a symmetry of the QHE vacua with excitations that are composite objects consisting of electrons bound to vortices.

In this sense the fractional charges seen in the quantum Hall effect are analogous to the Witten effect in 4-dimensional SUSY QCD.

This analogy can be exploited using an AdS/CFT approach by considering 4-dimensional bulk theory with enjoys a continuous $Sl(2, \mathbf{R})$ symmetry acting on its classical solutions. Without being specific about which sub-group, we argued in [28] that this classical $Sl(2, \mathbf{R})$ is then broken to a discrete modular sub-group by Dirac-Schwinger-Zwanziger quantisation, $Sl(2, \mathbf{R}) \to \Gamma \subset Sl(2, \mathbf{Z})/\mathbf{Z}_2$ which survives in the full quantum theory. We know that this can happen for some supersymmetric theories in the bulk [4]. The boundary CFT is then identified with a quantum Hall system and the parameters in bulk solution are related to exponents in CFT: in particular if the dilaton action has the weighting dictated by supersymmetry in the bulk the Lifshitz scaling exponent of the boundary theory is determined from the bulk solution to be $z = 5$ and this agrees remarkably well with the experimental values of the quantum Hall scaling exponents κ and κ'.

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