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Abstract

We present an approximate technique, based on the principles of multimode Gaussian optics and perturbation theory, for calculating the distortions that occur when an off-axis ellipsoidal mirror is placed in the path of a weakly diffracting beam. More specifically, we calculate the Gaussian-beam-mode scattering matrices of off-axis ellipsoidal mirrors. The technique can be applied when the phase errors across the surface of the mirror are small. In this case, power is only scattered into a few neighbouring modes, and simple closed-form expressions can be derived for the elements of the scattering matrix.

1. Introduction

Off-axis ellipsoidal mirrors are often used at submillimetre wavelengths for controlling the characteristics of free-space beams. Unfortunately, mirrors of this kind introduce cross-polar scattering and spatial aberrations of the types well known in classical optics [1–3]. These undesirable effects occur even at the wavelength for which the mirror was designed; that is to say, even at the wavelength for which the surface of the mirror correctly transforms the phase front of the spherically-expanding input beam into the phase front of the spherically-contracting output beam. The distortions are caused by the need to conserve flux at the surface of the mirror. For example, if the intention is to couple two circularly-symmetric beams, and if the mirror is regarded as an inclined phase-transforming plane, then the difficulty of matching the flux of the diverging input beam to the flux of the contracting output beam at every point on the surface of the mirror is clear. Normally one thinks of amplitude distortions, and therefore coupling efficiency, as being associated with the shape of the mirror; for long-focal-length systems, however, the coupling efficiency can, in a sense, be regarded as being intrinsic to the two undistorted beams and independent of the precise form of the phase-transforming surface over which the reflection occurs. In the case of a diffracting beam, the distortions are particularly awkward to analyse because the form of the beam changes over the region occupied by the mirror. In this paper, we study the behaviour of off-axis ellipsoidal mirrors in the context of

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Gaussian-beam modes. We show, through a process of elimination and the application of perturbation theory, that there is a simple first-order way of describing the field distortions that occur when an off-axis ellipsoidal mirror is placed in the path of a weakly diffracting beam.

It is well known that Gaussian beam modes are eigenfunctions of the Fresnel diffraction integral [4]. Consequently, a paraxial beam can be traced through an optical system from one plane to another simply by changing two parameters in the modal expansion: one describes the scale size of the beam and is called the Gaussian radius, and the other parameterises the form of the beam and is called the phase slippage. Once the mode coefficients of a particular beam are known, by evaluating the overlap integrals over a convenient surface, the beam profile at any other location can be found simply by tracing the Gaussian width and the phase slippage through the optical system to the plane of interest.

When analysing a submillimetre-wave system in terms of Gaussian modes one usually assumes that each optical element only changes the radius of curvature of the incoming phase front and that there is a one-to-one correspondence between the modes of the incident beam and the modes of the reflected beam [5]; that is to say there is some outgoing mode set, defined by the position and size of the output waist, for which the coupling matrix between the incident and reflected modes is diagonal. When this is not the case, for example when the optical element is a truncating stop, the behaviour can be described by a scattering matrix, the off-diagonal elements of which are non zero [6]. In the case of an off-axis ellipsoidal mirror, the incident modes become distorted on reflection, whatever the size and position of the output waist, and the incident modes cease to be a valid basis set in terms of which the propagating beam can be followed through the mirror. The problem is then one of determining how each of the incident modes is scattered into the ouput modes, or equivalently one must be able to re-express the incident modes after distortion in terms of a propagating basis set. In this paper, we show that for reasonable mode sets the phase errors caused by the input and output beams diffracting within the region occupied by the mirror can be ignored and that a perturbation technique can be used to determine simple analytical expressions for the elements of the scattering matrix. This work is part of an ongoing effort to find simple ways of describing the effects of off-axis mirrors on paraxial beams.

2. Preliminary considerations

Consider the situation where a beam of coherent radiation is incident on a curved off-axis mirror. The phase front radii of curvature of the input and output modes, at the point of intersection of their axes, are say R_1 and R_2 respectively. One would intuitively expect—by analogy with geometrical optics—that the ellipsoid of revolution shown in Fig. 1 and defined by the centres of curvature of the input and output wavefronts, C_1 and C_2, and the centre of the off-axis mirror section, O, to be a good approximation to the surface required to transform the phase front of the input mode set into the phase front of the output mode set with little phase distortion. In this section we show that this assumption is indeed correct. Even if, however, we had a perfect phase match between the two phase fronts, the off-axis angle ensures that, due to projection effects, there is mismatch between the fields of corresponding modes, or equivalently, that some power must be scattered from one mode into another regardless of the shape of the mirror. The approach to scattering taken in this paper involves an approximate perturbation expansion, over the

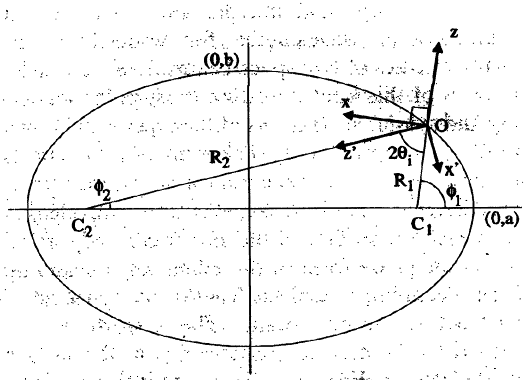

Fig. 1. The geometry of an off-axis ellipsoidal mirror.

surface of the mirror, of the incident component modes in terms of the true propagating modes of the reflected beam. Using this technique, we can calculate the scattering matix relating the complex mode coefficients of the input beam to the complex mode coefficients of the output beam. Moreover, we can derive simple analytical forms for the elements of the scattering matrix.

Throughout the paper, we shall treat fields as scalar and ignore cross-polar scattering. Let the modes of the incident beam have their waist at some plane A, and let the distance from A to O (the mirror centre) be d_1. Also, let the modes of the output beam have their waist at some plane B, and let the distance from O to B be d_2. The (x, y, z) coordinate axes are defined so that the z axis lies along OC_1 (see Fig. 1), with $z = 0$ at O and $z = -d_1$ at A. Similarly, the (x', y', z') coordinate system is defined so that the z' axis coincides with the optical axis of the reflected beam, OC_2, with $z' = 0$ at O and $z' = d_2$ at B. In Fig. 1 the (z, z') plane lies in the plane of the paper in such a way that the y' and y axes coincide. The angle of incidence, θ_i, is simply the angle between the axis of the input beam and the surface normal at the centre of the mirror.

On describing the incident beam in terms of a modal expansion we can write

$$E_i(x, y, z) = \sum_{mn} a_{mn} \psi^{(i)}_{mn}(x, y, Z)$$
$$= \sum_{mn} a_{mn} u_{mn}(x, y; W_i(Z))\, e^{j\phi^{(i)}_{mn}(x,y,Z)}, \qquad (1)$$

where $\psi^{(i)}_{mn}$ is a Gaussian mode of amplitude u_{mn} and phase $\phi^{(i)}_{mn}$, and a_{mn} is the complex mode coefficient. $Z = z + d_1$ is the distance measured from the position of the mode waist along the z-axis. Because the mirror is essentially an inclined phase-transforming plane, it is convenient to work in terms of Gaussian–Hermite modes. In this case u_{mn} is given by

$$u_{mn}(x, y; W_i(Z))$$
$$= \frac{H_m(\sqrt{2}x/W_i(Z))\, H_n(\sqrt{2}y/W_i(Z))}{W_i(Z)\sqrt{2^{m+n-1}m!n!\pi}}$$
$$\times \exp\left(-\frac{x^2 + y^2}{W_i^2(Z)}\right) \qquad (2)$$

where $H_m(s)$ represents a Hermite polynomial of order m in s, and $W_i(Z)$ is the Gaussian radius of the input beam at Z. $W_i(Z)$ is given by the usual relationship [7]

$$W_i(Z) = W_{0i}\left[1 + \left(\frac{Z\lambda}{\pi W_{0i}^2}\right)^2\right]^{1/2}. \qquad (3)$$

Notice that the above modes have been normalised to make the total generalised power $\int |\psi^{(i)}_{mn}|^2\, dx\, dy$ in each mode unity. The phase term $\phi^{(i)}_{mn}(x, y, Z)$ for the mnth mode is given by

$$\phi^{(i)}_{mn}(x, y, Z) = -k\left(Z + \frac{x^2 + y^2}{2R_i(Z)}\right)$$
$$+ (m + n + 1)\Delta\phi^{(i)}_0(Z), \qquad (4)$$

where $R_i(Z)$ is the phase curvature, and $\Delta\phi^{(i)}_0(Z)$ is the phase slippage for the incident beam. $R_i(Z)$ and $\Delta\phi^{(i)}_0(Z)$ are given by

$$R_i(Z) = Z\left[1 + \left(\frac{\pi W_{0i}^2}{\lambda Z}\right)^2\right], \qquad (5)$$

and

$$\Delta\phi^{(i)}_0(Z) = \tan^{-1}\left(\frac{\lambda Z}{\pi W_{0i}^2}\right) - \tan^{-1}\left(\frac{\lambda d_1}{\pi W_{0i}^2}\right), \qquad (6)$$

respectively. In this way the form of the field at any plane is completely parameterised by the Gaussian radius, $W_i(Z)$, the radius of curvature of the phase front, $R_i(Z)$, and the phase slippage $\Delta\phi^{(i)}_0(Z)$. It is important to realise that, in general, the radius of curvature of the mode set is not the same as the radius of curvature of the field. The radius of curvature of the mode set is a well-defined quantity whereas, because of structure in the phase of the field, the radius of curvature of the field is not.

It is also important to notice that the part of the slippage corresponding to propagation between the waist and the plane perpendicular to the beam at the centre of the mirror is included in the mode coefficients, whereas the phase slippage between the plane at the centre of the mirror and the actual surface is retained as part of the mode. In this way the mode coefficients of the input beam are referenced to the centre of the mirror. In the same way we reference the mode coefficients of the output beam to the centre of the mirror. From the point of view of generating computer software, this

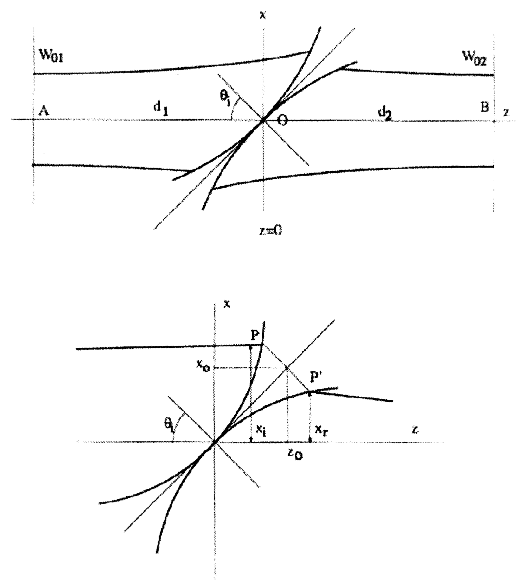

Fig. 2. The reflected beam represented as though in transmission.

scheme is a perfectly natural way of working because we can represent the diffraction of the input beam between its focus and the centre of the mirror by a scattering matrix which simply multiplies the mode coefficients by the appropriate phase slippage; we can then describe the amplitude distortion introduced by the mirror by a further scattering matrix; and finally we can describe the diffraction of the output beam between the centre of the mirror and its focus by yet another scattering matrix. In short, the reference plane for the scattering of modes at the mirror is taken to the plane perpendicular to the beam at the centre of the mirror.

The mode set for the reflected beam is most sensibly chosen so that (a) its axis of propagation is the same as the z'-axis in Fig. 1, (b) its Gaussian radius $W_r(Z')$, where $Z' = z' - d_2$, corresponds to that of the incident mode set at $O(W_i(O) = W_r(O))$, and (c) the phase curvatures of the incident and reflected mode sets satisfy the usual thin-lens formula at O:

$$\frac{1}{R_r(O)} = \frac{1}{R_i(O)} - \frac{1}{f}. \tag{7}$$

Since, however, z is not a constant over the mirror, W and R for both the incident and reflected modes vary across the surface. A mismatch will result between the phase and amplitude of an incident mode and the corresponding reflected mode. Thus, if the incident beam is described in terms of the modal sum $\Sigma a_{mn} \psi^{(i)}_{mn}$, and the reflected beam by an equivalent sum $\Sigma b_{mn} \psi^{(r)}_{mn}$, then a_{mn} will not in general equal b_{mn}. The two sets of mode coefficients can be regarded as being related through a scattering-matrix of the form

$$b_{ij} = \sum S_{mn,ij} a_{mn}. \tag{8}$$

The aim of this paper is to derive simple expressions for the elements of the scattering matrix.

To make the problem easier to visualise, we shall treat the reflected beam as though it is transmitted through the mirror. To obtain the correct "transmitted" beam, we reflect the true reflected beam and the surface of the mirror in the plane tangent to the surface of the mirror at the centre of the mirror O: this concept is illustrated in Fig. 2. The two coordinate axes (x, y, z) and (x', y', z') are then adjusted to coincide under this transformation. The optical path length between the input and output surfaces is clearly zero when the beam is considered in transmission; mathematically, the advancement of the phase front towards the outside of the mirror can be achieved by applying a phase transformation, which is dependent on the precise form of the mirror, over the tangent plane.

In the case of a diffracting beam, we have to be certain that the phase transformation provided by an off-axis ellipsoidal mirror is able to accommodate the changes in the radii of curvature of the incoming and outgoing phase fronts that occur within the region occupied by the mirror. Moreover, when calculating the amplitude distortions we have to consider the changes in the cross-sectional forms of the incoming and outgoing beams that occur within the region occupied by the mirror. In this paper, we show that the behaviour of an off-axis ellipsoidal mirror can, to first order, be described by an imaginary optical component, which provides a perfect phase transformation over a plane perpendicular to the unfolded beam at the centre of the mirror and which introduces amplitude distortions which can be characterised by a real-valued scattering matrix. More specifically, we will show that to first order the phase errors caused by the input beam diffracting within the region occupied by the mirror are cancelled by the phase errors caused by the output beam diffracting within the region occupied by the mirror. Let us proceed, therefore, by making the assumption, which we will show later to be correct, that at every point on the surface of the mirror the phase of each incident mode is approximately matched to the phase of the corresponding reflected mode and to the phases of the neighbouring modes into which power is scattered. In this way, we eliminate any phase curvature distortion introduced by the reflection of the incident beam, and we concentrate on amplitude mismatches due to projection effects.

2.1. Amplitude distortion

Our aim in this section is to express the amplitude distortion suffered by an incident Gaussian mode as a first-order perturbation expansion of the ideal reflected Gaussian modes. As pointed out above, the width of the incident beam at point P in Fig. 2, $W_i(P)$, is not the same as the width of the "transmitted" beam at point P', $W_r(P')$. At the centre of the mirror O, however the two beam radii are forced to be equal, $W_i(O) = W_r(O) = W_m$. Another complicating factor, which must be taken into account when calculating the amplitude distortions, is that the distance of a point on the input surface from the z-axis (such as P in Fig. 2) is different from the distance of the corresponding point on the output surface (P' in Fig. 2) from the z-axis.

Since the amplitude of an incident mode, $u_{mn}(P; W_i(P))$, and the amplitude of the corresponding undistorted reflected mode, $u_{mn}(P'; W_r(P'))$, are not equal at related points (P, P'), we express the amplitude of the incident mode, $u_{mn}(P; W_i(P))$, as a first order perturbation series of the transmitted mode amplitude $u_{ij}(P'; W_r(P'))$. To this end, let us denote the coordinates of P by (x_i, y_i, z_i) and the coordinates of P' by (x_r, y_r, z_r), as shown in Fig. 2. For the geometry chosen in Fig. 1, $y_i = y_r = y$; because, however, P and P' are not the same distances from the x- and z-axes, $x_i \neq x_r$ and $z_i \neq z_r$. We can write however,

$$u_{mn}(x_i, y_i; W_i) = u_{mn}(x_r + \Delta x, y; W_r + \Delta W). \tag{9}$$

Thus, there are effectively two contributions to the mismatch between the incident and reflected mode amplitudes, $u_{mn}(x_i, y; W_i(z_i))$ and $u_{mn}(x_r, y; W_r(z_r))$ respectively. These are (i) $x_r \neq x_i$ and (ii) $W_i(z_i) \neq W_r(z_r)$. As $\Delta x = x_i - x_r$ and $\Delta W = W_i(z_i) - W_r(z_r)$, we can write down a first-order Taylor expansion relating the mode amplitudes:

$$\begin{aligned} u_{mn}(x_i, y; W_i) \approx{} & u_{mn}(x_r, y; W_r) \\ & + \left(\frac{\partial u_{mn}}{\partial W}\right)(x_r, y; W_r)\,\Delta W \\ & + \left(\frac{\partial u_{mn}}{\partial x}\right)(x_r, y; W_r)\,\Delta x. \end{aligned} \tag{10}$$

In this expression we have ignored high-order terms. When the high-order terms are included it is clear that the sum of the terms containing derivatives constitutes an error function which when added to the output mode gives the corresponding input mode. Regardless of the functional forms of ΔW and Δx we can write the error function at any plane for which z is constant as a sum of output modes. The mode coefficients will be constant as we move over this plane, but will in general vary if we move to a different plane. It is also clear after some consideration that, because the ouput beam has a unique expansion in terms of the output mode set, if we restrict ourselves to moving over any surface, regardless of what the surface may be, then we can write the error function in terms of the output mode set, and the mode coefficients will not vary as we move over this surface. As a way forward, we will assume that for the functional forms of ΔW and Δx used in this paper the above statement is true even when we only consider the linear terms of the expansion. This assumption will turn out to be correct.

Expressions for the terms on the right hand side of the above equation are derived in the Appendix; it is shown that

$$\frac{\partial u_{mn}}{\partial W}\Delta W=\left[\frac{W_m\tan\theta_i}{f}\right]\times\frac{\partial}{\partial W}\left[\frac{x/WH_m(\sqrt{2}x/W)\,H_n(\sqrt{2}y/W)}{\sqrt{2^{m+n-1}m!n!\pi}}\times\exp\left(-\frac{x^2+y^2}{W^2}\right)\right], \tag{11}$$

and

$$\frac{\partial u_{mn}}{\partial x}\Delta x=\left[\frac{W_m\tan\theta_i}{2f}\right]\left(\frac{x^2+y^2}{W^2}\right)\times\frac{\partial}{\partial x}\left[\frac{H_m(\sqrt{2}x/W)\,H_n(\sqrt{2}y/W)}{\sqrt{2^{m+n-1}m!n!\pi}}\times\exp\left(-\frac{x^2+y^2}{W^2}\right)\right], \tag{12}$$

In the case of the first expression we have restricted ourselves to moving over the tangent plane of the mirror, whereas in the second expression we have restricted ourselves to moving over the surface of the mirror. Loosely speaking, the first expression describes the amplitude distortions caused by the inclination of the mirror whereas the second expression describes the amplitude distortions caused by the projection of the beams on the curved mirror surface. These approximations are valid as long as $(W_m\tan\theta_i/f)^2\ll 1$. Since, in general, for Gaussian-beam-mode theory to be a valid description of propagation the focal ratios of the beams must not be too small, the inequality will in fact be true for all reasonable quasi-optical systems.

We can now use the usual recursion relationships for H_n to generate an expansion in Hermite polynomials for the error terms:

$$\frac{\mathrm{d}H_n(s)}{\mathrm{d}s}=2nH_{n-1}(s), \tag{13}$$

$$H_{n+1}(s)=2sH_n(s)-2nH_{n-1}(s). \tag{14}$$

As described above, we wish to express the error function over the surface of the mirror as a sum of output modes, and we assume from the above argument that this expansion will give rise to a set of mode coefficients that are invariant over the surface of interest. We therefore write

$$\frac{\partial u_{mn}}{\partial W}\Delta W+\frac{\partial u_{mn}}{\partial x}\Delta x=\sum_{i,j}\alpha_{mn,ij}u_{ij}, \tag{15}$$

where $|\alpha_{mn,ij}|$ are all much less than one and independent, for a given surface, of position. Forming the derivatives and manipulating the resulting equations into the appropriate forms is a straightforward but tedious process. For convenience we used Mathematica as a symbolic manipulator to derive expansions for the error function in terms of the output modes ($\alpha_{mn,ij}$). Finally, because we are interested in the scattering of power between modes, rather than just the error function, we write

$$u_{mn}(x_i,y;W_i)=\sum_{i,j}S_{mn,ij}u_{ij}(x_r,y;W_r), \tag{16}$$

where $S_{mn,ij}=\delta_{i,m;j,n}+\alpha_{mn,ij}$. The expressions for the scattering coefficients derived in this way are given in Table 1. We shall discuss these equations later; for the moment, it is sufficient to observe that they all have exceedingly simple forms.

Table 1
Non-zero scattering matrix terms $S_{mn,ij}$, where $\beta=W_m\tan\theta_i/8f$, with f being the focal length of the mirror and W_m the beam radius at the mirror

i	j: $n-2$	n	$n+2$
$m-3$	0	$-\sqrt{m(m-1)(m-2)}\beta$	0
$m-1$	$-\sqrt{mn(n-1)}\beta$	$\sqrt{m}(2n-m+1)\beta$	$3\sqrt{m(n+1)(n+2)}\beta$
m	0	1	0
$m+1$	$-3\sqrt{(m+1)n(n-1)}\beta$	$(m-2n)\sqrt{(m+1)}$	$\sqrt{(m+1)(n+1)(n+2)}\beta$
$m+3$	0	$\sqrt{(m+3)(m+2)(m+1)}\beta$	0

We can now substitute the above expression for the amplitudes of the input modes into the modal expansion for the input beam across the surface of the mirror, and after changing the order of the summations, equate the result to the modal expansion for the output beam across the surface of the mirror. It is clear that the above scattering matrix completely describes the behaviour of the mirror, and casts the expansion of the output beam in terms of the output modes into the correct form, as long as at every point on the surface of the mirror the phase-front radius of curvature of the input mode set is transformed into the phase-front radius of curvature of the output mode set, and the phase error due to the phase-slippage term is approximately zero. In short $S_{mn,ij}$ cannot be regarded as scattering coefficients unless it can be shown that the modal phase fronts are perfectly matched by the phase transformation introduced by the mirror. We shall now consider each of the phase factors in turn.

2.2. Phase-front distortion

As the beam propagates along the z-axis, the phase curvature of the beam evolves in such a way that the centre of curvature does not remain fixed. This situation will cause a phase mismatch between the incident mode set and the reflected mode set, even if the reflecting surface is an off-axis ellipsoid, since for an ellipsoid the centres of curvature for the mirror surface (in other words the geometrical foci) are fixed in position. Of course, in the geometrical-optics limit ($\lambda \to 0$) the surface is a perfect phase transformer for any incident beam with a waist at one of the foci.

In general for any wavelength, it is easy to show that the rate at which R, the beam-mode phase curvature, varies with Z, the distance from the waist, is given by

$$\frac{\partial R}{\partial Z} = \left[1 - \left(\frac{\pi W_0^2}{\lambda Z}\right)^2\right] = \left[1 - \left(\frac{\lambda R}{\pi W^2}\right)^2\right]. \tag{17}$$

On the other hand, for an ellipsoidal mirror the geometrical foci (or centres of curvature) are fixed. In that case we can write $\partial R_g / \partial Z = 1$, where for the input beam, $R_g(z)$ is the distance of an on-axis point from C_1 (or C_2 for the reflected beam). Thus, even though the surface of the mirror is designed so that $R_{g,i}(0) = R_1 = R_i(O)$ for the incident beam and $R_{g,r}(O) = R_2 = R_r(O)$ for the reflected beam at the centre of the mirror (at point O in Fig. 1), away from $O, (z \neq 0)$, a phase match is not guaranteed. More precisely, there will be a phase error of

$$\Delta\phi(z) = \frac{\pi(x^2+y^2)}{\lambda}\left[\frac{1}{R_g(z)} - \frac{1}{R(z)}\right]. \tag{18}$$

At $z = 0$, $R_{g,i}(0) = R_i(0) = R_1$ so that $\Delta\phi(0) = 0$ for the input beam. Then, assuming that the fractional changes in the curvatures over the surface of the mirror, $(R_{g,i}(z) - R_1)/R_1$ and $(R_i(z) - R_1)/R_1$, are very much less than one, we can expand the first and second terms of the phase-error expression separately to first order and apply Eq. (17) to get, after some manipulation

$$\Delta\phi(z) \approx -\frac{\lambda(x^2+y^2)z}{\pi W_m^4}. \tag{19}$$

If we assume that almost all of the power is contained within the beam radius W_m, we have $x^2 + y^2 = W_m^2$, $\tan\theta_i = z/W_m$, and the maximum phase error for the input beam becomes

$$\Delta\phi(W_m \tan\theta_i) \approx -\frac{\lambda \tan\theta_i}{\pi W_m}. \tag{20}$$

For the reflected modes an identical expression is obtained except that sign is reversed; consequently, the total phase error associated with the phase-front radii of curvature changing within the region occupied by the mirror is, to first order, zero. Numerically, we have found this conclusion to be an accurate representation of actual behaviour. Thus, as expected, an ellipsoid of revolution is essentially a perfect phase transformer for any reasonable Gaussian mode.

It is important to remember that there is an implied assumption in the above analysis: it is assumed that the spherical phase front of a mode is adequately approximated by a parabolic phase front. In other words R is much larger than W such that

$$\exp\left(j\frac{\pi(x^2+y^2)}{\lambda R}\right) \approx \exp\left[j\frac{2\pi}{\lambda}\left(\sqrt{(x^2+y^2)+R^2} - R\right)\right]. \tag{21}$$

This approximation will break down in the extreme far-field; in that case, however, the propagation of the beam can be described in terms of geometrical optics, with $R(z) = z + d_1$, and the ellipsoid of revolution becomes the perfect phase-transforming surface.

To complete the analysis we need to show that the phase errors caused by the beam diffracting within the region occupied by the mirror can to first order be ignored. For the input mode set, the functional behaviour of the phase slippage of the fundamental with z is described by the relationship

$$\Delta\phi_0^{(i)}(z_i) = \tan^{-1}\left[\frac{\lambda(d_1 + z_i)}{\pi W_{0i}^2}\right] - \tan^{-1}\left[\frac{\lambda d_1}{\pi W_{0i}^2}\right] \approx \frac{\lambda z_i}{\pi W_m^2}, \tag{22}$$

where we have expanded the first term in z and noted that $\lambda \ll W_m$. Similarly one finds that the phase slippage for the transmitted beam is given by $\Delta\phi_0^{(r)}(z_r) \approx -(\lambda z_r/\pi W_m^2)$. Since for the mnth mode there is a total phase slippage of $(m + n + 1)\,\Delta\phi_0$, we can assume that there is no phase mismatch to first order between the mnth incident mode and the corresponding mnth reflected mode. This observation will be approximately true for the actual phase slippage as long as $(m + n + 1)$ is not too large. Now, when we consider the scattering of the mnth mode to reflected modes of order other than mn we do need to consider the phase slippage term because cancellation will not occur. More specifically, we are left with a maximum phase error of the form

$$\delta\phi_{mn,ij} = (m + n)\Delta\phi_0^{(i)}(z_i) + (i + j)\Delta\phi_0^{(r)}(z_r) \approx ((m - i) + (n - j))\left(\frac{\lambda \tan\theta_i x_i}{\pi W_m^2}\right), \tag{23}$$

with the approximation that $z_i \approx z_r \approx x_i \tan\theta_i$. It turns out (as discussed below) that i and j are only different from m and n by at most 3, and therefore the $\delta\phi$ term generates an error of order $(\lambda \tan\theta_i/\pi W_m)$: since $\exp(j\delta\phi_{ij,mn}) \approx 1 + j\delta\phi_{ij,mn}$. Since $\alpha_{mn,ij}$ is already small, this error will only generate second-order terms. Thus, even for the case of an inclined mirror, the phase errors can be neglected, as we set $\delta\phi_{mn,ij} \approx 0$. This implies only amplitude mismatch is important at the design wavelength and an input mode is scattered according to

$$\psi_{mn}^{(i)}(x_i, y, z_i; W_i, R_i) \Rightarrow \psi_{mn}^{(r)}(x_r, y, z_r; W_r, R_r) + \Sigma\alpha_{mn,ij}\psi_{ij}^{(r)}(x_r, y, z_r; W_r, R_r) \tag{24}$$

as required. We can interpret $\alpha_{mn,ij}$ as a perturbation error-matrix term, and $S_{mn,ij} = \delta_{mn,ij} + \alpha_{mn,ij}$, as a scattering-matrix term. In matrix notation $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{\alpha}$. The $S_{mn,ij}$ are tabulated in Table 1. As already indicated power is only scattered between neighbouring modes; thus i only takes on values of $m - 3$, $m - 1$, $m + 1$, and $m + 3$ and j values of $n - 2$, n, and $n + 2$. The total amount of power scattered from the mnth mode into neighbouring orders is approximately given by $\Sigma_{i \neq m, j \neq n}|S_{mn,ij}|^2$. A subtle but important point is that according to this first-order approximation, power is scattered into neighbouring modes but no power is lost from the original mode. This curious situation occurs because the scattering out of the original mode is not a first-order effect. It turns out that the best way to calculate the actual scattering coefficient is to subtract the power which is known to be lost to high-order modes. An analogous procedure is used regularly in quantum mechanics.

3. Discussion

In the preceding sections we have shown through a detailed process of elimination that an off-axis ellipsoidal mirror is an excellent approximation to the true surface that is required in order to focus a weakly diffracting beam. It has been shown that the phase errors introduced by the beam diffracting within the volume of the mirror can, to first order, be ignored. All of the distortions result from the need to conserve flux at the surface of the mirror. The analysis is complicated, but the result is extremely simple. In the context of Gaussian beam modes, we can represent an off-axis ellipsoidal mirror as a thin lens which introduces amplitude distortions. The amplitude distortions are characterised by a scattering-matrix, the elements of which are listed in Table 1. As a consequence, it is straightforward to propagate an

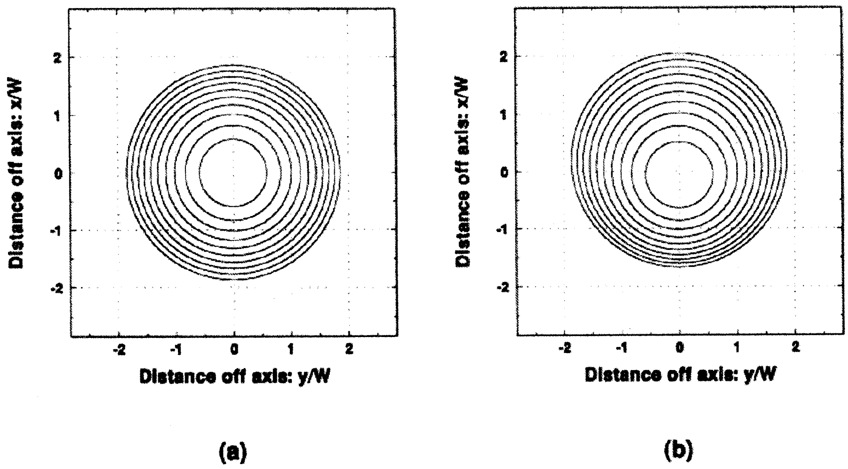

Fig. 3. Contour plots of the intensity in dB of (a) the incident beam and (b) the reflected beam at an off-axis ellipsoidal mirror in the case of the F3 Gaussian beam. The contour level interval is 3 dB. The first contour level is at −3 dB and the lowest level shown is −30 dB.

image through a mirror and study the emerging forms.

In many quasi-optical systems it is assumed that the propagating beam is adequately approximated by a pure Gaussiam mode. Any off-axis curved mirrors will tend to distort the Gaussian shape and introduce high-order modes. It is often of interest to quantify the power lost to the high-order modes. As can be seen from Table 1, in the case of the fundamental mode, power is scattered only into the (30) and (12) modes; the relevant scattering-matrix coefficients are $S_{00,30} = \sqrt{6}W_m \tan\theta_i/8f$ and $S_{00,12} = \sqrt{2}W_m \tan\theta_i/8f$. The total fraction of the power scattered from the fundamental is given by $|S_{00,30}|^2 + |S_{00,12}|^2 = [\tan\theta_i\, W_m/2\sqrt{2}f]^2$, in agreement with [2]. Moreover, we find that the scattering coefficients are in excellent agreement with those calculated using the more rigorous method described elsewhere [3]. Using $S_{00,30}$ and $S_{00,12}$, we can reconstruct the incident Gaussian and the distorted reflected beam at the plane defined by $z = 0$ (at the centre of the mirror), for the case when $W_m/f = 1/6$

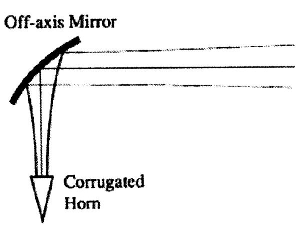

Fig. 4. Optical configuration showing an off-axis ellipsoidal mirror illuminated by a corrugated horn.

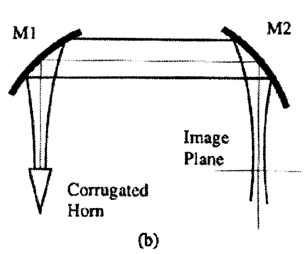

Fig. 5. Alternative optical configurations for a Gaussian beam telescope illuminated by a diffraction limited corrugated horn. The first mirror is in the far field of the horn.

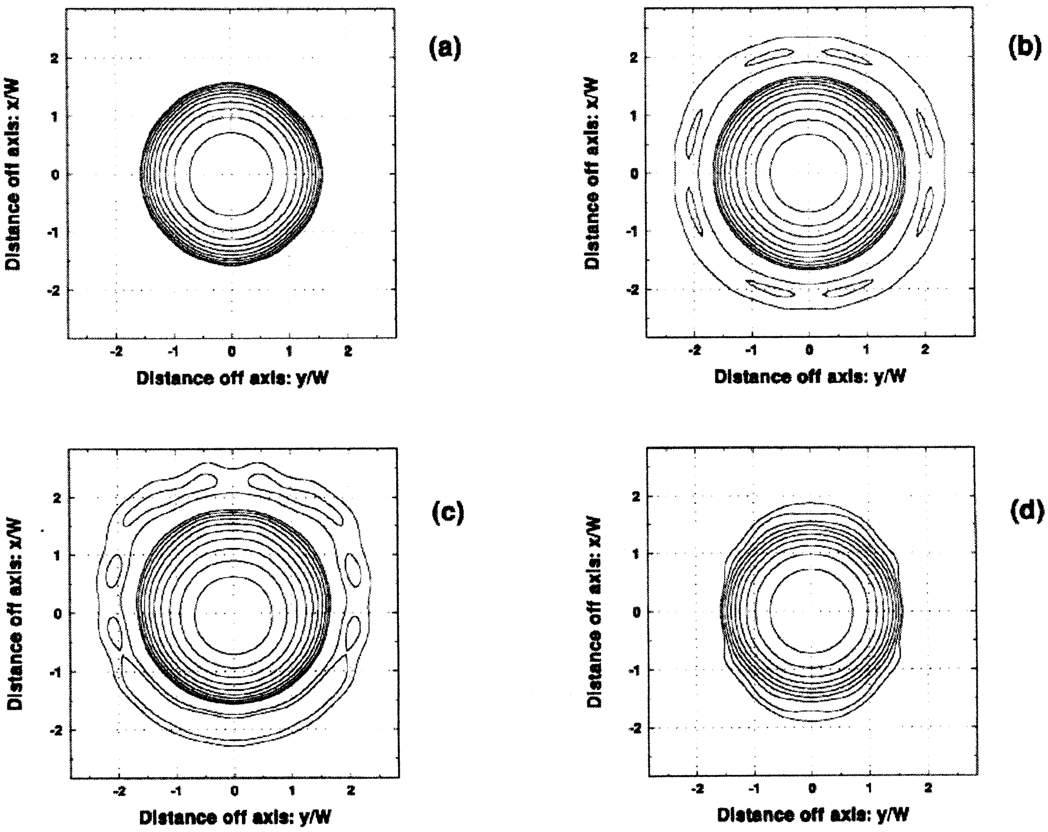

Fig. 6. Contour plots of the intensity in dB of the beam from an F5 diffraction limited corrugated horn illuminating an off-axis mirror as illustrated in Fig. 4. (a) Shows the beam at the horn mouth, (b) the beam incident on the off-axis mirror, (c) the beam reflected beams from the off-axis mirror and (d) the beam in the far-field of the mirror. The contour levels are the same as those in Fig. 3.

(F3) and $\theta_i = 45°$. In this way we regard the plane at the centre of the mirror as an effective aperture from which the output beam propagates. The results are shown in Fig. 3a and Fig. 3b as contour plots of intensity. The axes are normalised to the beam width at the centre of the mirror W_m. The output beam is distorted in the way one would expect.

A more interesting situation, and one of particular practical importance, occurs when the beam of a corrugated horn is reflected from an off-axis ellipsoidal mirror. Following Wylde [8], we describe the beam from the corrugated horn in terms of a small number of Gaussian modes. In our case, however, we describe the beam in terms of Hermite polynomials rather than Laguerre polynomials as was originally done by Wylde. In the case of Hermite–Gaussian modes essentially all of the power (99.9%) is contained in the modes having order of less than (10, 10). The two optical configurations we have chosen are shown in Fig. 4 and Fig. 5. The first, Fig. 4, consists simply of a corrugated horn illuminating an off-axis mirror. The second, Fig. 5, shows two Gaussian-beam telescope configurations of magnification unity ($f_1 = f_2$), where the two mirrors are separated by the sum of their focal lengths. In each case, we have assumed a diffraction-limited horn, and we have placed the aperture of the horn in the focal plane of the first mirror. It should be remembered that

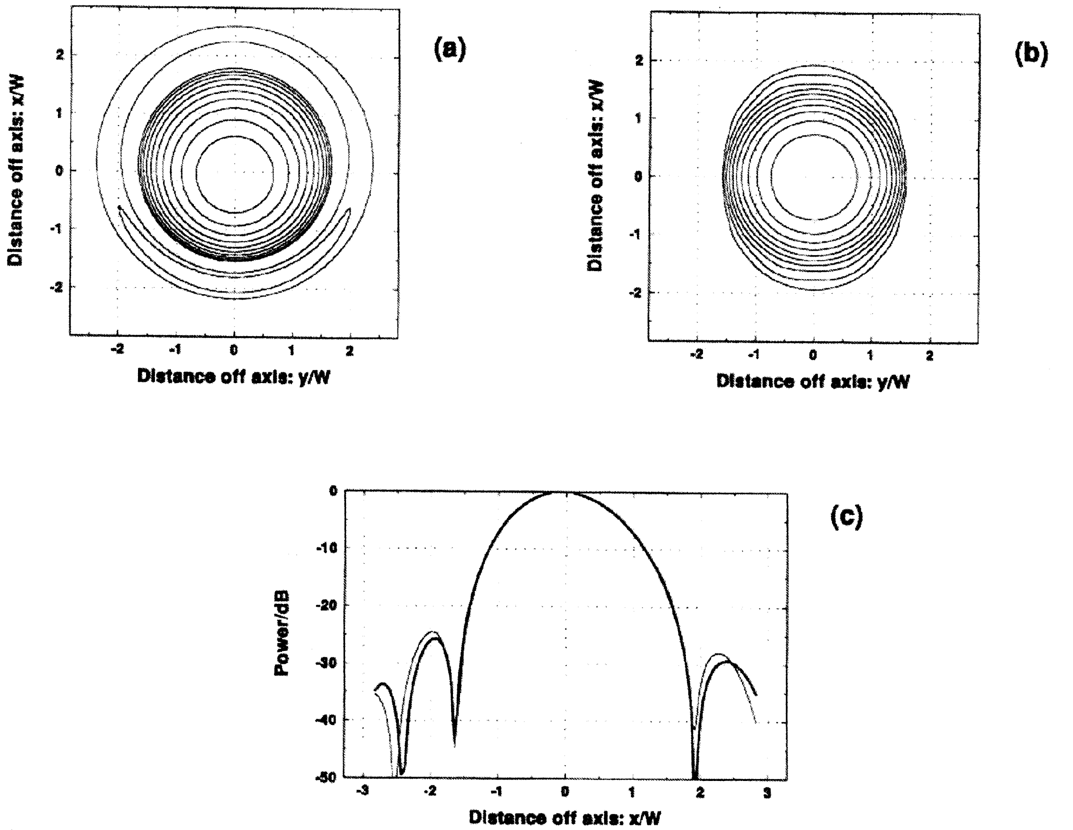

Fig. 7. Contour plots of the intensity in dB of the F5 diffraction limited corrugated horn beam (a) reflected from and (b) in the far-field of the off-axis mirror illustrated in Fig. 4 calculated using a projected aperture technique. (c) Shows the beam pattern comparison at the mirror with the beam illustrated in Fig. 6b. The contour levels in (a) and (b) are the same as those in Fig. 3.

for a diffraction-limited horn the position of the waist is at the aperture. We have also assumed that the horn produces an F5 beam ($W_m/f = 1/10$) and that the distance between the aperture of the horn and the mirror is 1000λ. The beam radius at the second focal plane of the first mirror is much larger than the input waist, and clearly the beam is highly collimated in that region.

In Fig. 6a, we show the power pattern at the aperture of the horn. The pattern shown was reconstructed by summing the appropriate set of Gaussian–Hermite modes. As expected, the calculated fields are well constrained within the aperture of the horn. In Fig. 6b, we show the intensity distribution incident on the first mirror. This field was reconstructed by using the incident mode set at the central reference plane of the first mirror. At this plane we should simply have the far-field pattern of a corrugated horn, or equivalently the Fourier transform of the field distribution over the aperture; indeed, the first sidelobe can easily be seen. In Fig. 6c, we show the scattered beam at the central reference plane. This intensity distribution was calculated by determining the scattering coefficients according to Table 1 and reconstructing the scattered field at the centre of the mirror. The beam is clearly skewed to one side and non-symmetrical sidelobes appear. Finally in Fig. 6d, we show the far-field power distribution of the horn-mirror combination. In this position we should see

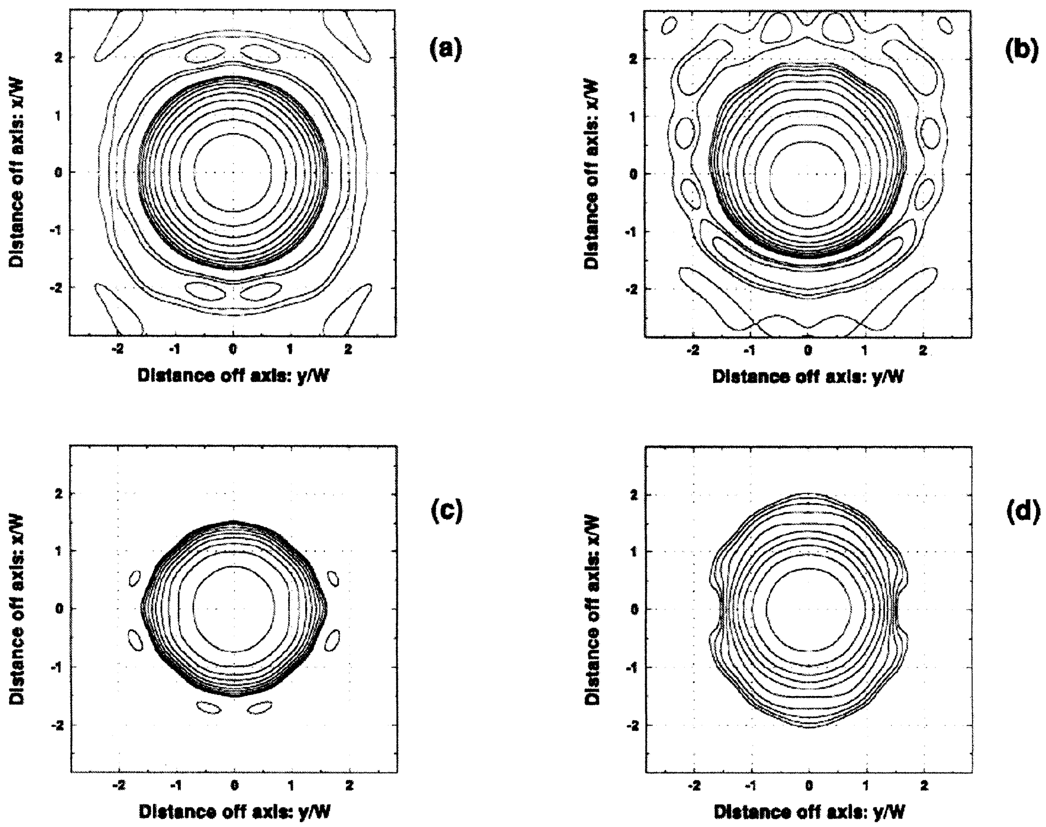

Fig. 8. Contour plots of the intensity in dB of the beams reflected from the second off-axis mirror (a) and (b) and the output image plane (c) and (d) in the alternative Gaussian beam telescope configurations shown in Fig. 5. The contour levels are the same as those in Fig. 3.

the Fourier transform of the field distribution at the second focus, which is very similar to the field distribution at the reference plane of the mirror. Clearly, the beam is no longer skewed, but the beam is elongated due to the distortions introduced by the mirror. All of these findings are consistent with the known behaviour of off-axis reflectors.

We have already compared the mode coefficients calculated using the above scheme with the mode coefficients calculated using a more rigorous method; we do, however, still require independent verification that the technique gives reasonable results. To this end we can consider the above example in the short-wavelength limit, in which case we have a paraboloidal mirror illuminated by a diffraction-limited F5 corrugated feedhorn. In this example, the mirror is effectively an infinite number of wavelengths away from the horn, and it is possible to determine the output field at the mirror using the usual projected-aperture technique [9,10]. In Fig. 7a, we show the power distribution for the fields in the focal plane of the scattering mirror; in Fig. 7b, we show the intensity of the Fourier transform of the field distribution, which of course gives the far-field power pattern of the horn-reflector combination. These distributions can be considered exact in this short-wavelength limit. We can now compare Fig. 7a and Fig. 6c and also Fig. 7b and Fig. 6d: the agreement is extraordinarily good considering the simplicity of the Gaussian-mode calculation. In Fig. 7c, we emphasise the agreement by showing cuts through the $y = 0$ axis of Figs. 7a and 6c. Much of the difference in the sidelobe structure is actually due to errors in the representation of the original beam rather than due to errors in the scattering matrix. We believe that by choosing mode sets that are more correctly chosen on the basis of sampling theory, we can improve the quality of the calculations yet further.

We now consider the two arrangements shown in Fig. 5. In case (b), the mirrors are orientated to minimise the total distortion; whereas in case (a), the mirrors are orientated to maximise the total distortion. In principle, both of these combinations should form an image of the aperture of the horn at the output waist of the second mirror [11]. In Fig. 8a, we show the reflected beam at the reference plane of the second mirror in the case when the distortions should be minimised. In Fig. 8c, we show the beam at the second focus. This contour plot should be identical to that shown in Fig. 6a. It is clear that the fields have returned to those of the aperture of the horn. In Fig. 8b, we show the reflected beam at the reference plane of the second mirror when the mirrors are orientated in a way which should maximise the distortions. In Fig. 8d, we show the resulting intensity distribution of the field at the second focus, and again this distribution should be compared with that of Fig. 6a. The incorrect orientation results in an elongation of the beam. The pinching across the middle of the beam is a real phenomenon. It is clear that one configuration (Fig. 5a) does indeed reduce the aberrations, while the other (Fig. 5b) increases the aberrations. If there had been a narrow waist between the two mirrors the effect of the two configurations would have been reversed [2].

4. Conclusions

We have shown through a detailed process of elimination that the phase transformation provided by an off-axis ellipsoidal mirror is an excellent approximation to the ideal surface that is required in order to focus a weakly diffracting beam. More specifically, we have shown that the phase errors incurred as a result of the input beam diffracting within the region occupied by the mirror are cancelled by the phase errors incurred as a result of the output beam also diffracting within the region occupied by the mirror. The distortions seen on the output beam are a consequence of having to conserve flux at the surface of the mirror, and can be regarded as a result of geometrically propagating through the mirror the amplitude distribution that is incident on the mirror. If we use Gaussian-beam modes to propagate a diffracting image, the field distortions can be included in a simple way by scattering power between neighbouring modes. The elements of the scattering matrix needed for this calculation have simple analytical forms. The technique can be used for assessing the distortions that occur when an off-axis ellipsoidal or parabolic mirror is used to focus a weakly diffracting beam. When incorporated in efficient multi-mode beam-propagation software it will provide a powerful tool for modelling the distortions that occur in real millimetre- and submillimetre-wave quasioptical systems.

Acknowledgements

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A. Appendix

A.1. Derivation of an expression for $(\partial u_{mn}/\partial W)\Delta W$

We begin by finding a suitable expression for ΔW. We will assume that the mismatch between the incident and transmitted beam widths is the same across the tangent plane as between the two mirror surfaces. This is a reasonable approximation for shallow mirrors. If we choose the origin of the z-axis to lie at O in Fig. 2 then W_i^2 can be expressed as

$$W_i^2(x,y,z) = W_{0i}^2\left[1+\left(\frac{\lambda(z+d_1)}{\pi W_{0i}}\right)^2\right] = W_m^2 + 2\left(\frac{\lambda}{\pi W_{0i}}\right)^2 d_1 z + \left(\frac{\lambda z}{\pi W_{0i}}\right)^2, \tag{A.1}$$

where W_{0i} is the incident beam-waist radius, which is located at $z = -d_1$. Similarly, we can write for the reflected beam

$$W_r^2(x,y,z) = W_{0r}^2\left[1+\left(\frac{\lambda(z-d_2)}{\pi W_{0r}}\right)^2\right] = W_m^2 - 2\left(\frac{\lambda}{\pi W_{0r}}\right)^2 d_2 z + \left(\frac{\lambda z}{\pi W_{0r}}\right)^2, \tag{A.2}$$

where W_{0r} is the reflected beam waist radius, which is located at $z = +d_2$. It is easy to show that $\pi W_m^2/\lambda R_i(O) = \lambda d_1/\pi W_{0i}^2$ and $\pi W_m^2/\lambda R_r(O) = -\lambda d_2/\pi W_{0r}^2$, which implies

$$\left(\frac{\lambda}{\pi W_{0i}}\right)^2 d_1 = \frac{W_m^2}{R_1}, \quad \left(\frac{\lambda}{\pi W_{0r}}\right)^2 d_2 = -\frac{W_m^2}{R_2}, \tag{A.3}$$

so that for corresponding points (x, y, z) on the tangent plane shown in Fig. 2, we can write

$$W_i^2 - W_r^2 = 2\left[\frac{z}{R_i(O)} - \frac{z}{R_r(O)}\right]W_m^2 + \left[\frac{z^2}{d_1 R_i(O)} + \frac{z^2}{d_2 R_0(O)}\right]W_m^2. \tag{A.4}$$

If we can assume that $\Delta W \ll W_m$ and also, therefore, that $W_m \approx W_i \approx W_r$, then $\Delta W \approx (W_i^2(z) - W_r^2(z))/2W_m$, and

$$\Delta W \approx z\left[\frac{1}{R_i(O)} - \frac{1}{R_0(O)}\right]W_m + O\left(\left(\frac{z}{f}\right)^2\right) \approx \frac{z}{f}W_m + O\left(\left(\frac{z}{f}\right)^2\right), \tag{A.5}$$

where we neglect terms of order $O((z_m/f)^2)$. Clearly if the mirror has a long focal ratio, $f \gg z$ for any point on the surface of the mirror. On the tangent plane $z = x \tan\theta_i$, which will be approximately true for the output surface. Thus we can express ΔW as

$$\Delta W = x\left(\frac{W_m \tan\theta_i}{f}\right). \tag{A.6}$$

Using this approximation for ΔW, we obtain

$$\frac{\partial u_{mn}}{\partial W}\Delta W = \left[\frac{W_m \tan\theta_i}{f}\right] \times \frac{\partial}{\partial W}\left[\frac{\frac{x}{W}H_m\left(\sqrt{2}\frac{x}{W}\right)H_n\left(\sqrt{2}\frac{y}{W}\right)}{\sqrt{2^{m+n-1}m!n!\pi}} \times \exp\left(-\frac{x^2+y^2}{W^2}\right)\right]. \tag{A.7}$$

A.2. Derivation of an expression for $(\partial u_{mn}/\partial x)\Delta x$

$\Delta x = x_i - x_r$, the "error" in x, occurs because corresponding points $P(x_i, y_i, z_i)$ and $P'(x_r, y_r, z_r)$, as shown in Fig. 2, for the input and output surfaces of the mirror have different coordinates. If the surface of the mirror is an ellipsoid of revolution and R_1 and R_2 are the two distances from the centre of the mirror to the geometrical foci, then clearly

$$2a = R_1 + R_2 = \sqrt{x_i^2 + y^2 + (R_1 + z_i)^2} + \sqrt{x_r^2 + y^2 + (R_2 - z_r)^2}. \tag{A.8}$$

Then assuming R_1 and R_2 are much greater than the dimensions of the mirror, this equation reduces to

$$z_r \approx z_i + \left(\frac{x_i^2+y^2}{2R_1} + \frac{x_r^2+y^2}{2R_2}\right). \tag{A.9}$$

But since (x_i, y, z_i) and (x_r, y, z_r) lie on a line perpendicular to the tangent plane, $x_i - x_r = (z_r - z_i)\tan\theta_i$; also Δx is small compared to x; one therefore obtains

$$\Delta x \approx \frac{\tan\theta_i}{2f}(x^2 + y^2). \tag{A.10}$$

Using this approximation for Δx, we obtain:

$$\frac{\partial u_{mn}}{\partial x}\Delta x = \left[\frac{W_m \tan\theta_i}{2f}\right]\left(\frac{x^2+y^2}{W^2}\right) \times \frac{\partial}{\partial x}\left[\frac{H_m\left(\sqrt{2}\frac{x}{W}\right)H_n\left(\sqrt{2}\frac{y}{W}\right)}{\sqrt{2^{m+n-1}m!n!\pi}} \times \exp\left(-\frac{x^2+y^2}{W^2}\right)\right]. \tag{A.11}$$

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