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Spectral properties of matrix polynomials in the max algebra

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A B S T R A C T

We consider the spectral properties of matrix polynomials over the max algebra. In particular, we show how the Perron–Frobenius theorem for the max algebra extends to such polynomials and illustrate the relevance of this for multistep difference equations in the max algebra. We also present a number of inequalities for the largest max eigenvalue of a matrix polynomial.

1. Introduction

The *max algebra*, which is the focus of this paper, consists of the nonnegative real numbers $\mathbb{R}_+$ equipped with the basic operations of multiplication, $a \otimes b = ab$, and maximization $a \oplus b = max\{a, b\}$. This algebraic system arises directly in applications such as the Viterbi algorithm [1] and has also been used to construct ranking vectors for analytic hierarchy processes [2]. Furthermore, the max algebra is isomorphic to the *max-plus* algebra, which provides a natural framework for analysing a broad class of discrete event systems arising in areas such as transportation and manufacturing. In fact, the max-plus algebra has been used in applications such as the design and analysis of bus and railway timetables as well as in the scheduling of high-throughput industrial processes [3].

Given their importance in applications, dynamical systems and matrices in the max and max-plus setting have received considerable attention. Many key results in these areas relate to extensions of the classical Perron–Frobenius theory [4,5] to the max setting. A very early result in this direction

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was obtained by Vorobyev in [6]. For more recent references giving details on results of this type and their applications, see [1,7] and references therein. For a recent reference focussing specifically on the Perron–Frobenius Theorem for the max algebra, see [8], wherein several proofs of this fundamental result were presented. Formally, the max Perron–Frobenius Theorem states the following; for an irreducible nonnegative matrix A, the maximal cycle geometric mean, $\mu(A)$, which is defined below, is the unique eigenvalue of A over the max algebra. Furthermore, the eigenvectors corresponding to $\mu(A)$ must be strictly positive in this case.

The classical power method was modified to obtain a technique for calculating the max eigenvalue and max eigenvectors of an irreducible nonnegative matrix in [9,10]. Conditions guaranteeing convergence of the power method were also given in these papers.

More recently, results relating classes of matrix norm, the maximal cycle geometric mean and asymptotic stability for a single matrix in the max algebra were presented in [11]. This line of research was then further extended to sets of matrices in [12,13]. Here, a max algebra version of the generalised spectral radius, which plays a central role in determining stability and convergence properties of discrete linear inclusions and nonhomogeneous matrix products was introduced and the generalised spectral radius theorem was extended to the max algebra.

In the present paper, we consider another aspect of the spectral theory for the max algebra. Specifically, we investigate matrix polynomials defined over the max algebra and obtain a set of results corresponding to those presented recently in [14] for nonnegative Perron polynomials. It should be emphasised that matrix polynomials over the max-plus algebra have previously been considered in connection with scheduling problems and timetable analysis [3]. The spectral properties of such polynomials have important implications for the stability of timetables with respect to the propagation of delays. For both the max and the conventional algebras, matrix polynomials are closely related to multistep difference equations. In [14], this relationship was exploited to derive a multistep version of the Fundamental Theorem of Demography for the conventional algebra. For a reference highlighting the role played by classical Perron–Frobenius theory in the Fundamental Theorem of Demography and other key results of population dynamics, see [15].

The layout of the paper is as follows: In Section 2, we introduce the main notation used throughout the paper as well as some preliminary results and define formally what is meant by a max matrix polynomial. In Section 3, in the spirit of [14], we show how to associate a companion matrix with a max matrix polynomial and show that there is a perfect correspondence between the eigenvalues and eigenvectors of the polynomial and those of the companion matrix. This allows us to apply the Perron–Frobenius theorem for the max algebra to obtain a corresponding result for matrix polynomials over the max algebra. In Section 4, we investigate the implications of these results for the convergence of multistep difference equations in the max algebra, while in Section 5, we derive a number of inequalities for the largest max eigenvalue of a max matrix polynomial in terms of the largest max eigenvalue of a fixed matrix naturally associated with the polynomial. Finally, in Section 6, we present our conclusions.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}_+$ denotes the set of nonnegative real numbers; $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers; $\mathbb{R}^{n\times n}$ stands for the space of $n\times n$ matrices with real entries. For $v \in \mathbb{R}^n$ and $1 \leqslant i \leqslant n$, v_i denotes the ith component of v. Similarly, for $A \in \mathbb{R}^{n\times n}$ and $1 \leqslant i, j \leqslant n$, a_{ij} refers to the (i, j)th entry of A. We use A^T and v^T for the transpose of $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$ respectively.

A vector $v \in \mathbb{R}^n$ is said to be positive if $v_i > 0$ for $1 \leqslant i \leqslant n$. This is denoted by $v > 0$. If $v_i \geqslant 0$ for $1 \leqslant i \leqslant n$, we write $v \geqslant 0$. Similarly, for $A \in \mathbb{R}^{n\times n}$, we say A is positive (nonnegative) and write $A > 0$ $(A \geqslant 0)$ if $a_{ij} > 0$ $(a_{ij} \geqslant 0)$ for $1 \leqslant i, j \leqslant n$. The set of nonnegative matrices is denoted by $A \in \mathbb{R}_+^{n\times n}$.

We call $A \in \mathbb{R}_+^{n\times n}$ reducible if there exists a permutation matrix P such that PAP^T has the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}_+^{k\times k}$, $A_{12} \in \mathbb{R}_+^{k\times(n-k)}$, $A_{22} \in \mathbb{R}_+^{(n-k)\times(n-k)}$ and 0 is the zero matrix in $\mathbb{R}_+^{(n-k)\times k}$ for $1 \leqslant k < n$. A matrix is said to be irreducible if it is not reducible. A nonnegative matrix A is *primitive* if and only if $A^n > 0$ for some positive integer n.

For $A \in \mathbb{R}^{n\times n}$, the weighted directed graph associated with A is denoted by $D(A)$. Formally, $D(A)$ consists of the finite set of vertices $\{1, 2, \ldots, n\}$ and there is a directed edge (i, j) from i to j if and only if $a_{ij} > 0$. It is standard that A is an irreducible matrix if and only if there is a directed path $i = i_1, i_2, \ldots, i_k = j$ between any two vertices i, j in $D(A)$, where (i_p, i_{p+1}) is an edge in $D(A)$ for $p = 1, \ldots, k-1$. The *weight* of a path $i = i_1, i_2, \ldots, i_k = j$ of length $k-1$ is given by $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$.

A cycle Γ of length k in $D(A)$ is a closed path of the form $i_1, i_2, \ldots, i_k, i_1$ where $i_1, i_2, \ldots, i_k$ are in $\{1, 2, \ldots, n\}$ and distinct. For such a cycle its cycle geometric mean is $\sqrt[k]{a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}}$. The maximal cycle geometric mean in $D(A)$ over all possible cycles is denoted by $\mu(A)$. Throughout the paper, we adopt the notation $\pi(\Gamma) = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$ for the product of a cycle Γ and $l(\Gamma) = k$ for the length of a cycle.

A cycle whose cycle geometric mean equals to $\mu(A)$ is called a *critical cycle*. Vertices that lie on some critical cycle are known as critical vertices. The *critical matrix* [9,10] of $A \in \mathbb{R}^{n\times n}$, denoted by A^C, is formed from the submatrix of A consisting of the rows and columns of A corresponding to critical vertices as follows. Set $a^C_{ij} = a_{ij}$ if (i, j) lies on a critical cycle and $a^C_{ij} = 0$ otherwise.

The max algebra consists of the set of nonnegative numbers together with the two basic operations $a \oplus b = \max(a, b)$ $a \otimes b = ab$. These operations extend to nonnegative matrices and vectors in the obvious manner. Throughout the paper, $A^j_\otimes$ denotes the jth power of A in the max algebra. Note that $(A \otimes B)^T = B^T \otimes A^T$.

The eigenequation in the max algebra is given by

$$A \otimes v = \lambda v, \tag{1}$$

where $v \geqslant 0$, $\lambda \geqslant 0$ and $(A \otimes v)_i = \max_{1\leqslant j\leqslant n} a_{ij}v_j$ for $i = 1, 2, \ldots, n$. If $v \geqslant 0$, $\lambda \geqslant 0$ satisfy (1), then v is a max eigenvector of A with corresponding max eigenvalue λ. For A in $\mathbb{R}_+^{n\times n}$, $\mu(A)$ is the largest max eigenvalue of A; this is still true in the case where $D(A)$ contains no cycles and $\mu(A) = 0$ [11]. If A is irreducible, then $\mu(A)$ is the unique max eigenvalue of A and there is a positive max eigenvector $v > 0$ corresponding to it [8]. The eigenvector v is unique up to a scalar multiple if and only if the critical matrix of A is irreducible.

3. Eigenvectors and eigenvalues for max matrix polynomials

In this paper, inspired by the work of [14] for nonnegative matrix polynomials over the conventional algebra, we consider the spectral properties of matrix polynomials defined over the max algebra. Formally, we consider polynomials given by

$$P(\lambda) = A_0 \oplus \lambda A_1 \oplus \cdots \oplus \lambda^{m-1}A_{m-1}, \tag{2}$$

where $A_0, A_1, \ldots, A_{m-1}$ are in $\mathbb{R}_+^{n\times n}$. We refer to $P(\lambda)$ as a max matrix polynomial of degree $m-1$. In analogy with the definitions for the conventional algebra [14], we say that

(i) $\kappa \geqslant 0$ is said to be a right max eigenvalue of $P(\lambda)$ with corresponding right max eigenvector $v \geqslant 0$ if $P(\kappa) \otimes v = \kappa^m v$. (κ, v) is then a right max eigenpair of $P(\lambda)$.
(ii) $\tau \geqslant 0$ is said to be a left max eigenvalue of $P(\lambda)$ with corresponding left max eigenvector $w \geqslant 0$ if $w^T \otimes P(\tau) = \tau^m w^T$. (τ, w) is then a left max eigenpair of $P(\lambda)$.

The key result of this section, which allows us to directly apply the Perron–Frobenius theorem for the max algebra to obtain corresponding statements for max matrix polynomials is Proposition 3.1 below. Essentially, as was done in [14] for the conventional algebra, this establishes a one to one correspondence between the max eigenpairs of the polynomial $P(\lambda)$ in (2) and the max eigenpairs of the companion matrix

$$C_P = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I \\ A_0 & A_1 & \dots & A_{m-2} & A_{m-1} \end{bmatrix} \in \mathbb{R}_+^{mn\times mn}. \tag{3}$$

Proposition 3.1. *Consider the max matrix polynomial $P(\lambda)$ given by (2) and the corresponding companion matrix given by C_P (3). Then $(\kappa, v) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a right max eigenpair of $P(\lambda)$ if and only if $(\kappa, \hat{v}) \in \mathbb{R}_+ \times \mathbb{R}_+^{mn}$ is a right max eigenpair of C_P, where*

$$\hat{v} = \begin{bmatrix} v \\ \kappa v \\ \vdots \\ \kappa^{m-1} v \end{bmatrix}. \tag{4}$$

Moreover, $(\tau, w) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a left max eigenpair of $P(\lambda)$ if and only if $(\tau, \hat{w}) \in \mathbb{R}_+ \times \mathbb{R}_+^{mn}$ is a left max eigenpair of C_P, where

$$\hat{w} = \begin{bmatrix} \frac{1}{\tau} A_0^T \otimes w \\ \left(\frac{1}{\tau 2} A_0^T \oplus \frac{1}{\tau} A_1^T\right) \otimes w \\ \vdots \\ \left(\frac{1}{\tau^{m-1}} A_0^T \oplus \frac{1}{\tau^{m-2}} A_1^T \oplus \cdots \oplus \frac{1}{\tau} A_{m-2}^T\right) \otimes w \\ w \end{bmatrix}. \tag{5}$$

Proof. It is a straightforward calculation to verify that $C_P \otimes \hat{v}$ is given by

$$\begin{bmatrix} \kappa v \\ \kappa^2 v \\ \vdots \\ P(\kappa) \otimes v \end{bmatrix}.$$

Hence, if (κ, v) is a right max eigenpair of $P(\lambda)$, it is immediate that $C_P \otimes \hat{v} = \kappa \hat{v}$.

For the converse, it is clear that any right eigenvector of C_P must be of the form (4). Then equating the last rows of $C_P \otimes \hat{v} = \kappa \hat{v}$, we have

$$P(\kappa) \otimes v = A_0 \otimes v \oplus \kappa A_1 \otimes v \oplus \cdots \oplus \kappa^{m-1} A_{m-1} \otimes v = \kappa^m v.$$

For the left eigenpair statement, we have $\hat{w}^T \otimes C_P$ given by

$$\begin{bmatrix} A_0^T \otimes w \\ \left(\frac{1}{\tau} A_0^T \oplus A_1^T\right) \otimes w \\ \vdots \\ \frac{P(\tau)^T}{\tau^{m-1}} \otimes w \end{bmatrix}^T.$$

Since (τ, w) is a left max eigenpair of $P(\lambda)$, it follows that $\hat{w}^T \otimes C_P = \tau \hat{w}^T$. For the converse, equating the last columns of $\hat{w}^T \otimes C_P$ and $\tau \hat{w}^T$, we see that

$$w^T \otimes \frac{P(\tau)}{\tau^{m-1}} = \tau w^T \Rightarrow w^T \otimes P(\tau) = \tau^m w^T. \quad \square$$

For any A in $\mathbb{R}_+^{n\times n}$, $\mu(A) = \mu(A^T)$. It follows from Proposition 3.1 that the largest right and left max eigenvalues of the polynomial $P(\lambda)$ coincide. We now define $\mu := \mu(P(\lambda))$ to be the largest right (or left) max eigenvalue of $P(\lambda)$. However, note that in general the set of right and left max eigenvalues of A and A^T need not coincide. For example,

$$A = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}. \tag{6}$$

The following result, which extends the Perron–Frobenius theorem to matrix polynomials over the max algebra now follows easily from combining Proposition 3.1 with Theorem 2 of [8].

Theorem 3.1. *Consider the max matrix polynomial $P(\lambda)$ given by (2) and let C_P be the corresponding companion matrix (3). Suppose that C_P is irreducible. Then $\mu(C_P)$, the maximal cycle geometric mean of C_P, is the only max eigenvalue of $P(\lambda)$. Moreover, writing $\mu := \mu(C_P)$, there exist positive vectors $v, w > 0$ in $\mathbb{R}^n_+$ such that $P(\mu) \otimes v = \mu^m v$ and $w^T \otimes P(\mu) = \mu^m w^T$.*

Remark 3.1. It has been pointed out in [14] that C_P will be irreducible if A_0 is irreducible. Note that the irreducibility of C_P only implies that the eigenvalue μ is unique; there may be multiple eigenvectors corresponding to μ. The following result describes a situation in which the eigenvector is also unique.

Theorem 3.2. *Let the max matrix polynomial $P(\lambda)$ be given by (2) and let C_P be the corresponding companion matrix (3). Further, let C_P^C denote the critical matrix of C_P. Then $P(\lambda)$ has unique left and right eigenvectors up to scalar multiples if and only if the graph of C_P^C is strongly connected.*

4. Multistep difference equations in the max algebra

The association of matrix polynomials with multistep difference equations was studied in [14] for the conventional algebra. In this section, we show how the max matrix polynomial $P(\lambda)$ given by (2) is related with multistep difference equations over the max algebra. As noted in the introduction, equations of this type have been previously studied in the max-plus setting [3].

Consider the multistep difference equation:

$$u_{j+m} = A_{m-1} \otimes u_{j+m-1} \oplus \cdots \oplus A_1 \otimes u_{j+1} \oplus A_0 \otimes u_j \quad (j = 0, 1, \ldots). \tag{7}$$

where $A_0, A_1, \ldots, A_{m-1} \in \mathbb{R}^{n\times n}_+$ are coefficient matrices and $u_0, u_1, \ldots u_{m-1} \in \mathbb{R}^n_+$ are initial values. As with multistep difference equations for the standard algebra, the system (7) is equivalent to the singlestep difference equation given by

$$x_{j+1} = C_P \otimes x_j \quad (j = 0, 1, \ldots). \tag{8}$$

This is seen by setting $x_j = \begin{bmatrix} u_j \\ u_{j+1} \\ \vdots \\ u_{j+m-1} \end{bmatrix} \in \mathbb{R}^{mn}_+$. For a given initial condition $x_0 \in \mathbb{R}^{mn}_+$, the solution of (8) is

$$x_j = C^j_{P_\otimes} \otimes x_0. \tag{9}$$

Hence, the solution of (7) can be written in the form:

$$u_j = \underbrace{[I \quad 0 \quad \cdots \quad 0]}_{\in \mathbb{R}^{n\times mn}_+} \otimes C^j_{P_\otimes} \otimes x_0. \tag{10}$$

Throughout this section, we assume that C_P is irreducible and that the critical matrix C_P^C is primitive. Therefore, as C_P^C is certainly irreducible, it follows that $P(\lambda)$ has unique left and right max eigenpairs.

Under the above assumptions, it follows from Theorem 2.2 in [9] that max powers of the normalized companion matrix $\frac{1}{\mu^j} C^j_{P_\otimes}$ converge in finitely many steps to a matrix C_∞. In fact, Theorem 2.2 of [9] established that for any irreducible $A \in \mathbb{R}_+^{n\times n}$ with A^C primitive, there is some $K \in \mathbb{R}_+$ and some $A_\infty \in \mathbb{R}_+^{n\times n}$ such that for $\forall k \geqslant K$,

$$\frac{A_\otimes^k}{\mu(A)^k} = A_\infty. \tag{11}$$

The following lemma restates the above convergence result of [9] in terms of the normalized max eigenvectors of A.

Lemma 4.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and A^C be primitive. Then there exists some $K > 0$ such that*

$$\frac{A_\otimes^j}{\mu(A)^j} = v \otimes w^T, \quad \textit{for } j \geqslant K, \tag{12}$$

where $v > 0$ and $w > 0$ are the unique right and left max eigenvectors of A satisfying $A \otimes v = \mu(A)v$, $A^T \otimes w = \mu(A)w$ and $v^T \otimes w = 1$.

Proof. It follows from (11) that there is some $K > 0$ and a matrix A_∞ such that $\frac{A_\otimes^k}{\mu(A)^k} = A_\infty$ for all $k \geqslant K$. Now calculate $A \otimes A_\infty$:

$$A \otimes A_\infty = A \otimes \frac{A_\otimes^K}{\mu(A)^K} = \mu(A)\frac{A_\otimes^{K+1}}{\mu(A)^{K+1}} = \mu(A)A_\infty.$$

It follows immediately that the columns of A_∞ are right max eigenvectors of A and hence that $A_\infty = v \otimes x^T$ for some $x \in \mathbb{R}_+^n$.

On the other hand,

$$A^T \otimes A_\infty^T = A^T \otimes \frac{(A_\otimes^K)^T}{\mu(A)^K} = A^T \otimes \frac{(A_\otimes^T)^K}{\mu(A)^K} = \mu(A)\frac{(A_\otimes^T)^{K+1}}{\mu(A)^{K+1}} = \mu(A)A_\infty^T.$$

But, $A_\infty = v \otimes x^T$ and hence,

$$A^T \otimes x \otimes v^T = \mu(A)x \otimes v^T \Rightarrow A^T \otimes x = \mu(A)x.$$

Thus, $x = \lambda w$ for some $\lambda \in \mathbb{R}_+$. It is explicit that $A_\infty \otimes A_\infty = A_\infty$. Since $A_\infty = \lambda v \otimes w^T$, we have

$$\lambda^2 v \otimes \underbrace{w^T \otimes v}_{1} \otimes w^T = \lambda v \otimes w^T \Rightarrow \lambda = 1.$$

Therefore, we conclude that $x = w$ and $A_\infty = v \otimes w^T$. □

In Theorem 4.2 of [14], a generalisation of the so-called Fundamental Theorem of Demography was derived. In the following result, we extend this to the max algebra.

Theorem 4.1. *Let $P(\lambda)$ be the max matrix polynomial given by (2); let C_P be irreducible and C_P^C be primitive. Let v and w be the right and left max eigenvectors of $P(\lambda)$ corresponding to μ normalized so that*

$$\begin{bmatrix} w^T \otimes \frac{A_0}{\mu} & w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}\right) & \cdots & w^T \end{bmatrix} \otimes \begin{bmatrix} v \\ \mu v \\ \cdots \\ \mu^{m-1} v \end{bmatrix} = 1. \tag{13}$$

Write u_j, $j = 0, 1, \ldots$ for the solution of the multistep difference equation (7) corresponding to a nonzero initial vector $x_0 = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \in \mathbb{R}_+^{mn}$.

Then there is some positive integer K such that for all $j \geqslant K$

$$\frac{u_j}{\mu^j} = \left[w^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{k=0}^{j-1} \frac{A_k}{\mu^{j-k}} \right) \otimes u_{j-1} \right) \right] \otimes v. \tag{14}$$

Proof. Let $\hat{v}$ and $\hat{w}$ be the right and left max eigenvectors of C_P given by (4), (5) respectively. Lemma 4.1 implies that there is some integer $K > 0$ such that for all $j \geqslant K$,

$$\frac{C^j_{P_\otimes}}{\mu^j} = \hat{v} \otimes \hat{w}^T = \begin{bmatrix} v \\ \mu v \\ \cdots \\ \mu^{m-1} v \end{bmatrix} \otimes \left[w^T \otimes \frac{A_0}{\mu} \quad w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) \quad \cdots \quad w^T \right]$$

$$= \begin{bmatrix} v \otimes w^T \otimes \frac{A_0}{\mu} & v \otimes w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) & \cdots & v \otimes w^T \\ \mu v \otimes w^T \otimes \frac{A_0}{\mu} & \mu v \otimes w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) & \cdots & \mu v \otimes w^T \\ \vdots & \vdots & & \vdots \\ \mu^{m-1} v \otimes w^T \otimes \frac{A_0}{\mu} & \mu^{m-1} v \otimes w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) & \cdots & \mu^{m-1} v \otimes w^T \end{bmatrix}.$$

The solution of (7) is given by $u_j = [I \quad 0 \quad \cdots \quad 0] \otimes C^j_{P_\otimes} \otimes x_0$. It immediately follows from the above calculation that for all $j \geqslant K$,

$$\frac{u_j}{\mu^j} = [I \quad 0 \quad \cdots \quad 0] \otimes \frac{C^j_{P_\otimes}}{\mu^j} \otimes x_0$$

$$= \left[v \otimes w^T \otimes \frac{A_0}{\mu} \quad v \otimes w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) \quad \cdots \quad v \otimes w^T \right] \otimes \begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ u_{m-1} \end{bmatrix}$$

$$= v \otimes \left(w^T \otimes \frac{A_0}{\mu} \otimes u_0 \oplus w^T \otimes \left(\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu} \right) \otimes u_1 \oplus \cdots \oplus w^T \otimes u_{m-1} \right).$$

Using the fact that $w^T = w^T \otimes \frac{P(\lambda)}{\mu^m} = w^T \otimes \left(\frac{A_0}{\mu^m} \oplus \frac{A_1}{\mu^{m-1}} \oplus \cdots \oplus \frac{A_{m-1}}{\mu} \right)$, we find that for all $j \geqslant K$,

$$\frac{u_j}{\mu^j} = \left[w^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{k=0}^{j-1} \frac{A_k}{\mu^{j-k}} \right) \otimes u_{j-1} \right) \right] \otimes v$$

as claimed. □

Note that the above result implies directly that

$$\lim_{j \to \infty} \frac{u_j}{\mu^j} = \left[w^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{k=0}^{j-1} \frac{A_k}{\mu^{j-k}} \right) \otimes u_{j-1} \right) \right] \otimes v. \tag{15}$$

which is a direct generalisation of Theorem 4.2 of [14].

5. Some bounds for μ

In this section, we explore the relationship between the max eigenvalues of the max matrix polynomial (2) and the max eigenvalues of the $n \times n$, nonnegative matrix

$$S := A_{m-1} \oplus A_{m-2} \oplus \cdots \oplus A_1 \oplus A_0. \tag{16}$$

Specifically, we present a number of results relating μ and $\mu(S)$ that are similar to those given in Proposition 3.8 of [14] for matrix polynomials over the conventional algebra.

Before deriving the main result of the section, which is Theorem 5.1 below, we first need to introduce some notation and a number of preliminary results.

Given $A_0, \ldots, A_{m-1}$, we write a_{ij}^p for the (i, j) entry of A_p for $0 \leqslant p \leqslant m-1$. Throughout this section, we shall write M for the *multigraph* associated with the set $A_0, \ldots, A_{m-1}$. Thus M consists of the vertices $\{1, \ldots, n\}$ with an edge of weight $a_{ij}^p$from i to j for every p for which $a_{ij}^p > 0$. In an abuse of notation we shall identify the edge with its weight a_{ij}^p in this case. A cycle in the multigraph M is then a sequence of vertices $i_1, i_2, \ldots, i_k, i_{k+1} = i_1$ and edges $a_{i_j i_{j+1}}^{p_j} > 0,\ 1 \leqslant j \leqslant k$ such that $i_1, \ldots, i_k$ are distinct and $p_1, \ldots, p_k$ are in $\{0, \ldots m-1\}$. The geometric mean of a cycle in M and the maximal cycle geometric mean $\mu(M)$ of M are defined analogously to the case of a simple graph. Critical cycles are defined for M in the obvious manner. For a cycle Γ in a graph or multigraph, we write $\pi(\Gamma)$ for the product of the weights of the edges in the cycle.

Lemma 5.1. *Let $\mu(M)$ denote the maximal cycle geometric mean of the multigraph associated with $A_0, A_1, \ldots, A_{m-1}$ and let $\mu(S)$ be the maximal cycle geometric mean of*

$$S = A_{m-1} \oplus \cdots \oplus A_0.$$

Then $\mu(M) = \mu(S)$.

Proof. First it is immediate that any cycle in $D(S)$ is also a cycle in M. This implies that $\mu(S) \leqslant \mu(M)$. On the other hand, if Γ_M is a critical cycle in M with product $a_{i_1 i_2}^{p_1} a_{i_2 i_3}^{p_2} \cdots a_{i_k i_1}^{p_k}$, it is clear that $i_1, i_2, \ldots, i_k, i_1$ is also a cycle in $D(S)$ and moreover from the definition of S,

$$s_{i_1 i_2} s_{i_2 i_3} \cdots s_{i_k i_1} \geqslant a_{i_1 i_2}^{p_1} a_{i_2 i_3}^{p_2} \cdots a_{i_k i_1}^{p_k}.$$

This implies that $\mu(S) \geqslant \mu(M)$. Hence $\mu(S) = \mu(M)$ as claimed. □

Before proceeding, note that the argument used above also shows that $\mu(M) = 0$ if and only if $\mu(S) = 0$.

The following result plays a central role in the proof of the main result of this section. It shows that there is a 1–1 correspondence between cycles in the multigraph M and cycles in the directed graph $D(C_P)$. In the proof of this result we write $c_{i,j}$ for the (i, j) entry of C_P.

Lemma 5.2. *Let Γ_M be a cycle in the multigraph M with cycle product $\pi(\Gamma_M)$ and length j. Then there exists a cycle Γ_C in $D(C_P)$ of length $k \geqslant j$ such that $\pi(\Gamma_C) = \pi(\Gamma_M)$. Conversely, for every cycle Γ_C in $D(C_P)$ of length k, there exists a cycle Γ_M in M with cycle product $\pi(\Gamma_M) = \pi(\Gamma_C)$ and length $j \leqslant k$.*

Proof. Let Γ_M be a cycle in M with product

$$\pi(\Gamma_M) = a_{i_1 i_2}^{p_1} a_{i_2 i_3}^{p_2} \cdots a_{i_j i_1}^{p_j}.$$

Note that for $1 \leqslant s \leqslant j$, the entry $a_{i_s i_{s+1}}^{p_s}$ corresponds to the entry in the companion matrix C_P given by $c_{(m-1)n+i_s, p_s n+i_{s+1}}$. Now note that the form of C_P means that for any p with $0 \leqslant p < m-1$, and any i with $1 \leqslant i \leqslant n$, there exists a simple path in $D(C_P)$ from the vertex $pn + i$ to $(m-1)n + i$. Further, all the entries of C_P used to construct this path are equal to one. It follows immediately from this that there exists a cycle Γ_C in $D(C_P)$ whose product is equal to $\pi(\Gamma_M)$ but whose length k may be greater than j (as extra edges of weight 1 may have been added to define the cycle in $D(C_P)$.)

For the converse, note that any cycle Γ_C of length k in $D(C_P)$ must contain at least one vertex corresponding to an index i with $(m-1)n + 1 \leqslant i \leqslant mn$ (an index from the bottom n rows of C_P). Suppose the product $\pi(\Gamma_C)$ contains j terms from the bottom n rows of C_P and is given by

$$c_{(m-1)n+i_1, p_1 n+i_2}\, c_{(m-1)n+i_2, p_2 n+i_3} \cdots c_{(m-1)n+i_j, p_j n+i_1}$$

(where we have omitted terms equal to one from the product).

Then the cycle Γ_M in M consisting of the vertices $i_1, \ldots, i_j, i_{j+1} = i_1$ and the edges with weights

$$a^{p_1}_{i_1 i_2}, a^{p_2}_{i_2 i_3}, \ldots, a^{p_j}_{i_j i_1}$$

has length j with $j \leqslant k$ and moreover, it is immediate that $\pi(\Gamma_M) = \pi(\Gamma_C)$. □

Again, note that the above argument shows that $\mu(M) = 0$ if and only if $\mu = 0$. Hence, from Lemma 5.1, $\mu(S) = 0$ if and only if $\mu = 0$. As all of the following results are trivial in the case where $\mu = \mu(S) = 0$, we henceforth assume that $\mu \neq 0$.

Corollary 5.1. *Let μ denote the largest max eigenvalue of the max matrix polynomial given by* (2) *and let $\mu(S)$ denote the maximal cycle geometric mean of the matrix S given by* (16). *Then there exist integers j_1, j_2, k_1, k_2 with $0 < j_1 \leqslant k_1,\ 0 < j_2 \leqslant k_2$ such that*

$$\mu(S)^{j_1/k_1} \leqslant \mu \leqslant \mu(S)^{j_2/k_2}. \tag{17}$$

Proof. First let Γ_M be a critical cycle in M of length j_1. Then the product of Γ_M is given by $\mu(M)^{j_1}$. From Lemma 5.2 there is a corresponding cycle, not necessarily critical, Γ_C in $D(C_P)$ of length $k_1 \geqslant j_1$ with the same cycle product. It follows from the definition of μ that $\mu(M)^{j_1/k_1} \leqslant \mu$.

On the other hand, let Γ_C be a critical cycle in $D(C_P)$ of length k_2. Then as above the cycle product of Γ_C is μ^{k_2} and there exists a (not necessarily critical) cycle in M of length $j_2 \leqslant k_2$ with the same cycle product. This implies that

$$\mu^{k_2/j_2} \leqslant \mu(M).$$

Rearranging this, we see that

$$\mu \leqslant \mu(M)^{j_2/k_2}.$$

As $\mu(M) = \mu(S)$ from Lemma 5.1 the result follows. □

Next, we present a numerical example to illustrate the result in Lemma 5.2.

Example 5.1. Let $P(\lambda)$ be given by

$$P(\lambda) = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix} \lambda \oplus \begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} \lambda^2. \tag{18}$$

Then, the corresponding companion matrix and S are as follows:

$$C_P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 4 & 2 & 3 & 5 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 3 \\ 4 & 5 \end{bmatrix}. \tag{19}$$

Consider the cycle Γ in M whose product is $\pi(\Gamma) = a^0_{12} a^1_{21} = s_{12}s_{21} = 12$ with $l(\Gamma) = 2$. Writing c_{ij} for the (i,j) entry in C_P, the following cycles in $D(C_P)$ both satisfy the conditions in the above lemma.

1. $\Gamma'_1 := 5, 2, 4, 6, 3, 5$. $\pi(\Gamma'_1) = c_{52}c_{24}c_{46}c_{63}c_{35} = s_{12}s_{21} = 12$ and $l(\Gamma'_1) = 5$.
2. $\Gamma'_2 := 6, 3, 5, 2, 4, 6$. $\pi(\Gamma'_2) = c_{63}c_{35}c_{52}c_{24}c_{46} = s_{21}s_{12} = 12$ and $l(\Gamma'_2) = 5$.

We are now able to state the main result of this section, which provides a max algebra version of Proposition 3.8 in [14].

Theorem 5.1. *Let $P(\lambda)$ be the max matrix polynomial in* (2) *and $S = P(1)$. Further, μ is the largest max eigenvalue of $P(\lambda)$ and $\mu(S)$ is the maximal cycle geometric mean of S. Then, the following hold.*

(i) $\mu(S) < 1$ *if and only if* $\mu < 1$.
(ii) $\mu(S) > 1$ *if and only if* $\mu > 1$.
(iii) $\mu(S) = 1$ *if and only if* $\mu = 1$.

Proof. This result follows immediately from the identity

$$\mu(S)^{j_1/k_1} \leqslant \mu \leqslant \mu(S)^{j_2/k_2}$$

established in Corollary 5.1. □

The following Corollary is obtained immediately from Corollary 5.1.

Corollary 5.2

$$\begin{aligned} &\mu(S) > 1 \Rightarrow \mu \leqslant \mu(S), \\ &\mu(S) < 1 \Rightarrow \mu \geqslant \mu(S), \\ &\mu(S) = 1 \Rightarrow \mu = \mu(S). \end{aligned} \tag{20}$$

Theorem 5.1 shows that $\mu = \mu(S)$ when $\mu(S) = 1$. In the next result, we give a necessary condition for $\mu = \mu(S)$ when $\mu(S) \neq 1$.

Corollary 5.3. *If* $\mu(S) \neq 1$ *and* $\mu = \mu(S)$, *then* $\mu = \mu(A_{m-1})$.

Proof. Consider $\mu = \mu(S)$.

Case 1: Let $\mu(S) > 1$. Using Corollary 5.1, we have $\mu \leqslant \mu^{j_2/k_2} \Rightarrow \mu^{1-j_2/k_2} \leqslant 1$. This is only possible when $j_2 = k_2$. This means that there is some critical cycle in $D(C_P)$ whose product only contains terms from the last n rows of C_P. This immediately implies that all the terms in this product are in A_{m-1}, so in this case $\mu = \mu(A_{m-1})$.

Case 2: Let $\mu(S) < 1$. As above, using Corollary 5.1, we have $\mu^{j_1/k_1} \leqslant \mu \Rightarrow 1 \leqslant \mu^{1-j_1/k_1}$. This is only possible when $j_1 = k_1$. As in Case 1 this implies that $\mu = \mu(A_{m-1})$. □

As a final point, we note that the converse of the previous result does not hold. Specifically, the example below has $\mu = \mu(A_{m-1})$ with $m = 2$ but $\mu \neq \mu(S)$.

Example 5.2

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.2 & 1 & 0.1 & 1 & 0.5 & 3 \\ 2 & 1 & 0.2 & 1.5 & 0.1 & 1 \\ 0.3 & 2 & 2 & 2 & 5 & 0.6 \end{bmatrix} \Rightarrow \mu = 2.8231, \tag{21}$$

$$A_1 = \begin{bmatrix} 1 & 0.5 & 3 \\ 1.5 & 0.1 & 1 \\ 2 & 5 & 0.6 \end{bmatrix} \Rightarrow \mu(A_1) = 2.8231, \tag{22}$$

$$S = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 2 & 5 & 2 \end{bmatrix} \Rightarrow \mu(S) = 3.1072. \tag{23}$$

6. Conclusions

Continuing a recent line of work on spectral properties of matrices over the max algebra, we have extended Perron–Frobenius theory for matrix polynomials to matrix polynomials defined over the max algebra. Specifically, we have shown that the basic theory for Perron matrix polynomials presented in [14] carries over to the max algebra. We have also derived convergence results for multistep difference equations over the max algebra analogous to those for the conventional algebra in [14]. A number of results relating the maximal max eigenvalue μ of a max matrix polynomial with the maximal cycle geometric mean of the matrix $S = P(1)$ have also been presented. An interesting possible direction for future work is to investigate the potential for extending the results here to sets of matrix polynomials over the max algebra. In particular, it would be interesting to investigate whether the results of Section 5 can be extended to the generalised max spectral radius studied in [12,13].

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References

[1] F. Baccelli, G. Cohen, G.J. Olsder, J.P. Quadrat, Synchronization and Linearity: An Algebra for Discrete Event Systems, October 2001. Free Web Edition: <http://www-roc.inria.fr/metalau/cohen/documents/BCOQ-book.pdf>.
[2] L. Elsner, P. van den Driessche, Max-algebra and pairwise comparison matrices, Linear Algebra Appl. 385 (2004) 47–62.
[3] B. Heidergott, G.J. Olsder, J. van der Woude, Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and Its Applications, Princeton University Press, 2005.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[5] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM Classics in Applied Mathematics, 1994.
[6] N.N. Vorobyev, The extremal algebra of positive matrices [in Russian], Elektron. Information Kybernet. 3 (1967) 39–72.
[7] R.A. Cuninghame-Green, Minimax Algebra, Lecture Notes in Economics and Math. Systems, vol. 166, Springer, Berlin, 1979.
[8] R.B. Bapat, A max version of the Perron Frobenius theorem, Linear Algebra Appl. 275–276 (1998) 3–18.
[9] L. Elsner, P. van den Driessche, On the power method in max algebra, Linear Algebra Appl. 302–303 (1999) 17–32.
[10] L. Elsner, P. van den Driessche, Modifying the power method in max algebra, Linear Algebra Appl. 332–333 (2001) 3–13.
[11] Y.Y. Lur, On the asymptotic stability of nonnegative matrices in max algebra, Linear Algebra Appl. 407 (2005) 149–161.
[12] A. Peperko, On the max version of the generalized spectral radius theorem, Linear Algebra Appl. 428 (2008) 2312–2318.
[13] Y.Y. Lur, A max version of the generalized spectral radius theorem, Linear Algebra Appl. 418 (2006) 336–346.
[14] P.J. Psarrakos, M.J. Tsatsomeros, A primer of Perron–Frobenius theory for matrix polynomials, Linear Algebra Appl. 393 (2004) 333–351.
[15] C.-K. Li, H. Schneider, Application of Perron–Frobenius theory to population dynamics, Math. Biol. 44 (2002) 450–462.