Stability Criteria for SIS Epidemiological Models under Switching Policies

Mustapha Ait Rami[*]   Vahid S. Bokharaie[†]   Oliver Mason[‡]   Fabian R. Wirth[§]

April 21, 2014

Abstract

We study the spread of disease in an SIS model for a structured population. The model considered is a time-varying, switched model, in which the parameters of the SIS model are subject to abrupt change. We show that the joint spectral radius can be used as a threshold parameter for this model in the spirit of the basic reproduction number for time-invariant models. We also present conditions for persistence and the existence of periodic orbits for the switched model and results for a stochastic switched model.

Keywords. Mathematical Epidemiology, SIS Model, Compartmental Model, Switched system, Disease Propagation, Endemic Equilibrium, Positive System, Extremal Norm, Lyapunov Exponents.

AMS subject classifications. 92D30,34D23,37B25

1   Introduction

In this paper a number of stability results are derived for switched compartmental epidemiological models of SIS (susceptible-infectious-susceptible) type. Such models are related to structured populations to which an infection graph is associated. The switched system aspect models changes is parameters, e.g. in infection or recovery rates. We derive uniform stability results for the disease free equilibrium, as well as instability results, study the existence of nontrivial periodic solutions and present some results on Markovian switching. Mathematically, the results rely on a combination of the theory of positive and monotone systems, Lyapunov theory for switched systems, tools from degree theory and Markov systems.

Mathematical epidemiology is concerned with the construction and analysis of mathematical models for disease propagation in a network of individuals; similar models can also be applied to study the spread of computer viruses. The role of mathematical modelling is particularly important in epidemiology as experimentation in this field is both impractical and unethical. As mentioned in [4]: "we need to develop models that will assist the decision-making process by helping to evaluate the consequences of choosing one of the alternative strategies available. Thus, mathematical models of the dynamics of a communicable disease can have a direct bearing on the choice of an immunisation programme, the optimal allocation of scarce resources, or the best combination of control or education technologies". To further underline the importance of the topic, it should be noted that in spite of the advances in vaccination and prevention of disease transmission in the past few decades, infectious and parasitic diseases are the second leading cause of death worldwide (after cardiovascular diseases) [39, Figure 4]. They are the leading cause of death in low-income countries [39, Table 2] and in children aged under five years [39, Figure 5].

Epidemiological models based on ordinary or partial differential equations typically divide the population into different epidemiological classes or compartments. For instance, such compartments may represent:

---

[*]Dept. Ingenieria de Sistemas y Automatica, Universidad de Valladolid, Valladolid, Spain (`aitrami@autom.uva.es`)

[†]Departement Werktuigkunde, KU Leuven, Leuven, Belgium (`vahid.bokharaie@kuleuven.be`)

[‡]Hamilton Institute, National University of Ireland Maynooth,Maynooth, Ireland (`oliver.mason@nuim.ie`)

[§]IBM Research Ireland, Damastown Industrial Estate, Mulhuddart, Dublin 15, Ireland (`fabwirth@ie.ibm.com`)

susceptible (S) individuals who are healthy but not immune to the disease; *infective* individuals (I), who are already infected and can transfer the disease to a susceptible; or *recovered* individuals (R) who have immunity to the disease. Some other more complex models also include compartments for exposed people who are infected but not yet infective and for infants with temporary inherited immunity. A model is typically represented by the initials of the epidemiological classes it incorporates.

In this manuscript, we are concerned with SIS models, in which all individual are considered to be either susceptible or infective. When susceptibles are in sufficient contact with infectives, they become infectives themselves. When infectives are restored to health, they re-join the susceptible compartment. SIS models have been used to model diseases that do not confer immunity on the survivors. Tuberculosis and gonorrhoea are two example diseases which are mathematically described using SIS models [12, 20, 30]. Computer viruses also fall into this category; they can be 'cured' by antivirus software, but without a permanent virus-checking program, the computer is not protected against the subsequent attacks by the same virus.

Our work in this paper seeks to build on the results presented in [17], where an SIS model, with constant parameters, that can describe heterogeneous populations was considered. The authors of [17] presented conditions for the stability of the disease-free equilibrium and for the existence and stability of a unique endemic equilibrium. These conditions were described in terms of the spectral radius of a matrix naturally associated with the system and are in the spirit of the more general model class studied in [37]. It has been recognized for some time that time-varying parameters play an important role in the dynamics of disease propagation; in particular, several authors have considered the impact of seasonal effects by analysing periodic SIS models [33, 27].

A major concern in the current paper is to study a switched version of the model in [17]. This allows us to consider the effect of sudden changes in the parameters of the model. Such changes can arise for a variety of reasons. For instance, public health authorities may implement a rapid vaccination programme or close schools or public transport systems. Such policies generate abrupt changes in key model parameters. It is worth pointing out that with modern communication systems it is possible for large groups of individuals to collectively alter their behaviour rapidly through online rumour spreading for instance. The sudden variation in model parameters that this would give rise to need not be periodic and provides some practical motivation for considering a switched model.

We show that the results of [17] may be generalized in a far reaching manner. Our first main result is that for switched systems uniform stability of the linearization implies uniform global asymptotic stability. The result may be formulated using the assumption of irreducibility, in which case they become direct extensions of [17]. In addition, we present conditions for the existence of instability and the existence of endemic periodic solutions even if the constituent systems of the switched system are all stable. In this context we will several times make use of the theory of monotone systems studied in [35, 9]. One particular contribution of this paper is to analyse the stability of the epidemic model in [17] when its parameters are subject to random Markovian switching. General dynamical systems with random Markovian switching were introduced and studied in [22] and [24]. In the literature, such systems are frequently called *jump systems* and are connected to a wide range of applications; for more details we refer to [29]. For the problem of uniform global asymptotic stability it is shown in [32] that the techniques of the present paper may be used to also treat SIR and SIRS models.

The layout of the paper is as follows: in Section 2, we present the preliminary results and basic properties which are used in the remainder of this manuscript. The basic SIS model of [17] and the switched system framework we study in this manuscript are described in Section 3. We describe results on Positive Switched Linear Systems and extremal norms in Section 4. Building on this, results characterizing global uniform asymptotic stability of the disease-free equilibrium of the switched SIS model in terms of the joint spectral radius are given in Section 5. In Section 6, we consider the case where the Disease Free Equilibrium (DFE) is a globally asymptotically stable equilibrium of all the subsystems and introduce a switching signal that gives rise to endemic behaviour. Section 7 examines the stability of the DFE of the switched SIS model when the switching signal is a Markov process. This case has already been studied in [3, 18], where the interpretation of the basic reproduction number is studied for the case of populations with age structure or with no structure. The case where each subsystem of the switched SIS model has an endemic equilibrium is

considered in Section 8 and conditions for the existence of a switching signal for which the DFE is a globally asymptotically stable equilibrium of the switched SIS model are described. In Section 9, we present our conclusions.

2 Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of n dimensional column vectors with real entries, respectively. For $x \in \mathbb{R}^n$ and $i = 1, \dots, n$, x_i denotes the ith coordinate of x. Similarly, $\mathbb{R}^{n \times n}$ denotes the space of $n \times n$ matrices with real entries and for $A \in \mathbb{R}^{n \times n}$, a_{ij} denotes the (i, j)th entry of A. The positive orthant in $\mathbb{R}^n$ is $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. The interior of $\mathbb{R}_+^n$ is denoted by $\text{int}(\mathbb{R}_+^n)$ and its boundary by $\text{bd}(\mathbb{R}_+^n) := \mathbb{R}_+^n \setminus \text{int}(\mathbb{R}_+^n)$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$. The absolute value $|x|$ of a vector $x \in \mathbb{R}^n$ is defined by $|x|_i := |x_i|, i = 1, \dots, n$.

A matrix $A \in \mathbb{R}_+^{n \times n}$ is called *irreducible* if for every nonempty proper subset K of $N := \{1, \cdots, n\}$, there exists an $i \in K$, $j \in N \setminus K$ such that $a_{ij} \neq 0$. When A is not irreducible, it is *reducible*. Also, for $x \in \mathbb{R}^n$, $\text{diag}(x)$ is the $n \times n$ diagonal matrix in which $d_{ii} = x_i$.

For $A \in \mathbb{R}^{n \times n}$, we denote the *spectrum* of A by $\sigma(A)$ and the spectral radius of A by $\rho(A)$. The notation $\mu(A)$ denotes the *spectral abscissa* of A which is defined as follows:

$$\mu(A) := \max\{\text{Re}(\lambda) : \lambda \in \sigma(A)\}.$$

A matrix $A \in \mathbb{R}^{n \times n}$ is called *Hurwitz*, if $\mu(A) < 0$.

It will be useful to study norms, which are adapted to the nonnegative setting. A norm on $\mathbb{R}^n$ is called *monotone* if $|x| \geq |y|$ implies $\|x\| \geq \|y\|$. This is equivalent to the requirement that $\|x\| = \| |x| \|$ for all $x \in \mathbb{R}^n$, see [5, Theorem 2], [21, Theorem 5.5.10]. Norms with the latter property are called *absolute* and we will use this name throughout the remainder of the paper.

The dual norm $\| \cdot \|^*$ of a norm $\| \cdot \|$ on $\mathbb{R}^n$ is defined by

$$\|y\|^* := \max\{\langle x, y \rangle \mid \|x\| \leq 1\}, \quad y \in \mathbb{R}^n. \tag{1}$$

It is known that a norm is absolute if and only if its dual norm is, [5, Theorem 1]. Given $\| \cdot \|$ and its dual $\| \cdot \|^*$, a pair of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a *dual pair*, if

$$\langle x, y \rangle = \|x\| \, \|y\|^*. \tag{2}$$

A vector y is called *dual* to $x \in \mathbb{R}^n$, if (x, y) is a dual pair. Note that the order is important: in a dual pair the first vector is evaluated with $\| \cdot \|$ and the second with its dual. The relation is not symmetric, in general. The following observation on the properties of dual vectors of absolute norms will be useful.

Lemma 2.1 *Let $\| \cdot \|$ be an absolute norm on $\mathbb{R}^n$ and $x \in \mathbb{R}_+^n$. If y is dual to x and $x_i > 0$, then $y_i \geq 0$. In particular, if $x, y \neq 0$ then $x_j y_j > 0$ for some index j.*

Proof By (2) and the assumption we have

$$\|x\| \, \|y\|^* = \langle x, y \rangle = \sum_{x_i > 0} x_i y_i \leq \sum_{x_i > 0} x_i |y_i| = \langle x, |y| \rangle \leq \|x\| \, \| \, |y| \, \|^*.$$

As $\| \cdot \|^*$ is absolute, we have $\|y\|^* = \| \, |y| \, \|^*$ and so equality throughout. This implies that all summands in $\sum_{x_i > 0} x_i y_i$ are nonnegative, which is the first assertion. The second claim follows as by assumption $\|x\| \|y\|^* > 0$ and so some of the terms in the sum $\sum_{x_i > 0} x_i y_i$ have to be positive. $\qquad \square$

It follows in particular, that if $\| \cdot \|$ is an absolute norm, and $x \gg 0$, then any dual vector y to x satisfies $y \geq 0$. A particular absolute norm is the usual infinity norm denoted by $\| \cdot \|_\infty$.

Monotone Systems

Throughout the paper, $\mathcal{W}$ is a neighbourhood of $\mathbb{R}_+^n$. Let $f : \mathcal{W} \to \mathbb{R}^n$ be a C^1 vector field. The *forward solution* of the nonlinear system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x^0. \tag{3}$$

with initial condition $x^0 \in \mathcal{W}$ at $t = 0$ is denoted by $x(t, x^0)$ and is defined on the maximal forward interval of existence $\mathcal{I}_{x^0} := [0, T_{max}(x^0))$. All autonomous nonlinear systems considered here have unique solutions defined on $[0, \infty)$.

The systems considered here are *positive systems*. A system is called positive if $x^0 \geq 0$ implies $x(t, x^0) \geq 0$ for all $t \geq 0$. It is well known [13] that the following property is necessary and sufficient for (3) to be positive

$$\forall x \in \mathrm{bd}\,(\mathbb{R}_+^n) : x_i = 0 \Rightarrow f_i(x) \geq 0. \tag{4}$$

In particular, a linear system $\dot{x} = Ax$ is positive if and only if A has nonnegative off-diagonal entries, i.e. $a_{ij} \geq 0$, $\forall i \neq j$. Matrices with this property are called *Metzler*, or sometimes negative Z-matrices or essentially nonnegative matrices. In this paper we stick to the former terminology.

As is standard, we say that the C^1 vector field $f : \mathcal{W} \to \mathbb{R}^n$ is *cooperative* on $\mathcal{U} \subseteq \mathcal{W}$ if the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is Metzler for all $a \in \mathcal{U}$. When we say that f is cooperative without specifying the set $\mathcal{U}$, we understand that it is cooperative on $\mathbb{R}_+^n$. It is well known that cooperative systems are monotone [35]. I.e., if $f : \mathcal{W} \to \mathbb{R}^n$ is cooperative on $\mathbb{R}_+^n$ then $x^0 \leq y^0$, $x^0, y^0 \in \mathbb{R}_+^n$ implies $x(t, x^0) \leq x(t, y^0)$ for all $t \geq 0$.

We will use the following lemma, which is an immediate consequence of [35, Prop. 3.2.1].

Lemma 2.2 *Let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative and $w \in \mathcal{W}$ be such that $f(w) \ll 0$ $(f(w) \gg 0)$. Then the trajectory $x(\cdot, w)$ of system (3) is decreasing (increasing) in t for $t \geq 0$ with respect to the order on $\mathbb{R}_+^n$, i.e. if $f(w) \ll 0$, then*

$$x(t, w) \ll x(s, w), \quad \text{for all} \quad 0 \leq s < t < T_{\max}(w).$$

If $f(w) \leq 0$ $(f(w) \geq 0)$, the trajectory will be non-increasing (non-decreasing).

Switched Systems

The main contributions of this manuscript extend results for time-invariant compartmental SIS models to switched SIS models. We now briefly recall some fundamental concepts related to switched nonlinear systems, [26, 34]. Let $\mathcal{W}$ be an open neighbourhood of $\mathbb{R}_+^n$. Consider a family $\{f_1, \ldots, f_m\}, m \in \mathbb{N}$, where $f_i : \mathcal{W} \to \mathbb{R}^n$ is C^1, $1 \leq i \leq m$. We assume that the associated autonomous systems $\dot{x} = f_i(x)$, are forward complete on $\mathbb{R}_+^n$, that is, the unique solution $x(\cdot, x^0)$ is defined on $[0, \infty)$ for all $x^0 \in \mathbb{R}_+^n$, $1 \leq i \leq m$.

Associated to the family $\{f_1, \ldots, f_m\}$ and a set of switching signals $\sigma : \mathbb{R}_+ \to \{1, \ldots, m\}$ we consider the switched system

$$\dot{x}(t) = f_{\sigma(t)}x(t) \quad \text{a.e.} \quad x(0) = x^0. \tag{5}$$

By $\overline{\mathcal{S}}$ we denote the set of measurable functions $\sigma(\cdot) : \mathbb{R}_+ \mapsto \{1, \cdots, m\}$. We note that for each $\sigma \in \overline{\mathcal{S}}$ the Carathéodory conditions are satisfied, so that existence and uniqueness of solutions is guaranteed, [11]. We refer to $\dot{x} = f_i(x)$ as the ith constituent system for $1 \leq i \leq m$.

The subset $\mathcal{S} \subset \overline{\mathcal{S}}$ denotes the set of all piecewise constant mappings $\sigma(\cdot) : \mathbb{R}_+ \mapsto \{1, \ldots, m\}$ that are continuous from the right and for which there exists some $\tau > 0$ such that $t - s \geq \tau$ for any two points of discontinuity t, s of σ.

Note that the set $\mathcal{S} \subset \overline{\mathcal{S}}$ is dense in $\overline{\mathcal{S}}$, considered as subsets of $L^\infty(\mathbb{R}_+, \mathbb{R})$ endowed with the weak* topology. As a consequence solutions of (5) for $\sigma \in \overline{\mathcal{S}}$ may be approximated, on any compact interval, arbitrarily well by solutions corresponding to $\sigma \in \mathcal{S}$. Throughout the paper, we use the notation $x(\cdot, x^0, \sigma)$ to denote the solution corresponding to $\sigma \in \overline{\mathcal{S}}$ and the initial condition $x^0 \in \mathbb{R}_+^n$.

The points of discontinuity $0 = t_1, t_2, \ldots$ of $\sigma \in \mathcal{S}$ are the *switching instants*.

By assumption for every $\sigma \in \mathcal{S}$, there is some positive constant τ such that $t_{i+1} - t_i \geq \tau$ for all i; for switched SIS models, where switching corresponds to some seasonal or diurnal change, this is entirely reasonable.

We next recall various fundamental stability concepts. As we are dealing with positive systems, all definitions are with respect to the state space $X = \mathbb{R}_+^n$.

Definition 2.1 *Let* $\sigma \in \overline{\mathcal{S}}$ *be given and let* $\bar{x}$ *be an equilibrium of the system (5). Then we say that the equilibrium point* $\bar{x}$ *is*

(i) stable, *if for every* $\varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\|x^0 - \bar{x}\| < \delta \Rightarrow \|x(t, x^0, \sigma) - \bar{x}\| < \varepsilon, \quad \forall t > 0.$$

(ii) unstable, *if it is not stable;*

(iii) asymptotically stable *if it is stable and there exists a neighbourhood* N, *relatively open in* $\mathbb{R}^n_+$, *of* $\bar{x}$ *such that*

$$x^0 \in N \Rightarrow \lim_{t \to \infty} x(t, x^0, \sigma) = \bar{x}.$$

The domain of attraction of $\bar{x}$ *is given by*

$$A(\bar{x}) := \left\{ x^0 \in \mathbb{R}^n_+ : x(t, x^0, \sigma) \to \bar{x}, \ as \ t \to \infty \right\}.$$

If $A(\bar{x}) = \mathbb{R}^n_+$, *then we say that* $\bar{x}$ *is globally asymptotically stable.*

When discussing uniform stability with respect to a set of switching signals, it is convenient to make use of $\mathcal{K}$ and $\mathcal{KL}$ functions (the definitions given above for a fixed σ can also be formulated equivalently in these terms). A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if $\alpha(0) = 0$ and α is strictly increasing. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for every fixed $t \geq 0$ and $\beta(r, t) \to 0$ as $t \to \infty$ for each fixed $r \geq 0$.

We say that $\bar{x}$ is a uniformly stable equilibrium of (5) with respect to $\mathcal{S}$ $(\overline{\mathcal{S}})$ if there exists a class $\mathcal{K}$ function α such that

$$\|x(t, x^0, \sigma) - \bar{x}\| \leq \alpha(\|x^0 - \bar{x}\|) \quad \forall t \geq 0 \tag{6}$$

for all $x^0 \in \mathbb{R}^n_+$ and all $\sigma \in \mathcal{S}$ $(\sigma \in \overline{\mathcal{S}})$.

In addition, $\bar{x}$ is a uniformly globally asymptotically stable equilibrium of (5) with respect to $\mathcal{S}$ $(\overline{\mathcal{S}})$ if there exists a class $\mathcal{KL}$ function β such that

$$\|x(t, x^0, \sigma) - \bar{x}\| \leq \beta(\|x^0 - \bar{x}\|, t) \quad \forall t \geq 0 \tag{7}$$

for all $x^0 \in \mathbb{R}^n_+$ and all $\sigma \in \mathcal{S}$ $(\sigma \in \overline{\mathcal{S}})$. They key concept here is that the rate of convergence as $t \to \infty$ is uniform across all switching signals in $\mathcal{S}$ $(\overline{\mathcal{S}})$, where this uniformity is guaranteed by the same $\mathcal{KL}$ function β for all σ in the set of signals.

Finally, we need some formulations of Lipschitz continuous Lyapunov functions that will be used in later proofs. Let $D \subset \mathbb{R}^n$ be open and $h : D \to \mathbb{R}$ be locally Lipschitz continuous. By Rademacher's theorem this implies that h is differentiable almost everywhere. The *Clarke generalized gradient* of h at $x \in D$ may then be defined by, [10], [14, Theorem II.1.2],

$$\partial_C h(x) = \text{conv} \left\{ p \in \mathbb{R}^n \mid \exists x_k \to x, \nabla h(x_k) \text{ exists }, \lim_{k \to \infty} \nabla h(x_k) = p \right\}. \tag{8}$$

In particular, it is an exercise to see that the Clarke gradient of a norm v at a point $x \neq 0$ is given by normed dual vectors, i.e.

$$\partial_C v(x) = \{y \in \mathbb{R}^n \mid v^*(y) = 1, y \text{ is dual to } x\}. \tag{9}$$

A proper, positive definite, Lipschitz continuous function V is a strict Lyapunov function for the switched system (5), if

$$\langle p, f_j(x) \rangle \leq -\alpha(\|x\|) \tag{10}$$

for all $p \in \partial_C V(x)$ and $j = 1, \ldots, m$ and some positive definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$. If a strict Lyapunov function exists, this implies uniform asymptotic stability of the origin with respect to $\sigma \in \overline{\mathcal{S}}$. If in (10) $-\alpha(\|x\|)$ has to be replaced by 0, then we speak of a nonstrict Lyapunov function, the existence of which implies uniform stability, [10, Theorem 4.5.5].

3 Problem Description

We now recall some results concerning the basic autonomous SIS model considered in [17].

The population of interest is assumed to be divided into n groups; each group is then further divided into two classes: infectives and susceptibles. Let $I_i(t)$ and $S_i(t)$ be the number of infectives and susceptibles at time t in group i for $i = 1, \ldots, n$, respectively. Also, let $N_i(t) = S_i(t) + I_i(t)$ be the total population of group i. The total population of each group is assumed constant: formally, $N_i(t) = N_i, \forall\, t \geq 0$.

The constants β_{ij} denote the rate at which susceptibles in group i are infected by infectives in group j for $i, j = 1, \cdots, n$. Further, γ_i is the rate at which an infective individual in group i is cured. As the total population of each group is constant, within each group the birth and death rates are equal and denoted by μ_i. Using the mass-action law, the basic SIS model is described as follows [17]:

$$
\begin{cases}
\dot{S}_i(t) = \mu_i N_i - \mu_i S_i(t) - \displaystyle\sum_{j=1}^{n} \beta_{ij} \frac{S_i(t) I_j(t)}{N_i} + \gamma_i I_i(t) \\
\dot{I}_i(t) = \displaystyle\sum_{j=1}^{n} \beta_{ij} \frac{S_i(t) I_j(t)}{N_i} - (\gamma_i + \mu_i) I_i(t)
\end{cases}
$$

Since the population of each group is constant, it is sufficient to know $I_i(t)$. If we set $x_i(t) = I_i(t)/N_i$ and $\tilde{\beta}_{ij} = \beta_{ij} N_j / N_i$ and $\alpha_i = \gamma_i + \mu_i$, we obtain

$$
\dot{x}_i(t) = (1 - x_i(t)) \sum_{j=1}^{n} \tilde{\beta}_{ij} x_j(t) - \alpha_i x_i(t),
$$

which can be written in the compact form:

$$
\dot{x} = [-D + B - \operatorname{diag}(x)B] x, \tag{11}
$$

where $D = \operatorname{diag}(\alpha_i)$ and $B = (\tilde{\beta}_{ij}) > 0$. This is the model considered in [25, 17]. In [17] and throughout this paper, we assume that $\alpha_i > 0$ for $1 \leq i \leq n$. This is biologically reasonable, as otherwise there would exist a compartment for which both the birth-rate and the rate at which infectives are cured were zero. This assumption has implications for the location of so-called endemic equilibria on which we shall elaborate below.

The following properties of (11) are easy to check.

(i) The compact set $\Sigma_n := \{x \in \mathbb{R}_+^n : x_i \leq 1, i = 1, \ldots, n\}$ is forward invariant under (11).

(ii) The origin 0 is an equilibrium point of (11). This is referred to as the *Disease Free Equilibrium* (DFE) of (11).

(iii) For every $x^0 \in \Sigma_n$, there exists a unique solution $x(t, x^0)$ of (11) defined for all $t \geq 0$.

If there exists an equilibrium point $\bar{x}$ in $\operatorname{int}(\mathbb{R}_+^n)$, we refer to $\bar{x}$ as an *endemic equilibrium*. Note that as $\alpha_i > 0$ for $1 \leq i \leq n$, it follows readily that any endemic equilibrium must lie in $\operatorname{int}(\mathbb{R}_+^n) \cap \operatorname{int}(\Sigma_n)$.

The basic reproduction number R_0, defined as the average number of secondary infections that occur when one infective is introduced into a completely susceptible host population, is fundamental in mathematical epidemiology. In line with [37], in [17] this is defined as $R_0 = \rho(D^{-1}B)$. From basic properties of Metzler matrices it is easy to see that $R_0 < 1$ if and only if $\mu(D + B) < 0$ [21].

The parameter R_0 is used to characterize the existence and stability of the equilibria of (11). The following result is Theorem 2.3 in [17].

Theorem 3.1 *Consider the system (11). Assume that the matrix B is irreducible. The DFE at the origin is globally asymptotically stable if and only if $R_0 \leq 1$.*

The next result considers the existence and stability of endemic equilibria and is a restatement of Theorem 2.4 of [17].

Theorem 3.2 *Consider the system (11) and assume that B is irreducible. There exists a unique endemic equilibrium $\bar{x}$ in $\mathrm{int}\,(\mathbb{R}^n_+)$ if and only if $R_0 > 1$. Moreover, in this case, $\bar{x}$ is asymptotically stable with region of attraction $\mathbb{R}^n_+ \setminus \{0\}$.*

Note that as Σ_n is forward invariant under (11), it follows readily that the unique endemic equilibrium discussed in Theorem 3.2 must be in Σ_n. Furthermore, as $\alpha_i > 0$ for $1 \leq i \leq n$, it follows readily that it must lie in $\mathrm{int}\,(\mathbb{R}^n_+) \cap \Sigma_n$.

4   Positive linear switched systems

To facilitate our later analysis of the linearisation of a switched version of the SIS model (11) we need some results on positive switched linear systems, which are not available in the literature. This is the topic of the present section. The material presented here is a consequence of results presented in [1]. As it is essential for the following arguments we include it for the benefit of the reader.

We make use of a formulation of the joint spectral radius for continuous time systems as described in [38]. Let $\mathcal{M} \subset \mathbb{R}^{n \times n}$ be a compact set of matrices. In particular, a finite set $\mathcal{M} = \{A_1, \cdots, A_m\}$ may be considered. The set $\mathcal{M}$ gives rise to the switched linear system

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad \sigma \in \overline{\mathcal{S}}. \tag{12}$$

If we fix $\sigma : \mathbb{R}_+ \to \mathcal{M}$, then the evolution operator $\Phi_\sigma(\cdot)$ (with initial time $t_0 = 0$) corresponding to (12) is given as the solution of the matrix differential equation

$$\dot{\Phi}_\sigma(t) = A_\sigma(t)\Phi_\sigma(t), \quad \Phi_\sigma(0) = I.$$

Considering the set of all evolution operators we define a matrix semigroup $\mathcal{H}$ as follows. The set of time t evolution operators is given by

$$\mathcal{H}_t := \{\Phi_\sigma(t) \mid \sigma : [0, t] \to \mathcal{M} \text{ measurable}\}$$

and $\mathcal{H} := \bigcup_{t \in \mathsf{R}_+} \mathcal{H}_t$, where we set $\mathcal{H}_0 := \{I\}$. Define the growth at time t by

$$\rho_t(\mathcal{M}) := \sup_{\sigma \in \overline{\mathcal{S}}} \frac{1}{t} \log \|\Phi_\sigma(t)\|.$$

Then the *joint Lyapunov exponent* of (12) is given by

$$\rho(\mathcal{M}) := \lim_{t \to \infty} \rho_t(\mathcal{M}). \tag{13}$$

We note that the definition of $\rho(\mathcal{M})$ does not depend on the choice $\mathcal{S}$ versus $\overline{\mathcal{S}}$. Similarly to the *joint spectral radius* of a set of matrices, the joint Lyapunov exponent can be used to characterise uniform asymptotic stability of linear switched systems; see [38] and references therein. For our purposes the following fact is sufficient.

Lemma 4.1 *The joint Lyapunov exponent $\rho(\mathcal{M}) < 0$ if and only if the origin is a uniformly asymptotically stable equilibrium of (12).*

Also while it is necessary for the uniform exponential stability of (12) that all the matrices in the convex hull $\mathrm{conv}\,(\mathcal{M})$ are Hurwitz, this is not a sufficient condition, even if $\mathcal{M}$ consists of Metzler matrices [16, 19]. In the analysis of linear inclusions extremal norms play an interesting role.

Definition 4.1 *Let $\mathcal{M}$ be a compact set of Metzler matrices. A norm v on $\mathbb{R}^n$ is called extremal for the associated semigroup $\mathcal{H}$, if for all $x \in \mathbb{R}^n$ and all $t \geq 0$ we have*

$$v(Sx) \leq \exp(\rho(\mathcal{M})t)\, v(x)\,, \quad \forall\, S \in \mathcal{H}_t\,. \tag{14}$$

For the analysis of the switched SIS model (11) it will be instrumental to analyse the existence of extremal norms for its linearisation. We will show the more general statement that for positive switched linear systems an absolute extremal norm exists, if an irreducibility property is satisfied.

Associated to a set of Metzler matrices $\mathcal{M}$ we consider the directed graph $\mathcal{G}(\mathcal{M}) = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edges defined for $i \neq j, 1 \leq i, j \leq n$ by

$$(i,j) \in E \quad :\Leftrightarrow \quad \exists A \in \mathcal{M} \text{ with } a_{ij} > 0\,. \tag{15}$$

Note that we explicitly do not define edges from a vertex i to itself. The graph $\mathcal{G}(\mathcal{M})$ is called strongly connected if for all i, j there is a path form i to j using edges in E.

Lemma 4.2 *Let $\mathcal{M}$ be a set of Metzler matrices. Then $\mathcal{G}(\mathcal{M})$ is strongly connected if and only if for all $i, j \in V$ there exist $k \in \mathbb{N}$, $(A_1, \ldots, A_k) \in \mathcal{M}^k$ and $t_1, \ldots, t_k > 0$ such that*

$$\left(e^{A_1 t_1} \ldots e^{A_k t_k}\right)_{ij} > 0\,.$$

Proof Follows by direct calculation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also find it useful to point out the following fact.

Lemma 4.3 *Let $\mathcal{M}$ be a set of Metzler matrices. Then $\mathcal{G}(\mathcal{M})$ is strongly connected if and only if the convex hull $\operatorname{conv}\mathcal{M}$ contains an irreducible matrix.*

We are now ready to formulate our main result for positive switched linear systems.

Proposition 4.1 *Let $\mathcal{M}$ be a compact set of Metzler matrices. If $\operatorname{conv}\mathcal{M}$ contains an irreducible element $\bar{M}$ then there exists a absolute extremal norm v for (12).*

Proof By convexity of norms, a norm v is extremal for $\mathcal{M}$ if and only if it is extremal for $\operatorname{conv}\mathcal{M}$, so that we may assume that $\mathcal{M}$ is convex. By considering $\mathcal{M} - \rho(\mathcal{M})I$ we may assume that $\rho(\mathcal{M}) = 0$.

We first show that under the assumptions $\mathcal{H}$ is bounded if $\rho(\mathcal{M}) = 0$. Assume this is not the case. As $\mathcal{M}$ is irreducible and convex, there exists an irreducible matrix $\bar{M} \in \mathcal{M}$. Then $\exp(\bar{M}) \in \mathcal{H}_1$ is positive, [6], and there exists a constant $c > 0$ such that for all $i, j = 1, \ldots, n$

$$\exp(\bar{M})_{ij} > c\,. \tag{16}$$

As $\mathcal{H}$ is assumed to be unbounded, we may choose $S = \Phi_\sigma(t, 0) \in \mathcal{H}_t$ such that $S_{\nu\mu} > 1/c$ for some indices ν, μ. It follows by direct calculation that $\left(\exp(\bar{M})S\right)_{\mu\mu} =: a > 1$ and as $\left(\exp(\bar{M})S\right)^k \in \mathcal{H}_{k(t+1)}$ we obtain $\rho(\mathcal{M}) \geq \log(a)/(t+1) > 0$. This contradiction proves of boundedness of $\mathcal{H}$.

As the generated semigroup is bounded, an extremal norm may be defined in the following way, see also [23]. Let $\|\cdot\|$ be absolute, then define for $x \geq 0$

$$v(x) := \sup\{\|Sx\| \mid S \in \mathcal{H}\}\,. \tag{17}$$

and extend this definition to $\mathbb{R}^n$ by setting $v(x) := v(|x|), x \in \mathbb{R}^n$. It is clear that v is positively homogeneous and positive definite (as $I \in \mathcal{H}$). As the matrices $S \in \mathcal{H}$ are all nonnegative it follows by absoluteness of $\|\cdot\|$ that for $0 \leq x < y$ we have $v(x) < v(y)$. As a consequence v is absolute because if $|x| < |y|$ then $v(x) = v(|x|) < v(|y|) = v(y)$.

The triangle inequality for v then follows from

$$v(x + y) = v(|x + y|) \leq v(|x| + |y|) = \sup\{\|S(|x| + |y|)\| \mid S \in \mathcal{H}\}$$
$$\leq \sup\{\||S|x|\| + \|S|y|\|\| \mid S \in \mathcal{H}\} \leq v(|x|) + v(|y|) = v(x) + v(y)\,.$$

Thus v is a norm. Further, from $0 \leq |Sx| \leq S|x|$, using the monotonicity of v (and the assumption $\rho(\mathcal{M}) = 0$) we have for all $x \in \mathbb{R}^n, S \in \mathcal{H}$ that

$$v(Sx) = v(|Sx|) \leq v(S|x|) = \sup\{\|TS|x|\| \mid T \in \mathcal{H}\} \leq v(|x|) = v(x)\,, \tag{18}$$

where we have used the definition of v in the final inequality. Now extremality of v follows. $\qquad\square$

We note the following consequence for stability theory of switched positive systems.

Corollary 4.1 *Let $\mathcal{M}$ be a compact set of Metzler matrices and consider the positive switched linear system* (12). *The following two conditions are each sufficient for the existence of an absolute norm v, which is a nonstrict Lyapunov function for* (12).

(i) $\rho(\mathcal{M}) = 0$ and $\operatorname{conv}\mathcal{M}$ contains an irreducible element,

(ii) $\rho(\mathcal{M}) < 0$.

Proof We note that $v(\Phi_\sigma(t)x) \leq v(x)$ for all $x \in \mathbb{R}^n, t \geq 0$ and all $\sigma \in \mathcal{S}$ is equivalent to the statement that $\langle y, Ax \rangle \leq 0$ for all dual pairs (y, x) and all $A \in \mathcal{M}$, see the comments after (10). The case (i) is then immediate from the previous Proposition 4.1. In case (ii) holds, we have that the switched linear system has a uniformly asymptotically stable equilibrium at $x^* = 0$ by Lemma 4.1. In particular, the semigroup $\mathcal{H}$ is bounded and we can without further ado use (17) to define an absolute norm. It follows as in (18) that the norm is a nonstrict Lyapunov function. $\qquad\square$

We note that in the case (ii) of the previous corollary it may or may not be possible to construct an extremal norm for $\mathcal{M}$.

5 Global Asymptotic Stability of the Disease Free Equilibrium

In this section, we present an extension of Theorem 3.1 to a switched SIS model. In place of R_0, we use the joint Lyapunov exponent to characterize the stability of the Disease Free Equilibrium (DFE). In [8], conditions for local asymptotic stability of the DFE based on the joint Lyapunov exponent (joint spectral radius) were presented. Our result here shows that the same conditions imply global asymptotic stability for all switching signals.

For the remainder of the paper, $D_1, \ldots, D_m$ are diagonal matrices in $\mathbb{R}^{n \times n}$ with positive entries along their main diagonals; $B_1, \ldots, B_m$ are nonnegative matrices in $\mathbb{R}^{n \times n}$ and $\mathcal{S}$ denotes the set of admissible switching signals introduced in Section 2.

We now consider the switched SIS system described by

$$\dot{x} = (-D_{\sigma(t)} + B_{\sigma(t)} - \operatorname{diag}(x)B_{\sigma(t)})x =: f_{\sigma(t)}(x(t)). \tag{19}$$

It follows easily from the properties of the constituent systems, outlined in Section 3, that for each $x^0 \in \Sigma_n$, $\sigma \in \overline{\mathcal{S}}$, there is a unique solution $x(t, x^0, \sigma)$ of (19) defined for $t \geq 0$, satisfying $x(0, x^0, \sigma) = x^0$. Further, it is clear that the set Σ_n is forward invariant under (19) for all $\sigma \in \overline{\mathcal{S}}$.

Linearising (19) about the origin, we obtain the linear switched system

$$\dot{x} = (-D_{\sigma(t)} + B_{\sigma(t)})x. \tag{20}$$

Let $\mathcal{M} = \{-D_1 + B_1, \ldots, -D_m + B_m\}$ and recall that $\rho(\mathcal{M})$ denotes the joint Lyapunov exponent of $\mathcal{M}$. An application of Lemma 4.1 to (20) yields

Lemma 5.1 *The joint Lyapunov exponent $\rho(\mathcal{M}) < 0$ if and only if the origin is a uniformly globally asymptotically stable equilibrium of (20) w.r.t. $\sigma \in \mathcal{S}$ (or $\sigma \in \overline{\mathcal{S}}$).*

We note the following comparison properties of system (19) and the linearisation of the system in $x = 0$ given by (20).

Lemma 5.2 *Consider (19) and its linearisation (20). Let $0 \le x^0 \le y^0$ and let $\varphi(t) := \varphi(t; x^0)$, resp, $\Phi_\sigma(t, 0)y^0$ be the solutions of (19) and (20). Then*

$$\varphi(t; x^0) \le \Phi_\sigma(t, 0)y^0 \,, \quad \forall \, t \ge 0 \,. \tag{21}$$

Proof The claim is immediate using variation of constants and the nonnegativity of Φ_σ and B_σ which yields

$$\begin{aligned}
\varphi(t; x^0) &= \Phi_\sigma(t, 0)x^0 - \int_0^t \Phi_\sigma(t, s)\text{diag}\,(\varphi(s))B_{\sigma(s)}\varphi(s)ds \\
&\le \Phi_\sigma(t, 0)x^0 \le \Phi_\sigma(t, 0)y^0 \,.
\end{aligned}$$

$\square$

The following theorem generalizes Theorem 3.1 to the case of switched systems and extends Theorem 3.1 even for the case of an autonomous system, i.e. for the situation considered in Theorem 3.1.

Theorem 5.1 *Assume that the linear switched system (20) is uniformly stable in $x^* = 0$ and admits an absolute norm as a nonstrict Lyapunov function. Then the disease free equilibrium $x^* = 0$ of (19) is uniformly globally asymptotically stable for $\sigma \in \overline{\mathcal{S}}$.*

Remark 5.1 *(i) In the case $\rho(\mathcal{M}) = 0$, by Corollary 4.1, the linear switched system (20) admits an absolute norm as a nonstrict Lyapunov function, provided an appropriate irreducibility condition holds. So just as in the case of Theorem 3.1 irreducibility and stability of the linearisation imply global asymptotic stability of the nonlinear switched system. This is in contrast to the general linearisation theory in which exponential stability of the linearisation, equivalently $\rho(\mathcal{M}) < 0$, is required in order to infer stability of the nonlinear system.*
(ii) Observe that most of the work in the proof of Theorem 5.1 is devoted to the case that the linearization is not uniformly asymptotically stable. The asymptotically stable case can be proved in a much simpler fashion by applying Lemma 5.2.
(iii) Theorem 5.1 also applies to autonomous systems: the case $m = 1$. Even in this case it generalizes Theorem 3.1: If the basic reproduction number $R_0 = 1$, then irreducibility of B implies the existence of an extremal norm, but the converse is false. Still under the assumption that $\mu(A) = 0$, it is not necessary that a Metzler matrix A is irreducible for an absolute extremal norm to exist. The necessary and sufficient condition for this is that the matrix is stable, i.e. that for eigenvalues $\lambda \in \sigma(A), \text{Re}(\lambda) = 0$, algebraic and geometric multiplicity coincide. While this is well known in the context of quadratic Lyapunov functions, we point out that the construction in (17) yields an absolute extremal norm also in this case. The following corollary is thus immediate.

Corollary 5.1 *Consider the time-invariant system (11). The disease free equilibrium $x^* = 0$ is globally asymptotically stable, if the matrix $-D + B$ is stable.*

Proof (of Theorem 5.1) If (20) is stable, then a nonstrict Lyapunov function $v(\cdot)$ on $\mathbb{R}^n$ has the property that for all switching signals $\sigma \in \overline{\mathcal{S}}$ and all initial conditions we have

$$v(\Phi_\sigma(t, 0)x^0) \le v(x^0) \,, \quad \forall \, t \ge 0 \,. \tag{22}$$

Step 1: We first show that v is a nonstrict Lyapunov function for (19) on $\mathbb{R}^n_+$.

Consider a vector $x \in \mathbb{R}_+^n, x \neq 0$ and a dual vector y with $v^*(y) = 1$ for x and fix a constituent system given by $(B_j, D_j), j = 1, \ldots, m$. As v is an absolute norm we have using (22) and (9) that

$$\langle y, (-D_j + B_j)x \rangle \leq 0 \, . \tag{23}$$

Further, as $x_i > 0$ implies $y_i \geq 0$ by Lemma 2.1 it follows that

$$\langle y, -\text{diag}\,(x)B_j x \rangle \leq 0, \tag{24}$$

so that we always have

$$\langle y, f_j(x) \rangle = \langle y, (-D_j + B_j)x - \text{diag}\,(x)B_j x \rangle \leq 0 \, . \tag{25}$$

In particular, this shows that (19) is uniformly stable with respect to $\sigma \in \overline{\mathcal{S}}$. It remains to show attractivity. **Step 2:** We now investigate under which conditions (25) may fail to be strict. By (23) and (24) this can only happen, if in both these equations equality holds. Assuming this we obtain from equality in (24)

$$0 = -\langle y, \text{diag}\,(x)B_j x \rangle = - \sum_{x_i y_i \neq 0} y_i x_i (B_j x)_i \, .$$

As $y_i \geq 0$, if $x_i > 0$, this implies that from $x_i > 0$ we can conclude $y_i(B_j x)_i = 0$. Plugging this into (23) we obtain that

$$0 = \langle y, (-D_j + B_j)x \rangle = \sum_{x_i > 0} -x_i y_i D_{j,ii} + \sum_{x_i = 0} y_i (B_j x)_i \, .$$

Again by Lemma 2.1 the first sum on the right hand side is negative. We can conclude that the second sum has to be positive to compensate this, so that there are $x_i = 0$ with $(B_j x)_i > 0$.
Step 3: To exploit the property established in Step 2, that solutions which are not infinitesimally decreasing have to move to the interior of the positive orthant we add a further term to the Lyapunov function. To this end let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with support contained in $(-\infty, 1]$ and so that

$$\psi(0) = 1 \, , \quad \psi(z) > 0 \, , \; z \in [0,1) \, , \quad \psi'(z) < 0 \, , \; z \in [0,1) \, ,$$

and that in addition ψ' is increasing on $[0,1]$.
For $\varepsilon \in (0,1)$ we denote $\psi_\varepsilon(z) := \psi(z + (1 - \varepsilon))$ and note that the support of ψ_ε is contained in $(-\infty, \varepsilon]$ and that $\eta(\varepsilon) := |\psi_\varepsilon'(0)| = \max_{z \in [0,\varepsilon]} |\psi_\varepsilon'(z)| = \max_{z \in [1-\varepsilon,1]} |\psi'(z)|$ tends to 0 as $\varepsilon \to 0$, because $\psi'(1) = 0$.
Fix $0 < \ell < L$. We aim to show that there exists $1 > \varepsilon(\ell, L) > 0$ so that for all $0 < \varepsilon < \varepsilon(\ell, L)$ the function

$$V_\varepsilon(x) := v(x) \left(1 + \sum_{i=1}^n \psi_\varepsilon(x_i) \right) = v(x) \left(1 + \sum_{x_i < \varepsilon} \psi_\varepsilon(x_i) \right) \tag{26}$$

is a strict Lyapunov function for (19) on the set $v^{-1}([\ell, L]) := \{x \in \mathbb{R}_+^n \mid \ell \leq v(x) \leq L\}$. As $\ell > 0$ may be chosen to be arbitrarily small and $L > 0$ arbitrary large, this shows that the disease free equilibrium is globally asymptotically stable, as explained in Step 5 below.
Step 4: We proceed to prove the claim formulated in the previous step. Fix $x \in v^{-1}([\ell, L])$. We wish to show that for $\varepsilon > 0$ sufficiently small, we have for all elements of the Clarke subgradient $p \in \partial_C V_\varepsilon(x)$ and $j = 1, \ldots, m$ that

$$\langle p, f_j(x) \rangle < -\theta < 0 \, , \tag{27}$$

where $\theta = \theta(x) > 0$ is a suitable constant. As ψ_ε is smooth it is easy to see that the elements of $\partial_C V_\varepsilon(x)$ are of the form

$$p = y \left(1 + \sum_{i=1}^n \psi_\varepsilon(x_i) \right) + v(x) \sum_{x_i < \varepsilon} \psi_\varepsilon'(x_i) e_i \, , \tag{28}$$

where e_i denotes the ith unit vector and $y \in \partial_C v(x)$, i.e. y is dual to x and $v^*(y) = 1$.

For the sake of estimation set $d_{\max} := \max\{D_{j,ii} \mid j = 1, \ldots, m, i = 1, \ldots, n\}$. We now distinguish two cases. First, if for fixed j we have $\max\{\langle y, f_j(x) \rangle \mid y \in \partial_C v(x)\} = -c < 0$, then choose $1 > \varepsilon > 0$ so that

$$L \, n \, d_{\max} \, \eta(\varepsilon)\varepsilon < \frac{c}{2} \, . \tag{29}$$

Then we obtain for $p \in \partial_C V_\varepsilon(x)$ given by (28) that

$$\langle p, f_j(x) \rangle = \left(1 + \sum_{i=1}^{n} \psi_\varepsilon(x_i)\right) \langle y, f_j(x) \rangle + v(x) \sum_{x_i < \varepsilon} \psi_\varepsilon'(x_i) \langle e_i, f_j(x) \rangle \tag{30}$$

$$\leq -c + v(x) \sum_{x_i < \varepsilon} \psi_\varepsilon'(x_i) \left(-D_{j,ii}x_i + (1 - x_i)(B_j x)_i\right) \, .$$

As $\psi_\varepsilon'(z) \leq 0$ for $z \geq 0$ and the summands in the second term are only nonzero for $x_i < 1$ we have by nonnegativity of $(B_j x)_i$

$$\leq -c + L \sum_{x_i < \varepsilon} \psi_\varepsilon'(x_i) \left(-D_{j,ii}x_i\right)$$

$$\leq -c + n \, L \, d_{\max} \, \eta(\varepsilon)\varepsilon < -\frac{c}{2} \, . \tag{31}$$

Secondly, consider the case $\max\{\langle y, f_j(x) \rangle \mid y \in \partial_C v(x)\} = 0$. Then we have seen in Step 2 that for some index k we have $x_k = 0$ and $(B_j x)_k > 0$. Choose $1 > \varepsilon > 0$ small enough so that

$$n \, d_{\max}\varepsilon < (B_j x)_k \, . \tag{32}$$

Then continuing from (30) we obtain

$$\langle p, f_j(x) \rangle \leq v(x) \sum_{x_i < \varepsilon} \psi_\varepsilon'(x_i) \left(-D_{j,ii}x_i + (1 - x_i)(B_j x)_i\right) \tag{33}$$

$$\leq v(x) \left(\psi_\varepsilon'(0)(B_j x)_k + \sum_{x_i < \varepsilon, i \neq k} \psi_\varepsilon'(x_i) \left(-D_{j,ii}x_i + (1 - x_i)(B_j x)_i\right)\right)$$

and using that $-\psi_\varepsilon'(0) = \eta(\varepsilon)$

$$\leq -\ell \, \eta(\varepsilon)\big((B_j x)_k - n \, d_{\max} \, \varepsilon\big) < 0 \, . \tag{34}$$

In both cases we have shown that we can satisfy the decrease condition (27) for all $j = 1, \ldots, m$, all $x \in v^{-1}([\ell, L])$ and all $p \in \partial_C V_\varepsilon(x)$ by choosing $1 > \varepsilon > 0$ small enough, but dependent on j, x. In particular, note that the conditions (29) and (32) show that this conclusion holds for all $\varepsilon > 0$ sufficiently small. Also, as the set-valued map $x \mapsto \partial_C V_\varepsilon(x)$ is upper semicontinuous with compact values, it follows that (27) holds on an open neighbourhood of a point x. As there are finitely many constituent systems and the set $v^{-1}([\ell, L])$ is compact, we can use continuity and an open cover/compactness argument to conclude that there is an $\varepsilon > 0$ such that the decrease condition holds uniformly for V_ε for all $x \in v^{-1}([\ell, L])$.

Step 5: In oder to show uniform attractivity, note first that by construction $v(x) \leq V_\varepsilon(x) \leq (1 + n)v(x)$. Fix $\delta > 0$. Then for any $x^0 \in \mathbb{R}_+^n$, we may choose $L > 0$ such that $v(x) \leq L$, and $\varepsilon = \varepsilon(L, \delta)$ such that V_ε is a strict Lyapunov function on $\{x \mid \delta \leq v(x) \leq L\}$. Let $c > 0$ be the uniform constant of decay on this set, i.e. a uniform bound $\langle p, f_j(x) \rangle \leq -c$ for all $x \in v^{-1}([\delta, L])$ and $p \in \partial_C V_\varepsilon(x)$. Then for any switching signal $\sigma \in \overline{\mathcal{S}}$ we have $V_\varepsilon(\varphi(t, x^0, \sigma)) \leq V_\varepsilon(x^0) - ct$ as long as $v(\varphi(t, x^0, \sigma)) \geq \delta$. It follows that uniformly $v(\varphi(t, x^0, \sigma)) \leq (n + 1)\delta$ for all $t \geq (V_\varepsilon(x^0) - \delta)/c$ and all σ. As $\delta > 0$ and x^0 are arbitrary this shows uniform attractivity. This completes the proof. $\square$

Remark 5.2 (Epidemiological Interpretation) *Understanding the stability properties of disease-free and endemic equilibria is a fundamental issue in mathematical epidemiology. Much of the literature on this topic investigates the existence of threshold parameters which can be used to identify potential epidemic outbreaks. The previous result suggests that the joint Lyapunov exponent may be used as a threshold parameter for time-varying epidemiological systems described by switched SIS models. Recall that the system matrices B_j and D_j are determined by the contact rates between different patches or subgroups in the population as well as by the birth and death rates and the rates at which infectives are cured. Theorem 5.1 applies to situations where these parameters are allowed to vary in time. Subject to the irreducibility assumption, it establishes that the disease-free equilibrium is globally asymptotically stable for every measurable switching signal provided the linearisation is stable. Thus writing ρ for the JLE, we have a threshold-type condition with the critical value being $\rho = 0$.*
An advantage of this result is that no precise knowledge of the temporal pattern underlying the parameter variation is required. In this sense, it allows us to conclude that a disease will eventually die out irrespective of how the key epidemiological parameters vary in time. Results of this nature are potentially useful for public health authorities as it may not in general be possible to know precisely when and how the contact patterns between different subgroups of a population will change. Of course, given that the conclusions are so strong and that the result essentially covers a "worst-case", it is possible that the conditions provided by Theorem 5.1 are practically conservative; however, given that we are dealing with questions of public health, this may not be so serious an objection.

6   Persistence and Periodic Orbits

In the previous section, we considered the asymptotic stability of the Disease Free Equilibrium (DFE) for a switched SIS model, all of whose constituent systems possess a globally asymptotically stable DFE. We now describe conditions in which all constituent systems have a GAS DFE but for which there exist switching laws that give rise to endemic behaviour. Throughout the section, $D_1, \ldots, D_m$ are diagonal matrices with positive entries along the main diagonal and $B_1, \ldots, B_m$ are *irreducible* nonnegative matrices. We assume throughout that for $1 \le j \le m$, $\mu(-D_j + B_j) < 0$ so that the DFE is globally asymptotically stable for each constituent system of (19). We first show that if there is a matrix R in the convex hull conv $\{-D_1 + B_1, \ldots, -D_m + B_m\}$ with $\mu(R) > 0$, then there exists a switching signal $\sigma \in \mathcal{S}$ for which the associated system (19) is strongly persistent. As Σ_n is forward invariant under (19) for all switching signals, in this section we take Σ_n to be the state space.

We first recall a result concerning averaging for time-varying differential equations, which will prove very useful for the analysis in this section. Consider the time-varying differential equation

$$\dot{x}(t) = f(x, t), \quad x(0) = x^0, \tag{35}$$

where $f : \Sigma_n \times \mathbb{R}_+ \to \mathbb{R}^n$ is K-Lipschitz in x and measurable in t. Furthermore, we assume that f is bounded by r and periodic with period T, so that $f(x, t + T) = f(x, t)$ for all $t \in \mathbb{R}_+$, $x \in \Sigma_n$. Define the averaged system

$$\dot{x} = f_0(x), \quad x(0) = x^0, \quad \text{where } f_0(x) = \frac{1}{T} \int_0^T f(x, s) ds. \tag{36}$$

Combining Theorem 4.1 of [2] with Remark 7.1 of the same paper establishes the following fact.

Theorem 6.1 *Let $x(t, x^0)$, $y(t, x^0)$ denote the solutions of the systems (35), (36) respectively. Then for $t \in [0, 1]$,*

$$\|x(t, x^0) - y(t, x^0)\|_\infty < T(re^{2K}(2 + K)).$$

In applying Theorem 6.1 to the switched system (19), we shall take $f(x, t)$ to be of the form

$$f(x, t) = f_{\sigma(t)}(x)$$

for some $\sigma \in \mathcal{S}$. As each f_j in the definition of (19) is C^1 on $\mathbb{R}^n$, it follows that for any σ, $f_{\sigma(t)}(x)$ is K-Lipschitz in x and bounded for x in the state space Σ_n.

Proposition 6.1 *Consider the switched SIS model (19). Let $\mu(-D_j + B_j) < 0$ for $1 \leq j \leq m$ and assume that there exists some $R \in \mathrm{conv}\left\{-D_1 + B_1, \ldots, -D_m + B_m\right\}$ with $\mu(R) > 0$. Then there exists $\sigma \in \mathcal{S}$ such that for all $x^0 > 0$, $1 \leq i \leq n$*

$$\liminf_{t \to \infty} x_i(t, x^0, \sigma) > 0.$$

Proof By assumption, R is in $\mathrm{conv}\left\{-D_1 + B_1, \ldots, -D_m + B_m\right\}$ and hence can be written as

$$R = \kappa_1(-D_1 + B_1) + \ldots + \kappa_m(-D_m + B_m), \tag{37}$$

where $\kappa_j \geq 0$ and $\sum_{j=1}^m \kappa_j = 1$. Define $\hat{D} = \kappa_1 D_1 + \cdots + \kappa_m D_m$, $\hat{B} = \kappa_1 B_1 + \cdots + \kappa_m B_m$ and consider the autonomous SIS model given by

$$\dot{x}(t) = (-\hat{D} + \hat{B})x - \mathrm{diag}(x)\hat{B}x. \tag{38}$$

For any $T > 0$, we can define a periodic switching signal $\sigma \in \mathcal{S}$ as follows.

$$\begin{aligned}
\sigma(t) &= 1 & \text{for } 0 \leq t < \kappa_1 T \tag{39} \\
\sigma(t) &= j & \text{for } \sum_{k=1}^{j-1} \kappa_k T \leq t < \sum_{k=1}^{j} \kappa_k T, \;\; 2 \leq j \leq m.
\end{aligned}$$

Finally, $\sigma(t + T) = \sigma(t)$ for all $t \geq 0$.
For ease of notation, we shall write

$$f_{\sigma(t)}(x) = (-D_{\sigma(t)} + B_{\sigma(t)})x - \mathrm{diag}(x)B_{\sigma(t)}x, \tag{40}$$

and similarly we write

$$\hat{f}(x) = (-\hat{D} + \hat{B})x - \mathrm{diag}(x)\hat{B}x. \tag{41}$$

From Theorem 3.2, there is an endemic equilibrium $\hat{x}$ for

$$\dot{x}(t) = \hat{f}(x(t)) \tag{42}$$

which is asymptotically stable with region of attraction $\Sigma_n \setminus \{0\}$. Let $\phi(\cdot, x^0)$ denote the solution of (42) with initial condition x^0.

As $\mu(-\hat{D} + \hat{B}) > 0$, there exists some $v \gg 0$ with $(-\hat{D} + \hat{B})v \gg 0$. As the second term in (41) is quadratic in x, by suitably scaling v (by a constant less than 1), we can ensure that $\hat{f}(v) \gg 0$. It follows from Lemma 2.2 that $\phi(t, v)$ is increasing. We show that this implies that the solution $x(\cdot, v, \sigma)$ satisfies $\liminf_{t \to \infty} x_i(t, v, \sigma) > 0$ for $1 \leq i \leq n$. Note first that for all x

$$\hat{f}(x) = \frac{1}{T} \int_0^T f_{\sigma(s)}(x)ds$$

It follows from Theorem 6.1 that for any $\varepsilon > 0$, we can choose T to ensure that

$$\|x(s, x^0, \sigma) - \phi(s, x^0)\|_\infty < \varepsilon$$

for all $s \in [0, 1]$, $x^0 \in \Sigma_n$. In particular, as $\phi(s, v)$ is increasing, we can guarantee by appropriate choice of ε that $x(s, v, \sigma) \gg 0$ for all $s \in [0, 1]$ and that $x(1, v, \sigma) \gg v$. We can without loss of generality assume that $T = \frac{1}{N_0}$ for some integer N_0. Let $\delta = \min_i \inf\{x_i(s, v, \sigma) : 0 \leq s \leq 1\}$. Then $\delta > 0$.
By our choice of periodic σ (with period $T = 1/N_0$), $x(1 + t, v, \sigma) = x(t, x(1, v, \sigma), \sigma)$ for $0 \leq t \leq 1$. As $f_{\sigma(t)}(\cdot)$ is cooperative for any fixed t, it follows that

$$x(1 + t, v, \sigma) \gg x(t, v, \sigma)$$

for $0 \leq t \leq 1$ and hence $x_i(1 + t, v, \sigma) \geq \delta$ for $0 \leq t \leq 1$, $1 \leq i \leq n$. Iterating, we see that

$$x_i(t, v, \sigma) \geq \delta$$

for all $t \geq 0$, $1 \leq i \leq n$. As $f_{\sigma(t)}$ is cooperative for any t, it immediately follows that for all $x^0 \geq v$ we have $x(t, x^0, \sigma) \geq \delta$ for all $t \geq 0$. Suppose we are given some x^0 with $0 \ll x^0 \ll v$. We can choose some $\lambda < 1$ such that $\lambda v \ll x^0$. From the form of each f_j, it is easy to see that for all $t \geq 0$, $\lambda < 1$ and $x \geq 0$,

$$f_\sigma(t)(\lambda x) \geq \lambda f_{\sigma(t)}(x).$$

Now define $z(t) = \lambda x(t, v, \sigma)$. Clearly

$$\dot{z}(t) = \lambda f_{\sigma(t)}(x(t, v, \sigma)) \leq f_{\sigma(t)}(z(t)).$$

It now follows from results on differential inequalities (see for instance Theorem A.19 of [36]) that $z(t) \leq x(t, \lambda v, \sigma)$ for $t \geq 0$. Hence,

$$x(t, \lambda v, \sigma) \geq \lambda x(t, v, \sigma)$$

for $t \geq 0$. This immediately implies that $x_i(t, x^0, \sigma) \geq \lambda \delta$ for all $t \geq 0$, $1 \leq i \leq n$.
Thus far, we have shown that for all initial conditions $x^0 \gg 0$,

$$\liminf_{t \to \infty} x_i(t, x^0, \sigma) > 0$$

for $1 \leq i \leq n$. To finish the proof, note that for any $x^0 > 0$, as each B_i is irreducible, it follows that $x(t, x^0, \sigma) \gg 0$ for all $t > 0$ (see Theorem 4.1.1 of [35]). It is now simple to adapt the above argument to show that in this case also $\liminf_{t \to \infty} x_i(t, x^0, \sigma) > 0$ for $1 \leq i \leq n$. This completes the proof. □

Remark 6.1 (Epidemiological Interpretation) *Proposition 6.1 illustrates how switched SIS models can exhibit more complicated behaviour than autonomous models. In particular, it shows that even when each constituent system has a globally asymptotically stable disease free equilibrium, it is possible for the disease to persist in each population subgroup. When the conditions of the proposition are satisfied, there exists a periodic switching rule σ such that $\sup_{T>0}(\inf_{t \geq T} x_i(t)) > 0$. Epidemiologically, this has the following interpretation. For this switching rule, not only is the disease free equilibrium not asymptotically stable; there is some time T_0 such that a positive fraction of the population of every subgroup is infected at all times after T_0. This reflects classical endemic behaviour where the disease becomes a "fact of life" in the population.*

The next result shows that under the same assumptions as in Proposition 6.1, there exists $\sigma \in \mathcal{S}$ for which (19) admits a periodic orbit in $\mathrm{int}\,\mathbb{R}_+^n$. To show this result, we use some facts from the degree theory of continuous mappings. For background on this topic, see [15, 28]. In particular, the facts we use are drawn from Theorem 2.1.2 and Proposition 2.1.3 of [15]. For convenience, we now recall the main points required in our analysis.
Let Ω be an open region in $\mathbb{R}^n$ and let F be a continuous function from $\overline{\Omega}$ into $\mathbb{R}^n$. If $F(x) \neq 0$ for all $x \in \mathrm{bd}(\Omega)$ and $F(\hat{x}) = 0$ for some $\hat{x} \in \Omega$, then the degree $\deg(F, \Omega, 0) \neq 0$. Conversely, if $F(x) \neq 0$ for all $x \in \mathrm{bd}(\Omega)$ and $\deg(F, \Omega, 0) \neq 0$ then there is some $\hat{x}$ in Ω with $F(\hat{x}) = 0$. Furthermore, if G is another continuous mapping from $\overline{\Omega}$ into $\mathbb{R}^n$ and

$$\max_{x \in \overline{\Omega}} \|F(x) - G(x)\|_\infty < \inf_{x \in \mathrm{bd}(\Omega)} \|F(x)\|_\infty$$

then $\deg(F, \Omega, 0) = \deg(G, \Omega, 0)$.

Theorem 6.2 *Consider the switched SIS model (19). Let $\mu(-D_j + B_j) < 0$ for $1 \leq j \leq m$ and assume that there exists some $R \in \mathrm{conv}\{-D_1 + B_1, \ldots, -D_m + B_m\}$ with $\mu(R) > 0$. Then there exists $\sigma \in \mathcal{S}$ such that (19) admits a periodic orbit*

$$x(t + 1, x^0, \sigma) = x(t, x^0, \sigma) \quad \forall t \geq 0.$$

Proof As in the proof of Proposition 6.1, let R be given by (37). Also for $T > 0$, let σ in $\mathcal{S}$ be the switching signal defined by (39). Let $f_{\sigma(t)}$ and $\hat{f}$ be given by (40) and (41) respectively. Finally, we use $\phi(\cdot, x^0)$ and $x(\cdot, x^0, \sigma)$ to denote the solutions of (42) and (40) respectively.

Consider the two continuous mappings defined for $x^0 \in \Sigma_n$ by

$$S_1(x^0) := \int_0^1 \hat{f}(\phi(s, x^0))ds, \quad S_2(x^0) := \int_0^1 f_{\sigma(s)}(x(s, x^0, \sigma))ds.$$

It follows from Theorem 3.2 that there exists $\hat{x}$ in int (Σ_n) with $\hat{f}(\hat{x}) = 0$. Moreover, $\hat{x}$ is an asymptotically stable equilibrium of (42) with region of attraction $\Sigma_n \setminus \{0\}$. This implies that $S_1(\hat{x}) = 0$ and, moreover that $S_1(x) \neq 0$ for $x \in \text{int}(\Sigma_n) \setminus \{\hat{x}\}$. In particular, we can choose some bounded open neighbourhood $\Omega \subset \text{int}\,\Sigma_n$ of $\hat{x}$ such that $S_1(z) \neq 0$ for all $z \in \text{bd}(\Omega)$. Let

$$\varepsilon = \min\{\|S_1(z)\|_\infty : z \in \text{bd}(\Omega)\}.$$

As $S_1(z) - S_2(z) = \phi(1, z) - x(1, z, \sigma)$, it follows from Theorem 6.1 that we can choose $T = (1/N_0)$ for some positive integer N_0 such that

$$\max_{z \in \overline{\Omega}} \|S_1(z) - S_2(z)\|_\infty < \varepsilon.$$

It now follows from the remarks on degree theory given before this theorem that $\deg(S_1, \Omega, 0) = \deg(S_2, \Omega, 0)$ and hence that there exists some $x^1 \in \Omega$ with $S_2(x^1) = 0$. It follows immediately that

$$x(1, x^1, \sigma) = x(0, x^1, \sigma)$$

and as σ is T-periodic with $T = 1/N_0$, we have that $x(t+1, x^1, \sigma) = x(t, x^1, \sigma)$ for all $t \geq 0$. As $\mu(-D_j + B_j) < 0$ for $1 \leq j \leq m$, (19) possesses no equilibrium in int (Σ_n). Hence, it follows that x^1 gives rise to the claimed periodic orbit. $\qquad\square$

Remark 6.2 *Similarly to Proposition 6.1, Theorem 6.2 illustrates a form of endemic behaviour that can emerge in the switched model under certain conditions. When a matrix R with $\mu(R) > 0$ exists there is some switching scheme (pattern of time-variation in the system parameters) for which a periodic solution exists. Epidemiologically, such a solution amounts to a repetitive seasonal pattern in the number of infectives in the population; the numbers may decrease for some time but eventually return to their initial values and the pattern then simply repeats itself for all time. As we noted in Remark 6.2, this reflects the disease persisting and becoming a feature of life for the population.*

7 Markovian Switching

We now examine the stability of the switched SIS model (19) in which the switching parameter $\sigma : \mathbb{R}_+ \to \{1, ..., m\}$ is given as the realization of a right-continuous random process, in particular, a piecewise deterministic process. One way of specifying such a process is by prescribing the rates $\pi_{ij}(t)$ describing the evolution of the probabilities to switch from i to j, see [31] for details.

Let $(\Omega, \mathcal{B}, \mathbf{P})$ be a probability space, we suppose that the switching signals σ are realizations of a Markov process described by the following transition probabilities

$$\mathbf{P}\{\sigma(t + \Delta) = j | \sigma(t) = i\} = \begin{cases} \pi_{ij}(t)\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \pi_{ii}(t)\Delta + o(\Delta) & \text{else,} \end{cases} \tag{43}$$

where $\Delta > 0$ and $\lim\limits_{\Delta \to 0} o(\Delta)/\Delta = 0$. The matrix $\Pi(t) = [\pi_{ij}(t)]$ is the matrix of the transition probability rates and its components satisfy $\pi_{ij} \geq 0$ for $i \neq j$, and $\pi_{ii} = -\sum\limits_{j \neq i}^m \pi_{ij}$.

In order to be precise about the stability of the random system (19), we need to define some stability concepts. We denote by $\mathbf{E}(x(t))$ the expectation of $x(t)$.

Definition 7.1 *The switched SIS model (19) under a random switching σ is said to be*

(i) *mean stable if* $\lim\limits_{t \to +\infty} \mathbf{E}(x(t)) = 0, \;\; \forall x(0) \in \mathbb{R}_+^n.$

(ii) *mean-square stable if* $\lim\limits_{t \to +\infty} \mathbf{E}(x(t)^T x(t)) = 0, \;\; \forall x(0) \in \mathbb{R}_+^n.$

(iii) *L_1-stable if* $\|x\|_1 := \int_0^{+\infty} \mathbf{E}(\sum\limits_{i=1}^n x_i(t))dt < +\infty, \;\; \forall x(0) \in \mathbb{R}_+^n.$

(iv) *L_2-stable if* $\|x\|_2 := \int_0^{+\infty} \mathbf{E}(x^T(t)x(t))dt < +\infty, \;\; \forall x(0) \in \mathbb{R}_+^n.$

Now, define $\delta_i(\cdot)$ as the random processes given by the indicator function of the Markovian switching process σ, i.e.

$$\delta_i(t) = 1 \;\; \text{if } \sigma(t) = i, \qquad \delta_i(t) = 0 \;\; \text{if } \sigma(t) \neq i \,.$$

The following lemma provides a connection between the switched SIS model (19) and a special deterministic system.

Lemma 7.1 *The state $x(t)$ of the stochastic switched system (19) satisfies the differential equation*

$$\dot{\xi}_i(t) = (-D_i + B_i)\xi_i - \mathbf{E}[\delta_i(t)\mathrm{diag}\,(x(t))B_i)x(t)] + \sum_{j=1}^m \pi_{ji}(t)\xi_j(t), \;\; i = 1, \ldots, m, \qquad (44)$$

where $\xi_i(t) = \mathbf{E}(\delta_i(t)x(t))$.

Proof Follows using the generalized Itô formula for Markov jumps, see e.g. [7]. □

Note that since $\sum\limits_{i=1}^m \delta_i(t) = 1$, the expectation of any trajectory $x(t)$ of the switched SIS model (19) is given by

$$\mathbf{E}(x(t)) = \sum_{i=1}^m \xi_i(t). \qquad (45)$$

Remark 7.1 *We can deduce from Lemma 7.1 that the switched SIS model (19) is positive if and only if the system (44) is positive. This can be easily shown from the fact that $B_i \geq 0$, for $i = 1 \ldots, m$ and $\pi_{ij} \geq 0$ for $i \neq j$.*

In the sequel, we shall investigate the stability of the deterministic system (44) in the state variable $\xi(t) = [\xi_1(t) \ldots \xi_m(t)]^T$. This will allow us to derive stability conditions for the associated switched SIS model (19).

Remark 7.2 *An immediate consequence of the relation (45) is that the switched SIS model (19) is mean stable if and only if the deterministic system (44) is globally asymptotically stable. Indeed, since $x(t) \geq 0$, we also have $\xi_i(t) = \mathbf{E}(\delta_i(t)x(t)) \geq 0$. Hence, $\mathbf{E}(x(t)) = \sum\limits_{i=1}^m \xi_i(t)$ goes to zero if and only if all $\xi_i(t)$ go to zero. Also, it can easily be seen that the switched SIS model (19) is L_1-stable if and only if the deterministic system (44) is L_1-stable.*

For ease of notation, we shall use the following matrices

$$\mathcal{A}_\Pi := \begin{bmatrix} -D_1 + B_1 & & 0 \\ & \ddots & \\ 0 & & -D_m + B_m \end{bmatrix} + \Pi \otimes I \qquad (46)$$

$$\mathcal{B}(x(t)) := \begin{bmatrix} \mathbf{E}[\delta_1(t)\mathrm{diag}\,(x(t))B_1 x(t)] \\ \vdots \\ \mathbf{E}[\delta_m(t)\mathrm{diag}\,(x(t))B_m x(t)] \end{bmatrix}, \text{ where } \Pi \otimes I = \begin{bmatrix} \pi_{11}I & \ldots & \pi_{1m}I \\ \vdots & \ddots & \vdots \\ \pi_{m1}I & \ldots & \pi_{mm}I \end{bmatrix}$$

is the Kronecker product of Π and the identity. Thus, the system (44) can be expressed in compact form as

$$\dot{\xi}(t) = \mathcal{A}_\Pi \xi(t) - \mathcal{B}(x(t)), \text{ where } \xi(t) = [\xi_1(t)\ldots\xi_m(t)]^T. \tag{47}$$

We now derive L_1 stability conditions for the switched SIS model (19).

Theorem 7.1 *Assume that $\Pi(\cdot)$ is bounded, that is, there exists a constant Metzler matrix $\bar{\Pi}$ such that $\Pi(t) \le \bar{\Pi}$ for all $t \ge 0$. Then, the switched SIS model (19) is L_1-stable if the matrix $\mathcal{A}_{\bar{\Pi}}$ is Hurwitz.*

Proof Taking into account Remark 7.2, it suffices to prove that the system (47) is L_1-stable. The proof uses the well-known condition that a Metzler matrix $\mathcal{A}_{\bar{\Pi}}$ is Hurwitz if and only if $\mathcal{A}_{\bar{\Pi}}^{-1} \le 0$. Now, since $\dot{\xi}(t) = \mathcal{A}_\Pi \xi(t) - \mathcal{B}(x(t))$, we obtain

$$\xi(t) - \xi(0) = \int_0^t \mathcal{A}_\Pi \xi(s) - \mathcal{B}(x(s))ds.$$

As $\mathcal{B}(x(t)) \ge 0$ and $\mathcal{A}_\Pi \le \mathcal{A}_{\bar{\Pi}}$ we have $\xi(t) - \xi(0) \le \int_0^t \mathcal{A}_{\bar{\Pi}}\,\xi(s)ds$. The assumption that $\mathcal{A}_{\bar{\Pi}}^{-1} \le 0$ leads to

$$-\mathcal{A}_{\bar{\Pi}}^{-1}\xi(t) + \mathcal{A}_{\bar{\Pi}}^{-1}\xi(0) \le -\int_0^t \xi(s)ds.$$

Since $-\mathcal{A}_{\bar{\Pi}}^{-1}\xi(t) \ge 0$ we obtain

$$\int_0^t \xi(s)ds \le -\mathcal{A}_{\bar{\Pi}}^{-1}\xi(0), \ \forall t > 0.$$

The above inequality shows that the integral $\int_0^t \xi(s)ds$ is bounded and as it is nondecreasing, so that the integral $\int_0^{+\infty} \xi(t)dt$ exists and the proof is complete. $\qquad\square$

Since L_1 stability implies L_2 stability, then we can deduce from Theorem 7.1 that if $\mu(\mathcal{A}_{\bar{\Pi}}) < 0$, then the switched SIS model (19) is L_2-stable. Indeed, it is also mean stable and mean-square stable.

Remark 7.3 *We emphasize that we cannot apply Theorem 3.1 directly to system (44) because it involves a different structure. Also, Theorem 7.1 establishes an L_1 stability condition that also holds for the deterministic system (11) (when $\Pi = 0$).*

8 Stabilization of the Disease Free Equilibrium by Switching

In this section, we assume that each constituent system of (19) has an endemic equilibrium, which is asymptotically stable with region of attraction $\Sigma_n \setminus \{0\}$. We shall show how results from the literature on stabilization of switched linear systems can be applied to define switching laws that asymptotically stabilise the Disease Free Equilibrium (DFE).

Theorem 8.1 *Consider the switched SIS model (19). Assume that $\mu(-D_j + B_j) > 0$ for $1 \le j \le m$. Suppose that there exists some $R \in \mathrm{conv}\,\{-D_1 + B_1, \ldots, -D_m + B_m\}$ for which $\mu(R) < 0$. Then, there exists some $T > 0$ and a periodic switching law $\sigma \in \mathcal{S}$ with period T such that the DFE of (19) is GAS.*

Proof Let $R = \sum_{j=1}^{m} \kappa_i(-D_j + B_j)$ satisfy $\mu(R) < 0$ where $\kappa_j \geq 0$ for all j and $\sum_{j=1}^{m} \kappa_j = 1$. For $T > 0$, consider the periodic switching signal $\sigma \in \mathcal{S}$ given by (39). We claim that it is possible to choose T such that the DFE of (19) is GAS. To show this, we consider the associated switched linear system (20); we denote the solution of (20) by $\tilde{\psi}(t, x^0)$. It is well known that the existence of a Hurwitz convex combination of the system matrices $-D_j + B_j$ is sufficient for the existence of a stabilizing switching law (usually state-dependent) for such systems. (See for example [26, 34]). We include the proof here in the interests of completeness.

The averaged system (36) corresponding to (20) is given by

$$\dot{x} = Rx. \tag{48}$$

As $\mu(R) < 0$, we can choose some $v \gg 0$ with $v_i > 1$ for all i and $Rv \ll 0$. The solution $z(t, v)$ of (48) is decreasing by Lemma 2.2 and using Theorem 6.1 we can ensure that $\tilde{\psi}(T, v) \ll v$ by choosing T sufficiently small. Thus, for such a T, we can find some α with $0 < \alpha < 1$ such that

$$\tilde{\psi}(T, v) \leq \alpha v. \tag{49}$$

From the construction of σ and the linearity of (20), it follows that for $0 \leq t \leq T$,

$$\tilde{\psi}(T + t, v) = \tilde{\psi}(t, \tilde{\psi}(T, v)) \leq \alpha \tilde{\psi}(t, v).$$

Iterating the previous identity, we see readily that $\tilde{\psi}(t, v) \to 0$ as $t \to \infty$. As $v_i \geq 1$ for all i, it immediately follows from the monotonicity of the system (20) that $\tilde{\psi}(t, x^0) \to 0$ as $t \to \infty$ for any $x^0 \geq 0$ with $\|x^0\|_\infty \leq 1$. The result now follows from a simple application of Lemma 5.2. □

Remark 8.1 (Epidemiological Interpretation) *The epidemiological significance of the previous result relates to the development of control strategies for epidemic outbreaks based on switching policies. In essence, if the conditions of Theorem 8.1 are satisfied, then it is possible to asymptotically eradicate the disease by choosing a suitable switching signal. This means that even when each constituent model has an endemic equilibrium, we can drive the system to the disease free state by varying the contact patterns between population subgroups in a suitable manner, for instance.*

The stabilizing switching strategy described in the above result is a time-dependent switching signal in the set $\mathcal{S}$. It is also possible to prove the existence of state-dependent switching strategies following arguments similar to those in [26, Chapter 3].

9 Conclusion and Future Work

Building on the work of [17], we have described several results concerning compartmental SIS models with parameters subject to switching. In particular, Theorem 5.1 generalizes a main result of [17], and shows that stability combined with an appropriate notion of irreducibility for the linearisation is sufficient for asymptotic stability of the disease free equilibrium of the switched nonlinear system describing the epidemic dynamics. As highlighted in the text, this generalises the result in [17] even for the time-invariant case. Our work also indicates how endemic behaviour can emerge for systems constructed by switching between models, each of which has a globally asymptotically stable disease free equilibrium. Specifically, we have described conditions for the disease to be persistent and for the existence of periodic endemic orbits. Results for systems subject to Markovian switching have also been presented.

There are several natural questions arising from the work described here. We briefly highlight some of these. While we have provided conditions for the existence of periodic orbits in Theorem 6.2, it is natural to ask what stability properties this orbit possesses. Is it guaranteed to be locally or globally attractive? If not, then what additional requirements will render it so? Another question is whether or not the relaxed sufficient condition for global asymptotic stability of the DFE given in Corollary 5.1 is also necessary. Finally, it would be very interesting to investigate whether the work of the current paper can be extended to switched SIR epidemic models.

Acknowledgement

References

[1] M. Ait Rami, V. S. Bokharaie, O. Mason, and F. Wirth, *Extremal norms for positive linear inclusions*, in Proc. 20th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS 2012, Melbourne, Australia, 2012.

[2] Z. Artstein, *Averaging of time-varying differential equations revisited*, Journal of Differential Equations, 243 (2007), pp. 146–167.

[3] N. Bacaër and M. Khaladi, *On the basic reproduction number in a random environment*, J. Math. Biol., 67 (2012), pp. 1729–1739.

[4] N. T. J. Bailey, *The Mathematical Theory of Epidemics*, Griffin, London, 1957.

[5] F. Bauer, J. Stoer, and C. Witzgall, *Absolute and monotonic norms*, Numerische Mathematik, 3 (1961), pp. 257–264.

[6] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, SIAM, Philadelphia, PA, USA, 1987.

[7] T. Björk, *Finite dimensional optimal filters for a class of Ito-processes with jumping parameters*, Stochastics, 4 (1980), pp. 167–183.

[8] V. S. Bokharaie, O. Mason, and F. Wirth, *Spread of epidemics in time-dependent networks*, Proc. 19th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS 2010, (2010).

[9] I. Chueshov, *Monotone Random Systems*, Springer–Verlag, Berlin, 2002.

[10] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1998.

[11] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[12] K. L. Cooke and J. A. Yorke, *Some equations modelling growth processes and gonorrhea epidemics*, Mathematical Biosciences, 16 (1973), pp. 75–101.

[13] P. De Leenheer, *Stabiliteit, regeling en stabilisatie van positieve systemen*, PhD thesis, University of Gent, 2000.

[14] V. Demyanov and A. M. Rubinov, *Constructive Nonsmooth Analysis*, Verlag Peter Lang, Frankfurt Berlin, 1995.

[15] F. Facchinei and J. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. 1, Springer-Verlag, Berlin, 2003.

[16] L. Fainshil, M. Margaliot, and P. Chigansky, *On the stability of positive linear switched systems under arbitrary switching laws*, IEEE Transactions on Automatic Control, 54 (2009), pp. 897–899.

[17] A. Fall, A. Iggidr, G. Sallet, and J. Tewa, *Epidemiological models and Lyapunov functions*, Math. Model. Nat. Phenom., 2 (2007), pp. 62–68.

[18] A. Gray, D. Greenhalgh, X. Mao, and J. Pan, *The SIS epidemic model with Markovian switching*, Journal of Mathematical Analysis and Applications, 394 (2012), pp. 496–516.

[19] L. Gurvits, R. Shorten, and O. Mason, *On the stability of switched positive linear systems*, IEEE Transactions on Automatic Control, 52 (2007), pp. 1099–1103.

[20] H. W. Hethcote and J. A. York, *Gonorrhea Transmission and Control*, vol. 56 of Lectures Notes in Biomathematics, Springer-Verlag, New York, NY, 1984.

[21] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, NY, USA, 1985.

[22] I. Kats and N. Krasovskii, *On the stability of systems with random parameters*, J. Appl. Math. Mec., 24 (1960), pp. 1225–1246.

[23] V. S. Kozyakin, *Algebraic unsolvability of problem of absolute stability of desynchronized systems*, Autom. Rem. Control, 51 (1990), pp. 754–759.

[24] N. Krasovskii and E. Lidskii, *Analytical design of controllers in systems with random attributes*, Automation and Remote Control, 22 (1961), pp. 1021–1025.

[25] A. Lajmanovic and J. Yorke, *A deterministic model for gonorrhea in nonhomogeneous population*, Math. Biosci., 28 (1976), pp. 221–236.

[26] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, MA, USA, 2003.

[27] X. Liu and X.-Q. Zhao, *A periodic epidemic model with age-structure in a patchy environment*, SIAM Journal of Applied Mathematics, 71 (2011), pp. 1896–1917.

[28] N. Lloyd, *Degree Theory*, Cambridge University Press, 1978.

[29] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, New York, NY, 1990.

[30] M. E. J. Newman, *The structure and function of complex networks*, SIAM review, (2003), pp. 167–256.

[31] J. Norris, *Markov Chains*, Cambridge University Press, Cambridge, 2008.

[32] S. Pröll, *Stability of switched epidemiological models*, Master's thesis, Institute for Mathematics, University of Würzburg, Würzburg, Germany, 2013.

[33] C. Rebelo, A. Margheri, and N. Bacaër, *Persistence in seasonally forced epidemiological models*, Journal of Mathematical Biology, 64 (2012), pp. 933–949.

[34] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, *Stability theory for switched and hybrid systems*, SIAM Review, 49 (2007), pp. 545–592.

[35] H. A. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*, American Mathematical Society, Providence, RI, USA, 1995.

[36] H. L. Smith and H. R. Thieme, *Dynamical Systems and Population Persistence*, American Mathematical Society, Providence, RI, USA, 2011.

[37] P. van den Driessche and J. Watmough, *Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci., (2002), pp. 29–48.

[38] F. Wirth, *The generalized spectral radius and extremal norms*, Lin. Alg. Appl., 342 (2002), pp. 17–40.

[39] World Health Organization (WHO), *The global burden of disease: 2004 update.* http://www.who.int/healthinfo/global_burden_disease/2004_report_update/en/index.html, 2008. Last Retrieved: 05 December 2011.