Abelian and non-Abelian Hopfions in all odd dimensions

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Abstract. We extend the definition of the topological charge pertaining to the CP^1 (*i.e.* O(3)) Skyrme-Fadde'ev Hopfion on $\mathbb{R}^3$, to candidates for topological charges of $\mathbb{C}P^n$ sigma models on $\mathbb{R}^{2n+1}$, for all n. For this, the Abelian composite connections of the $\mathbb{C}P^n$ sigma models are employed. In higher dimensions ($n \geq 1$) it turns out that such charges, described by the non-Abelian composite connections of suitable Grassmannian sigma models, can also be constructed. A concrete discussion of the non-Abelian case for $n = 2$ is presented.

1. Introduction

Hopfions are static "soliton like" solutions to the field equations of nonlinear sigma models. With the usual solitons namely instantons of the Yang-Mills (YM) systems in all even dimensions and of Skyrmions of nonlinear sigma models in all dimensions, monopoles of YM-Higgs (YMH) systems in all dimensions three and higher, and vortices of YMH and gauged sigma models systems in two dimensions Manton & Sutcliffe (2004), have finite energy and are topologically stable. Likewise, Hopfions have finite energy and are topologically stable. To date, the only example are the solutions Faddeev & Niemi (1996) and Battye & Sutcliffe (1998) to the Skyrme-Fadde'ev $O(3)$ sigma model on $\mathbb{R}^3$. For a review see Manton & Sutcliffe (2004), Radu & Volkov (2008).

The salient qualitative difference between the afore mentioned solitons and the Hopfion is this. The topological charge densities of the former are by construction total divergence. For Yang-Mills (YM) and complex sigma models, and YMH models, they are $Chern - Pontryagin$ densities or their decendants, respectively. For the nonlinear sigma models with real sphere-valued fields they are "essentially total divergence". Such densities are not explicitly total diveregence. But when they are subjected to the variational princliple, with the sigma model constraint taken account of by a Lagrange multiplier term, they yield no nontrivial Euler-Lagrange equations. They share this property with total divergence densities. In the case of the Hopfion, the topological charge density is instead the $Chern - Simons$ (CS) density, which is **not** total divergence. It follows that Hopfions can exist only in odd space dimensions, where CS densities are defined.

The Chern-Simons density pertaining to Hopfions, which is defined in terms of the gauge connection and curvature, depends on the scalar sigma model fields. These connections and curvatures are the *composite* quantities constructed from the sigma model fields appropriate

to any given example. The most transparent way of doing this is to employ a "complex sigma model" (*e.g.* complex projective, quaternionic, Grassmannian, *etc.*) enabling the definitions of the *composite* connection and curvature necessary for the definition of the CS density.

Given that the topological charge of the Hopfion is the volume integral of the Chern-Simons density, the lower bound on the energy of the Skyrme-Fadde'ev Hopfion on $\mathbb{R}^3$ was established in Vakulenko & Kapitansky (1979), using a method that employs Sobolev space norms. Note that the analysis of Vakulenko & Kapitansky (1979) is not a proof of existence, and that this Hopfion is constructed numerically Faddeev & Niemi (1996); Battye & Sutcliffe (1998). A proof of existence can be found in Lin & Yang (2004).

To qualify as a topological charge density, this Chern-Simons density must become a total divergence which it is not *a priori*. This can be achieved by subjecting it to suitable symmetries which render it total divergence. This is perhaps the most striking difference between the usual solitons and Hopfions. The topological charges of the former exist subject to no symmetries, while that of the Hopfions exist only for systems constrained by the appropriate symmetry. Our aim in the present talk is strictly restricted to this task, namely to find suitable symmetries which render the Chern-Simons density in the given (odd) dimension, a total divergence.

To this end, our prescription here to achieve this aim is the imposition of multi-azimuthal symmetries. Specifically, it is the imposition of azimuthal symmetry in each of the n, 2-planes in $\mathbb{R}^{2n+1}$. (This is the case with the familiar Skyrme-Fadde'ev Hopfion on $\mathbb{R}^3$ with axial symmetry imposed in the $x-y$ plane.) Multi-azimuthal symmetry in $\mathbb{R}^{2n+1}$ eliminates n azimuthal angles, each in one of the n, 2-dimensional subspaces (planes), resulting in $(n+1)$-dimensional residual subsystems.

Our prescription hinges on positing an adequate Ansatz that results in the residual Chern-Simons density in a $(n+1)$-dimensional space being parametrised by $(n+1)$ independant functions of the $(n+1)$ residual 'coordinates'. The equality of the number independant functions in the residial system, and the number of the residual 'coordinates' can result in the residual CS being total divergence.

Finally, it should be noted that this aim of rendering the residual Chern-Simons density a total divergence can be achieved only when the sigma model constraint is satisfied. This is automatic if the Ansatz in question is parametrised in terms of functions that satisfy this constraint. Often however it is convenient to use more relaxed parametrisations, in which case the corresponding criterion is to show that the residual CS density is "essentially total divergence", subjecting this density to the variational principle taking account of the (residual) constraint *via* a Lagrange multiplier term, results in *trivial* Euler-Lagrange equations.

The above account of topological charges of Hopfions relies entirely on the existence of suitable composite connections and curvatures of the nonlinear sigma model in question. In this sense, both Abelian and non-Abelian systems can be considered. It is this context that we have used the nomenclature of Abelian and non-Abelian Hopfions in the title. The talk consists of two main sections, **2** and **3**, describing our prescriptions for Abelian and non-Abelian Hopfions respectively. Each of these sections is subdivided in a subsection defining the appropriate nonlinear sigma model, followed by the imposition of symmetries yielding the desired toplogical charge densities. Section **4** gives a brief summary.

2. Abelian Hopfions on $\mathbb{R}^{2n+1}$

The description of the arbitrary n case in the Abelian case is much more transparent than the non-Abelian case that follows. We describe the three cases $n = 1, 2, 3$, which point to the generic case readily.

2.1. CP^n models on $\mathbb{R}^{2n+1}$

We start with the generic structure of models that can support Abelian Hopfion on $\mathbb{R}^{2n+1}$. These are the $\mathbb{C}P^n$ sigma models on $\mathbb{R}^{2n+1}$ described by complex $n-$tuplets

$$
Z = \begin{bmatrix} z_1 \\ z_2 \\ .. \\ .. \\ z_{n+1} \end{bmatrix} \equiv z_a \quad ; \quad \bar{Z} = \begin{bmatrix} \bar{z}^1 \\ \bar{z}^2 \\ .. \\ .. \\ \bar{z}^{n+1} \end{bmatrix} \equiv \bar{z}^a, \quad \text{with} \quad a = 1, 2, ..., n+1\,, \tag{1}
$$

subject to the constraint

$$
Z^\dagger Z \equiv \bar{z}^a z_a = 1\,, \tag{2}
$$

taking their values in $\frac{U(n+1)}{U(n) \times U(1)}$, such they are described by $2n$ real parameters that parametrise Z on $\mathbb{R}^{2n+1}$. In (1), $\bar{z}^a$ is the complex conjugate of z_a, transforming with an index that is *contravariant* to the *covariant* index of z_a, and $Z^\dagger$ in (2) is the transpose of $\bar{Z}$. This leads to the definition of the *projection operator*

$$
P = \left(\mathbb{1} - Z\,Z^\dagger \right) \equiv \left(\delta_a{}^b - z_a\,\bar{z}^b \right)\,. \tag{3}
$$

The most interesting feature of these models is that when the field Z is subjected to a *local $U(1)$* gauge transformation $g = e^{i\Lambda(x)}$, the constraint (2) is invariant under

$$
Z \to Z\,g\,. \tag{4}
$$

As a consequence the quantity defined as

$$
B_i = i\,Z^\dagger \partial_i Z \quad , \quad i = 1, 2, ..., 2n+1 \tag{5}
$$

transforms like an *Abelian composite connection* under $g(\Lambda)$,

$$
B_i \to B_i \pm \partial_i \Lambda\,,
$$

which lead to the definitions of the covariant derivative of Z and and the *Abelian curvature* of this connection,

$$
\begin{align}
D_i Z &= \partial_i Z + B_i Z\,, \tag{6} \\
G_{ij} &= \partial_i B_j - \partial_j B_i\,, \tag{7}
\end{align}
$$

with $D_i Z$ transforming covariantly under the action of g, and G_{ij} invariantly.

The Abelian Chern-Simons (CS) density on $\mathbb{R}^{2n+1}$ is then readily defined in terms of the quantities (7) and (6).

$$
\Omega_{\text{CS}} \simeq \varepsilon_{i_1 i_2 ... i_{2n+1}} B_{i_{2n+1}} G_{i_1 i_2} G_{i_3 i_4} \ldots G_{i_{2n-1} i_{2n}}\,. \tag{8}
$$

It is these densities, subject to the appropriate symmetries, that play the role of topological charges stabilising Hopfions. That Ω_{CS} is not a total divergence is clear.

Subjecting the density

$$
\Omega_{\text{CS}} + \lambda\,(1 - Z^\dagger Z)
$$

to arbitrary variations of $Z^\dagger$ (or Z), where the Lagrange multiplier term accounts for the costraint (2), one has

$$
\varepsilon_{i_1 i_2 ... i_{2n+1}} D_{i_{2n+1}} Z\, G_{i_1 i_2} G_{i_3 i_4} \ldots G_{i_{2n-1} i_{2n}} = 0\,. \tag{9}
$$

Eqn. (9) trivialises only under the appropriate symmetries, *i.e.* (8) becomes a density that is "essentially total divergence", as required of a topological charge density.

2.2. Residual Abelian CS density subject to n-fold azimuthal symmetry

We treat the three cases $n = 1, 2, 3$ below by positing an Ansatz which imposes the appropriate symmetries that renders (8) a total divergence.

In the $n = 1$ case, the field (1) is subjected to axial symmetry in the (x_1, x_2) plane of $\mathbb{R}^3$,

$$
Z = \left[\begin{array}{c} a + ib \\ c\, e^{in\varphi} \end{array} \right] \equiv \left[\begin{array}{c} \sin \frac{f}{2}\, e^{i\alpha} \\ \cos \frac{f}{2}\, e^{in\varphi} \end{array} \right]
\tag{10}
$$

where the functions a, b, c, f and α all depend on both $\rho = \sqrt{|x_\alpha|^2}$ and $z \equiv x_3$, $\alpha = 1, 2$.

Substituting the azimuthally symmetric Ansatz (10) in the Abelian Chern–Simons density on $\mathbb{R}^3$

$$
\Omega_{\text{CS}}^{(3)} = \varepsilon_{mij}\, B_m\, G_{ij}\,.
\tag{11}
$$

yields the simple expression

$$
\Omega_{\text{CS}}^{(3)} = -4\,\frac{n}{\rho}\, c \cdot \det \left| \begin{array}{ccc} a & b & c \\ a_{,\rho} & b_{,\rho} & c_{,\rho} \\ a_{,z} & b_{,z} & c_{,z} \end{array} \right|\,.
\tag{12}
$$

The usual notation $a_{,\rho} = \partial_\rho a$, *etc.* is used here. It is easy to verify that this quantity is "essentially total divergence" when varied subject to the constraint $a^2 + b^2 + c^2 = 1$. Further, using the polar parametrisation in (10), it becomes a total divergence, since the constraint is automatically satified in that case.

Requiring the field configurations in question have the asymptotic values

$$
\lim_{r \to \infty} f(r, \theta) = 0\,, \quad \lim_{r \to \infty} \alpha(r, \theta) = m\,\pi\,,
\tag{13}
$$

one finds the topological charge

$$
Q = \frac{8}{3}\, i\, n\, m\, \pi^2\,.
\tag{14}
$$

In the $n = 2$ case, the field Z on $\mathbb{R}^5$ is subjected to the bi-azimuthal symmetry

$$
Z = \left[\begin{array}{c} a + ib \\ c_1\, e^{in_1\varphi} \\ c_2\, e^{in_2\chi} \end{array} \right] \equiv \left[\begin{array}{c} \sin \frac{1}{2} f\, e^{i\alpha} \\ \cos \frac{1}{2} f\, \sin g\, e^{in_1\varphi} \\ \cos \frac{1}{2} f\, \cos g\, e^{in_2\chi} \end{array} \right]
\tag{15}
$$

where $\rho = \sqrt{|x_\alpha|^2}$, $\sigma = \sqrt{|x_A|^2}$ with $\alpha = 1, 2$, $A = 3, 4$ and $z \equiv x_5$. φ and χ are the azimuthal angles in the (x_1, x_2) and (x_3, x_4) planes respectively, (n_1, n_2) being the winding (vortex) numbers of these planes respectively. All functions (a, b, c, d) or $(f, , g, \alpha)$ in (15) depend on (ρ, σ, z).

Substitution of the bi-azimuthally symmetric Ansatz (15) in the Abelian Chern–Simons density on $\mathbb{R}^5$

$$
\Omega_{\text{CS}}^{(5)} = \varepsilon_{mijkl}\, B_m\, G_{ij}\, G_{kl}
\tag{16}
$$

yields the simple expression

$$
\Omega_{\text{CS}}^{(5)} = 32\,\frac{n_1 n_2}{\rho\sigma}\, c_1 c_2 \cdot \det \left| \begin{array}{cccc} a & b & c_1 & c_2 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} \end{array} \right|\,.
\tag{17}
$$

which when varied subject to the constraint $a^2 + b^2 + c_1^2 + c_2^2 = 1$ turns out to be "essentially total divergence". Again, using the polar parametrisation in (15), this becomes a total divergence.

Requiring the field configurations in question have the asymptotic values

$$\lim_{r \to \infty} f = 0 \quad , \quad \lim_{r \to \infty} g = \theta \quad , \quad \lim_{r \to \infty} \alpha = m\,\pi\,, \tag{18}$$

where $0 \le \theta \le \frac{\pi}{2}$ is the polar angle in the (ρ, σ) quarterplane, one finds the topological charge

$$Q_{\text{CS}}^{(5)} = -12\, n_1\, n_2\, m\, \pi^3\,. \tag{19}$$

Finally, in the $n = 3$ case, one subjects the field Z on $\mathbb{R}^7$ is to the tri-azimuthal symmetry

$$Z = \begin{bmatrix} a + ib \\ c_1\, e^{in_1\varphi} \\ c_2\, e^{in_2\chi} \\ c_3\, e^{in_3\xi} \end{bmatrix} \equiv \begin{bmatrix} \sin\frac{1}{2}f\, e^{i\alpha} \\ \cos\frac{1}{2}f\, \sin g\, \cos h\, e^{in_1\varphi} \\ \cos\frac{1}{2}f\, \sin g\, \sin h\, e^{in_2\chi} \\ \cos\frac{1}{2}f\, \cos g\, e^{in_3 l\xi} \end{bmatrix} \tag{20}$$

in terms of the variables $\rho = \sqrt{|x_\alpha|^2}$, $\sigma = \sqrt{|x_A|^2}$, $\tau = \sqrt{|x_a|^2}$ with $\alpha = 1, 2$, $A = 3, 4$, $a = 5, 6$ and $z \equiv x_7$. φ, χ and ξ are the azimutal angles in the (x_1, x_2), (x_3, x_4) and (x_5, x_6) planes respectively, (n_1, n_2, n_3) being the winding (vortex) numbers of each plane respectively. All functions (a, b, c, d, e) or $(f, , g, h, \alpha)$ in (20) depend on (ρ, σ, τ, z).

Substitution of the tri-azimuthally symmetric Ansatz (20) in the Abelian Chern–Simons density on $\mathbb{R}^7$

$$\Omega_{\text{CS}}^{(7)} \;=\; \varepsilon_{pijklmn}\, B_p\, G_{ij}\, G_{kl}\, G_{mn} \tag{21}$$

yields the simple expression

$$\Omega_{\text{CS}}^{(7)} = 96\, \frac{n_1\, n_2\, n_3}{\rho\sigma\tau}\, c_1\, c_2\, c_3 \cdot \det \begin{vmatrix} a & b & c_1 & c_2 & c_3 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} & c_{3,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} & c_{3,\sigma} \\ a_{,\tau} & b_{,\tau} & c_{1,\tau} & c_{2,\tau} & c_{3,\tau} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} & c_{3,z} \end{vmatrix}\,. \tag{22}$$

which when varied subject to the constraint $a^2 + b^2 + c_1^2 + c_2^2 + c_3^2 = 1$ turns out to be "essentially total divergence". Again, using the polar parametrisation in (20), this becomes a total divergence.

Requiring the field configurations in question have the asymptotic values

$$\lim_{r \to \infty} g = \theta_1 \quad , \quad \lim_{r \to \infty} h = \theta_2 \quad , \quad \lim_{r \to \infty} \alpha = m\,\pi\,, \tag{23}$$

where $0 \le \theta_1 \le \frac{\pi}{2}$ and $0 \le \theta_2 \le \frac{\pi}{2}$ are the polar angles of the octant sphere (ρ, σ, τ), one finds the topological charge

$$Q_{\text{CS}}^{(7)} = 60\, i\, n_1\, n_2\, n_3\, m\, \pi^4\,. \tag{24}$$

From the results (12), (17) and (22) for $n = 1$, $n = 2$ and $n = 3$, the form of the corresponding result for arbitary n follows by induction. The appropriate imposition of symmetry is the application of azimuthal symmetry in each of the n planes in $\mathbb{R}^{2n+1}$. The resulting reduced subsystem will now be an $n + 1$ dimensional system of PDE's, parametrised by $n + 2$ functions

$$a\,, \; b\,, \; c_1\,, \; c_2\,, \ldots, c_n$$

only $n+1$ of which are independent, subject to the sigma model constraint

$$a^2 + b^2 + c_1^2 + c_2^2 + \cdots + c_{n-1}^2 + c_n^2 = 1 \,. \tag{25}$$

The reduced Chern-Simons density will then take the form

$$\Omega_{\text{CS}}^{(2n+1)} \simeq \frac{n_1 \, n_2 \ldots n_n}{\rho_1 \rho_2 \ldots \rho_n} \, c_1 \, c_2 \, \ldots c_n \cdot \det \begin{vmatrix} a & b & c_1 & c_2 & . & . & c_n \\ a_{,\rho_1} & b_{,\rho_1} & c_{1,\rho_1} & c_{2,\rho_1} & . & . & c_{n,\rho_1} \\ a_{,\rho_2} & b_{,\rho_2} & c_{1,\rho_2} & c_{2,\rho_2} & . & . & c_{n,\rho_2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{,\rho_n} & b_{,\rho_n} & c_{1,\rho_n} & c_{2,\rho_n} & . & . & c_{n,\rho_n} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} & . & . & c_{,z} \end{vmatrix} \,, \tag{26}$$

where of course $z = x_{n+1}$.

Again by induction, it follows that the charges (14), (19) and (24) for $n = 1$, $n = 2$ and $n = 3$, that

$$Q_{\text{CS}}^{(2n+1)} \simeq i^{(2n+1)} \, n_1 \, n_2 \ldots n_n \, m \, \pi^{n+1} \,, \tag{27}$$

where $n_1, n_2, \ldots, n_n$ are the vorticities in the n, 2-planes in $\mathbb{R}^{2n+1}$.

3. non-Abelian Hopfions on $\mathbb{R}^{2n+1}$

The description of the arbitrary n case in the non-Abelian case is much less transparent than the Abelian case that preceded. We describe the two cases $n = 1, 2$ in detail, demonstrating that the Chern-Simons densities become "essentially total divergence" when subjected to the appropriate symmetries. But our discussion of the $n = 3$ case is less detailed and is aimed at giving a convincing demonstration of this property, only qualitatively.

What is more, is we shall see (in detail) that the non-Abelian $n = 1$ case reduces to the corresponding Abelian case.

3.1. $2^{n+1} \times 2^n$ Grassmannian models on $\mathbb{R}^{2n+1}$

Here, we start with the definition of models that may support non-Abelian Hopfions on $\mathbb{R}^{2n+1}$. These are Grassmannian sigma models on $\mathbb{R}^{2n+1}$ described by $2^{n+1} \times 2^n$ scalar fields with complex entries

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{28}$$

where z_1 and z_2 are complex $2^n \times 2^n$ matrices subject to the constraint

$$Z^\dagger Z = \mathbb{1}_{2^n \times 2^n} \,. \tag{29}$$

and the $2^{n+1} \times 2^{n+1}$ quantity

$$\Pi = \left(\mathbb{1} - Z \, Z^\dagger \right) \tag{30}$$

is a projection operator.

This constraint is invariant under the action of the local *unitary* non-Abelian gauge transformation g acting on Z

$$Z \to Z \, g \quad , \quad Z^\dagger \to g^\dagger \, Z^\dagger \,. \tag{31}$$

Here, the unitary matrix g is chosen to be an element of $SO(2n+2)$ in the $2^n \times 2^n$ chiral Dirac matrix representation.

The invariance (31) of the constraint (29) leads to the definition of the non-Abelian (anti-Hermitian) *composite* connection

$$B_i = Z^\dagger \partial_i Z \,. \tag{32}$$

transforming like

$$B_i \to g^{-1} B_i \, g + g^{-1} \partial_i \, g \,.$$

There follow the definitions of the covariant derivative of Z and the (composite) non-Abelian curvature

$$D_i Z \;=\; \partial_i Z - Z \, B_i \tag{33}$$

$$G_{ij} \;=\; \partial_{[i} B_{j]} + [B_i, B_j] \tag{34}$$

which under the action of g transform covariantly as

$$D_i Z \;\to\; D_i Z \, g \,, \tag{35}$$

$$G_{ij} \;\to\; g^{-1} G_{ij} \, g \,. \tag{36}$$

Unlike in the Abelian case where the Chern-Simons density (8) can be defined compactly for all $2n+1$ dimensions, in the non-Abelian case the expression for arbitrary n is rather formal and less transparent. For this reason we list here the non-Abelian Chern-Simons densities on $\mathbb{R}^{2n+1}$, up to $n=3$.

These are, for $(n=1\,, D=3)$ and $(n=2\,, D=5)$

$$\Omega_{\mathrm{CS}}^{(1)} = \varepsilon_{ijk} \mathrm{Tr}\, B_k \left(G_{ij} - \frac{2}{3} B_i B_j \right) \,, \tag{37}$$

and

$$\Omega_{\mathrm{CS}}^{(2)} = \varepsilon_{ijklm} \, \mathrm{Tr}\, B_m \left(G_{ij} G_{kl} - G_{ij} B_k B_l + \frac{2}{5} B_i B_j B_k B_l \right) \,, \tag{38}$$

respectively. For $n \geq 3$ there are multiple distinct definitions for the CS density, each characterised by the number of traces in its definition. For $(n=3\,, D=7)$, there are two possibilities; one definition with a single trace and another one with double trace. These are

$$\Omega_{\mathrm{CS}}^{(3)} \;=\; \varepsilon_{ijklmnp} \mathrm{Tr}\, B_p \Big(G_{ij} G_{kl} G_{mn} - \frac{4}{5} G_{ij} G_{kl} B_m B_n - \frac{2}{5} G_{ij} B_k G_{lm} B_n$$

$$+ \frac{4}{5} G_{ij} B_k B_l B_m B_n - \frac{8}{35} B_i B_j B_k B_l B_m B_n \Big) \,, \tag{39}$$

$$\tilde{\Omega}_{\mathrm{CS}}^{(3)} \;=\; \varepsilon_{ijklmnp} \mathrm{Tr}\, B_p \left(G_{mn} - \frac{2}{3} B_m B_n \right) \cdot (\mathrm{Tr}\, G_{ij} G_{kl}) \,. \tag{40}$$

Subjecting these densities to variations of $Z^\dagger$ (or Z)

$$\Omega_{\mathrm{CS}}^{(n)} + \Lambda \, (\mathbb{1} - Z^\dagger Z)$$

Λ being the (now matrix valued) Lagrange multiplier, the resulting *nontrivial* gauge covariant equations in the above examples are

$$\varepsilon_{ijk} \, D_k Z \, G_{ij} \;=\; 0 \tag{41}$$

$$\varepsilon_{ijklm} \, D_m Z \, G_{ij} \, G_{kl} \;=\; 0 \tag{42}$$

$$\varepsilon_{ijklmnp} \, D_p Z \, G_{ij} \, G_{kl} \, G_{mn} \;=\; 0 \tag{43}$$

$$\varepsilon_{ijklmnp} \, (\mathrm{Tr}\, G_{ij} G_{kl}) \cdot D_p Z \, G_{mn} \;=\; 0 \tag{44}$$

which trivialise only under the appropriate symmetries, *i.e.* $\Omega_{\mathrm{CS}}^{(n)}$ become "essentially total divergence", whence they qualify as candidates for topological charge densities.

3.2. Residual non-Abelian CS densities subject to n-fold azimuthal symmetry

As in the Abelian case, we treat the three cases $n = 1, 2, 3$ below by positing suitable Ansätze that impose the appropriate symmetries that qualify the densities (37), (38) and (39)-(40) as topological charge densities. We also use the same notation for the residual coordinates as in the Abelian case.

The $n = 1$ case is special in the sense that the topological charge density in that case trivialises to the corresponding Abelian one.

Before examining each n example in detail, we define the spin matrices employed in stating the respective Ansätze. As in the Abelian case presented above, the symmetries imposed will be the azimuthal symmetries in each of the n planes in $\mathbb{R}^{2n+1}$.

The chiral Dirac representations of the $SO(2n + 2)$ algebra are

$$\Sigma_{ij}^{(+)} = -\frac{1}{4} \Sigma_{[i} \tilde{\Sigma}_{j]} \quad \text{or} \quad \Sigma_{ij}^{(-)} = -\frac{1}{4} \tilde{\Sigma}_{[i} \Sigma_{j]} \tag{45}$$

in terms of the spin matrices

$$\Sigma_i = \frac{\mathbb{1} + \Gamma_{2n+3}}{2} \Gamma_i \quad \text{and} \quad \tilde{\Sigma}_i = \frac{\mathbb{1} + \Gamma_{2n+3}}{2} \Gamma_i \,,$$

Γ_{2n+3} being the chiral matrix in $2n + 2$ dimensions.

In the $n = 1$ case, the field (28) is subjected to axial symmetry in the (x_1, x_2) plane of $\mathbb{R}^3$,

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a\,\mathbb{1} + 2\,b\,\Sigma_{34} \\ c\,n^\alpha\,\tilde{\Sigma}_\alpha \end{bmatrix} \,, \tag{46}$$

the functions (a, b, c) depending on (ρ, z) as in the Abelian case and with the unit vector n^α

$$n^\alpha = \begin{pmatrix} \cos n\,\varphi \\ \sin n\,\varphi \end{pmatrix} \tag{47}$$

n being the winding (vortex) number in the (x_1, x_2) plane.

It turns out that the Ansatz (46) leads to an Abelian composite connection $B_i = (B_\alpha, B_z)$, so our prescription cannot supply a non-Abelian Hopfion in three dimensions.

$$\begin{aligned} B_\alpha &= 2\left[(a\,b_{,\rho} - b\,a_{,\rho})\,\hat{x}_\alpha + \frac{n}{\rho}\,c^2\,(\hat{x}\varepsilon)_\alpha\right]\Sigma_{12} \,, \\ B_z &= 2\,(a\,b_{,z} - b\,a_{,z})\,\Sigma_{12} \,, \end{aligned}$$

whose commutators

$$[B_i, B_j] = [B_i, B_z] = 0.$$

The composite curvature G_{ij} is then Abelian, and coincides with the previously constructed Abelian case.

In the $n = 2$ case, the field (28) is subjected to azimuthal symmetry in the (x_1, x_2) and (x_3, x_4) planes of $\mathbb{R}^5$,

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a\,\mathbb{1} + 2\,b\,\Sigma_{56} \\ c_1\,n_1^\alpha\,\tilde{\Sigma}_\alpha + c_2\,n_2^A\,\tilde{\Sigma}_A \end{bmatrix} \,, \tag{48}$$

the functions (a, b, c_1, c_2) depending on (ρ, σ, z) as in the corresponding Abelian case, with $\alpha = 1, 2$, $A = 3, 4$ and $z \equiv x_5$. φ and χ are the azimuthal angles in the (x_1, x_2) and (x_3, x_4) planes respectively, such that the unit vectors n^α and n_2^A are parametrised as

$$n_1^\alpha = \begin{pmatrix} \cos n_1\,\varphi \\ \sin n_1\,\varphi \end{pmatrix} \quad n_2^A = \begin{pmatrix} \cos n_2\,\chi \\ \sin n_2\,\chi \end{pmatrix} \,, \tag{49}$$

(n_1, n_2) being the winding (vortex) numbers in each of the two planes respectively.

Substitution of (48) in the CS density (38) yields

$$\Omega_{\text{CS}}^{(5)} = 96\, i\, \frac{n_1 n_2}{\rho \sigma}\, c_1\, c_2\, [-4 + 3(c_1^2 + c_2^2)] \cdot \det \begin{vmatrix} a & b & c_1 & c_2 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} \end{vmatrix}. \tag{50}$$

Note that this non-Abelian density (50) differs qualitatively from the corresponding Abelian one (17) due to the appearance of the prefactor

$$[-4 + 3(c_1^2 + c_2^2)].$$

When (50) is varied subject to the constraint $a^2 + b^2 + c_1^2 + c_2^2 = 1$, it turns out to be "essentially total divergence". using the polar parametrisation in (15), this becomes a total divergence.

Again, requiring the field configurations in question have the asymptotic values (18) as in the Abelian case, one finds the topological charge

$$Q_{\text{CS}}^{(5)} = -12\, n_1\, n_2\, m\, \pi^3.$$

4. Summary

The scope of this talk is restricted to the consideration of the topological charges of possible Hopfions in $2n + 1$ dimensions. Since the topological charges are the volume integrals of Chern-Simons densities in the appropriate dimension. The latter however are not total divergence, and to qualify as topological charge densities they must be subjected to the appropriate symmetries that render them total divergences. This is the main technical task carried out.

The prescription used to achieve this is the imposition of azimuthal (axial) symmetry in each of the n, 2-dimensional planes inside $\mathbb{R}^{2n+1}$.

The Chern-Simons densities in question are defined my means of the composite connections of the complex sigma model appropriate to the dimension. Composite connections can be both Abelian and non-Abelian, depending on the choice of sigma model. Here, the Abelian connections employed in $2n + 1$ dimensions are those of the $\mathbb{C}P^n$ models, while the non-Abelian ones pertain to $2^{n+1} \times 2^n$ (complex) Grassmannian sigma models. We have hence used the nomenclature of Abelian and non-Abelian Hopfions in the title.

In the Abelian case, it has been possible to carry out this task for the case of arbitrary n, by induction. In the non-Abelian case, our consideration here have been restricted to $n = 2$, *i.e.* $\mathbb{R}^5$, due to technical complexity. It turns out also, that $n = 2$, *i.e.* 5, is the lowest dimension that can admit a non-Abelian Hopfion topological charge.

Post Script The above invoked criterion, namely that the Chern-Simons density reduces to total divergence, eliminates the possibility of $U(1)$ gauging of the CP^1 Hopfion on $\mathbb{R}^3$.

The Covariant derivative of the $U(1)$ gauged CP^1 field is

$$D_i Z = \partial_i Z + i(B_i \mathbb{1} + A_i \sigma^3) Z$$

where A_i is the Maxwell connection, and B_i the composite connection (5). This extended connection and its curvature are

$$B_i \mathbb{1} + A_i \sigma^3$$
$$G_{ij} \mathbb{1} + F_{ij} \sigma^3 \,, \quad F_{ij} = \partial_{[i} A_{j]} \,.$$

The resulting Chern-Simons density is

$$\tilde{\Omega}_{\text{CS}} = \Omega_{\text{CS}}^{(3)}[B_i] + \varepsilon_{ijk}\, A_i\, F_{ij}\,, \tag{51}$$

where $\Omega_{\text{CS}}^{(3)}$ is the CS density for the composite connection B_i, (11).

While $\Omega_{\text{CS}}^{(3)}$ in (51) reduces to a total divergence when subjected to symmetry, the second term in (51) is does not. Consequently, the CS density of the $U(1)$ gauged CP^1 model does not qualify as a topological charge density of that (putative) Hopfion.

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